9.23.) a)

Note: assuming θ^2 is variance

MLE of θ^2:

$$L(\theta,\theta) = (2\pi\theta^2)^{-n/2} \exp\left[-\frac{1}{2\theta^2}\sum_{i=1}^{n}(X_i)^2\right]$$

$$l(\theta,\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta^2) - \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$$

$$\frac{dl}{d\theta^2} = -\frac{n}{2\theta^2} - \left[\frac{1}{2}\left(\sum_{i=1}^{n}(X_i)^2\right)\left(\frac{-1}{(\theta^2)^2}\right)\right]$$

$$= \frac{1}{2\theta^2}\left[\frac{1}{\theta^2}\sum_{i=1}^{n} X_i^2 - n\right] = 0$$

$$\Rightarrow \hat{\theta}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Is $\hat{\theta}_{MLE}$ biased?

$$E[\hat{\theta}^2] = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \frac{1}{n}\sum_{i=1}^{n} Var(X_i) + E(X_i)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} Var(X_i) + 0^2 = \frac{n\theta^2}{n} = \theta^2$$

Thus $\hat{\theta}_{MLE}$ is unbiased.

9.28) b.) Let $\sigma^2 = \theta$

$$\ell(\theta) = -\frac{1}{2}\ln\theta - \frac{\sum_{i=1}^{n} x_i^2}{2\theta}$$

$$\ell'(\theta) = -\frac{1}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2}$$

$$I(\theta) = -E\left[\ell''(\theta)\right] = -E\left[\frac{1}{2\theta^2} - \frac{\sum_{i=1}^{n} x_i^2}{\theta^3}\right]$$

$$= \frac{n}{2\theta^2} = \frac{n}{2\sigma^4}$$

CRLB: $Var(T) \geq \frac{1}{n \cdot n/2\sigma^4} = \frac{2\sigma^4}{n^2}$

$$Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = \cancel{\sum_{i=1}^{n} E(X_i^2)\,\theta^2} =$$

$$= \frac{1}{n^2} 2 \cdot \theta^2 = \frac{2\theta^2}{n^2}$$

Therefore it achieves the CRLB and is unbiased, thus it is a UMVUE

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1.) BE.) 9.21.)

a.) $CRLB = \frac{[\tau'(p)]^2}{n I(p)}$ $\quad \tau'(p) = 1$

$I(p) = \frac{d}{dp^2} \ln f(x_i, p) = -E\left[\frac{d}{dp^2} \ln\left(\frac{1}{p}\right) p^x (1-p)^{n-x}\right]$

$= -E\left\{\frac{d}{dp^2}\left[\ln\left(\frac{1}{p}\right) + x\ln(p) + (1-x)\ln(1-p)\right]\right\}$

$= -E\left\{\frac{d}{dp}\left[\frac{x}{p} - \frac{(1-x)}{1-p}\right]\right\}$

$= -E\left[-\frac{X}{p^2} - \frac{1-X}{(1-p)^2}\right]$

$= \frac{1}{p(1-p)}$

$\Rightarrow CRLB = \frac{1}{n\left(\frac{1}{p(1-p)}\right)} = \frac{p(1-p)}{n}$

9.21.) b.) $CRLB = \frac{[\tau'(p(1-p))]^2}{nI(p)}$

again, $I(p) = \frac{1}{p(1-p)}$

$$\tau'(p(1-p)) = \frac{d}{dp}(p-p^2) = 1-2p$$

$$\Rightarrow CRLB = \frac{(1-2p)^2}{n\left(\frac{1}{(1-p)p}\right)} = \frac{p(1-p)(1-2p)^2}{n}$$

c.) Consider the sample mean $\bar{X}$

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

$= \frac{1}{n} \cdot np = p$, Thus $\bar{X}$ is unbiased

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2} \cdot np(1-p) = \frac{p(1-p)}{n}$$

Thus $\bar{X}$ obtains the CRLB, thus it is a UMVUE

9.26.) a.) $L(X,\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$

$\ell(X,\theta) = -n\log(\theta)$

$\frac{d\ell}{d\theta} = \frac{-n}{\theta} = 0$

since the derivative is $\frac{-n}{\theta}$, and $\ell(X,\theta) = -n\log(\theta)$, which $L(X,\theta) = \frac{1}{\theta^n}$, this likelihood is maximized by $X_{n:n}$, the maximum statistic.

b.) MME

$E(X) = \int_0^\theta x\frac{1}{\theta}dx = \frac{\theta}{2}$

$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\theta}{2} \Rightarrow \hat{\theta}_{MME} = 2\bar{X}$

9.26.) c.) The pdf of the maximum is

$$F_{X_{min}}(x) = \left(\frac{x}{\theta}\right)^n$$

$$f_{X_{min}}(x) = \frac{1}{\theta^n} n x^{n-1}$$

$$E(X) = \int_0^\theta \frac{x}{\theta^n} n x^{n-1} dx$$

$$= \int_0^\theta \frac{n}{\theta^n} x^n dx = \frac{n}{n+1} \left(\frac{x}{\theta}\right)^{n+1} \cdot \frac{1}{\theta^n} \Big|_0^\theta$$

$$= \frac{n}{n+1} \theta$$

Note that $\frac{n}{n+1}\theta \neq \theta$, thus X_{min} is biased

9.26.) d.) $E[2\bar{X}_n] = \frac{2}{n} E\left[\sum_{i=1}^{n} X_i\right]$

$$= \frac{2}{n} \sum_{i=1}^{n} E[X_i] = \frac{2}{n} n \cdot \frac{\theta}{2}$$

$$= \theta$$

Thus $\tilde{\theta}$ is unbiased.

9.26.) e.) The MSE of $\hat{\theta}$,

$E((X_{min} - \theta)^2)$ will be larger than the MSE of $\tilde{\theta}$

$E((2\bar{X}_n - \theta)^2)$, this is because $\hat{\theta}$ is biased and $\tilde{\theta}$ is unbiased

9.31) a) $\hat{\theta} = X_{min}$

$\lim_{n\to\infty} E(X_{min}) = \lim_{n\to\infty} \frac{n}{n+1}\theta = \theta$, thus by Defn. 9.4.3 $\hat{\theta}$ is asymptotically unbiased

$Var(X_{min}) = E(X_{min}^2) - E(X_{min})^2$

$E(X_{min}^2) = \int_0^\theta \frac{n}{\theta^n} x^{n+1} dx = \frac{n}{n+2}(x)^{n+2} \cdot \frac{1}{\theta^n}\Big|_0^\theta$

$= \frac{n}{n+2}\theta^2$

$\lim_{n\to\infty} Var(X_{min}) = \lim_{n\to\infty} \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = 0$

Thus by Theorem 9.4.1, since it is asym. unbiased and $Var(X_{min}) \to 0$ as $n \to \infty$, $\hat{\theta}$ is MSE consistent

9.3 1.) b.) $\lim_{n\to\infty} E(2\bar{X}_n) = \lim_{n\to\infty} \theta = \theta$

Thus by definition 9.4.3 $\tilde{\theta}$ is asymptotically unbiased

Since $X_i \sim \text{Unif}(0, \theta)$,

$$Var(X_i) = \frac{\theta^2}{12}$$

Thus $Var(2\bar{X}_n) = 4\,Var(\bar{X}_n) = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}$

$\lim_{n\to\infty} Var(\tilde{\theta}) = \lim_{n\to\infty} \frac{\theta^2}{3n} = 0$

Thus by Theorem 9.4.1 since it is asym. unbiased and $Var(2\bar{X}_n) \to 0$ as $n \to \infty$, $\tilde{\theta}$ is MSE consistent.

9.38) $X_i \sim BIN(1, p)$

MLE of p

$$L(p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$\ell(p) = \log p \sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)$$

$$\frac{d\ell(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} \stackrel{set}{=} 0$$

$$\Rightarrow \hat{p} = \sum_{i=1}^{n} x_i / n = \bar{X}_n$$

$$\frac{d^2\ell(p)}{dp^2} = -\frac{\sum_{i=1}^{n} x_i}{p^2} - \frac{\sum_{i=1}^{n}(1-x_i)}{(1-p)^2} < 0$$

Thus the MLE of p is $\hat{p} = \bar{X}_n$

By the CLT, $\bar{X}_n \xrightarrow{d} N\left(p, \frac{p(1-p)}{n}\right)$

2.) a.) pdf of $GAM(\theta, 2)$: $\frac{1}{\theta^2 \cdot 2} x e^{-x/\theta}$

$$L(x|\alpha,\beta) = \frac{1}{\theta^{2n} \cdot 2^n} \prod_{i=1}^{n} x_i \, e^{-\sum_{i=1}^{n} x_i / \theta}$$

$$\ell(x,\alpha,\beta) = \sum_{i=1}^{n} \log(x_i) - n\log(2) - 2n\log(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$\frac{d\ell}{d\theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \stackrel{set}{=} 0$$

$$\Rightarrow \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = \frac{2n}{\theta}$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{2n}$$

b.) Note that $\frac{\sum_{i=1}^{n} x_i}{2n} = \frac{\bar{X}_n}{2}$

By the WLLN, $\bar{X}_n \xrightarrow{P} E(X_i) = 2\theta$

By the CMT, $\frac{\bar{X}_n}{2} \Rightarrow \frac{2\theta}{2} = \theta$

$\Rightarrow$ The MLE of θ simply consistent

3) a.) It is not appropriate to compare performance by computing relative efficiency, since the MLE estimator is biased.

b.) Since both the MME and MLE of θ are MSE consistent as shown in 9.3.1., they are both simply consistent, and by 9.4.1 they are asymptotically unbiased. Therefore we can use asymptotic relative efficiency

$$are(T_n, T_n') = are(\hat{\theta}, \tilde{\theta})$$

$$= \lim_{n \to \infty} \frac{V(\tilde{\theta})}{V(\hat{\theta})} = \theta^2 \left(\frac{n}{(n+2)(n+1)^2} \right) \cdot \frac{3n}{\theta^2}$$

$$= 0$$

Thus $\tilde{\theta}$ is asymptotically efficient

c.) We wouldn't want to use either metric for finite-sample performance, as we can't use relative efficiency and the asymptotic relative efficiency requires taking the limit. Instead, we could evaluate the MSE for $\hat{\theta}$ and $\tilde{\theta}$ and see which estimator has the smaller MSE

d.) We can use the asymptotic relative efficiency to compare the large-sample performance of the estimators.

4.) a) MLE of p

$$L(p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$\ell(p) = \log p \sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)$$

$$\frac{d\ell(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} \stackrel{set}{=} 0$$

$$\Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\frac{d^2\ell(p)}{dp^2} = -\frac{\sum_{i=1}^{n} x_i}{p^2} - \frac{\sum_{i=1}^{n}(1-x_i)}{(1-p)^2} < 0$$

Thus the MLE of p is $\hat{p} = \sum_{i=1}^{n} x_i / n$

By the Invariance property of MLEs, the MLE of $p(1-p)$ is

$$\frac{\sum_{i=1}^{n} x_i}{n}\left(1 - \frac{\sum_{i=1}^{n} x_i}{n}\right)$$

4b.) Note that the MLE of $p(1-p)$ is $\bar{X}_n - \bar{X}_n^2$

$$E(\bar{X}_n - \bar{X}_n^2) = E(\bar{X}_n) - E(\bar{X}_n^2)$$

$$= p - (Var(\bar{X}_n) + (E(\bar{X}_n))^2)$$

$$= p - \left(\frac{p(1-p)}{n} + p^2\right)$$

$$= p - \left(\frac{p - p^2 + np^2}{n}\right)$$

$$= \frac{2p^2 - p^2 + np^2}{n} \neq p(1-p)$$

Thus The MLE found in part (a) is biased, and so it cannot be an UMVUE

4c.) No, this does not ~~contrapt~~ contradict the result on 3(b), since the result in 4b is biased, it cannot be the UMVUE. This violates our assumptions.