Solving the Poisson equation with SOR (successive over-relaxation):

Given the Poisson equation,

$\nabla^2 \psi = \zeta$

which can be expressed by

$$\left[\frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2}\right] = \zeta_{i,j}$$

Let the residual defined as

$$\mathrm{R}_{i,j} = \left[\frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{(\Delta y)^2}\right] - \zeta_{i,j}$$

Thus,

$$(\Delta y)^2(\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}) + (\Delta x)^2(\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}) = (\Delta x)^2(\Delta y)^2\zeta_{i,j}$$

$$\psi_{i,j}\{-2[(\Delta x)^2 + (\Delta y)^2]\} + (\Delta x)^2\left(\psi_{i,j+1} + \psi_{i,j-1}\right) + (\Delta y)^2(\psi_{i+1,j} + \psi_{i-1,j}) \\ = (\Delta x)^2(\Delta y)^2\zeta_{i,j}$$

$$\psi_{i,j} + \frac{(\Delta x)^2\left(\psi_{i,j+1} + \psi_{i,j-1}\right) + (\Delta y)^2(\psi_{i+1,j} + \psi_{i-1,j})}{\{-2[(\Delta x)^2 + (\Delta y)^2]\}} = \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}}{\{-2[(\Delta x)^2 + (\Delta y)^2]\}}$$

The solution can be obtained by the iteration method with the updated vorticity $\zeta_{i,j}^{n+1}$ by rewriting it as an iterative form

$$\psi_{i,j}^{(m+1)} - \psi_{i,j}^{(m)} + \psi_{i,j}^{(m)} + \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)}\right) + (\Delta y)^2(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)})}{\{-2[(\Delta x)^2 + (\Delta y)^2]\}} \\ = \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{\{-2[(\Delta x)^2 + (\Delta y)^2]\}}$$

where *m* indicates the *m*th iteration and $\psi_{i,j}^{(0)} = \psi_{i,j}^{n}$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)}\right) + (\Delta y)^2(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)})}{2[(\Delta x)^2 + (\Delta y)^2]} - \psi_{i,j}^{(m)} \\ + \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{\{-2[(\Delta x)^2 + (\Delta y)^2]\}}$$

$$\psi_{i,j}^{(m+1)}$$

$$= \psi_{i,j}^{(m)}$$

$$+ \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)}\right) + (\Delta y)^2(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)}) - 2[(\Delta x)^2 + (\Delta y)^2]\psi_{i,j}^{(m)} - (\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2 + (\Delta y)^2]}$$

With the ω factor to accelerate the convergence rate

$$\psi_{i,j}^{(m+1)}$$

$$= \psi_{i,j}^{(m)}$$

$$+ \omega\frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)}\right) + (\Delta y)^2(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)}) - 2[(\Delta x)^2 + (\Delta y)^2]\psi_{i,j}^{(m)} - (\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2 + (\Delta y)^2]}$$

i.e.,

$$\psi_{i,j}^{(m+1)}$$

$$= \psi_{i,j}^{(m)}$$

$$+ \omega\frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} - 2\psi_{i,j}^{(m)}\right) + (\Delta y)^2(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 2\psi_{i,j}^{(m)}) - (\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2 + (\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)}$$

$$= \psi_{i,j}^{(m)}$$

$$+ \omega\frac{\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} - 2\psi_{i,j}^{(m)}\right)/(\Delta y)^2 + (\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 2\psi_{i,j}^{(m)})/(\Delta x)^2 - \zeta_{i,j}^{n+1}}{2[(\Delta x)^2 + (\Delta y)^2]/[(\Delta x)^2(\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)}$$

$$= \psi_{i,j}^{(m)}$$

$$+ \omega\frac{(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 2\psi_{i,j}^{(m)})/(\Delta x)^2 + \left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} - 2\psi_{i,j}^{(m)}\right)/(\Delta y)^2 - \zeta_{i,j}^{n+1}}{2[(\Delta x)^2 + (\Delta y)^2]/[(\Delta x)^2(\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega \frac{[(\Delta x)^2 (\Delta y)^2]}{2[(\Delta x)^2 + (\Delta y)^2]} \mathrm{R}_{i,j}^{(m)}$$

where

$$\mathrm{R}_{i,j}^{(m)} = \frac{(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 2\psi_{i,j}^{(m)})}{(\Delta x)^2} + \frac{\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} - 2\psi_{i,j}^{(m)}\right)}{(\Delta y)^2} - \zeta_{i,j}^{n+1}$$

For $\Delta x = \Delta y = \Delta$,

$$\psi_{i,j}^{(m+1)}$$
$$= \psi_{i,j}^{(m)} + \omega \frac{\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} + \psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 4\psi_{i,j}^{(m)} - \Delta^2 \zeta_{i,j}^{n+1}\right)}{4} \qquad (1)$$

1) Solving the Poisson equation requires and the updated vorticity as $\zeta_{i,j}^{n+1}$ and the conditions on the lateral boundaries given (as a choice) by

$\mathrm{u} = -\frac{\partial \psi}{\partial \mathrm{y}}$ on the northern and southern boundaries, and

$\mathrm{v} = \frac{\partial \psi}{\partial \mathrm{x}}$ on the western and eastern boundaries; both can be expressed by

$\mathrm{u}_{i,b}^{n} = -\frac{\psi_{i,b+1}^{(m)} - \psi_{i,b-1}^{(m)}}{2\Delta y}$ $\qquad (b = 1, ny)$ $\qquad$ (2a) $\qquad$ and

$v_{b,j}^{n} = -\frac{\psi_{b+1,j}^{(m)} - \psi_{b-1}^{(m)}}{2\Delta x}$ $\qquad (b = 1, nx)$ $\qquad$ (2b).

Substituting (2a) and (2b) for the undefined grid values in (1).

2) The iterations will proceed until the residual reaches the criterion or the maximum relative change in $\psi_{i,j}^{(m)}$ is less than a very small value such that

$$\max[(\psi^{(m+1)} - \psi^{(m)})/\overline{\psi}^{(m)}] < \varepsilon$$

where ε can be 0.1% or smaller.
Successive over-relaxation ($\omega > 1$) under-relaxation ($\omega < 1$) can be applied to faster the convergence rate.

Note that with the given Neumann boundary conditions for the Poisson equation, the solution if existing plus any constant is also a solution. Thus, there is no unique solution! How can we do under such a situation?

3) Finally, the updated streamfunction is given by

$$\psi_{i,j}^{n+1} = \psi_{i,j}^{(m+1)}$$

and the updated velocity is given by

$$u_{i,j}^{n+1} = -\frac{\left(\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}\right)}{2\Delta y}$$ and

$$v_{i,j}^{n+1} = \frac{\left(\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}\right)}{2\Delta x}$$