MATH 138 LECTURE 3

1 Properties of Definite Integrals

Suppose f and g are integrable on $[a, b]$.

1. $\int_a^b f(x)dx = 0$ (from the definition of the integral)

2. Let c be a real number. Then the function cf is integrable on $[a, b]$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

3. $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

4. Max-min Inequality:
If $m <= f(x) <= M$ for every x in $[a, b]$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

5. Domination:
If $f(x) \geq g(x)$ for every x in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
In particular, if $f(x) \geq 0$ for every x in $[a, b]$, then $\int_a^b f(x)dx \geq 0$.

6. The function $|f|$ is integrable and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

7. Order of Integration:
$\int_a^b f(x)dx = -\int_a^b f(x)dx$ (by definition)

8. Additivity:
Suppose I is an interval containing a, b, and c.
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Suppose c lies outside $[a, b]$. By additivity,

$$
\begin{aligned}
\int_a^c f(x)dx &= \int_a^b f(x)dx + \int_b^c f(x)dx \\
\implies \int_a^b f(x)dx &= \int_a^c f(x)dx - \int_b^c f(x)dx \\
&= \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}
$$

Proof of the Max-min Inequality
Let $P = \{t_0, ..., t_n\}$ be a partition of $[a, b]$. For $i = 1, 2, ..., k$, let c_i be a point in $[t_{i-1}, t_i]$. Then,

$$m\Delta t_i \leq f(c_i)\Delta t_i \leq M\Delta t_i$$

$$\sum_{i=1}^{n} m\Delta t_i \leq \sum_{i=1}^{n} f(c_i)\Delta t_i \leq \sum_{i=1}^{n} M\Delta t_i$$

$$\sum_{i=1}^{n} M\Delta t_i = M\sum_{i=1}^{n} \Delta t_i = M(b-a)$$

Similarly, $\sum_{i=1}^{n} n\Delta t_i = m(b-a)$.

All Riemann sums satisfy $m(b-a) \leq \sum_{i=1}^{n} f(c_i)\Delta t_i \leq M(b-a)$.

Hence the limit, that is the integral, satisfies $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Proof of Property 6
Recall that $|c| \leq c \leq |c|$. Then $-|f(x)| \leq f(x) \leq |f(x)|$.

Thus, $-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.

EXERCISE: Suppose f is continuous on $[-1, 7]$. If $\int_{-1}^{1} f(x)dx = 0$ and $\int_{-1}^{7} f(x)dx = 5$. Compute $\int_7^1 f(x)dx$.

If $f(x) \geq 0$ for every x in $[a, b]$, then the integral $\int_a^b f(x)dx$ is equal to the region below the graph of $y = f(x)$ and above the x-axis, between $x = a$ and $x = b$.

Suppose $f(x) \leq 0$ for each x in $[a, b]$, then

$\int_a^b f(x)dx$ is the negative of the area of the region above the graph of $y = f(x)$ and below the x-axis, between $x = a$ and $x = b$.

In other words, $\int_a^b f(x)dx$ is the area of the region under the graph of f, above the x-axis, between $x = a$ and $x = b$ subtract the area of the graph of f, below the x-axis, between $x = a$ and $x = b$.

2 Average Value of f

Suppose f is continuous on $[a, b]$. Then, the average value of f is defined to be

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Average Value Theorem (Mean Value Theorem for Definite Integrals)
Suppose f is continuous on $[a, b]$. Then there exists a point c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

EXERCISE: Suppose f is continuous on $[a, b]$ where $a \neq b$. If $\int_a^b f(x)dx = 0$, then prove that there is a point c in $[a, b]$ such that $f(c) = 0$.

3 Summary

Properties of the Definite Integral:
They have similar properties to the Riemann sums.

Important Properties:
1. Domination: If $f(x) \geq g(x)$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

2. Squeeze Theorem: $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

3. Average Value: If f is continuous on $[a, b]$, then the average value of f is defined to be $\frac{1}{b-a}\int_a^b f(x)dx$. This is the mean value theorem for integrals.