MATH 138 LECTURE 2

1 Riemann Sums

Recall: Summation Notation
Let a_n and b_n be sequences. Then,

$$\sum_{i=1}^{n} a_i = a_1 + ... + a_n \tag{1}$$

$$\sum_{i=1}^{n} Ra_i = R\sum_{i=1}^{n} a_i \text{ where } R \in \mathbb{R} \tag{2}$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{3}$$

Important Summations

$$\begin{aligned}\sum_{i=1}^{n} i &= 1 + 2 + 3 + ... + n \\ &= \frac{n(n+1)}{2}\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= 1^2 + 2^2 + ... + n^2 \\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

2 Definitions

Recall:

$P = \{t_0, ..., t_n\}$ is a partition of $[a, b]$. $\Delta t_i = t_i - t_{i-1}$
$|P| = max(\Delta t_1, ..., \Delta t_n)$

If $p^{(n)}$ is the regular n-partition of $[a,b]$, then $|p^{(n)}| = \dfrac{b-a}{n}$. For $i = 1, 2, ..., n$. Select a point c_i in $[t_{i-1}, t_i]$.Construct a rectangle from $[t_{i-1}, t_i]$ to the point $(c_i, f(c_i))$.

Riemann Sum of f With Respect to p:
The sum S $= \sum_{i=1}^{n} f(c_i)\Delta t_i$ is called a **Riemann sum of f with respect to** p.

Right-hand Riemann Sum of f with Respect to p:
If $c_i = t_i$, then $\sum_{i=i}^{n} f(c_i)\Delta t_i$ is called a **Right-hand Riemann Sum of f with Respect to** p. The right-hand Riemann sum of f with respect to $P^{(n)}$ is denoted by R_n.
$R_n = \sum_{i=1}^{n} f(c_i)\Delta t_i = \sum_{i=1}^{n} f(t_i)\dfrac{b-a}{n} = \sum_{i=1}^{n} f(a + \dfrac{i(b-a)}{n})\dfrac{b-a}{n}$

Example 1: $f(x) = 2x^2 + 1, a = 0, b = 6$. Compute R_n:

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(\frac{6i}{n})\frac{6}{n} \\
&= \frac{6}{n}\sum_{i=1}^{n} f(\frac{6i}{n}) \\
&= \frac{6}{n}\sum_{i=1}^{n}(2(\frac{6i}{n})^2 + 1) \\
&= \frac{6}{n}\sum_{i=1}^{n}(\frac{72i^2}{n^2} + 1) \\
&= \frac{6}{n}(\frac{72}{n^2}\sum_{i=1}^{n} i^2 + n) \\
&= \frac{6}{n}(\frac{72}{n^2}(\frac{n(n+1)(2n+1)}{6}) + n)
\end{aligned}$$

Therefore, $R_n = \dfrac{72(n+1)(2n+1)}{n^2} + 6$. We can also see that:
$\lim_{n\to\infty} R_n = 150$.

Left-hand Riemann Sum of f with Respect to p:
If $c_i = t_{i-1}$, then $L = \sum_{i=1}^{n} f(c_i)\Delta t_i$ is called the **Left-hand Riemann Sum of f with Respect to** p.

$L_n = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i = \sum_{i=1}^{n} f(a + \dfrac{(i-1)(b-a)}{n})\dfrac{b-a}{n}$.

EXERCISE Example 2: Compute L_n for $f(x) = 2x^2 + 1, a = 0, b = 6$ and check that $\lim_{n\to\infty} L_n = 150$.

Integrability on $[a, b]$:
We say that a bounded function f is **integrable on** $[a, b]$ if there exists a real number I such that the following property holds.

If P_i is a sequence of partitions of$[a, b]$ with $\lim_{i\to\infty} |P_i| = 0$ and S_i is a sequence of Riemann sums with respect to P_i, then $\lim_{i\to\infty} S_i = I$.

I is called the **integral of f from a to b**.

Notation:

$\int_a^b f(x)dx$

a and b are called the limits of integration.
f is called the integrand.
x is called the variable of integration.

THEOREM: Integrability Theorem for Continuous Functions
If f is a continuous function on $[a, b]$, then f is integrable on $[a, b]$.
$\int_a^b f(x)dx = \lim_{n\to\infty} S_n$, where S_n is any Riemann sum associated to $P^{(n)}$.

In particular, $\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$.

From the integrability theorem for continuous functions, we can clearly see that $\int_0^6 2x^2 + 1dx = 150$.

3 Summary

The definite integral by using Riemann sums invluve the splitting of the interval into n partitions and letting $n \to \infty$.

As n gets bigger, the rectangles get smaller and we approach a limit of the true area.

This can be applied only if the function is continuous or has a finite amount of discontinuities (refer to the Integrability Theorem for Continuous Functions).