MATH 138 LECTURE 4

1 Fundamental Theorem of Calculus (Part 1):

Suppose f is continuous on an interval I containing a point a. For each x in I, let

$$F(x) = \int_a^x f(t)dt$$

Then, F is differentiable on I and $F'(x) = f(x)$ or $\frac{d}{dx}(\int_a^x f(t)dt) = f(x)$.

Proof:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ \frac{F(x+h)-F(x)}{h} &= \frac{1}{h}(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt) \\ &= \frac{1}{h}(\int_a^{x+h} f(t)dt + \int_x^a f(t)dt) \\ &= \frac{1}{h}(\int_x^a f(t)dt + \int_a^{x+h} f(t)dt) \\ &= \frac{1}{h}\int_x^{x+h} f(t)dt \text{ (by additivity)} \end{aligned}$$

This is just the average value of f over $[x, x+h]$.

By the average value theorem, there is a point c in $[x, x+h]$ such that $f(c) = \frac{F(x+h)-F(x)}{h}$. As $h \to 0$, then $c \to x$ and $f(c) \to f(x)$. Thus,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$$

Example 1: Find $\dfrac{d}{dx}(\int_0^x sin(t)dt)$.

$$\begin{aligned}\frac{d}{dx}(\int_0^x sin(t)dt) &= \frac{d}{dx}(\int_a^x f(t)dt) \\ &= f(x) \\ &= sin(x)\end{aligned}$$

Example 2: Find $\dfrac{d}{dx}(\int_0^{x^2} sin(t)dt)$.

Let $u = x^2$. Then,

$$\begin{aligned}y(u) &= \int_0^u f(t)dt \\ &= \int_0^u sin(t)dt\end{aligned}$$

Additionally,

$$\begin{aligned}y(x) &= \int_0^{x^2} \\ &= y(u(x))\end{aligned}$$

We can see that $y(u) = \int_0^u sin(t)dt$.

By the fundamental theorem of calculus, $\dfrac{dy}{du} = sin(u)$.

$$\begin{aligned}\frac{dy}{du} &= sin(u) \\ \frac{du}{dx} &= \frac{d}{dx}x^2 = 2x\end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ \frac{dy}{dx} &= sin(u)\frac{du}{dx} \\ &= sin(x^2) \cdot 2x \end{aligned}$$

Therefore, $\frac{d}{dx}\int_0^{x^2} sin(t)dt = 2xsin(x^2)$.

Extended Version of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

Example 3: Compute the derivative. $\frac{d}{dx}\int_0^{\sqrt{x}} sin(t^2)dt$

Let $f(t) = sin(t^2)$, $g(x) = 0$, and $h(x) = \sqrt{x}$. Then $f(h(x)) = sin(x)$.

Thus, $\frac{d}{dx}\int_0^{\sqrt{x}} sin(t^2)dt = sin(x) \cdot \frac{1}{2\sqrt{x}}$.

2 Displacement

If $s(t)$ denotes the displacement of an object at time t and $v(t)$ denotes the velocity, then $\frac{ds}{dt} = v(t)$. Displacement equals the area under the graph of the velocity function.

$$F(x) = \int_a^x f(t)dt, \text{ then } F'(x) = f(x).$$

3 The Antiderivative

Antiderivative: A function G is said to be an antiderivative of a function g if $G'(x) = g(x)$.

Example 4: Find the antiderivative of $cos(x)$.

The antiderivative of $cos(x)$ is $sin(x)$ because $\frac{d}{dx}sin(x) = cos(x)$. Since the antiderivative is not unique, we write that the antiderivative is

$$sin(x) + C, \text{ where } C \text{ is a constant.}$$

If $F(x)$ is an antiderivative of $f(x)$, then every antiderivative of $f(x)$ is of the form $F(x) + C$.

Indefinite Integral: The indefinite integral of $f(x)$ is the collection of all antiderivatives of f.

Notation: $\int f(x)dx$

Function	Indefinite Integral
1	$x + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$ln(\lvert x \rvert) + C$
e^x	$e^x + C$
$cos(x)$	$sin(x) + C$
$sin(x)$	$-cos(x) + C$
$sec^2(x)$	$tan(x) + C$
$csc^2(x)$	$-cot(x) + C$
$sec(x)tan(x)$	$sec(x) + C$

4 Summary

Using the **Fundamental Theorem of Calculus Part I**, we can find the original function through differentiating its integral. In general, remember

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

Antiderivatives are functions that when derived, equal the original function. In general, $F'(x) = f(x)$.

Antiderivatives, unlike derivatives, are not unique. Thus a family of antiderivatives take the form $F(x) + C$.