Algebraic topology review

Symplectic geometry. 2024 Winter Term. Heidelberg University
Course taught by J.-Pr. Agustín Moreno*

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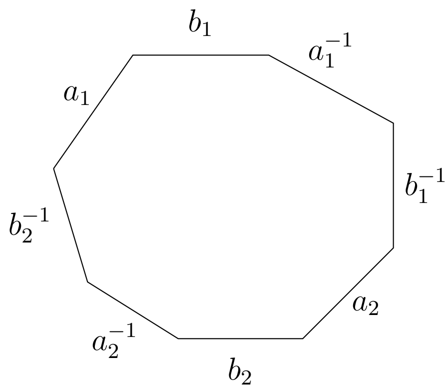
This sheet is **optional** and will not be subject to grading. It is meant as a reminder of some useful notions from algebraic topology. None of the material in this sheet is examinable nor strictly necessary to follow the course.

Problems

Exercise 1. Consider a polygon with $4g$ edges which are grouped into g tuples, each consisting of four consecutive edges labeled in clockwise order by $a_k, b_k, a_k^{-1}, b_k^{-1}$ for $1 \leq k \leq g$ (see the figure below ofr the case $g = 2$). By identifying the edges according to the labeling, one obtains a closed orientable surface Σ_g of genus g. Compute $H_1(\Sigma_g)$.



Exercise 2. Define the *unreduced suspension* ΣX of a space X to be the quotient space of $[0, 1] \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. Show that there is a natural isomorphism $\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)$.
Hint: consider the two cones $C_+X = \{[t, x] \in \Sigma X \mid \ t \geq \frac{1}{2}\}$ and $C_-X = \{[t, x] \in \Sigma X \mid t \leq \frac{1}{2}\}$

Exercise 3. Given two disjoint connected n-manifolds M_1 and M_2, their connected sum $M_1 \sharp M_2$ is defined by deleting the interiors of of cosed n-balls $B_1 \subseteq M_1$ and $B_2 \subseteq M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some hemeomorphism

*For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or ruscelli.francesco1@gmail.com between them. Show that for a closed connected orientable n-manifolds M_1, M_2 there are isomorphisms

$$H_i(M_1) \oplus H_i(M_2) \cong H_i(M_1 \sharp M_2)$$

for $0 < i < n$.

Exercise 4. Let $f\colon M \to N$ be a map between connected closed orientable manifolds and suppose there is a ball $B \subseteq N$ such that $f^{-1}(B)$ is the disjoint union of open ball $B_1, \ldots, B_k \subseteq M$ which each het mapped hemeomorphically onto B. Show that the degree of f is $\sum \epsilon_i$, where $\epsilon_i = \pm 1$ according to whether $f_{B_i}\colon B_i \to B$ preserves or reverses local orientations induced by the fundamental classes $[M]$ and $[N]$.