Problem Sheet #4

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises.

Please, hand in this property before **Friday Nov. 8** (either in person at the exercise class, or by email at alimoge@mathi.uni-heidelberg.de)

Problems

Exercise 1. Show that the exponential map

$$\exp\colon \mathfrak{sp}(2n) \to \mathrm{Sp}(2n)$$

is not surjective.

Hint: consider the matrix $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2)$.

Exercise 2. Show that if $A \in \mathrm{Sp}(2n)$ is diagonalizable, then it is symplectically diagonalizable, i.e. there exists $N \in \mathrm{Sp}(2n)$ such that NAN^{-1} is diagonal.

Exercise 3. A *contact structure* on a $(2n+1)$-dimensional manifold M is a smooth 1-form α such that $\alpha \wedge (d\alpha)^{2n}$ is a volume form on M. The kernel $\xi = \ker \alpha$ is a *maximally nonintegrable* 2-plane distribution on M, i.e. it is nowhere integrable, not even locally (recall the Frobenius integrability condition: $\alpha \wedge d\alpha = 0$). α is called a contact form and (M, ξ) is called a contact structure. When we want to make the 1-form α explicit we write $(M.\alpha)$.
Let (M, ξ) be a contact manifold.

1. Show that the 1-form $\alpha_0 = dx_{2n+1} + \sum_{i=1}^{n} x_i dx_{n+i}$ makes $\mathbb{R}^{2n+1}$ into a contact manifold.

2. Show that the contact condition does not depend on the the contact form.

3. Let α be a contact form and consider the subbundle $\xi \to M$ of TM. Show that the restriction

$$d\alpha|_{\xi}\colon \xi \times \xi \to \mathbb{R}$$

is nondegenerate.

For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or ruscelli.francesco1@gmail.com

4. Show that there exists a unique vector field R_α such that

$$\begin{cases} i_{R_\alpha} d\alpha = 0, \\ \alpha(R_\alpha) = 1. \end{cases}$$

R_α is called the *Reeb vector field* of α.

Exercise 4. (Contact Darboux Theorem) The goal of this exercise is to show that, much like symplectic manifolds, contact manifolds do not have local invariants.
Let α be a contact form on $\mathbb{R}^{2n+1}$.

1. Show that you can choose linear coordinates around 0 such that on $T_0\mathbb{R}^{2n+1}$:

$$\begin{cases} \alpha(\partial_z) = 1, \quad i_{\partial_z} d\alpha = 0, \\ \partial_{x_j}, \partial_{y_j} \ \forall\, 1 \leq j \leq n, \quad d\alpha = \sum_{j=1}^n dx_j \wedge dy_j. \end{cases}$$

2. Define $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$ and consider the family

$$\alpha_t = (1-t)\alpha_0 + t\alpha$$

for $t \in [0,1]$. Show that α_t is a contact form for all t small enough.

3. Use the Moser trick to find an isotopy ψ_t of a neighborhood of the origin such that $\psi_t^*\alpha_t = \alpha_0$ (**Hint:** write the vector field generating ψ_t as $X_t = H_t R_{\alpha_t} + Y_t$ with $Y_t \in \ker \alpha_t$. What conditions do we need on Y_t and H_t?)