Monotonicity of c $\Longrightarrow$ $c(B_\varepsilon, \omega) \leq c(B_{\varepsilon'}, \omega)$ if $\varepsilon \leq \varepsilon'$.

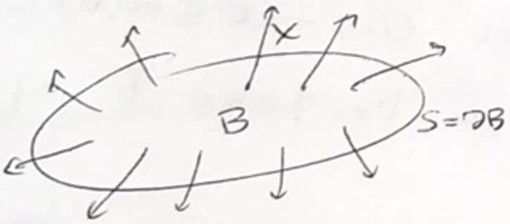
Denote $C(\varepsilon) = c(B_\varepsilon, \omega)$, so that $\varepsilon \mapsto C(\varepsilon)$ is monotone increasing.

<u>Def.</u> S_{ε^*} is called of c-Lipschitz-type if there exist $L, \mu > 0$ such that $C(\varepsilon) \leq C(\varepsilon^*) + L(\varepsilon - \varepsilon^*)$ $\forall\, \varepsilon^* \leq \varepsilon \leq \varepsilon^* + \mu$.

<u>Exercise</u>. Show that this notion does depend on the choice of family modeled on S_{ε^*}.

<u>Example</u>. Suppose in a nbd of $S = \partial B$ there exists a Liouville vector field, i.e. a vector field X satisfying

$$\begin{cases} L_X \omega = \omega \\ X \pitchfork S. \end{cases}$$



Then, S is of c-Lipschitz type.

<u>Exercise</u> Prove this.

<u>Thm</u> Assume $c(M, \omega) < \infty$. If the cpt hypersurface $S \subseteq (M, \omega)$ bounds a symplectic mfd and is of c-Lipschitz type, then $P(S) \neq \emptyset$.

<u>Proof.</u> by assumption, $\exists$ (S_ε) family modeled on $S = S_0$ with

$$C(\varepsilon) \leq C(0) + L\varepsilon \quad \forall\, 0 \leq \varepsilon \leq \mu.$$

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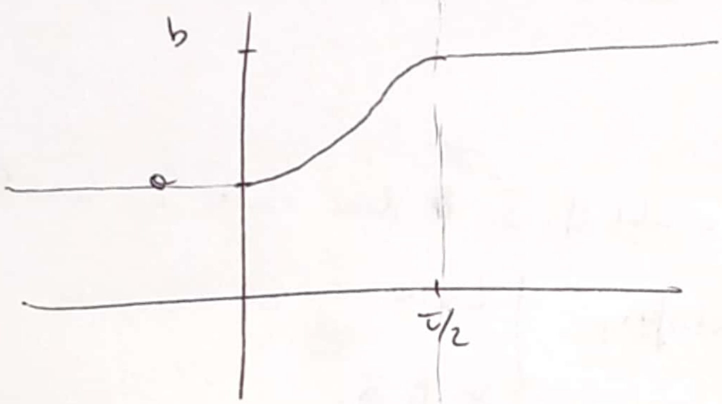
Define the set $\mathcal{F}_\tau$ of functions $f : \mathbb{R} \to (C(0) - \tau L, \infty)$ for $0 < \tau < \mu$. with the following restrictions:

$$\begin{cases} f(s) = a & \text{if } s \le 0 \\ f(s) = b & \text{if } s \ge \frac{\tau}{2} \\ 0 < f'(s) \le c & \text{if } 0 < s < \frac{\tau}{2} \end{cases}$$

with
$$\begin{cases} C(0) - L\tau \le a \le C(0) \\ C(0) + 2L\tau \le b \le C(0) + 3L\tau \\ c \text{ large enough, but independent of } \tau. \end{cases}$$
(we are going to be more specific at the end of the proof).



$f \in \mathcal{F}_\tau$.

Then $\mathcal{F}_\tau \neq \emptyset$. By definition of $c_0(B_0)$ there exists an admissible function $H \in \mathcal{H}_a(B_0, \omega)$ with oscillation $C(0) - L\tau \le m(H) < C(0)$.

Choose $f \in \mathcal{F}_\tau$ with $a = m(H)$ and define the function F by

$$\begin{cases} F(x) = H(x) & \text{if } x \in B_0 \\ F(x) = f(\varepsilon) & \text{if } x \in S_\varepsilon, \quad 0 \le \varepsilon \le \tau. \\ F(x) = b & \text{if } x \notin \overline{B_\tau}. \end{cases}$$

Then, $F \in \mathcal{H}(B_\tau, \omega)$ and $m(F) = b \ge C(0) + 2L\tau > C(0) + L\tau \ge C(\tau)$

C_0-Lipschitz condition.

By definition, $\exists$ nonconstant periodic orbit $x(t)$ of X_F with period $0 < T \le 1$ contained in B_τ.

Note that B_0 is invariant under the flow of X_F. Since $F|_{B_0} = H$ is admissible, we see that $x(t) \in B_\tau \setminus \overline{B_0}$ $\forall t$.

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$\Rightarrow \exists\, \varepsilon \in (0, \frac{\pi}{2})$ such that $x(t) \in S_\varepsilon \ \forall t$.

This argument works for every $0<\tau<\mu$.

By choosing a sequence $\tau_j \longrightarrow 0$, we get sequences F_j, ε_j and periodic orbits $x_j(t)$ satisfying

$$\begin{cases} \dot{x}_j = X_{F_j}(x_j). \\ x_j(t) \in S_{\varepsilon_j}, \quad \varepsilon_j \longrightarrow 0. \\ 0 < T_j \leq 1. \end{cases}$$

Now, consider the nbd U of S fibrated by (S_ε). Define a function K on U by $K(x) = \varepsilon$ if $x \in S_\varepsilon$.

Note that $F_j(x) = f_j(K(x))$ $\forall x \in S_\varepsilon$, $0 \leq \varepsilon \leq \tau_j$.

In particular, for those points x, we get $X_{F_j}(x) = f_j'(K(x))\, X_K(x)$.

$\rightsquigarrow$ the periodic orbits x_j satisfy $\begin{cases} \dot{x}_j(t) = f'(\varepsilon_j)\, X_K(x_j(t)) \\ x_j(0) = x_j(T_j) \end{cases}$, $0 < T_j \leq 1$.

Reparametrize: $y_j(t) := x_j\left(\frac{t}{f'(\varepsilon_j)}\right)$

$$\rightsquigarrow \begin{cases} \dot{y}_j(t) = X_K(y_j(t)) \\ K(y_j(t)) = \varepsilon_j \end{cases}$$

The periods of the y_j's are given by $T_j f'(\varepsilon_j)$. Choose c large enough by independent of τ, for instance $c = 10L \Rightarrow$ the periods of the y_j's are uniformly bounded. $\Rightarrow P(S) \neq \emptyset$.

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Remark

Suppose S is a cpt, connected hypersurface in $(\mathbb{R}^{2n}, \omega_0)$. Then S is orientable (there are a few ways of seeing this, I might put one in the exercises) and it separates $\mathbb{R}^{2n}$ into two components, one bounded and the other one unbounded.

The bounded component has finite capacity since we can embed it in some big enough ball. If S admits a parametrized family and is of co-Lipschitz type, we can apply the previous theorem.

II Hypersurfaces of contact type

This is another property of surfaces that ensures the existence of closed characteristics.

Let $S \subseteq (M,\omega)$ be a cpt hypersurface.

Def. We say that S is of contact type if $\exists$ Liouville vector field X defined in a nbd of S, i.e. a vector field satisfying

$$\begin{cases} \mathcal{L}_X \omega = \omega \\ X \pitchfork S. \end{cases}$$

Brief detour into contact geometry

Let M^{2n+1} be an orientable $(2n+1)$-dimensional mfd.

Def. A contact structure on M^{2n+1} is a cooriented hyperplane distribution $\xi \subseteq TM$ that is maximally nowhere integrable. This means that if $\alpha \in \Omega^1(M)$ is such that $\ker\alpha = \xi$ (this is possible by coorientability), then $\alpha \wedge (d\alpha)^n$ is a volume form.

This means that there does not exist any open set on which ξ can be integrated.

Examples (i) $(\mathbb{R}^{2n+1}, \alpha = dz + \sum_{i=1}^{n} x_i dy_i)$.

(ii) $S^{2n+1} \subseteq (\mathbb{R}^{2n+2}, \omega_0)$ with the contact structure $\ker \omega_0|_{S^{2n+1}}$.

Exercise Show that if $S \subseteq (M, \omega)$ is of contact type, then the form $\alpha := (i_X \omega)|_S$ is a contact form on S.

Exercise Prove that strictly convex hypersurfaces are of contact type.

The contact type condition is nice because it gives a special parametrized family modeled on S. Indeed, the flow φ^t of X is defined for $|t| < \varepsilon$ (ε small enough by compactness of S) and it defines a diffeo $\varphi^t : S \times (-\varepsilon, \varepsilon) \longrightarrow U$ onto a nbd of S.

Since $\mathcal{L}_X \omega = \omega$, we deduce that $\varphi^{t*}\omega = e^t \omega$.

Using this, it's easy to see that ~~$d\varphi^t : TS \to TS_t$ restricts to a~~ $d\varphi^t : \mathcal{L}_S \longrightarrow \mathcal{L}_{S_t}$ is a bundle isomorphism.

This means that φ^t induces a bijection $P(S) \longrightarrow P(S_t)$

We can extrapolate a definition out of this:

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Def. A cpt hypersurface $S \subseteq (M,\omega)$ is called stable if there exists a parametrized family modeled on S having the property that the associated diffeo $\psi : S \times I \to U$ induces bundle isomorphisms

$$d\psi_\varepsilon : \mathcal{L}_S \longrightarrow \mathcal{L}_{S_\varepsilon}.$$

We can thus rephrase the existence theorem for closed characteristics as follows:

Thm. Assume $S \subseteq (M,\omega)$ admits a nbd U with $c_0(U,\omega) < \infty$. If S is stable, then $P(S) \neq \emptyset$.

Example A stable surface need not be of contact type. Consider a closed symplectic mfd (N,ω_1) and

$$M = (N \times \mathbb{R}^2, \underbrace{\omega_1 \oplus \omega_0}_{=\omega}).$$

Let $S := \{(x,v) \mid \|v\| = 1\} \subseteq M$ be a cpt hypersurface.

and define the parametrization $\psi_\varepsilon(x,v) = (x, \varepsilon v)$.

$\rightsquigarrow S_\varepsilon = \{(x,v) \mid \|v\| = \varepsilon\}$.

Clearly, S is stable. However, we claim it is not of contact type.
If it were, we would be able to find a 1-form α on S such that
$d\alpha = j^*\omega$, where $j : S \hookrightarrow M$ is the inclusion.
Let $i : N \hookrightarrow N \times \{(1,0)\}$ be the inclusion. Then we have

$$\underbrace{i^* d\alpha}_{= d(i^*\alpha)} = i^* j^* \omega = (ji)^*\omega = \omega_1 \implies \omega_1 \text{ is exact, contradiction (as } N \text{ is closed).}$$

XIV

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