Problem Sheet #3

Symplectic geometry. 2024 Winter Term. Heidelberg University
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Please solve the following problems. Show all your work and justify your answers. The dagger † denotes optional exercises.

Please, hand in this property before **Friday Nov. 8** (either in person at the exercise class, or by email at alimoge@mathi.uni-heidelberg.de)

Problems

Let V be an even-dimensional real vector space with a non-degenerate 2-form ω. Given a vector subspace $S \subset V$, we define:

$$S^\omega := \{v \in V \mid \omega(v,w) = 0 \ \ \forall w \in S\} \tag{1}$$

Furthermore, we make the following definitions:

- S is symplectic if $S \cap S^\omega = \{0\}$.
- S is isotropic if $S \subseteq S^\omega$.
- S is co-isotropic if $S \supseteq S^\omega$.
- S is **Lagrangian** if $S = S^\omega$.

Exercise 1. Prove the following:

1. S is symplectic $\iff$ S^ω is symplectic $\iff$ $\omega|_S$ is non-degenerate.
2. S is isotropic $\iff$ $\omega|_S \equiv 0$.
3. S is co-isotropic $\iff$ S^ω is isotropic.
4. S is Lagrangian $\iff$ $\omega|_S \equiv 0$ and $\dim S = \frac{1}{2}\dim S$.

Exercise 2. Consider $M = \mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$.

A diffeomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a **symplectomorphism** if $f^*\omega_0 = \omega_0$, i.e it preserves the symplectic form.

*For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or fruscelli@mathi.uni-heidelberg.de

1. Compute ω_0^n, and show that symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ are volume-preserving.

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian, X_H its Hamiltonian vector field (i.e the unique vector field such that $\omega_0(X_H, \cdot) \equiv \mathrm{d}H$), and $\phi_H^t : M \to M$ the flow of X_H.

2. Let $\psi := \phi^{t=1}$ be the time 1 map of the flow. Show that ψ is a symplectomorphism.

3. Show that there is a bijection:

$$\big\{\text{Fixed points of } \psi\big\} \xleftrightarrow{1:1} \big\{\text{Periodic orbits of the flow}\big\} \tag{2}$$

Exercise 3. Let $T^\star Q$ be a cotangent bundle, with coordinates q_i, p_i, and standard symplectic form ω. Consider a 1-form α on Q, and write $\mathrm{Graph}(\alpha)$ its graph as a function $Q \to T^\star Q$ (where Q is viewed as the zero section in $T^\star Q$).

1. Show that $\mathrm{Graph}(\alpha)$ is Lagrangian in $T^\star Q \iff \alpha$ is closed.

2. Let (M, ω) be a symplectic manifold, and $H : M \to \mathbb{R}$ a time-independent Hamiltonian. Show that if N is a Lagrangian contained in a regular level set of H, then N is invariant under the Hamiltonian flow.

3. Say now that $H : M \times \mathbb{R} \to \mathbb{R}$ is allowed to be time-dependent, and define:

$$\begin{aligned}
&\widehat{M} := M \times \mathbb{R} \times \mathbb{R} \\
&\widehat{H} := \widehat{H}(m, h, t) := H(m, t) - h \\
&J \in \mathrm{End}(T\widehat{M}) \text{ s.t } J|_M \text{ is almost complex, and } J\partial_h = \partial_t
\end{aligned}$$

   Show that $\hat{\omega} := \omega - \mathrm{d}h \wedge \mathrm{d}t$ defines a symplectic structure on $\widehat{M}$, and that the Hamiltonian vector field of $\widehat{H}$ is given by:

$$X_{\widehat{H}} = X_{H_t} + \partial_t + (\partial_t H)\partial_h$$

4. Take M as above, assume the symplectic form is exact (i.e $\omega = \mathrm{d}\lambda$); and define the 1-form $\alpha := \lambda - H\mathrm{d}t$. Show that the Lagrangian submanifolds $\widehat{N} \subset \widehat{W}$ lying in the energy level set $\{\widehat{H} = 0\}$ are exactly those submanifolds $\widehat{N} \subset \{\widehat{H} = 0\}$ such that $\alpha|_{\widehat{N}}$ is closed.

5. Let $\widehat{N} \subset M \times \mathbb{R}$ be Lagrangian, like in 4. Show that for every t, $N_t := \widehat{N} \cap (M \times \{t\})$ is Lagrangian in $M \times \{t\}$.

Exercise 4. (Algebraic topology parenthesis) This exercise is a prerequisite for Exercise 5, where we will define a famous loop invariant from Symplectic Geometry.

Consider the spaces:

$$\begin{aligned}
U_n &:= \big\{U \in \mathcal{M}(\mathbb{C}^n) \mid UU^\dagger = U^\dagger U = \mathrm{id}\big\} \\
O_n &:= \big\{O \in \mathcal{M}(\mathbb{R}^n) \mid OO^t = O^t O = \mathrm{id}\big\}
\end{aligned}$$

as well as $SU_n := \ker\{\det : U_n \to (\mathbb{C}^\star, \times)\}, SO_n := \ker\{\det : O_n \to (\mathbb{C}^\star, \times)\}$; and where ^{t} denotes the transpose, and † the Hermitian conjugate.

We recall from linear algebra that any matrix in SO_n can be turned into a block diagonal matrix: $D = D_1 \oplus \cdots \oplus D_n$, where either $D_i = (\,1\,)$, or $D_i \in SO_2$; and from algebraic topology that a fibration $F \hookrightarrow E \twoheadrightarrow B$ induces a long exact sequence in homotopy:

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \dots$$

Our goal in this exercise is to show the following: $\boxed{\forall n \geq 2 : \pi_1(SU_n/SO_n) = 0}$.

1. Show that it suffices to show that SU_n is simply connected, and that SO_n is path-connected.
2. Show that, for $n \geq 2$, SO_n is path-connected.
3. Show that SU_{n+1} acts transitively on $\mathbb{S}^{2n+1}$. Deduce that there exists a fibration $SU_n \hookrightarrow SU_{n+1} \twoheadrightarrow \mathbb{S}^{2n+1}$.
4. Deduce that $\forall n \geq 2 : \pi_1(SU_n/SO_n) = 0$ (it might be helpful to use the identification $SU_2 \cong \mathbb{S}^3$).

Exercise 5. (The Maslov index) Let $V = \mathbb{C}^n$, and define $\mathbf{\Lambda}$ to be the space of Lagrangians in $\mathbb{C}^n$. Recall from lectures that $\mathbf{\Lambda} \cong U_n/O_n$.

1. Show that the map $\rho : U_n/O_n \to \mathbb{S}^1 : u \mapsto (\det u)^2$ is well-defined.
2. Show that ρ descends to an isomorphism $\rho_\star : \pi_1(U_n/O_n) \xrightarrow{\cong} \mathbb{Z}$, and deduce that one can associate a homotopy invariant $\mu \in \mathbb{Z}$ to any loop of Lagrangians in $\mathbb{C}^n$. This μ is called the **Maslov index**.