We have seen that the Hofer–Zehnder capacity c_0 is indeed a capacity. We are now going to use this to obtain results of dynamical nature.

Consider a symplectic mfd (M,ω) and a Hamiltonian function $H \in C^\infty(M,\mathbb{R})$.

Suppose $S = H^{-1}(c)$ is a regular level set for some $c \in \mathbb{R}$ and assume S is cpt.

$\Rightarrow$ S is a ~~compact~~ closed submfd of M of codimension 1.

and $TS = (\ker dH)\big|_S$.

It is easy to see that the Hamiltonian vector field X_H is tangent to S. Indeed, $dH(X_H) = -\omega(X_H, X_H) = 0$ on S.

<u>Problem</u> Does X_H admit closed orbits on S?

First, note that the existence of closed orbits does not depend on the choice of H.

Indeed, suppose $S = \{H \equiv c\} = \{F \equiv c\}$ for two Hamiltonians $H, F \in C^\infty(M,\mathbb{R})$ with $dH, dF \neq 0$ on S.

Then, $\forall x \in S$: $\ker d_xH = \ker d_xF \Leftrightarrow d_xF = \rho(x)\, d_xH$ for a nonvanishing smooth function ρ on S.

$\Rightarrow X_F = \rho X_H$ on S.

If φ^t is the flow of X_H on S, ψ^s is the flow of X_F on S, we have $\psi^s(x) = \varphi^t(x)$ $\forall x \in S$,

where $t = t(x,s)$ is a function determined by the ODE

$$\begin{cases} \dfrac{dt}{ds} = \rho(\varphi^t(x)). \\ t(x,0) = x. \end{cases}$$

$\Rightarrow X_H, X_F$ have the same flow lines and, in particular, the same periodic orbits.

Remark
There is a geometric way of viewing this problem, that also shows independence of H.

Let $S \subseteq (M^n, \omega)$ be any codimension 1 submfd.

Define $\mathcal{L}_S = \ker \omega|_S$. By nondegeneracy of ω on M and the fact that $\dim S = 2n-1$, $\mathcal{L}_S$ is a line bundle.

If S is a cpt regular level set of some $H: M \to \mathbb{R}$,

$X_H(x) \in \mathcal{L}_S(x)$ $\forall x \in S$. (as we showed earlier).

(note that the condition $dH \neq 0$ on S implies $X_H \neq 0$ on S).

$\Rightarrow \mathcal{L}_S$ is orientable (in particular trivial).

Conversely, suppose $\mathcal{L}_S \to S$ is orientable. We will construct a function $H: U \to \mathbb{R}$, where U is a nbd of S such that $S = H^{-1}(0)$ is a regular level set.

II

Pick an almost complex structure on J compatible with ω.
In particular, $\langle \cdot , \cdot \rangle = \omega(\cdot , J \cdot)$ is a scalar product.

Let $N_S = \{ \xi \in T_x M \mid \langle \xi , v \rangle = 0 \ \forall v \in TS \}$ be the normal bundle of S. Note that the map

$$\mathcal{L}_S \to N_S \qquad \text{is a bundle isomorphism.}$$
$$\xi \mapsto J\xi$$

$\mathcal{L}_S$ is trivial by assumption $\Rightarrow$ N_S is trivial.
Pick a nonvanishing section $z : S \to N_S$ and define

$$\varphi : S \times (-\varepsilon, \varepsilon) \to M$$
$$(x,t) \longmapsto \exp_x(t z(x))$$

This is a diffeomorphism onto a nbd U of S if $\varepsilon > 0$ is small enough (we are using the fact that S is cpt).

If $F : S \times (-\varepsilon, \varepsilon) \to \mathbb{R}$, the desired Hamiltonian is given by
$$(x,t) \longmapsto t$$

$$H := F\varphi^{-1} : U \to \mathbb{R}.$$

~~[illegible]~~ $\mathcal{L}_S$ is called "characteristic line bundle" of S.

Our problem of finding closed orbits of X_H on S is thus purely geometric, i.e. it is equivalent to finding an embedded circle $P \subseteq S$ ~~[illegible]~~ such that $TP = \mathcal{L}_S|_P$. Such a circle is called a "closed characteristic".

Note that our construction provides a nbd of S that is foliated by hypersurfaces diffeomorphic to S. This prompts the following definition:

<u>Def.</u> Let S be a cpt hypersurface in (M,ω). A parametrized ~~set~~ family of hypersurfaces ~~of~~ modelled on S is a diffeomorphism

$$\psi : S \times I \to U \subseteq M, \quad I \text{ open interval containing } 0 \in \mathbb{R}$$

such that $\psi(x,0) = x \quad \forall x \in S$.

We are going to denote ~~this~~ such a family by $(S_\varepsilon)_{\varepsilon \in I}$.

~~We have thus shown the following~~

Rephrasing our work so far, we have shown that the following statement are equivalent:

(i) $\mathcal{L}_S \to S$ is orientable

(ii) $N_S \to S$ is orientable

(iii) S is orientable

(iv) There exists a parametrized family of hypersurfaces modelled on S.

(v) $\exists\, H : \underset{M}{\overset{U}{\cap}} \to \mathbb{R}$, U nbd of S satisfying $dH \neq 0$ on S.

Our ~~[illegible]~~ search for closed characteristics starts with the following theorem by Hofer and Zehnder.

<u>Thm [Hofer-Zehnder]</u>
Let S be a cpt hypersurface and (S_ε) a parametrized family of surfaces modelled on S. Let $P(S_\varepsilon)$ be the set of closed characteristics on S_ε. Then, if $c_0(U,\omega) < \infty$, there exists a dense set $\Sigma \subseteq I$ such that $P(S_\varepsilon) \neq \emptyset \quad \forall \varepsilon \in \Sigma$.

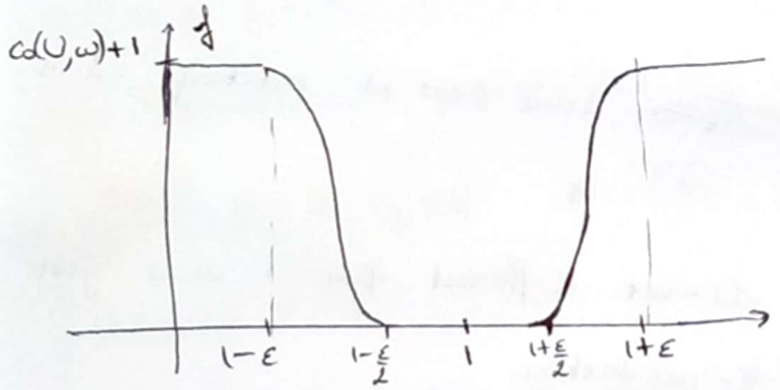
Proof: we are going to construct a special Hamiltonian on U that belongs to the set $\mathcal{H}(U,\omega)$ of functions used to define c_0.

~~Let S_λ be a surface in the family~~

~~Suppose~~ If $I = \{\lambda \mid 1-\rho < \lambda < 1+\rho\}$ for some $\rho > 0$, choose $0<\varepsilon<\rho$.

Choose a smooth function $f: \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} f(s) = c_0(U,\omega)+1 & \text{for } s \le 1-\varepsilon,\ s \ge 1+\varepsilon \\ f(s) = 0 & \text{for } 1-\frac{\varepsilon}{2} \le s \le 1+\frac{\varepsilon}{2}. \\ f'(s) < 0 & \text{for } 1-\varepsilon < s < 1-\frac{\varepsilon}{2}. \\ f'(s) > 0 & \text{for } 1+\frac{\varepsilon}{2} < s < 1+\varepsilon. \end{cases}$$



Define $F: U \to \mathbb{R}$, $x \mapsto f(H(x))$. $\Rightarrow$ F is constant on each S_λ and $F \in \mathcal{H}(U,\omega)$.

Note that the oscillation $m(F) = \max(F) - \min(F) = c_0(U,\omega)+1 > c_0(U,\omega)$.

By definition of c_0, there exists a nonconstant periodic orbit $x(t)$ having period $0<T\le 1$ of the system $\dot{x} = X_F(x)$, $x \in U$.

It is easy to show that $X_F(x) = f'(H(x))\, X_H(x)$. $(*)$

Moreover, $H(x(t))$ is constant w.r.t. Indeed,

$$\frac{d}{dt} H(x(t)) = dH\big(X_F(x(t))\big) = -\omega\big(X_H(x(t)), X_F(x(t))\big) = 0.$$

$\rightsquigarrow$ ~~that~~ $H(x(t)) \equiv \lambda$.

Since $x(t)$ is nonconstant, we have (in view of $(*)$)

$1-\varepsilon < \lambda < 1-\frac{\varepsilon}{2}$ or $1+\frac{\varepsilon}{2} < \lambda < 1+\varepsilon$.

If $f'(\lambda) = \tau \neq 0$, define $y : \mathbb{R} \to S_\lambda$, $y(t) = x\left(\frac{t}{\tau}\right)$.

y has period τT and satisfies $\dot{y} = X_H(y)$.

$\rightsquigarrow$ y is periodic [redacted] orbit of X_H on S_λ. ~~[illegible]~~

By construction, $|\lambda - 1| < \varepsilon$. ~~[illegible]~~ Since $\varepsilon > 0$ is arbitrary, λ is arbitrarily close to 1.

To get the statement for any other element different from 1 in I just replace 1 with element and repeat the construction.

~~Question we have~~ <u>Question</u> We have found solutions on a dense set of S_λ's. Does ~~the~~ there exist a solution on $S_1 = S$?

If we know that the periods T_j of the orbits x_j on S_{λ_j} for $\lambda_j \longrightarrow 1$ are uniformly bounded, then the answer is positive

Let us make this more precise.

Let $g = \langle , \rangle$ be a Riemannian metric on M.

If $x(t)$ is a period solution, we define its length

$$\ell(x) = \int_0^T \|\dot{x}(t)\|\, dt$$

Possibly after shrinking U, we can assume $\frac{1}{C} \le \|X_H\| \le C$ on U.

for some $C > 0$.

$$\Rightarrow \quad \frac{T_j}{C} \le \ell(x_j) \le C T_j \quad \forall j.$$

<u>Proposition</u>. Let $\lambda_j \to 1$ and assume (T_j) is bounded. Then $S = S_1$ admits a periodic solution.

<u>Proof</u>: normalize the periods to 1 by defining $y_j(t) = x_j(T_j t)$, $t \in [0,1]$.

$$\Rightarrow \begin{cases} \dot{y}_j(t) = T_j X_H(y_j(t)). & (*) \\ H(y_j(t)) \equiv \lambda_j. \end{cases}$$

Note that $T_j X_H(y_j(t))$ is bounded by assumption. By the Arzelà–Ascoli theorem,

$\Rightarrow (y_j)$ is equicts and bounded.

we can assume $y_j \xrightarrow{C^0} y$. By using $(*)$, we see that this convergence is actually in the C^∞-topology. $\rightsquigarrow$ we get a 1-periodic solution y, with $H(y(t)) = 1$.

i.e. $\dot{y}(t) = X_H(y(t))$.

If the period T of y is not 0, we are done.

Suppose $T = 0$. $\Rightarrow$ $y_j \to y^*$, where $y^* \in S$ is a point.

$\Rightarrow X_H(y_j(t)) \to X_H(y^*). =: V$.

Since S is a regular level set, $X_H \neq 0$ on $S \Rightarrow V \neq 0$.

Note that $\langle X_H(y_j(t)), V \rangle \geq (1-\varepsilon)\|V\|^2$ for large j and $\varepsilon > 0$ small.

$\Rightarrow \frac{1}{T_j}\langle \dot{y}_j(t), V \rangle \geq (1-\varepsilon)\|V\|^2$ (this makes sense in local coordinates and we are using the standard euclidean product)

However, $0 = \int_0^{T_j} \frac{1}{T_j}\langle \dot{y}_j(t), V\rangle dt = \int_0^{T_j} \frac{d}{dt}\left(\frac{1}{T_j}\langle y_j(t), V\rangle\right) dt = T_j(1-\varepsilon)\|V\|^2$

$\Rightarrow \|V\| = 0$, contradiction ■

<u>Remark</u> We can apply Hofer-Zehnder to cpt hypersurfaces in $(\mathbb{R}^{2n}, \omega_0)$. Since we can always embed such a surfaces in large enough balls, which have finite capacity, we can apply the theorem.

We are now going to restrict the class of hypersurfaces we consider in order to be able to apply the previous proposition

~~Assume S is the boundary of some~~ We are going to consider two classes (actually, the second is a subclass of the first).

(I): Let $S \subseteq (M, \omega)$ be a cpt hypersurface and assume S is the boundary of some compact symplectic mfd $(B, \omega) \subseteq (M, \omega)$.

Let (S_ε) be a parametrized family of surfaces modeled on S.

$\rightsquigarrow$ S_ε bounds a mfd B_ε. Assume the parametrization is such that $\varepsilon < \varepsilon' \Rightarrow B_\varepsilon \subseteq B_{\varepsilon'}$.