Problem Sheet #2

Symplectic geometry. 2024 Winter Term. Heidelberg University
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October 21, 2024

You are allowed to hand in solutions to this Problem Sheet by pairs of two students, if you desire. Please, solve all the exercises, and show your working.

Please, hand in this problem sheet before **Friday Nov. 1** (either in person at the exercise class, or by email at alimoge@mathi.uni-heidelberg.de).

Problems

Exercise 1. (Hamiltonian flow) Recall the definition of the usual gradient:

Definition 0.1. Let (N, g) be a Riemannian manifold, and $f : N \to \mathbb{R}$. Its gradient is defined as the unique vector field ∇f satisfying $g(\nabla f, \cdot) = \mathrm{d}H$.

1. Why is ∇f well-defined and unique?
2. Let (M, ω) be a symplectic manifold. Show that a function $H : M \to \mathbb{R}$ naturally induces a vector field X_H. We call X_H the Hamiltonian vector field.
3. Say $M = \mathbb{R}^{2n}$, with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, and with symplectic form:

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i$$

   Express X_H in terms of ∇H and J_0. Here, $J_0 = i \oplus \cdots \oplus i$ denotes the standard complex structure on $\mathbb{R}^{2n} \cong \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$, where:

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in GL(\mathbb{R}^2)$$

4. Still on $(\mathbb{R}^{2n}, \omega_0)$, show that the flow of X_H describes the Hamiltonian equations of motion, from classical physics. This flow is called the Hamiltonian flow.

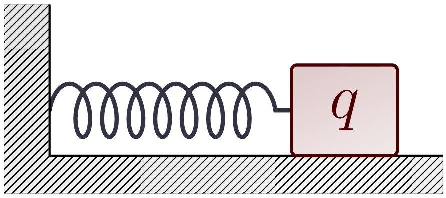
**For comments, questions, or potential corrections on the exercise sheets, please email alimoge@mathi.uni-heidelberg.de, or fruscelli@mathi.uni-heidelberg.de*

Exercise 2. (In physics) Let us now study a concrete example. Assume we have an object moving in position space $\mathbb{R}^n$, with position $q(t)$. We denote its momentum by $p(t) := \dot{q}(t)$, which is a vector living in a different copy of $\mathbb{R}^n$, momentum space. Then, our object is described by coordinates $(q, p) \in \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ (phase space).

In classical mechanics, an object is fully determined by its total energy:

$$H = \text{Kinetic Energy } + \text{ Potential Energy} = \frac{1}{2}\|p\|^2 + V(q)$$

This is a function $\mathbb{R}^{2n} \to \mathbb{R}$, which we call the **Hamiltonian**, and where V is such that $F = -\nabla V$, where F is the force applied on the system.
From now on, assume we work on the real half-line, and have a spring with origin at 0, as well as a mass attached to the spring, whose position we denote by q.



By Hooke's law, the force on the mass is given by $F = -kq$, where $k > 0$ is a constant.

1. Compute X_H, and draw the Hamiltonian flow of this system on phase space (*without solving the differential equations).*

Let $q_1, \dots, q_n, p_1, \dots, p_n$ be coordinates on $\mathbb{R}^{2n}$, and let $f, g : \mathbb{R}^{2n} \to \mathbb{R}$ be differentiable functions (to which one may add time-dependence, if they wish). Then we define their **Poisson bracket**:

$$\{f, g\} := \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

2. Given ω_0 the standard symplectic form on $\mathbb{R}^{2n}$, show that $\{f, g\} = \omega_0(X_f, X_g)$.

3. Prove the following result from classical mechanics:

   Proposition. Let H be a Hamiltonian describing a physical system. A time-independent function $f = f(q, p)$ is constant along the motion iff $\{f, H\} \equiv 0$.

4. Say f is now time-dependent. Compute $\mathrm{d}f/\mathrm{d}t$ along physical trajectories of the system (i.e, solutions to the Hamiltonian equations of motion).

   Deduce that f is constant along the motion iff $\{f, H\} + \partial_t f = 0$.

Exercise 3. Let V be an arbitrary, even-dimensional vector space. An **almost complex structure** is a map $J \in GL(V)$ such that $J^2 = -\mathrm{id}$. A linear symplectic form ω is a non-degenerate 2-form on V. And we define the following spaces:

$$\mathcal{S}(V) := \{\text{Linear symplectic forms on } V\}$$
$$\mathcal{J}(V) := \{\text{Almost complex structures on } V\}$$

Given $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$, we say that J is **compatible** with ω if:

- J is an isometry with respect to ω (i.e $J^\star\omega = \omega$)
- $\forall v \in V \backslash \{0\} : \omega(v, Jv) > 0$

And we write $\mathcal{J}(V, \omega) = \{J \in \mathcal{J}(V) \mid J \text{ is compatible with } \omega\}$.

1. Show that J is compatible with $\omega \iff$ the expression $g_J := \omega(\cdot, J\cdot)$ defines an inner product on V.
2. Show that J is compatible with $\omega \iff$ it is an isometry with respect to g_J.

In Exercise 1, we defined the standard complex structure J_0 on $\mathbb{R}^{2n}$. And in [Problem Sheet 1, Ex. 1], you proved that if you take g_0 to be the Euclidean inner product on $\mathbb{R}^{2n}$, then the associated linear symplectic form must be:

$$\omega_0 = \sum_{i=1}^{n} v^i \wedge w^i \tag{1}$$

where $\{v_1, w_1, \ldots, v_n, w_n\}$ is a basis for $\mathbb{R}^{2n}$, and $\{v^1, w^1, \ldots, v^n, w^n\}$ is the dual basis. In other words, $\boxed{g_0 = \omega_0(\cdot, J_0)}$.

3. Show that the following three statements are equivalent:

- J is compatible with ω.
- There exist $u_1, \ldots, u_n \in V$ such that $\{u_1, Ju_1, u_2, Ju_2, \ldots, u_n, Ju_n\}$ form a basis of V, and such that:

$$\forall i, j : \omega(v_i, Jv_j) = \delta_{ij}, \;\; \omega(v_i, v_j) = 0 = \omega(Jv_i, Jv_j)$$

- There is a vector space isomorphism $\phi : \mathbb{R}^{2n} \xrightarrow{\cong} V$ such that $\begin{cases} \phi^\star\omega = \omega_0 \\ \phi^\star J = J_0 \end{cases}$

Exercise 4. (The symplectic group) Consider an inner product space (V, g) with basis $\mathcal{B} = \{v_1, w_1, \ldots, v_n, w_n\}$, and consider the linear symplectic form ω_0 from (1).

Then, we define the symplectic group on V:

$$Sp(V) := \{A \in GL(V) \mid A^*\omega_0 = \omega_0\} \tag{2}$$

where the pullback $A^*\omega_0$ is defined as $\omega(A_0\cdot, A_0\cdot)$. We write J the standard complex structure on V. In other words, we have $Jv_i = w_i$ and $Jw_i = -v_i$.

1. Show that $A \in Sp(V) \iff A^tJA = J$.
2. Argue that $Sp(V)$ is a Lie group.

Recall that the Lie algebra $\mathfrak{g}$ of a Lie group G is defined as T_1G, where 1 is the identity in G. Write $\mathfrak{sp}(V) = \text{Lie}(Sp(V))$.

3. Let $\psi(t)$ be a path in $Sp(V)$. Differentiate $\psi(t)$. Deduce that:

$$B \in \mathfrak{sp}(V) \iff B^tJ + J^tB = 0 \tag{3}$$