Degree Theory

1 Degree theory in finite dimensions

This is adapted from [1] Recall the local inverse thm:

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function and assume that $\nabla f(x_0)$ is invertible. Then there exists an open neighborhood U of x_0 and an open neighborhood V of $f(x_0)$ such that the inverse function $f^{-1} : V \to U$ exists and belongs to C^1.*

Let X, Y be open, paracompact (separable, all covers admit a locally finite refinement) smooth manifolds of dimensions n and k, $n \geq k$ respectively. Let f be a map $f : X \to Y$ which is C^{n-k+1}.

Definition 1. *A point $x_0 \in X$ is a regular point for f if $(\nabla f(x_0))$ has maximal rank k. A point which is not regular is called a critical point. A point $y \in Y$ is a critical value if the preimage $f^{-1}(\{y\})$ contains a critical point. Otherwise, y is called a regular value.*

Theorem 2. *Sard's theorem: If $f \in C^{n-k+1}$, $f : X \to Y$ like above, then the set of its critical values has measure zero in Y.*

Proof. "Measure zero" in Y is well defined in a chart. We only give the proof for $n = k$. Enough to prove when X is a closed cube with sides parallel to the axes in $\mathbb{R}^n$ and with side of size L. We subdivide the cube in small cubes of side $\frac{L}{N}$, with sides parallel to the axes. If x and x_0 belong to the same small cube Q then

$$f(x) = f(x_0) + (\nabla f(x_0))(x - x_0) + o\left(\frac{L}{N}\right)$$

because the first derivatives of f are continuous in X. If the point x_0 is critical for f then $\det(\nabla f(x_0)) = 0$, and therefore the image of Q lies in a cylinder of base in a plane of dimension $n-1$ and base area $\leq C\left(\frac{L}{N}\right)^{n-1}$ and height $o\left(\frac{L}{N}\right)$. As there are at most N^n cubes containing critical points, their image under f is contained in a set whose volume is of the order $No\left(\frac{1}{N}\right)$. This converges to zero as $N \to \infty$.

Now we recall some notation from differential geometry. If we have an n form in $\mathbb{R}^n$ it is given locally by

$$\mu = f dy$$

where $dy = dy^1 \wedge \cdots \wedge dy^n$ is the volume form and f is a real valued function. The pull back of μ under a change of variables ϕ is

$$\phi^*(\mu) = (f \circ \phi)(x) \det J_\phi(x) dx$$

where J_ϕ is the Jacobian of ϕ. By the change of variables formula

$$\int_Y \mu = \operatorname{sgn} J_\phi \int_X \phi^* \mu \tag{1}$$

Let $X \subset X_0$ where X_0 is a smooth paracompact manifold of dimension n and X is an open subset with compact closure $\bar{X} = X \cup \partial X$ in X_0. Let $\phi : \bar{X} \to Y$ be a continuous map which is C^1 in X, to the smooth n dimensional paracompact manifold Y. Let $y_0 \in Y \setminus \phi(\partial X)$. From the local inverse theorem, the preimage

$$\phi^{-1}\left(\{y_0\}\right) = \{x \in \bar{X} \mid \ \phi(x) = y_0\}$$

is a discrete set (consists of isolated points). Because $\bar{X}$ is compact, this set is finite.

Definition 2. *If y_0 is a regular value of ϕ then*

$$d(y_0) = \sum_{j=1}^{k} sgn\, J_\phi(x_j) \tag{2}$$

where

$$\phi^{-1}\left(\{y_0\}\right) = \{x_1, \ldots, x_k\}$$

We say that a coordinate patch Ω of a point $y_0 \in Y$ is "nice" if there are suitable coordinates $g : \Omega \to \mathbb{R}^n$ so that $g(\Omega)$ is a cube.

Definition 3. *Let $\mu = f(y)dy$ be a smooth n form on Y with support contained in a nice coordinate patch Ω of $y_0 \in Y$, with $\Omega \subset Y \setminus \phi(\partial X)$ and $\int_Y \mu = 1$. Then we set*

$$deg\,(\phi, X, y_0) = \int_X \phi^* \mu \tag{3}$$

Differential forms of the kind above will be called "admissible". The fact that $\deg(\phi, X, y_0)$ is well defined is a consequence of the following lemma.

Lemma 1. *Let $\mu = f(y)dy$ be a smooth form on Y with $\int_Y \mu = 0$ and with $\operatorname{supp}\mu$ contained in a nice coordinate patch Ω. Then there exists an $n-1$-form ω whose support is included in Ω and such that $\mu = d\omega$.*

Indeed, given the lemma, if ν and μ are admissible for y_0 and ϕ in X then, because $\nu - \mu = d\omega$ and because $\phi^*(\nu - \mu) = \phi^*(d\omega) = d\phi^*\omega$, the integrals of $\phi^*\nu$ and $\phi^*\mu$ are equal by Green's theorem

$$\int_X d(\phi^*\omega) = 0.$$

Proof of Lemma 1 Without loss of generality we may assume that the support of μ is included in a cube Q. We must show that we can find g_j supported in Q such that

$$f = \sum_{j=1}^{n} \partial_j g_j$$

The proof is by induction. If $n = 1$, then $g_1 = \int_{-\infty}^{y} f(z)dz$ satisfies $dg_1 = f dy$. Now suppose the lemma is true in n dimensions. Let $y^{n+1} = t$, $(y, t) = (y^1, \ldots, y^n, t)$ and set

$$m(y) = \int_{-\infty}^{\infty} f(y, t)dt.$$

Now $\int m(y)dy = 0$, so, by induction, there exist $g_1, \ldots, g_n$ such that

$$m(y) = \sum_{j=1}^{n} \partial_j g_j(y)$$

and g_j are supported in the projection of the cube. Let $\tau(t)$ be a smooth function supported on the corresponding side of the cube, with

$$\int_{-\infty}^{\infty} \tau(t)dt = 1.$$

Consider $f(y,t) - \tau(t)\mu(y)$. Because its integral in t vanishes,

$$g(y,t) = \int_{-\infty}^{t} (f(y,s) - \tau(s)m(y))ds$$

has support in Q and obeys

$$\partial_t g(y,t) = f(y,t) - \tau(t)m(y).$$

Thus

$$f(y, y^{n+1}) = \partial_{n+1} g(y, y^{n+1}) + \sum_{j=1}^{n} \partial_j (g_j(y)\tau(y^{n+1}))$$

which finishes the proof.

1.1 Properties of the degree

Proposition 1. *For y_1 close to y_0,*

$$deg\,(\phi, X, y_0) = deg\,(\phi, X, y_1).$$

Proof. Indeed, if μ is admissible for ϕ in X for y_0, it is also admissible for ϕ in X for y_1. Because the degree is an integer, it is locally constant and therefore is constant on connected components of $Y \setminus \phi(\partial X)$.

Proposition 2. *If y_0 is a regular point for ϕ then*

$$deg\,(\phi, X, y_0) = d(y_0)$$

Proof. There are disjoint neighborhoods V_j of x_j, the points which comprise $\phi^{-1}(\{y_0\})$, such that ϕ is one-to-one on them. Then if $N = \cap_{j=1}^{k} \phi(V_j)$, then N is a neighborhood of y_0, and if μ is admissible with support in N then

$$\deg\,(\phi, X, y_0) = \textstyle\int \phi^* \mu = \sum_{j=1}^{k} \int_{V_j} \phi^* \mu = \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \int_{\phi(V_j)} \mu$$
$$= \textstyle\sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) \int_Y \mu = \sum_{j=1}^{k} \operatorname{sgn} J_\phi(x_j) = d(y_0).$$

It follows that $\deg\,(\phi, X, y_0)$ is an integer equal to $d(y)$ for ant regular value y belonging to the same connected component of $Y \setminus \phi(\partial X)$ as y_0.

Proposition 3. *Homotopy invariance. Consider a one parameter family of maps $\phi_t : \bar{X} \to Y$, continuous on $\bar{X} \times [0,1]$ and with $\phi_t \in C^1(X)$ for each $t \in [0,1]$. Assume that $y_0 \notin \phi_t(\partial X)$ holds for each $t \in [0,1]$. Then $deg\,(\phi_t, X, y_0)$ does not depend on t.*

Proof. We take a small neighborhood of y_0 which avoids the compact set $\phi(\partial X \times [0,1])$. Let μ be admissible for all ϕ_t, $t \in [0,1]$ and y_0 in X. Then

$$\deg(\phi_t, X, y_0) = \int \phi_t^*(\mu)$$

is continuous and integer valued, so it is constant.

We can generalize this by allowing y_0 to depend continuously on t and having a relatively open set $A \subset X \times [0,1]$ with compact closure. If y_t does not belong to $\phi_t((\partial A)_t)$ where $A_t = \{x \in X;\ (x,t) \in A\}$ and $(\partial A)_t = \{x \in X;\ (x,t) \in \partial A\}$, then $\deg(\phi_t, A_t, y_t)$ is constant.

Proposition 4. *Let X_i be a sequence of disjoint open sets contained in the interior of X. Let $y_0 \notin \phi\left(\bar{X} \setminus \cup_i X_i\right)$. Then $deg(\phi, X_i, y_0) = 0$ for all but finitely many i, and*

$$deg(\phi, X, y_0) = \sum_i deg(\phi, X_i, y_0).$$

Proof. Let N be an open neighborhood of y_0 not intersecting $\phi(\bar{X} \setminus \cup_i X_i)$ (because the latter is compact, hence closed). Then we take a regular value $y \in N$. The degrees are computed at y, and y has a finite number of preimages. A particular case is

Proposition 5. *Excision. Let $K \subset \bar{X}$ be closed. If $y_0 \notin \phi(K) \cup \phi(\partial X)$ then*

$$deg(\phi, X, y_0) = deg(\phi, X \setminus K, y_0).$$

Proof. We apply the previous proposition with $X_1 = X \setminus K$.

Proposition 6. *Let X, Y be manifolds of dimension n and X', Y' of dimension m and $\phi : X \to Y$ and $\phi' : X' \to Y'$ be such that the degrees are defined at y and y' respectively. Then*

$$deg(\phi \times \phi', X \times X', (y, y')) = deg(\phi, X, y) \times deg(\phi', X', y')$$

Proof. If μ and μ' are admissible for ϕ and ϕ' and y and y' then $\mu \times \mu'$ is admissible for $\phi \times \phi'$ and (y, y') at $X \times X'$ and

$$\int (\phi \times \phi')^*(\mu \times \mu') = \int \phi^* \mu \cdot \int \phi'^* \mu'$$

A few remarks about the degree. First, if the map ϕ is one-to-one and preserving the orientation and if $y_0 \in \phi(X) \cap (Y \setminus \phi(\partial X))$ then $\deg(\phi, X, y_0) = 1$. If $y_0 \notin \phi(\bar{X})$ then $\deg(\phi, X, y_0) = 0$. If $\partial X = \emptyset$, X is compact and Y is connected and not compact, then the degree vanishes at any $y \in Y$.

Extension to continuous maps. If $\phi_n \to \phi$ uniformly in $\bar{X}$, then for large enough n, the degrees $\deg(\phi_n, X, y_0)$ are independent of n. Indeed, the property $y_0 \notin \phi(\partial X)$ implies that there exists a neighborhood N of y_0 such that $\phi_n(\partial X) \cap N = \emptyset$ for large enough n. If $dist(\phi_i(\partial X), y_0) \geq \delta > 0$, $i = 1, 2$, then $(1-t)\phi_1(x) + t\phi_2(x) = \phi_1(x) + t\psi(x)$ with $\psi(x)$ uniformly small on ∂X, and therefore the homotopy cannot touch ∂X. Note that the convergence in C^0 does not imply continuity of the degree, but the homotopy invariance does. Note also that the degree depends only on values of ϕ on ∂X: all continuous extensions of ϕ to the whole $\bar{X}$ have the same degree. (same proof: if we have two continuous extensions, then the homotopy described above does not touch the boundary).

Theorem 3. *Let $\phi : X \to Y$, $\phi \in C(\bar{X})$). Let Ω be a connected component of $Y \setminus \phi(\partial X)$ and μ a smooth $n-form$ in Y with compact support in Ω and with $\int_Y \mu \neq 0$. Then*

$$deg\,(\phi, X, \Omega) = \frac{\int_X \phi^* \mu}{\int_Y \mu}$$

The proof follows by establishing the relation first for measures supported in nice coordinate patches, then using a partition of unity, cross multiplying (using Lemma 1) and summing.

Theorem 4. *Let $\phi : \bar{X} \to Y$, $\psi : Y \to Z$ be continuous. Let Ω_i be the connected components of $Y \setminus \phi(\partial X)$ having compact closure in Y. Then, for $z \notin \psi \circ \phi(\partial X)$ we have*

$$deg\,(\psi \circ \phi, X, z) = \sum_i deg\,(\phi, X, \Omega_i)\, deg\,(\psi, \Omega_i, z)$$

and the sum on the right hand side is finite.

Proof. WLOG: $\phi, \psi \in C^1$ and z is a regular value for both $\psi \circ \phi$ and for ψ. Then,

$$\begin{aligned}
&\deg(\psi \circ \phi, X, z) = \textstyle\sum_{\psi\circ\phi(x)=z} \operatorname{sign} J_{\psi\circ\phi}(x) \\
&= \textstyle\sum_{\psi(\phi(x))=z} \operatorname{sign} J_{\psi(\phi(x))} \operatorname{sign} J_{\phi(x)} \\
&= \textstyle\sum_{\psi(y)=z} \operatorname{sign} J_{\psi(y)} \sum_{\phi(x)=y} \operatorname{sign} J_{\phi}(x) \\
&= \textstyle\sum_{\psi(y)=z} \operatorname{sign} J_{\psi(y)} \deg(\phi, X, y).
\end{aligned}$$

Note that if y belongs to a connected component of $Y \setminus \phi(\partial X)$ whose closure is not compact, then $\deg(\phi, X, y) = 0$ so the sum is restricted to the connected components whose closure is compact. Then

$$\begin{aligned} &\deg(\psi \circ \phi, X, z) = \textstyle\sum_{\psi(y)=z} \operatorname{sign} J_{\psi(y)} \sum_i \deg(\phi, X, \Omega_i) \\ &= \textstyle\sum_i \deg(\phi, X, \Omega_i) \deg(\psi, \Omega_i, z). \end{aligned}$$

2 Applications

Let B be the closed unit ball in $\mathbb{R}^n$.

Proposition 7. *Let $\phi : B \to \mathbb{R}^n$ be continuous and such that $\phi(x)$ never points opposite to x on ∂B, i.e.,*

$$\phi(x) + tx \neq 0, \qquad \forall t \geq 0, x \in \partial B.$$

Then $\phi(x) = 0$ has a solution inside B.

Proof. Indeed $t\phi(x) + (1-t)x$ does not vanish for any $t \in [0, 1]$ and $x \in \partial B$. Therefore $\deg(\phi, B, 0) = 1$.

Note that the same result holds for $-\phi$, i.e. if $\phi(x)$ never points in the same direction as x on ∂B. In particular, if $(\phi(x), x) \leq 0$ on ∂B then ϕ has a fixed point in B.

Proposition 8. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy*

$$\lim_{x \to \infty} \frac{(\phi(x), x)}{|x|} = \infty.$$

Then ϕ is onto.

Indeed, because $\phi(x) - y$ still satisfies the assumption, it is enough to prove that $\exists x,\ \phi(x) = 0$. But because $(\phi(x), x) \geq 0$ for $|x| \geq R$ we see that, if $\phi(x) \neq 0$ on $|x| = R$, we obtain a function which never points in the opposite direction of $\phi(x)$ on $|x| = R$, and we may use Proposition 7.

Theorem 5. *If $F : B \to \mathbb{R}^n$ is continuous and $F(\partial B) \subset B$ then F has a fixed point.*

Proof. Assume that there is no fixed point on the boundary. Let $\phi = x - F(x)$. Then, $0 = \phi(x) + tx = (1 + t)x - F(x)$ is impossible for $x \in \partial X$, $t \geq 0$. (If $t > 0$ this would send $F(x)$ outside B.) We apply Prop 7.

A variant of Brouwer's fixed point:

Theorem 6. *(Brouwer fixed point) A continuous map f from a closed convex set in $\mathbb{R}^n$ to itself has a fixed point.*

Proof. We first prove the result in the case when K is the closure of an open bounded convex set Ω. In that case, WLOG 0 is in the interior of the open set. We consider $\phi(x) = x - f(x)$. If we assume that $0 \notin \phi(\partial\Omega)$ then $0 \notin \phi_t(\partial\Omega)$ where $\phi_t(x) = x - tf(x)$. Indeed, if $x = tf(x)$ for $0 \leq t < 1$ and $x \in \partial\Omega$ then $tf(x)$ is on one hand in $\partial\Omega$ and on the other hand $(1-t)0 + tf(x) \in \Omega$ for $t < 1$ because 0 is in the interior and $f(x) \in K$. This is easily seen by taking a tiny ball B_r around zero so that its dilate by $\frac{1}{t}$ is still included in Ω. That produces $tf(x) + z \in K$ for $|z| < r$. We conclude by degree theory ϕ has a zero in Ω.

The general case is done by considering convolution with a mollifier ϕ_ϵ. The function $f_\epsilon = \mathbf{1}_K f * \phi_\epsilon$ is supported in $K_\epsilon = \{x | \operatorname{dist}(x, K) \leq \epsilon\}$ which is the closure of the open bounded convex set $\Omega_\epsilon = \{x | \operatorname{dist}(x, K) < \epsilon\}$, and f_ϵ maps K_ϵ to itself. A convergent subsequence of fixed points of f_ϵ converges as $\epsilon \to 0$ to a fixed point of f in K.

Theorem 7. *There is no continuous function $f : B \to \partial B$ so that $f_{|\partial B} = I$*

Indeed, if there were such a function, then $f_t(x) = (1-t)f(x) + tx$ would be a homotopy to I such that $0 \notin f_t(\partial B)$. Therefore $\deg(f, B, 0) = 1$, but that is impossible because $0 \notin f(B)$, so $\deg(f, B, 0) = 0$.

Theorem 8. *Borsuk's Theorem. Let X be a bounded open subset of $\mathbb{R}^n$ symmetric about the origin and such that $0 \in X$. Let $\psi : \partial X \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd ($\psi(-x) = -\psi(x)$). Then the deg$(\psi, X, 0)$ is odd.*

Proof in [1].

The next result is needed for the Leray-Schauder degree.

Proposition 9. *Let Ω be an open bounded set in $\mathbb{R}^n$ and consider $\mathbb{R}^n$ as a direct sum $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ with $n = n_1 + n_2$ so that any x in $\mathbb{R}^n$ has a unique decomposition $x = x_1 + x_2$ with $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. We consider a map of the form $f = x + \phi(x)$, with $\phi : \overline{\Omega} \to \mathbb{R}^{n_1}$. Suppose that $y \in \mathbb{R}^{n_1}$ and $y \notin f(\partial\Omega)$. Then*

$$deg(f, \Omega, y) = deg(f_{|\,\Omega_1}, \Omega_1, y)$$

where $\Omega_1 = \Omega \cap \mathbb{R}^{n_1}$.

Proof. We may assume that $f \in C^1(\Omega)$ and $y = 0 \in \mathbb{R}^{n_1}$. Let $\psi_j(x_j)$ be smooth compactly supported functions in $\mathbb{R}^{n_j}$ supported near the origin for $j = 1, 2$ and with normalized integrals $\int_{\mathbb{R}^{n_j}} \psi_j(x_j) dx_j = 1$. Then

$$\deg (f, \Omega, 0) = \int_{\mathbb{R}^n} f^*(\psi_1(x_1)\psi_2(x_2)dx).$$

Now $J_f(x) = \det(I + \nabla_{x_1}\phi(x_1 + x_2))$ so that

$$\deg (f, \Omega, 0) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} \psi_1(x_1 + \phi(x_1))\psi_2(x_2) \det(I + \nabla_{x_1}\phi(x_1 + x_2))dx_1 dx_2$$

We replace ψ_2 by a sequence of functions tending to the delta function, without changing the equality. We obtain

$$\begin{aligned} &\deg (f, \Omega, 0) = \textstyle\int_{\mathbb{R}^{n_1}} \psi_1(x_1 + \phi(x_1)) \det(I + \nabla_{x_1}\phi(x_1)dx_1 \\ &= \deg (f_{|\,\Omega_1}, \Omega_1, 0) \end{aligned}$$

References

[1] L. Nirenberg, Topics in Nonlinear Functional Analysis, CIMS, 1973-1974.

Degree in Infinite Dimensions

1 Schauder fixed point

Warning: Brouwer's Thm is false in infinite dimensions. Example: $\ell_2(\mathbb{N})$, with unit closed ball B. Then

$$f : B \to \partial B, \qquad f(x) = (\|x\|^2 - 1, x_1, x_2, \dots)$$

is continuous, and if it had a fixed point, the fixed point equations would be $x_1 = 0$, $x_2 = x_1$, $\dots, x_{n+1} = x_n$, so the fixed point would be 0, but it had to have norm equal to 1.

Definition 1. *A continuous function $F : S \subset X \to X$, where X is a Banach space, is compact if it maps bounded closed sets to relatively compact sets (sets whose closure is compact)*

Theorem 1. *Let $f : S \to X$ where S is closed and bounded in the Banach space X. Then f is compact iff it is a uniform limit of continuous finite range maps.*

Proof. If f is compact then $K = \overline{f(S)}$ is compact. Given $\epsilon > 0$ there exist $x_1 \dots x_{j(\epsilon)} \in K$ such that the balls B_i of centers x_i and radii ϵ cover K. Let ψ_i be a partition of unity for K subordinated to the cover, i.e $\psi_i \geq 0$ is supported in B_i and $\sum_i \psi_i = 1$ on K. Let

$$f_\epsilon(x) = \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))x_i$$

Then $f_\epsilon(x)$ belongs to the convex hull of x_i and

$$\|f(x) - f_\epsilon(x)\| \leq \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))\|f(x) - x_i\| \leq \epsilon$$

The argument in the other direction is an exercise.

Theorem 2. *(Schauder fixed point). Let S be a closed, convex, bounded subset of a Banach space X, and let $f : S \to S$ be a compact map. Then f has a fixed point.*

Proof. Consider $f_\epsilon(x)$ defined above, and let X_ϵ be the finite dimensional linear spaced spanned by x_i, $i = 1, \dots j(\epsilon)$. Since S is convex and $f_\epsilon(S)$ is contained in the convex hull of $f(S)$ we have $f_\epsilon : S \to S \cap X_\epsilon$. Therefore f_ϵ maps the closed bounded set $S \cap X_\epsilon$ to itself. This is a subset of X_ϵ so we may apply the finite dimensional Brouwer fixed point theorem, and find $x_\epsilon \in X_\epsilon \cap S$ such that $x_\epsilon = f_\epsilon(x_\epsilon)$. Now $f_\epsilon(x_\epsilon)$ has a convergent subsequence by the relative compactness of $f(S)$. Passing to the limit and using $x_\epsilon - f(x_\epsilon) = f_\epsilon(x_\epsilon) - f(x_\epsilon)$, we finish the proof.

2 Leray-Schauder Degree

If X is a Banach space and $\phi = I - K$ where $K : \overline{\Omega} \to X$ is a compact transformation, then we the image under $\phi(S)$ of a closed bounded set is closed. Indeed, if $y_n = \phi(x_n)$ with $x_n \in S$ converges to $y \in X$ then, because S is bounded and K is compact we may extract a subsequence, relabeled x_n, such that $Kx_n \to z$, and then $x_n = \phi(x_n) + Kx_n$ converges to $x = y + Kz$. By continuity, $y = x - Kz$.

If $y_0 \notin \phi(\partial\Omega)$, then it is at positive distance δ from $\partial\Omega$. We take an ϵ-approximation K_ϵ of K with range in X_ϵ, a finite dimensional subspace of X such that $y_0 \in X_\epsilon$. If $\epsilon \le \frac{\delta}{2}$ then $y_0 \notin \phi_\epsilon(\partial\Omega)$ where $\phi_\epsilon = I - K_\epsilon$. We consider

$$\phi_{\epsilon|\ X_\epsilon \cap \overline{\Omega}} : X_\epsilon \cap \overline{\Omega} \to X_\epsilon$$

Definition 2.

$$deg\,(\phi, \Omega, y_0) = deg\left(\phi_{\epsilon|\ X_\epsilon \cap \overline{\Omega}}, \Omega \cap X_\epsilon, y_0\right)$$

This is well defined by the last proposition in the chapter on finite dimensional degree. That means that we may change the finite dimensional space X_ϵ, and we may also change the finite range approximation K_ϵ. This follows by first placing both approximation ranges in a common (larger) finite dimensional space, and the using homotopy.

We note that if $y_0 \notin \phi(\overline{\Omega})$ then $\deg\,(\phi, \Omega, y_0) = 0$. All results in the chapter on finite dimensional degree are valid. In particular $\deg\,(\phi, \Omega, y_0)$

depends only on the homotopy class of $\phi : \partial\Omega \to X \setminus \{y_0\}$, where the homotopy is of the form $\phi_t = I - K_t$, with K_t continuous in $t \in [0,1]$ and compact for each t. In particular, the image of an open set under a one-to-one map $\phi = I - K$ is open.

3 First elementary applications

First, an application of Schauder's fixed point theorem. Let $K(s,t)$ be a continuous function and let

$$Ku(s) = \int_0^1 K(s,t)f(t,u(t))dt$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Taking $X = C([0,1])$ we have that K is a compact map on any ball $\|u\| \leq R$. By the Schauder fixed point, there exists u constinuous, such that

$$u(s) = Ku(s).$$

Indeed we want to find R such that K maps the ball of radius R into itself. Now, let $M = \sup|f|$ and $L = \sup|K|$. The range of K obeys $\|Ku\| \leq ML$, so that if we take $R \geq ML$ we are done.

We recall from functional analysis that if K is a *linear* compact operator then $I - K$ is Fredholm of index zero. That is, range is closed, of finite codimension, kernel is finite dimensional, and

$$\dim \ker(I-K) = \operatorname{codim} \operatorname{Range}(I-K).$$

We recall here also $P(x,D)$ linear elliptic operators in Sobolev spaces and Hölder spaces, and embeddng theorems.

Now an application involving elliptic operators. Let $P = P(x,\partial)$ be an elliptic operator of order m

$$P(x,D)u = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u$$

with principal symbol

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

that does not vanish for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We consider boundary conditions on $\partial\Omega$ that are good: $Bu = 0$ on $\partial\Omega$ imply that the $P : X \to Y$ is a Fredholm operator (kernel finite dimensional, closed range with finite dimensional codimension. In many cases the index of P is zero, i.e. the dimension of the kernel equals the dimension of the coimage. Examples are the Laplacian with Neumann or Dirichlet BC.

Now we consider a sublinear function $g(x, \partial^\alpha u)$ with $|\alpha| \le m - 1$, satisfying

$$|g(x, \partial^\alpha u)| \le C(1 + \sum_{|\alpha|\le 1} |\partial^\alpha u|)^r$$

with $r < 1$, uniformly for $x \in \overline{\Omega}$ and arbitrary entries $\partial^\alpha u \in \mathbb{R}^M$ where M is the number of such things. We consider the equation

$$P(x, D)u = g(x, \partial^\alpha u)$$

with boundary conditions $Bu = 0$. We assume that the index of P is zero and P is injective. Then there exists a $C^\infty(\overline{\Omega})$ solution. (Assuming the boundary, and all coefficients are smooth all the way to the boundary).

The idea of the proof is to take $I - P^{-1}g(x, \partial^\alpha u)$ and apply degree theory. We may choose the space $X = C^{m-1}(\overline{\Omega}) \cap \{Bu = 0\}$.

The steps of the proof are instructive. First we establish a priori estimates. For example, we can look at $W^{m,p}(\Omega)$, $p > n$, and assuming a solution, obtain uniform bounds

$$\|u\|_{m,p} \le C_{m,p}$$

with constant independent of anything. This comes from $r < 1$ and ellipticity. We could have had a fully nonlinear equation here (right hand side depending on all m derivatives). Then we show that this means that solutions have to belong to a fixed ball of X. This uses Sobolev embedding and $p > n$ and the fact that the right hand side sees $m - 1$ derivatives only. Then we take a stricly larger ball $B \subset X$. There are no solution on the boundary of this ball. Also, by embeddings, $K(u) = P^{-1}g(x, \partial^\alpha u)$ is compact (because its range is bounded in the Hölder space $C^{m-1,\gamma}(\Omega)$, with $\gamma = 1 - \frac{n}{p}$. By homotopy to I vis $I - tK$, the degree $\deg(I - K, B, 0) = 1$, and therefore there is a solution. Smoothness follows by bootstrapping.

This was sublinear, but set the stage. Here is a semilinear example that is not trivial: the existence of steady solutions of Navier-Stokes equations with arbitrary forcing in both 2 and 3 dimensions.

The equation

$$Au + B(u,u) = f$$

where A is the Stokes operator and $B(u,v) = \mathbb{P}(u\cdot\nabla v)$ has solutions $u\in V$ for any $f\in L^2(\Omega)^d$ with $\mathbb{P}f = f$.

Here Ω is an open bounded set with smooth boundary, $d = 2,3$ and $\mathbb{P}$ is the projector on divergence-free functions in L^2. We recall notations: V is the closure of the space of divergence-free $C_0^\infty(\Omega)$ vectors in the topology of $H^1(\Omega)^d$, $d = 2,3$. The Stokes operator is $A = -\mathbb{P}\Delta$ with domain $\mathcal{D}(A) = V\cap H^2(\Omega)^d$. The function

$$K(u) = A^{-1}B(u,u) : V \to V$$

is compact. This follows because $A^{-\frac{3}{4}}B(u,u)$ is continuous

$$\|A^{-\frac{3}{4}}B(u,v)\|_V \le C\|u\|_V\|v\|_V$$

(see [2]). For any $t\in[0,1]$, the equation

$$u + tK(u) = tA^{-1}f$$

has no solutions on the boundary of the ball $B_R = \{u \mid \|u\|_V < R\}$ for $R > \|A^{-1}f\|_V$. Indeed, any solution in V obeys

$$\|u\|_V^2 = t\langle A^{-1}f, u\rangle_V.$$

Therefore, $\phi(u) = u + K(u) - A^{-1}f$ obeys $\deg(\phi, B_R, 0) = 1$ and the equation has solution in B_R.

Finally, for a quasilinear example: Damped and driven Euler equations in 2D.

Consider a bounded domain $\Omega\subset\mathbb{R}^2$. Consider a time independent force $F\in H^1(\Omega)$ and a positive constant $\gamma > 0$. Then there exist $H^1(\Omega)$ solutions of the damped Euler equations

$$\gamma u + u\cdot\nabla u + \nabla p = F, \quad \operatorname{div} u = 0$$

in Ω with $u\cdot n = 0$ on $\partial\Omega$.

The proof starts by adding artificial viscosity, thus producing a semilinear equation. We take the vorticity-stream formulation of the equation, $\omega = \Delta\psi$, $u = \nabla^\perp\psi$. The vorticity equation is

$$\gamma\omega + u\cdot\nabla\omega = f$$

with $f = \nabla^\perp \cdot F$. This we want to solve in L^2. We take first $\nu > 0$ and seek solutions of

$$-\nu\Delta\omega + \gamma\omega + u \cdot \nabla\omega = f$$

with the artificial boundary condition $\omega = 0$ at $\partial\Omega$. We should think of this as being

$$\nu\Delta^2\psi + \gamma(-\Delta\psi) + J(\psi, \Delta\psi) = f$$

where $J(f, g) = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$ is the Poisson bracket. The boundary conditions are $\psi = \Delta\psi = 0$ at $\partial\Omega$. (These are "good").

We start by showing there exist solutions at fixed ν. Then we pass to the limit as $\nu \to 0$. At fixed ν.

References

[1] L. Nirenberg, Topics in Nonlinear Functional Analysis, CIMS, 1973-1974.

[2] P. Constantin, C. Foias, Navier-Stokes Equations, U. Chicago Press, 1988.