GROWTH IN AN UNSTRUCTURED POPN

BALANCING EQN:

$$K(1) = K(0) + B(0) - D(0) \left[+ NM(0)\right]$$

$$= K(0)\left[1 + \underbrace{\frac{B(0)}{K(0)}}_{CBR} - \underbrace{\frac{D(0)}{K(0)}}_{CDR}\right]$$

and then

$$K(2) = K(1) + B(1) - D(1)$$

$$= K(1)\left[1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)}\right]$$

$$= K(0)\left[1 + \frac{B(0)}{K(0)} - \frac{D(0)}{K(0)}\right]\left[1 + \frac{B(1)}{K(1)} - \frac{D(1)}{K(1)}\right]$$

if CBR & CDR stay fixed over time

$$A = \left[1 + \frac{B}{K} - \frac{D}{K}\right]$$

$$K(1) = A\,K(0)$$

$$K(2) = A\,K(1) = A \cdot (A K(0)) = A^2 K(0)$$

$$\vdots$$

$$\underline{\underline{K(t) = A^t K(0)}}$$

Define the growth rate R as the slope of the log of popn size over time

$$\log(A^t) = t \log(A)$$

$K(t) = A^t K(0) \rightarrow \log K(t) = \log K(0) + t\underbrace{\log(A)}$

measure / summary of popn growth

$R = \log(A)$

since $e^R = A$

so we have

$$K(t) = (e^R)^t K(0) = e^{Rt} K(0)$$

EXPONENTIAL GROWTH (or DECAY)

$R > 0 \rightarrow$ popn will increase in size exponentially over time

$R = 0 \rightarrow$ popn doesn't change in size

$R < 0 \rightarrow$ popn will decrease in size over time

Leslie Matrix

- focus on humans.
- we'll assume $\begin{cases} \text{5 year age groups} & J = 10 \text{ total groups} \\ & \text{so } 0\text{-}4, 5\text{-}9, \dots, 45\text{-}49 \\ \text{only females} \\ \text{no migration ("closed")} \end{cases}$

- entry a_{ij} of the Leslie Matrix tells us how many future members of age group i there will be, per member of age group j

$a_{i \leftarrow j}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1J} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{J1} & & a_{JJ} \end{pmatrix} \begin{pmatrix} {}_5K_0(0) \\ {}_5K_5(0) \\ \vdots \\ {}_5K_{45}(0) \end{pmatrix}$$

EXAMPLE:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$$

entry a_{12}, which says there is 1 future member of the 1st age gp for each current member of the second age gp

a_{32}

let $K(0) = \underbrace{\begin{pmatrix} {}_5K_0(0) \\ {}_5K_5(0) \\ \vdots \\ {}_5K_{45}(0) \end{pmatrix}}_{J \text{ entries}}$

$$\underbrace{{}_5K_5(1)}_{\text{projected \# people in age group 5-10 (second age group)}} = \underbrace{{}_5K_0(0)}_{\text{people currently aged 0 to 5}} \times \underbrace{a_{21}}_{\text{avg \# future members of } i^{th} \text{ age group per person in first age gp}} + {}_5K_5(0) \times a_{22} + \cdots + {}_5K_{45}(0) \times a_{2J}$$

$$= \sum {}_5K_{x(j)}(0)\, a_{2j}$$

$x(2) = 5$

this is the same as the dot (vector) product of second row of Leslie Matrix and the starting population vector.

... so, multiplying A by $K(0)$ gives us the projected population $K(1)$

$$K(1) = A K(0)$$

Human Leslie Matrices

$$\begin{pmatrix} \boxed{\text{Kids} \quad \text{Kids} \quad \cdots \quad \text{Kids}} \\ \text{survivors} & 0 & & 0 \\ 0 & \text{survivors} & & \\ & & \text{survivors} & 0 \\ 0 & & 0 & \ddots \end{pmatrix}$$

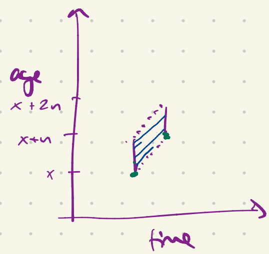
first row: comes from reproduction

subdiagonal: about survival

① SURVIVORS

$$A_{j(x)+1,\, j(x)} = \frac{{}_nL_{x+n}}{{}_nL_x}$$

A_{21} or A_{32} or A_{43} ⋮



$$\frac{\ell(x+n) + \ell(x+n+\varepsilon) + \dots}{\ell(x) + \ell(x+\varepsilon) + \dots} = \frac{{}_nL_{x+n}}{{}_nL_x}$$

② KIDS

$$A_{1,j(x)} = \frac{{}_nL_0}{2\ell_0} \times \frac{n}{n} \left({}_nF_x + \frac{{}_nL_{x+n}}{{}_nL_x}\, {}_nF_{x+n} \right) f_{fab}$$

this accounts for

- women aging into next age interval & experiencing those fertility rates
- not all women survive the projection interval
- not all babies survive the projection interval

How to project more than one time step?

Say we want to project 50 years (10 time steps)

$$K(10) = \underbrace{A \cdots A}_{10 \text{ times}} K(0) = A^{10} K(0)$$

or more generally

$$K(t) = A^t K(0).$$

KEY ASSUMPTION:
mortality & fertility remain constant
i.e. Leslie Matrix does not change over time

STABLE POPN RELATIONSHIPS

In a stable popn

- AGE DISTRIBUTION
- GROWTH RATE
- BIRTH RATE
- DEATH RATE

all become fixed (unchanging) over time, independent of starting conditions

AGE STRUCTURE will be given by

$$\underbrace{{}_nK_x(t)}_{\text{\# people in age gp } x \text{ to } x+n \text{ at time } t} = \underbrace{B(t)}_{\text{\# births at time } t} \underbrace{e^{-rx}}_{\text{factor by which \# births has changed over } x \text{ years}} \underbrace{\frac{{}_nL_x}{l_0}}_{\text{survival of newborns born } x \text{ years ago to age } x \text{ to } x+n}$$

$(e^r)^{-x}$

PDF of a length-biased sample

$$f(x_{lb}) \propto x\, f(x)$$

$$f(x_{lb}) = K\, x\, f(x)$$ we know a pdf has to integrate to 1

$$\int_0^\infty K x f(x)\,dx = 1$$

$$K \int_0^\infty x f(x)\,dx = 1 \iff \boxed{\int_0^\infty x f(x)\,dx} = \frac{1}{K}$$

$$\mu_x = \frac{1}{K}$$

$$f(x_{lb}) = \frac{x f(x)}{\mu_x}.$$

$E[X^2]$ recall that $Var[X] = \sigma_x^2 = E[X^2] - E[X]^2$

$$\iff E[X^2] = \sigma_x^2 + E[X]^2$$

$$= E[X]^2\left[1 + \underbrace{\frac{\sigma_x^2}{E[X]^2}}_{= cv^2}\right]$$

$$= \mu^2\left[1 + \frac{\sigma_x^2}{\mu^2}\right]$$

$$= \mu^2[1 + cv^2(x)]$$

$$cv = \frac{\sigma_x}{E[X]}$$

Avg lifespan of the living

- $d(x)$: # deaths in a cohort at age x
- Density of cohort deaths $f(x) = \dfrac{d(x)}{\ell(0)}$
- $\ell(x)$: # survivors to age x

- Avg lifespan in a cohort $e(0) = \dfrac{\int_0^\omega \ell(x)\,dx}{\ell(0)}$

$$= \frac{\int_0^\omega x\, d(x)\,dx}{\int_0^\omega d(x)\,dx}$$

- denominator $\ell(0) = \int_0^\omega d(x)\,dx$

- numerator
$$\int_0^\omega \ell(x)\,dx = \int_0^\omega x\, d(x)\,dx$$

2 ways of writing total lifespan of a cohort

sum over all ages of death of cohort members of
dying at age x • lifespan of those dying at age x

- $f_\ell(x) = \dfrac{x\, d(x)}{\int_0^\omega \ell(a)\,da}$ ← # living people when snapshot is taken

$$= x \cdot \frac{f(x)\cdot \ell(0)}{e(0)\cdot \ell(0)} = \boxed{x\,\frac{f(x)}{e(0)}}.$$

... so ALL = length-biased sample of cohort lifespans

$$= \mu_x\left[1 + cv^2(x)\right] = e(0)\left[1 + cv_x^2\right]$$