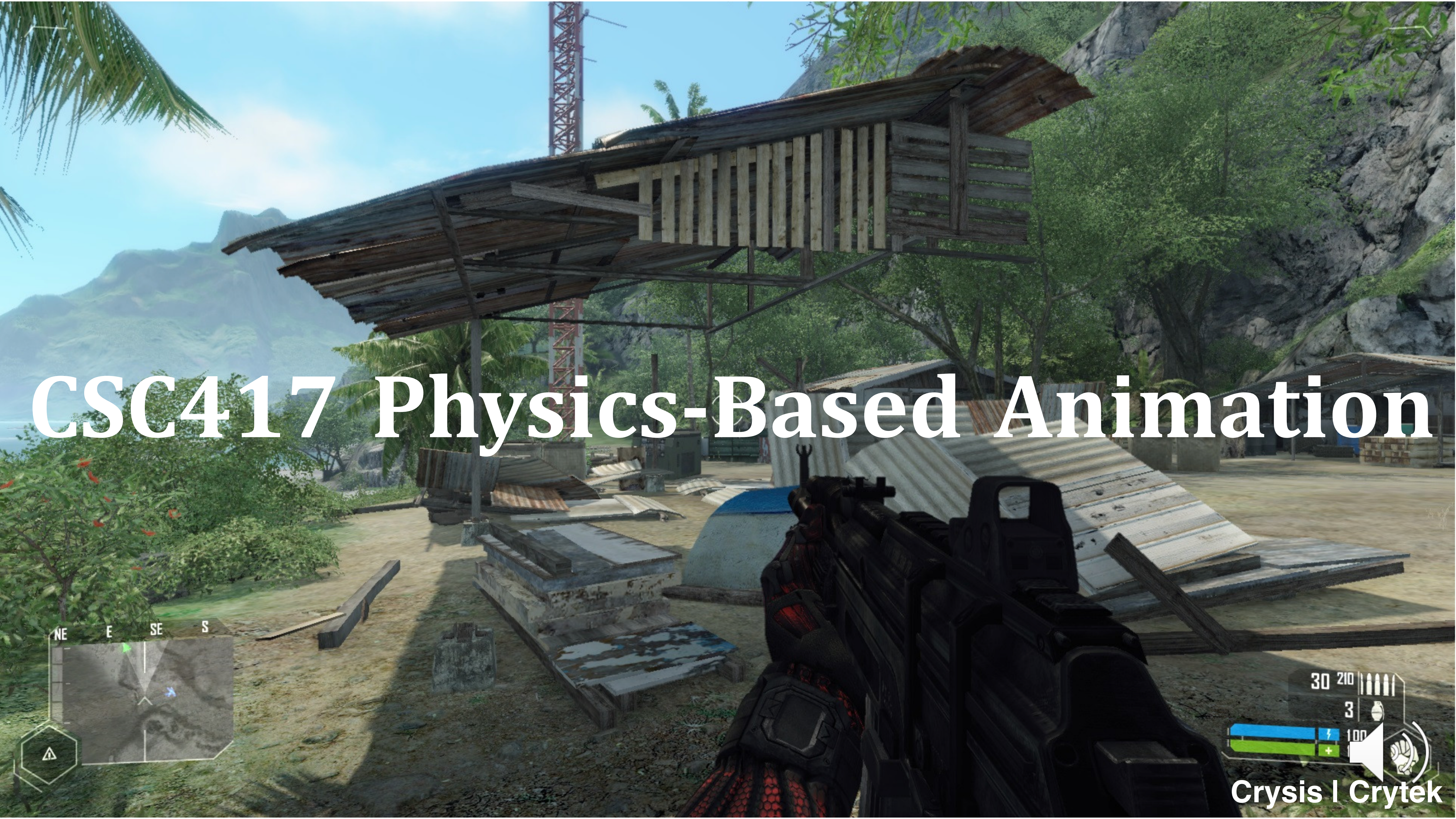
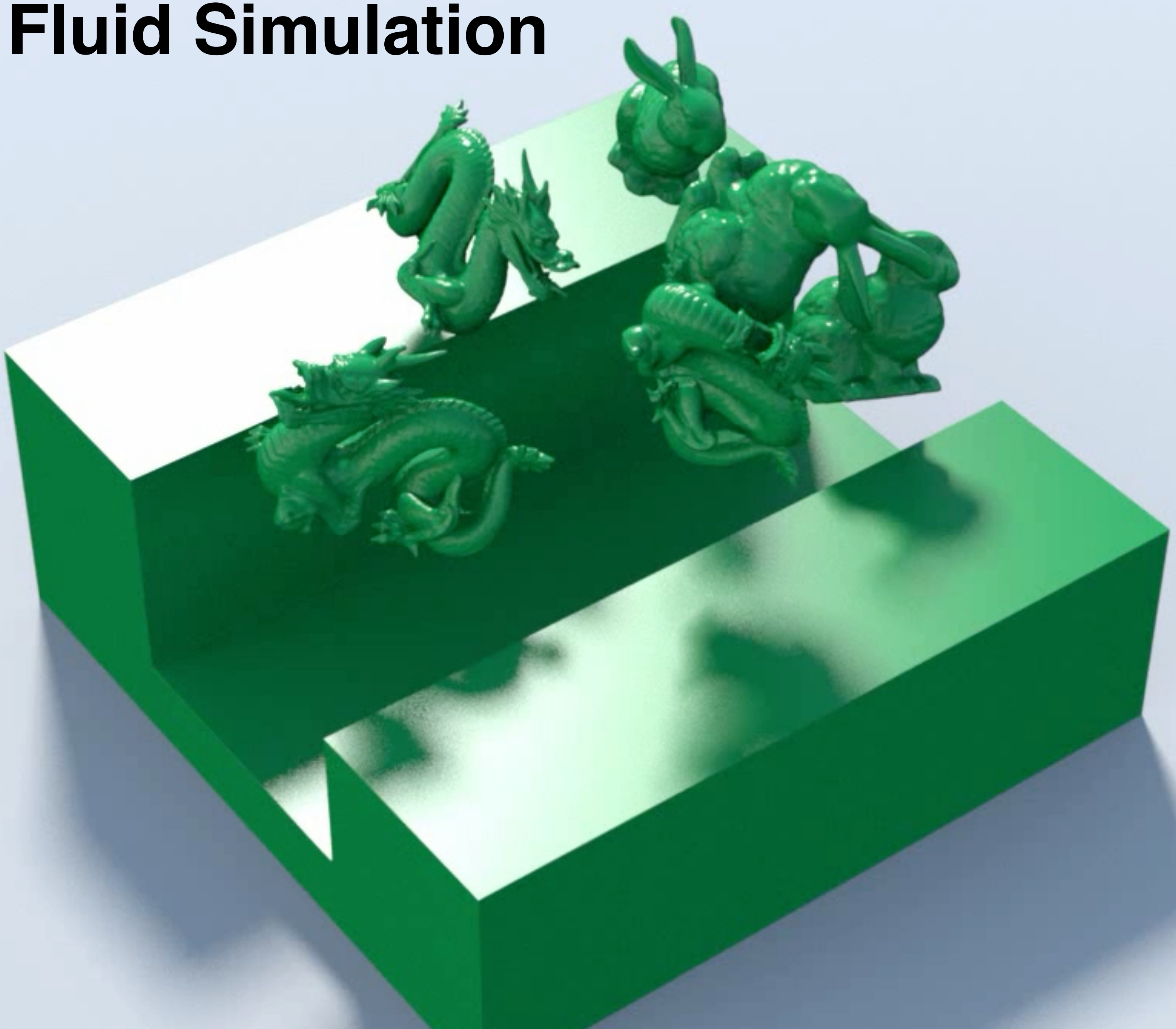
CSC417 Physics-Based Animation



Crysis | Crytek

Last Video: Rigid Body Simulation with Contact

This Video: Fluid Simulation



Goldade et al | A Practical Method for High-Resolution Embedded Liquid Surfaces

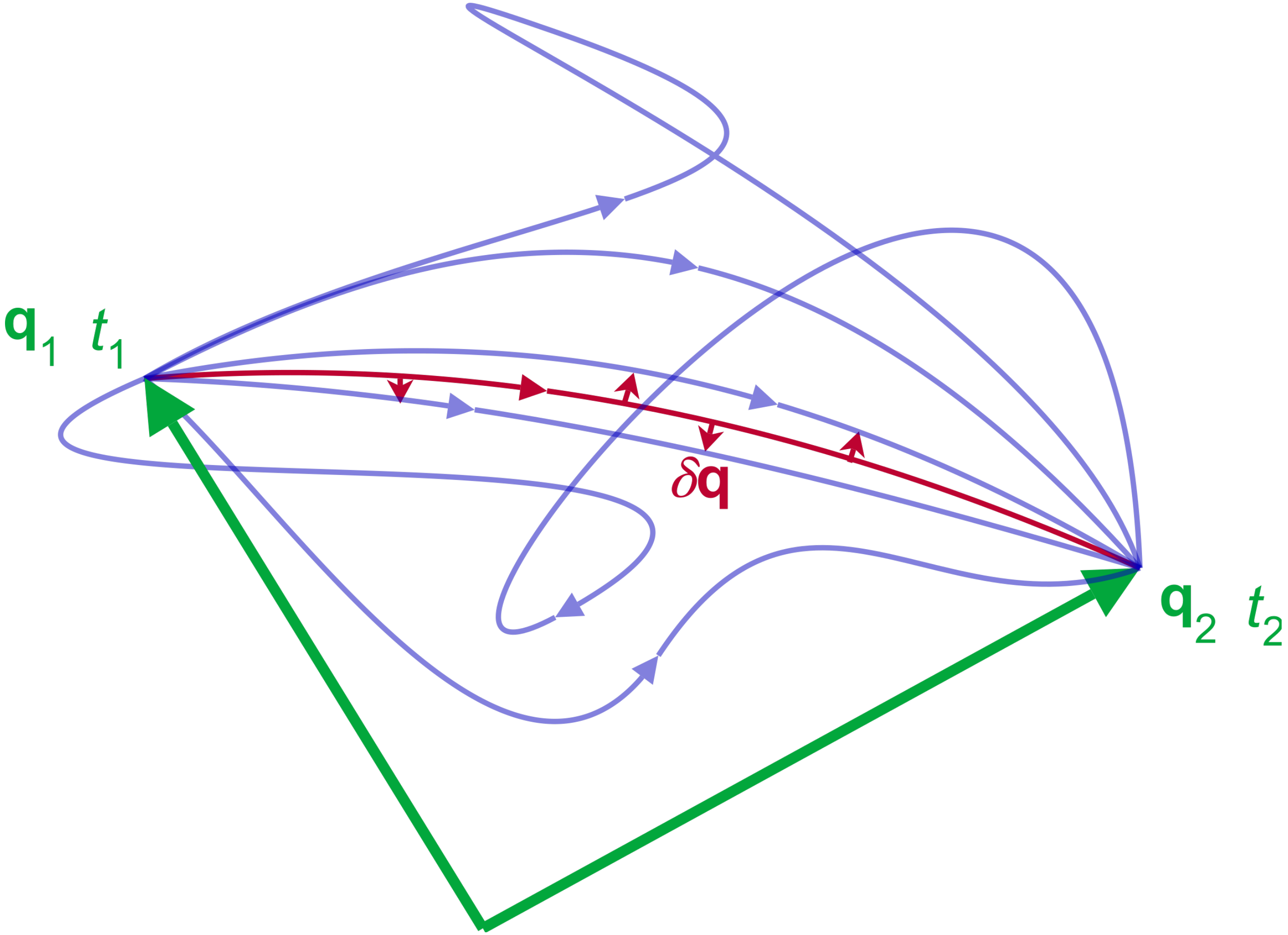
SECOND EDITION

Fluid Simulation for Computer Graphics

Robert Bridson

CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

The Principle of Least Action



Assume you know the end points, find the path between them by finding a stationary point of the **ACTION**

Gottfried Wilhelm Leibniz
(was pretty good at mat[illegible])

Vectorial Mechanics

Newton's Laws

momentum $\underbrace{m}_{\text{mass}}\overbrace{\mathbf{v}}^{\text{velocity}}$

time rate-of-change of the momentum = force

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{f}$$

for constant mass

$$m\overbrace{\frac{d\mathbf{v}}{dt}}^{\text{acceleration}} = \mathbf{f}$$

Newton's Laws

momentum $\underbrace{m}_{\text{mass}}\overbrace{\mathbf{v}}^{\text{velocity}}$

time rate-of-change of the momentum force

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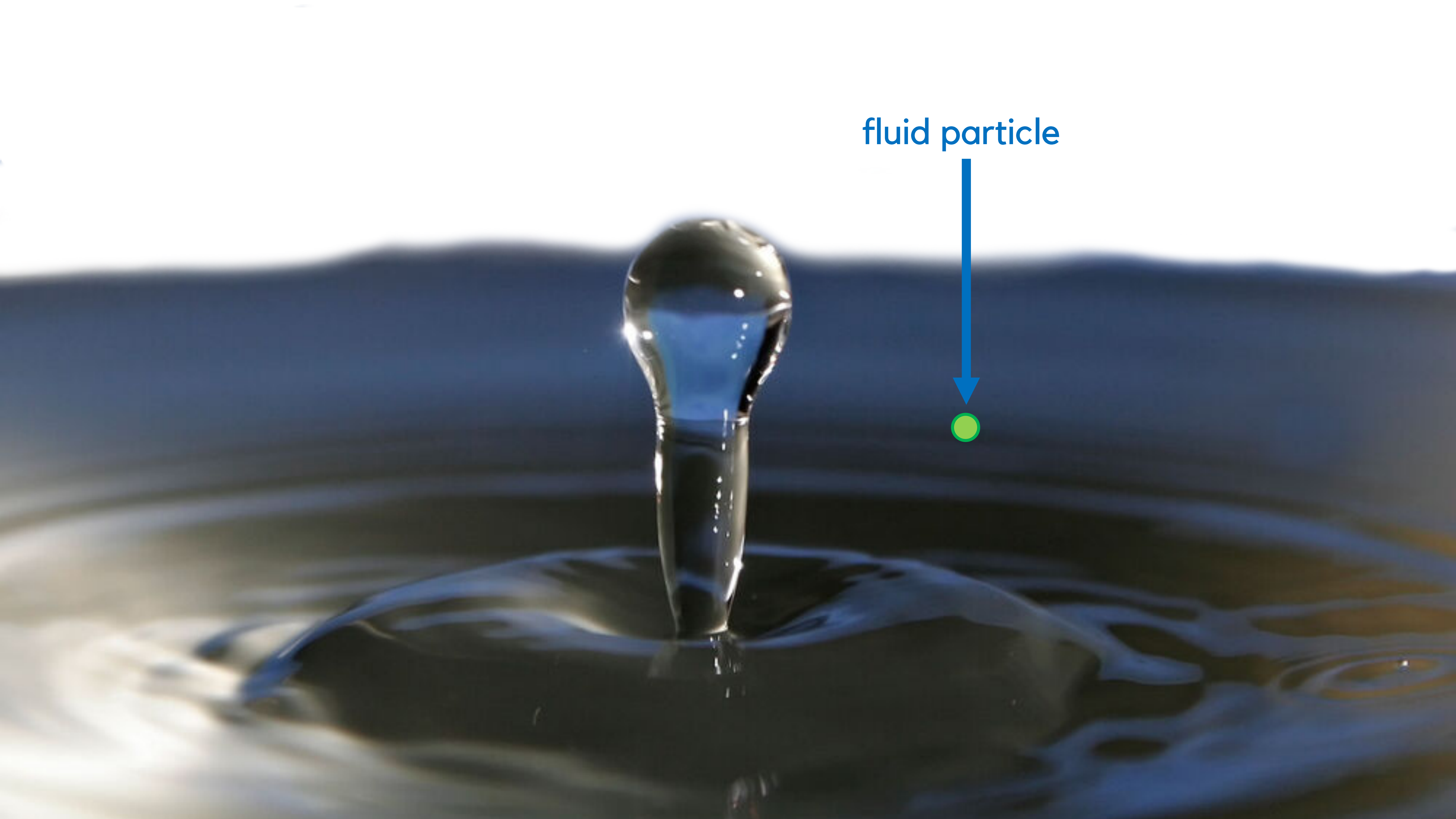
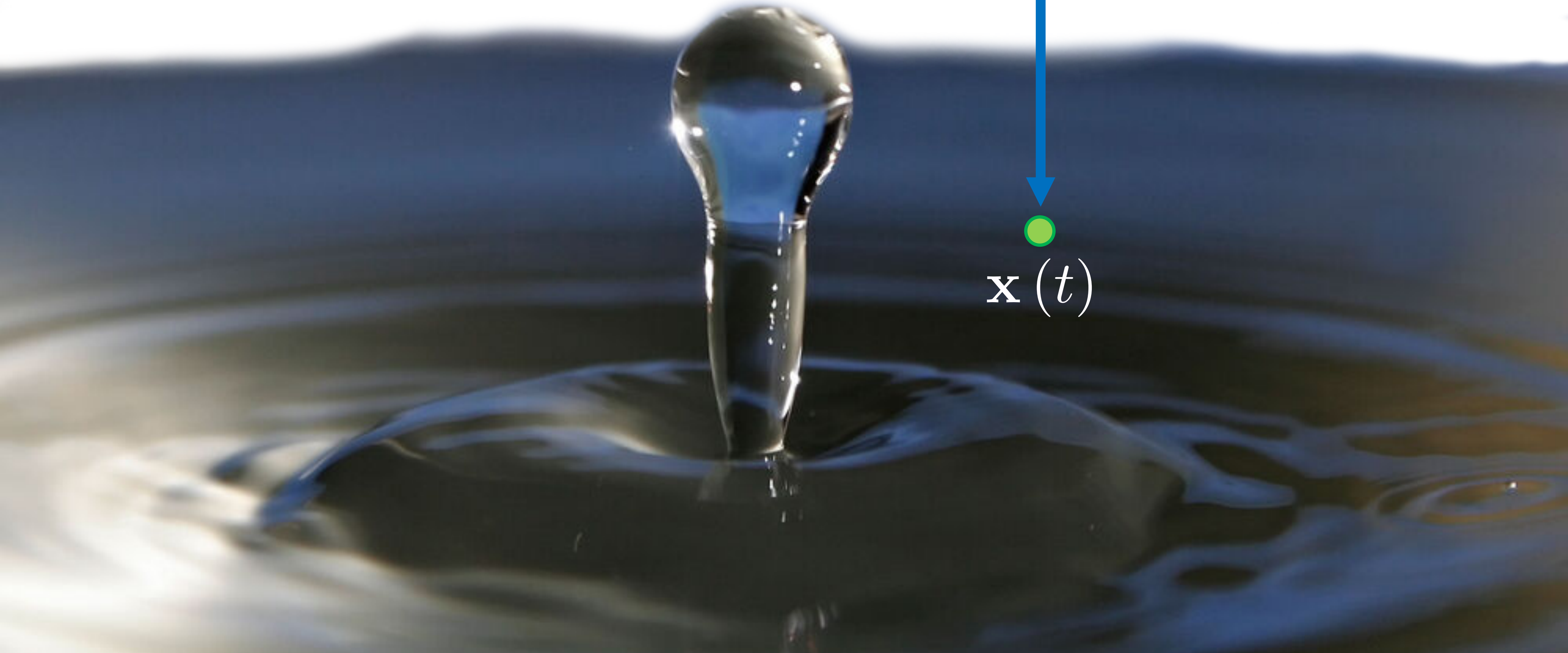
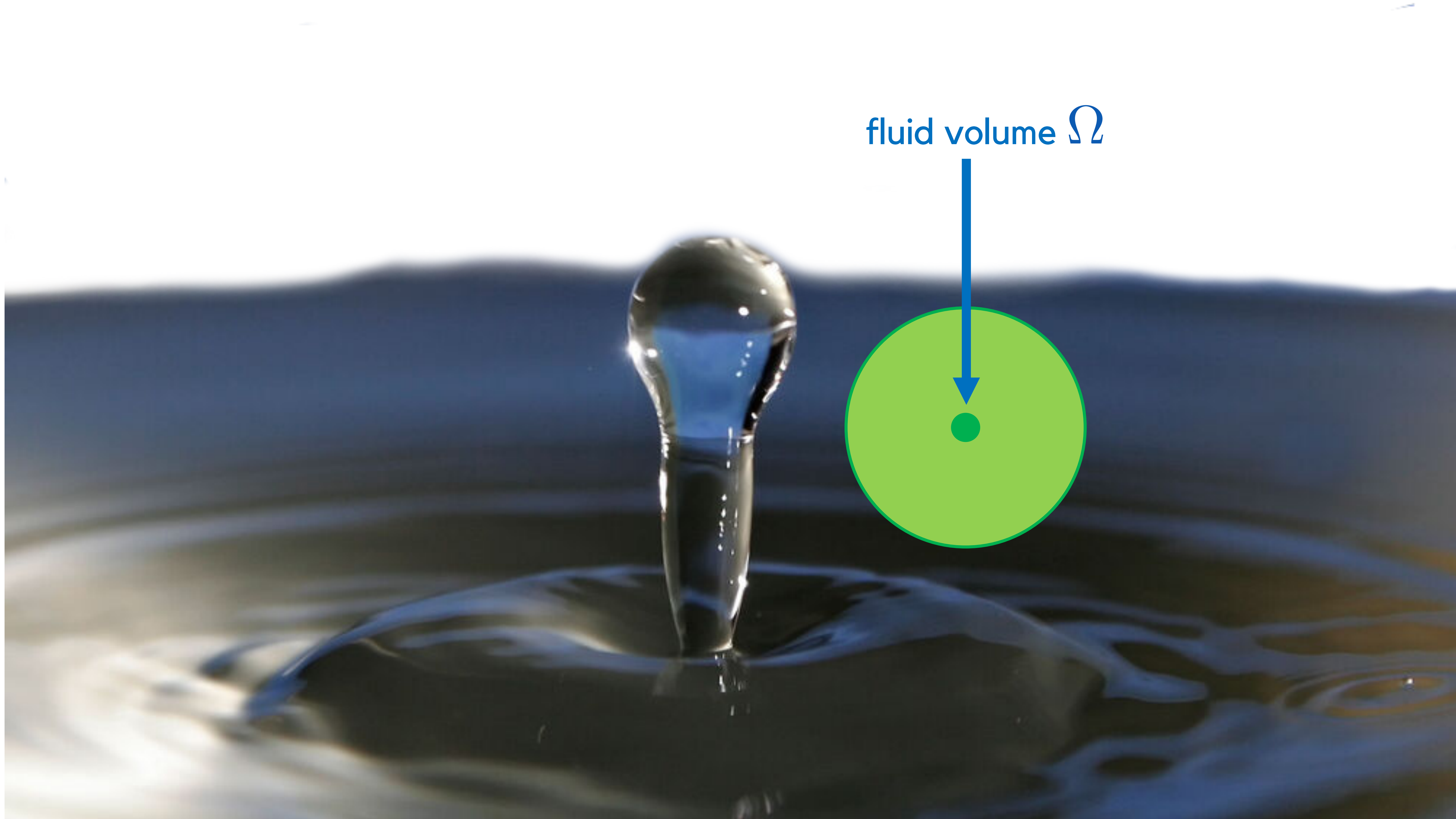
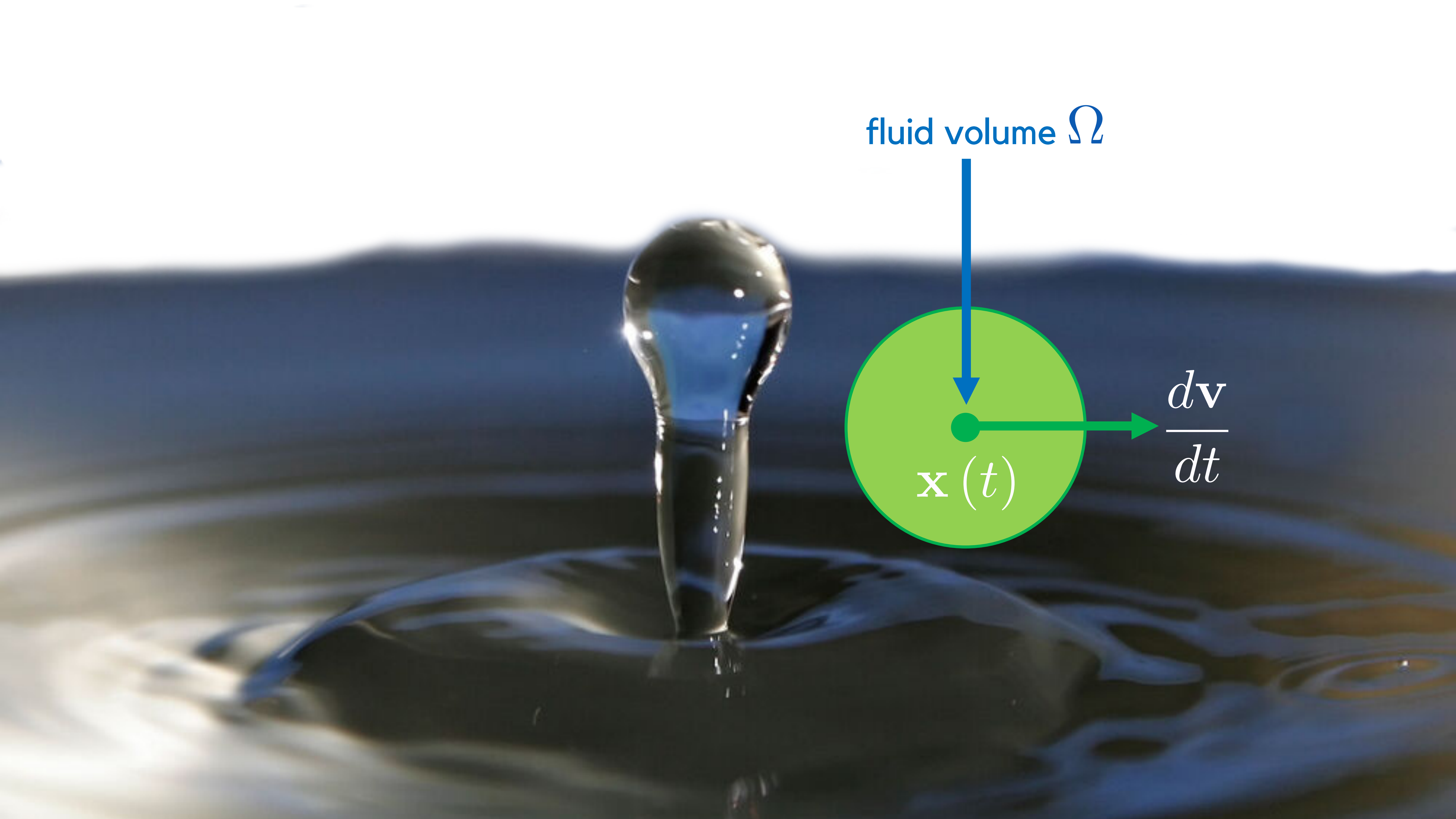
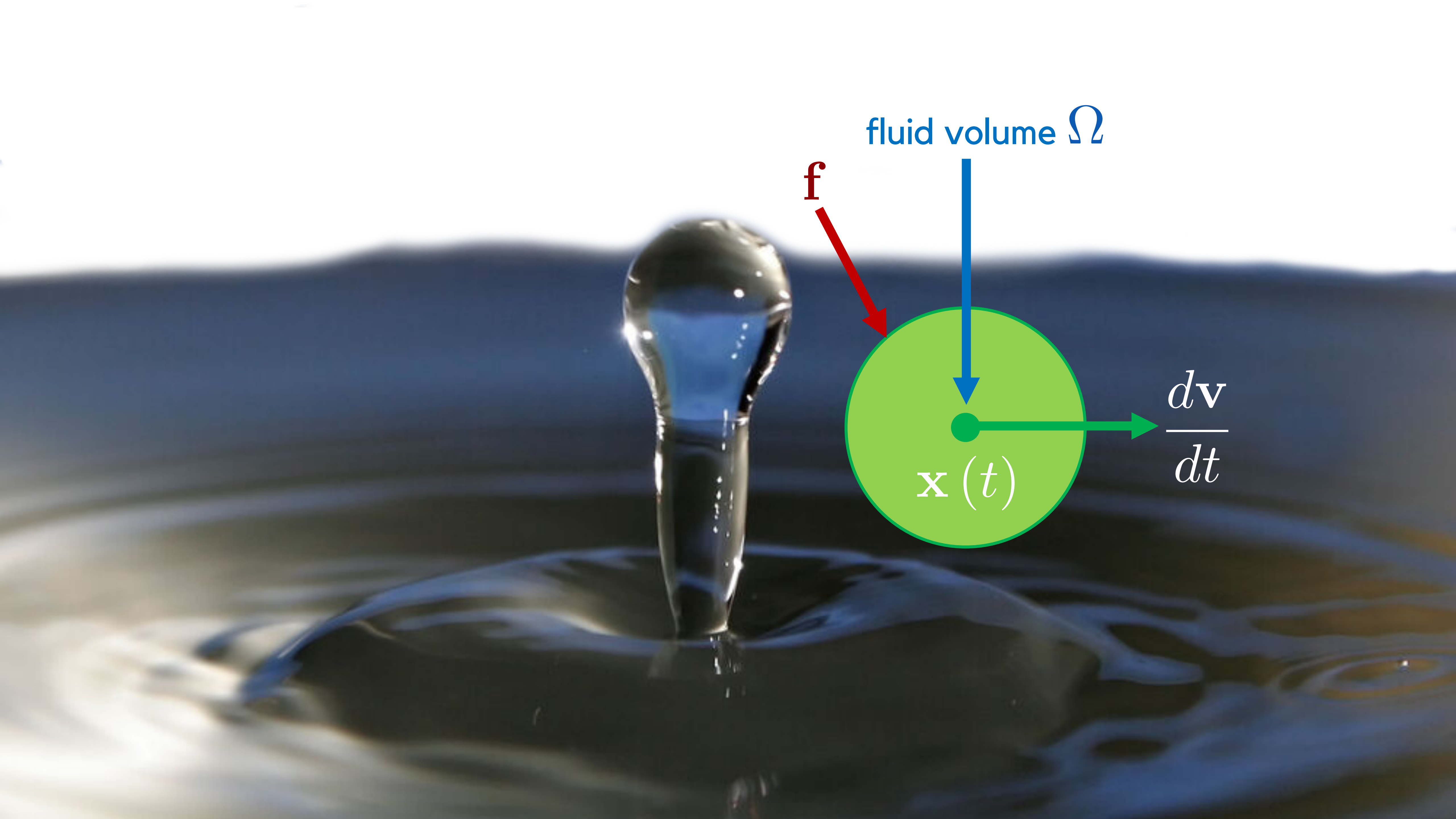

fluid particle

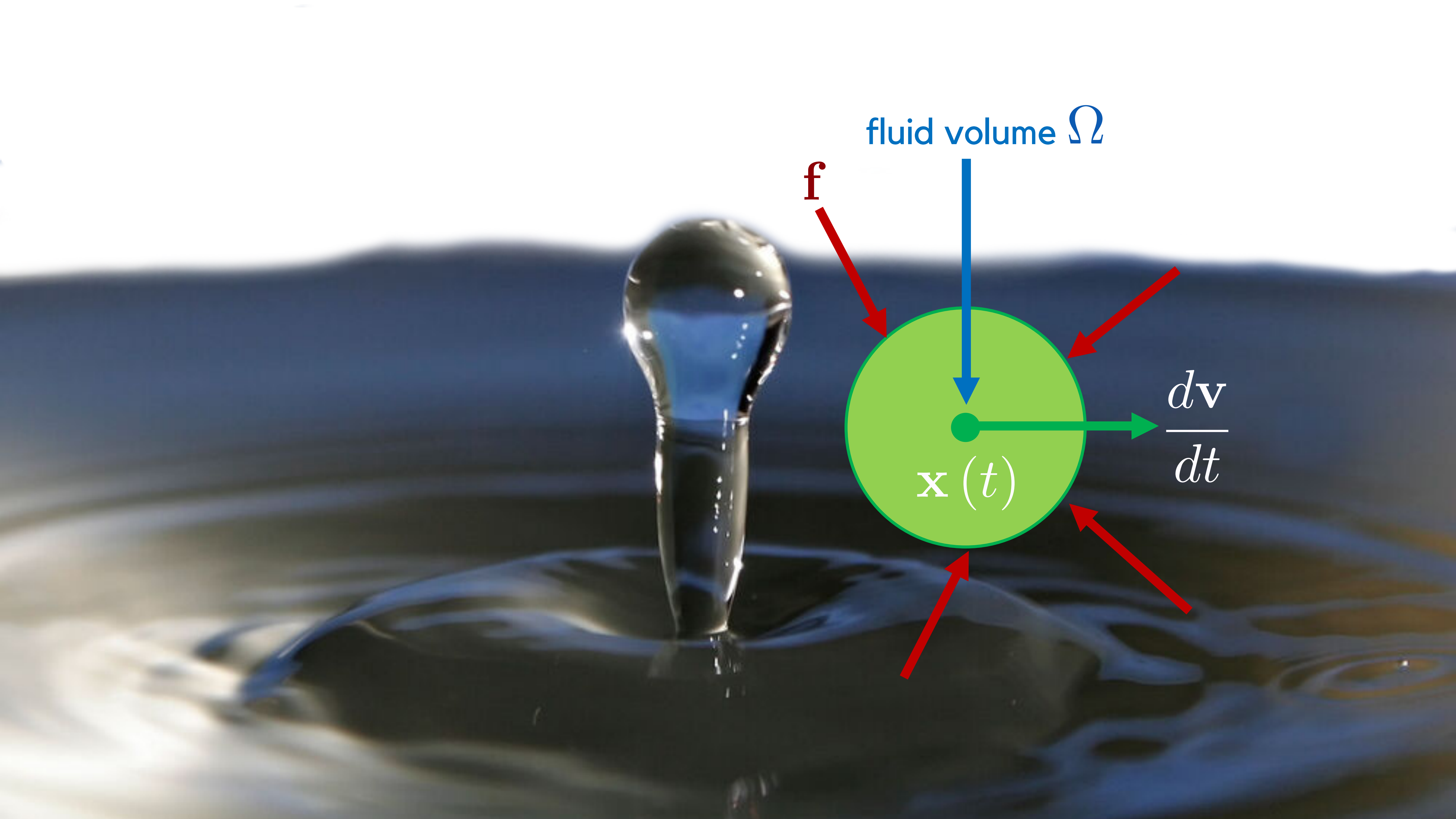
$$m\frac{d\mathbf{v}}{dt} = \mathbf{f}$$

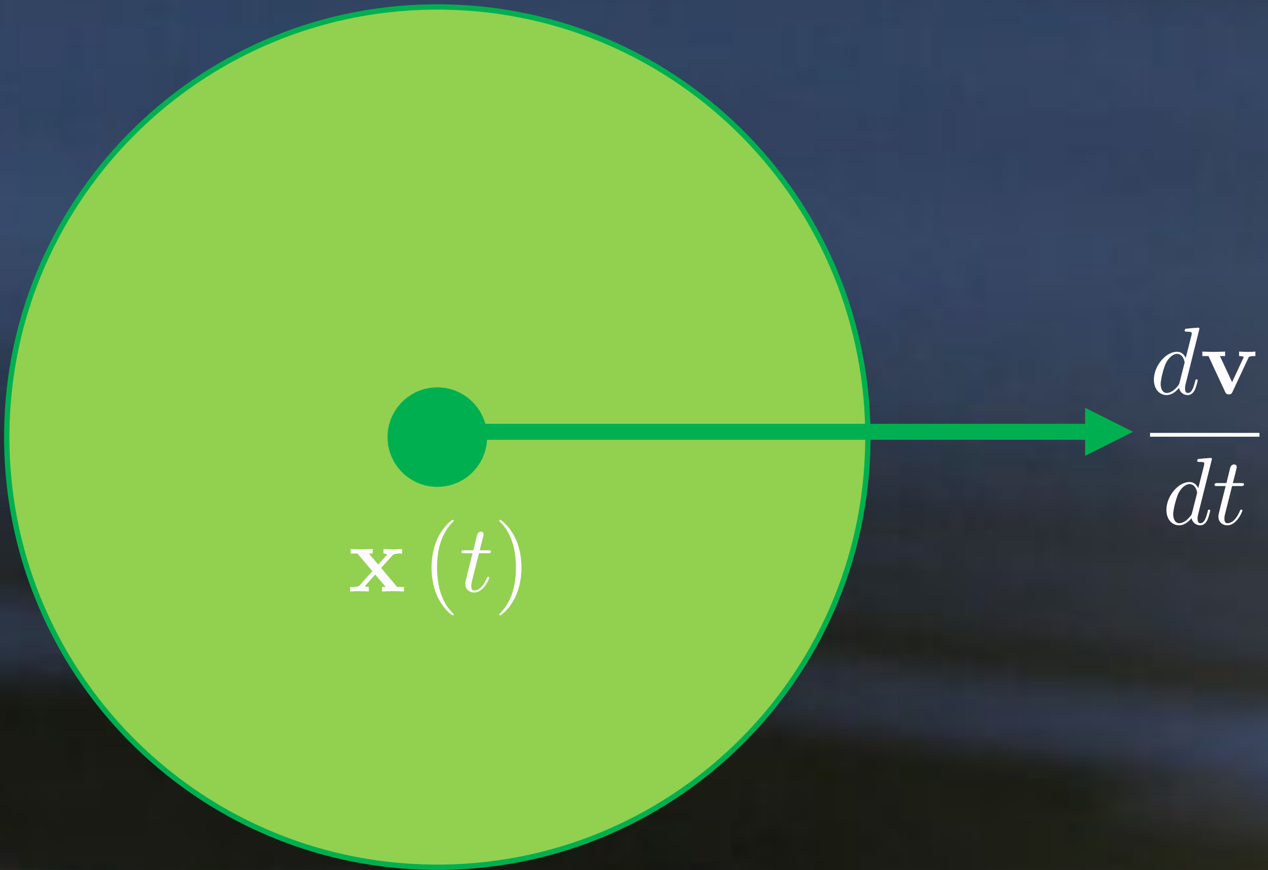



fluid volume Ω


fluid volume Ω
$\mathbf{x}(t)$
$\frac{d\mathbf{v}}{dt}$


fluid volume Ω
$\mathbf{f}$
$\mathbf{x}(t)$
$\frac{d\mathbf{v}}{dt}$


fluid volume Ω
$\mathbf{f}$
$\frac{d\mathbf{v}}{dt}$
$\mathbf{x}(t)$

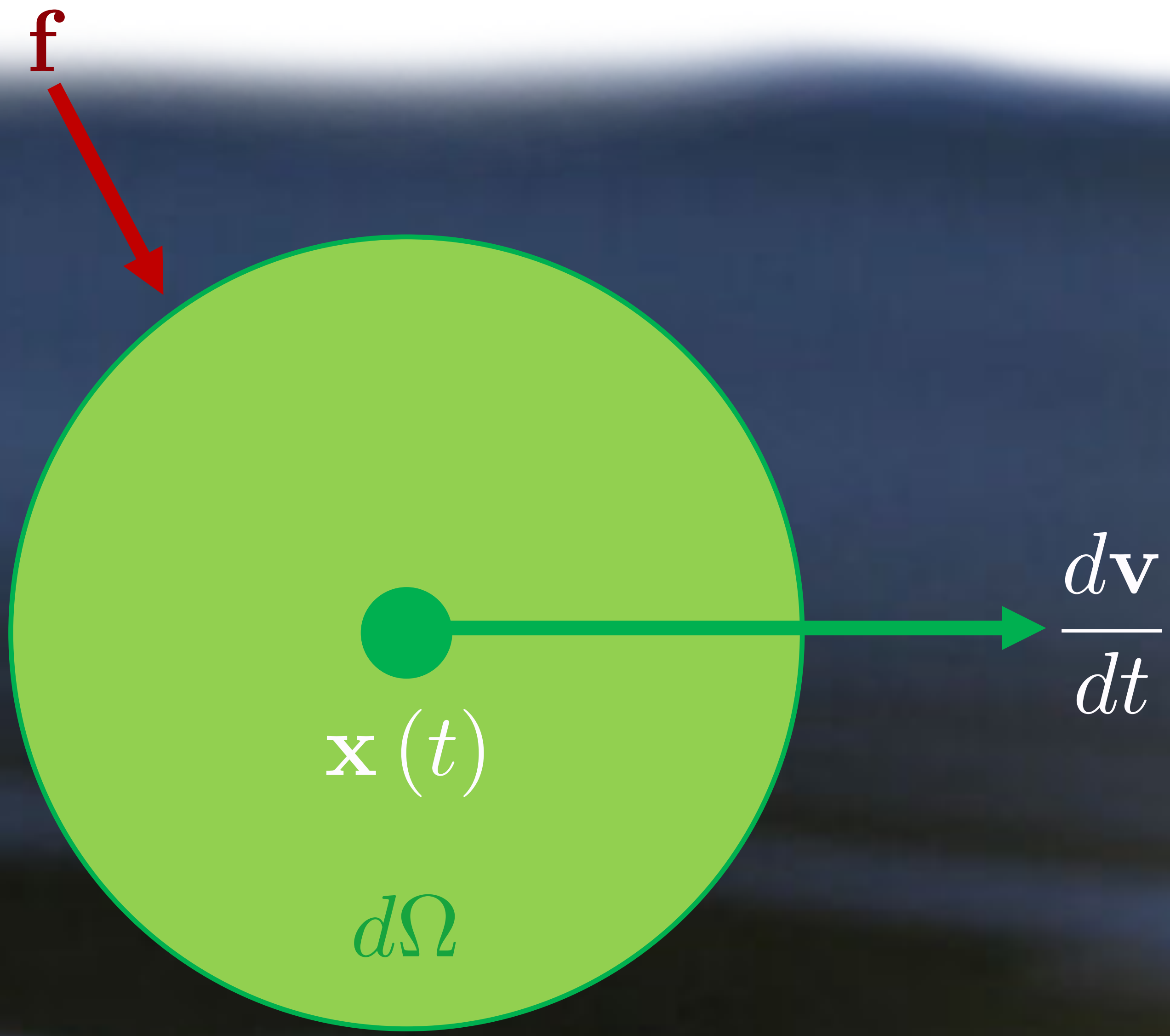


$$m\frac{d\mathbf{v}}{dt} = \mathbf{f}$$

mass = density X volume

$$\rho\frac{d\mathbf{v}}{dt}d\Omega = \mathbf{f}$$

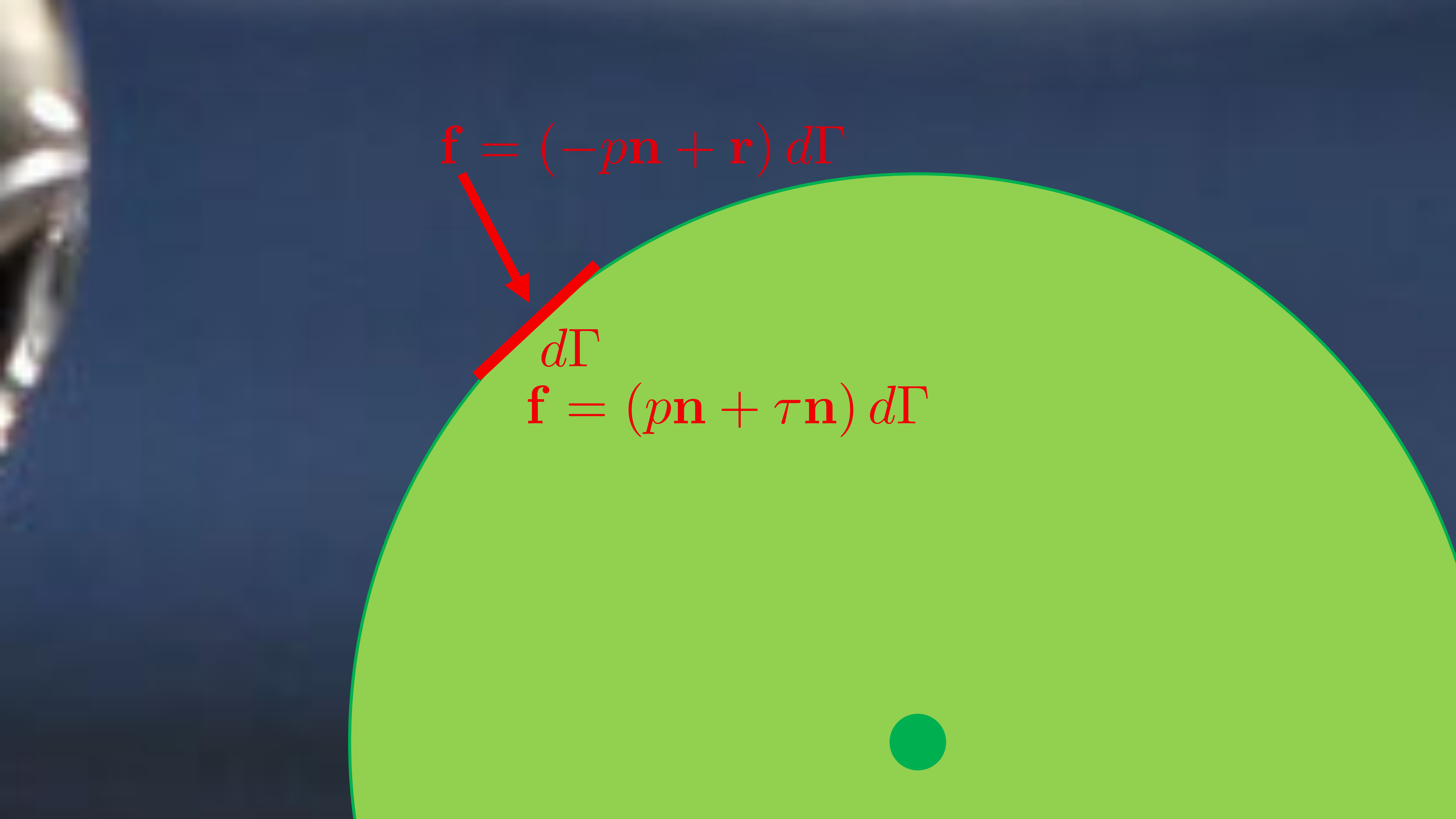
density

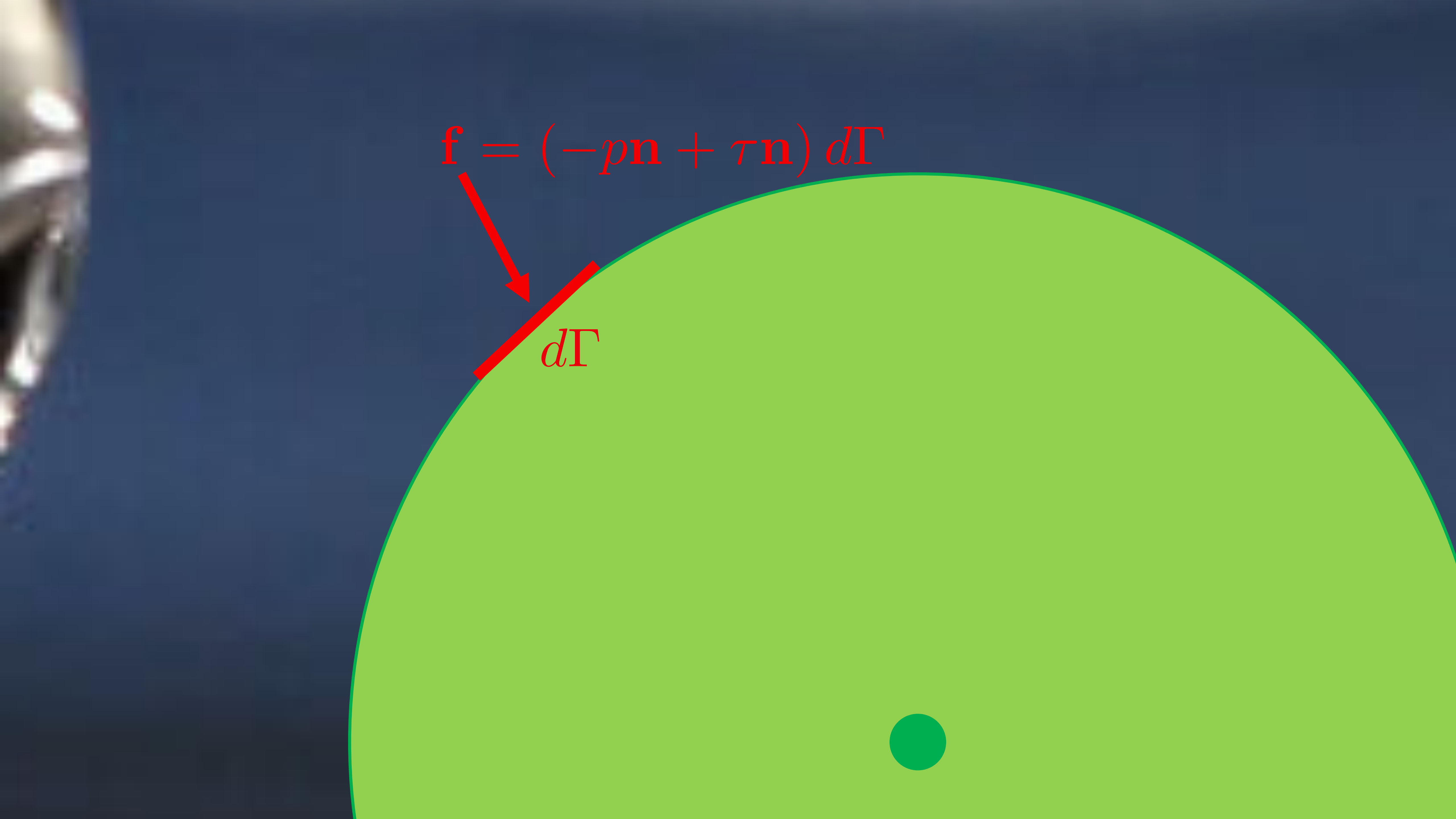


$$\int_{\Omega} \rho \frac{d\mathbf{v}}{dt} \, d\Omega = \mathbf{f}$$

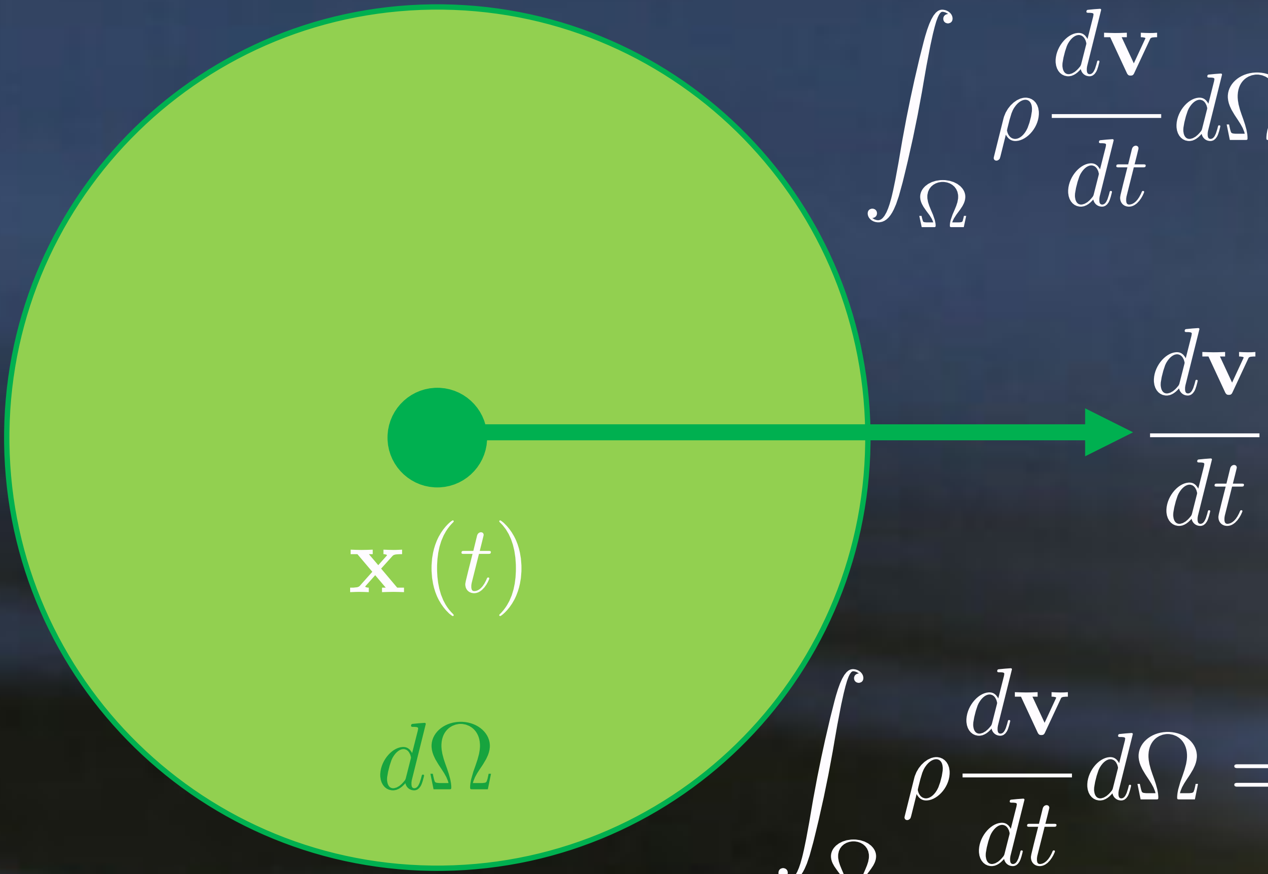

f
$d\Gamma$
surface area


traction
$\mathbf{f} = \mathbf{t}\, d\Gamma$
$d\Gamma$


$\mathbf{f} = (-p\mathbf{n} + \mathbf{r})\, d\Gamma$
$d\Gamma$
$\mathbf{f} = (p\mathbf{n} + \tau\mathbf{n})\, d\Gamma$


$\mathbf{f} = (-p\mathbf{n} + \tau\mathbf{n})\, d\Gamma$
$d\Gamma$

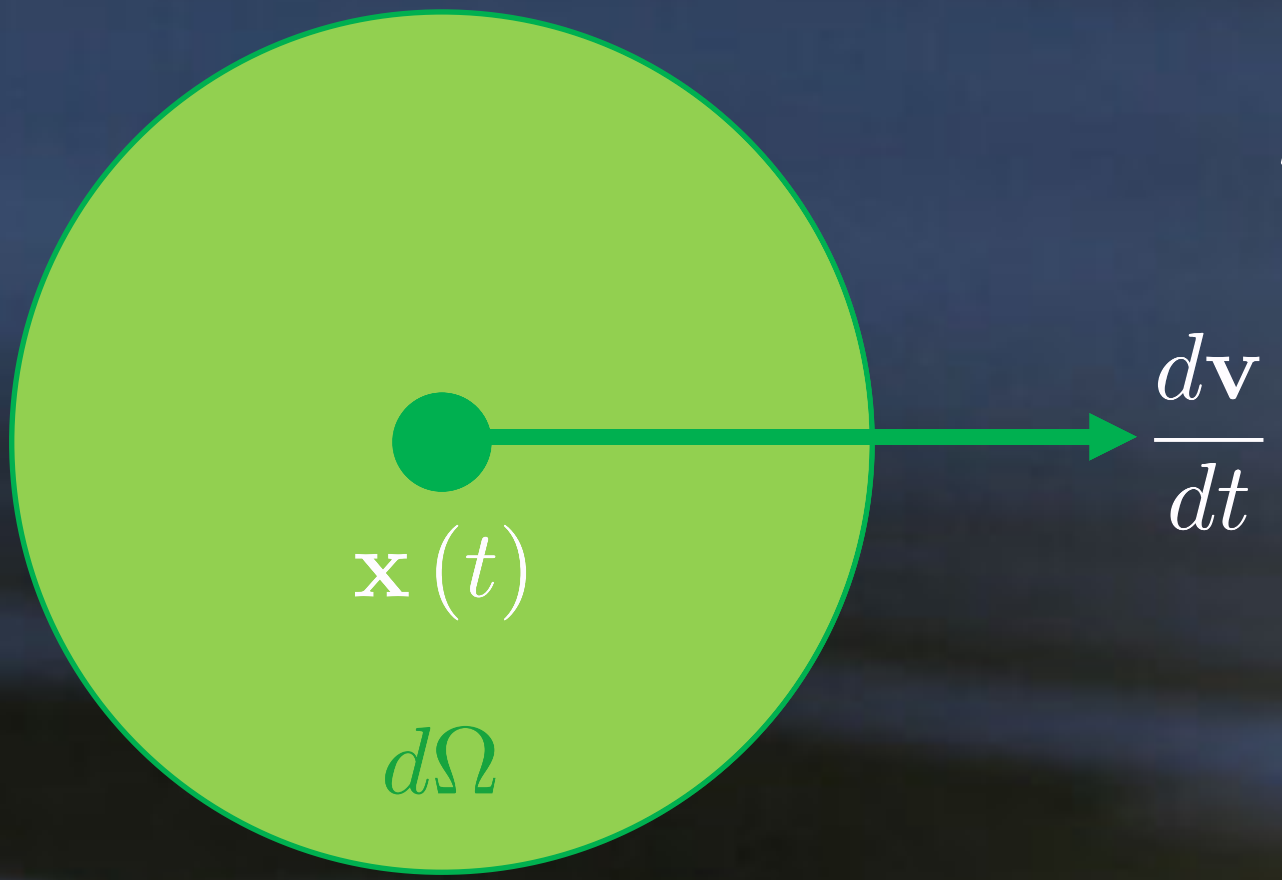
$$\int_{\Omega} \rho \frac{d\mathbf{v}}{dt} d\Omega = \int_{\Gamma} \left(-p\mathbf{n} + \tau\mathbf{n}\right) d\Gamma$$



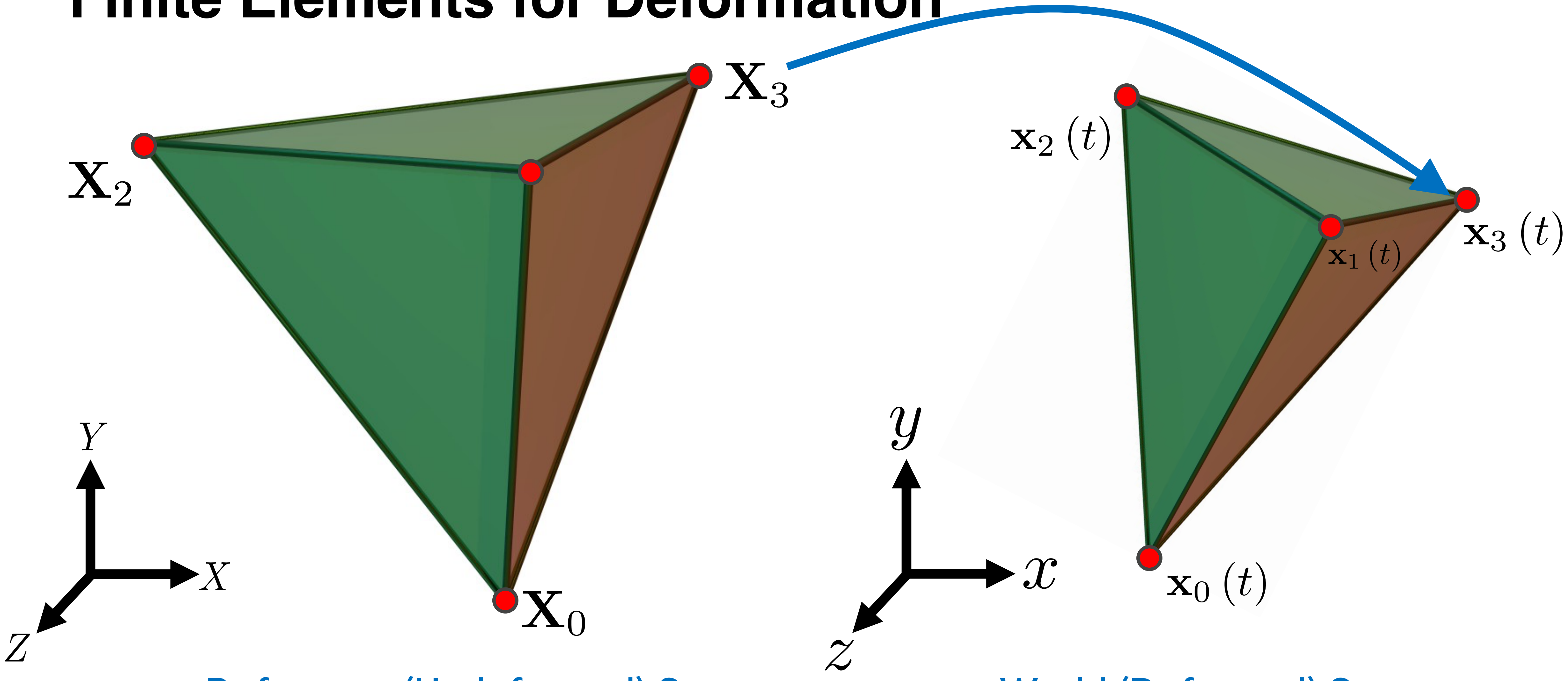
Green's Identity

$$\int_{\Omega} \rho \frac{d\mathbf{v}}{dt} d\Omega = \int_{\Omega} \left(-\nabla p + \nabla \cdot \tau\right) d\Gamma$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \tau$$

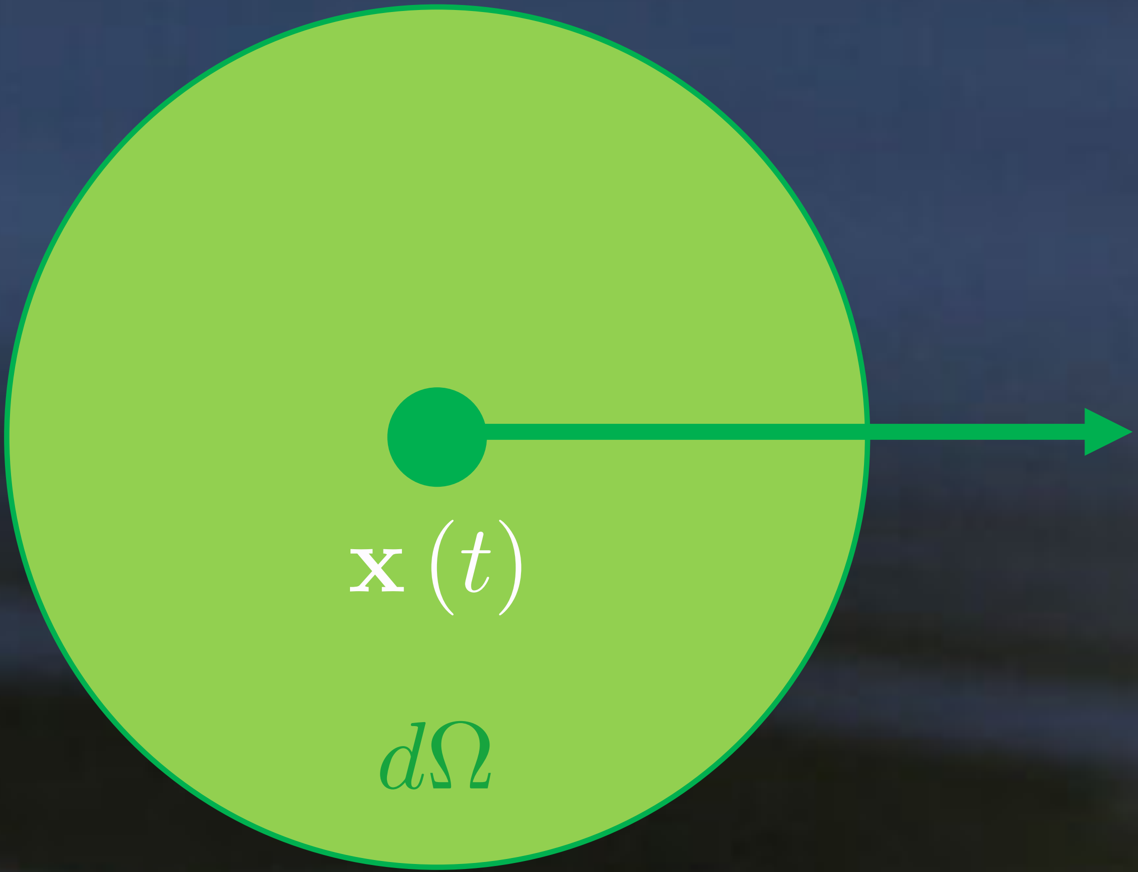


Finite Elements for Deformation



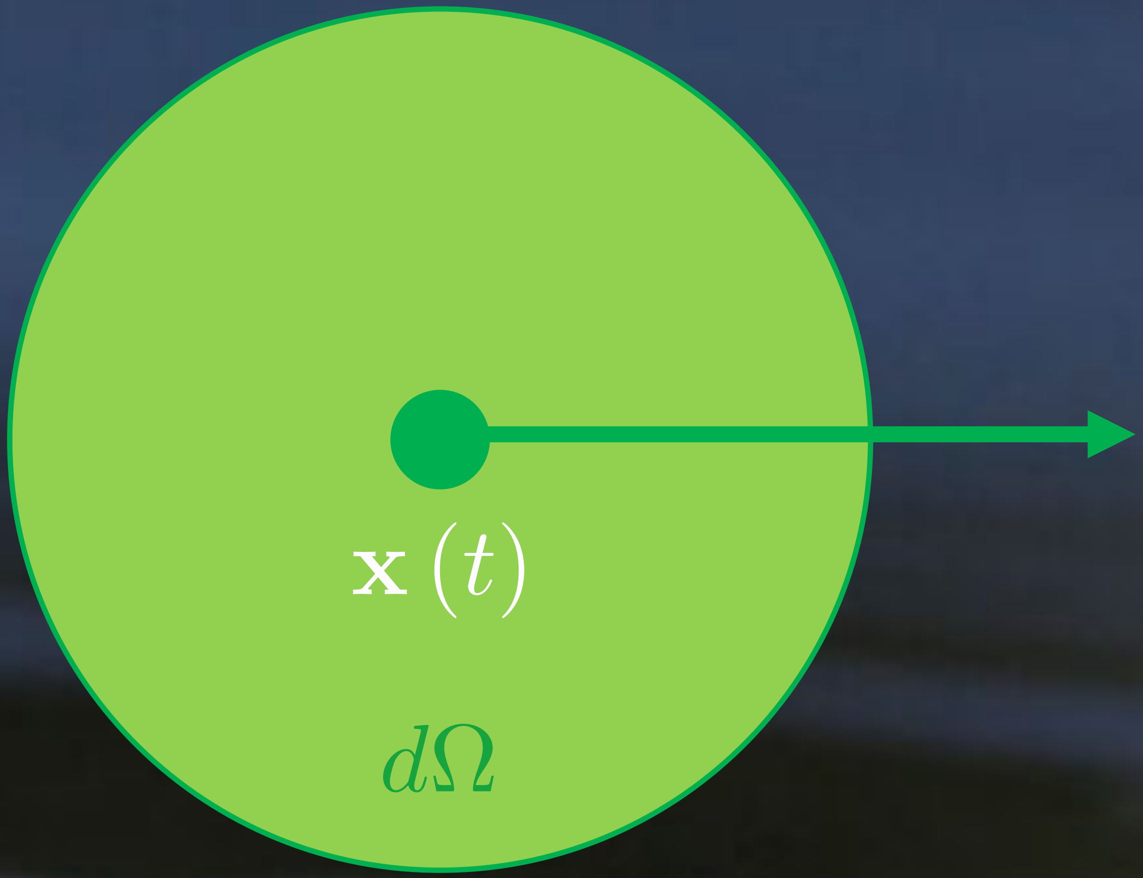
Reference (Undeformed) Space

World (Deformed) Space



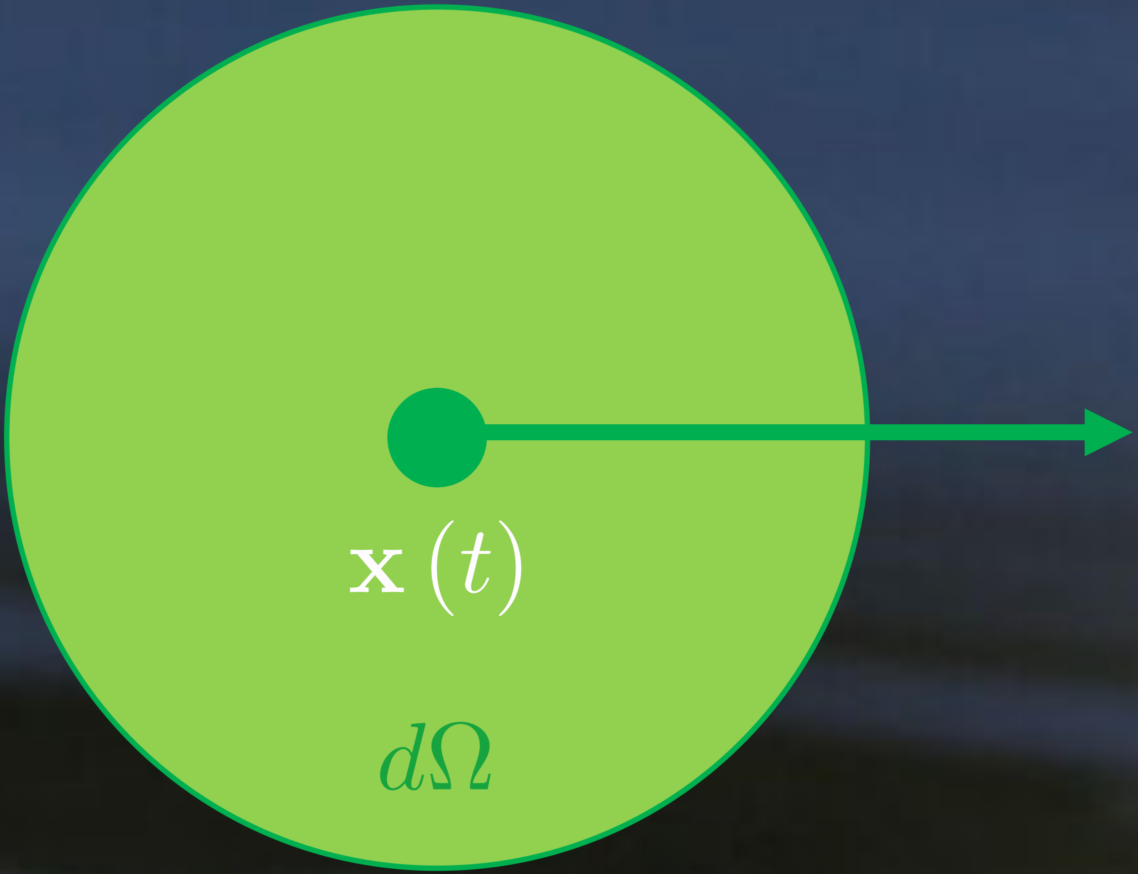
$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\mathbf{v}\left(\mathbf{x}\left(t\right), t\right)$$

Material Derivative



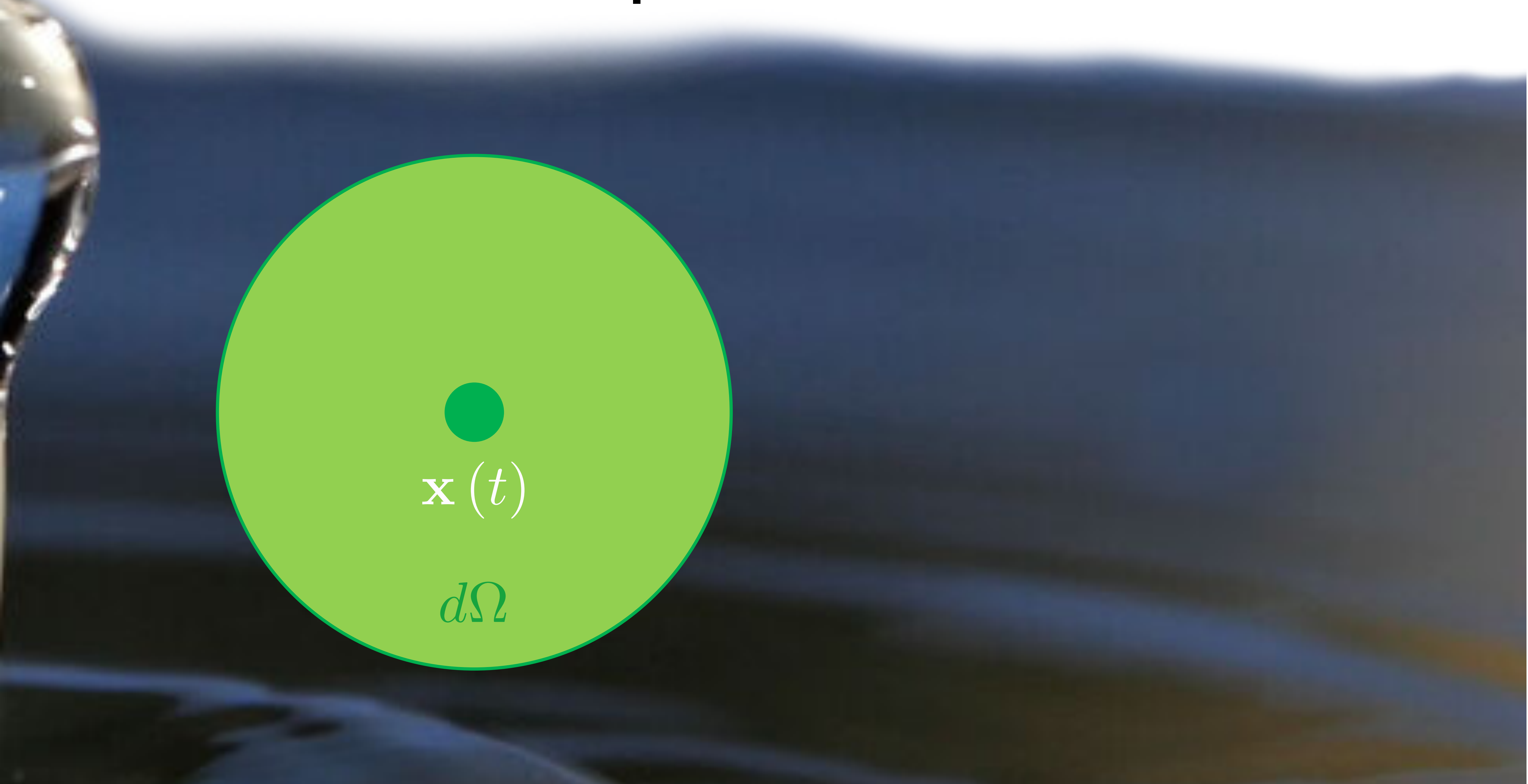
$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\mathbf{v}\left(\mathbf{x}\left(t\right), t\right)$$

Material Derivative

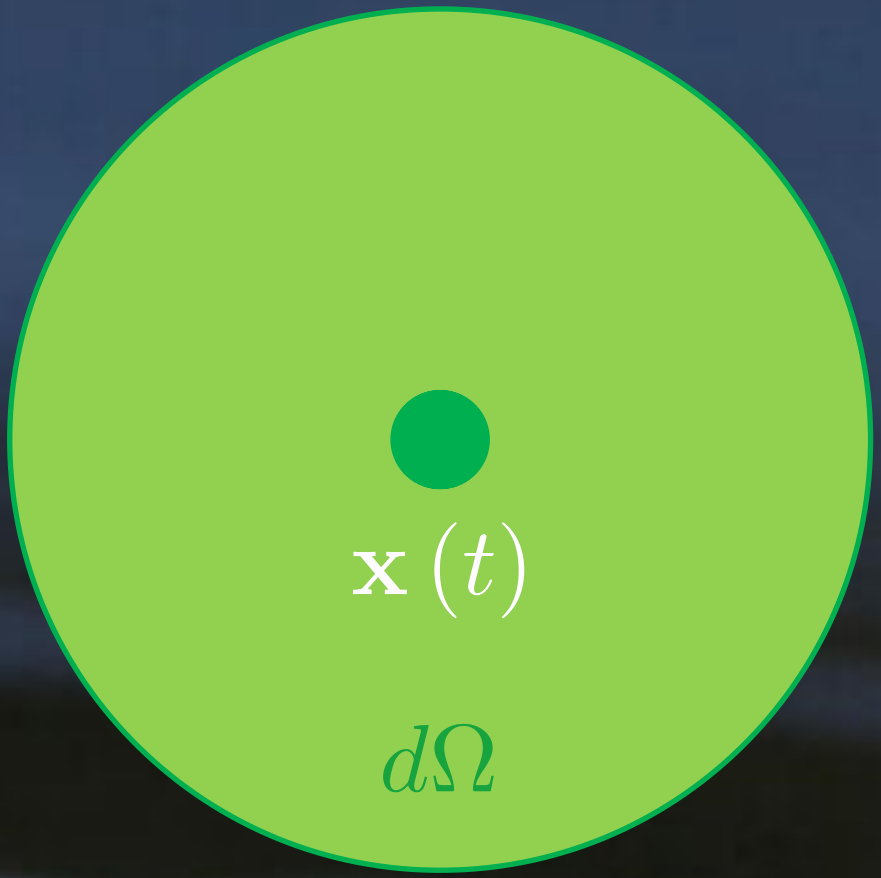


$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v}$$

The Navier-Stokes Equations



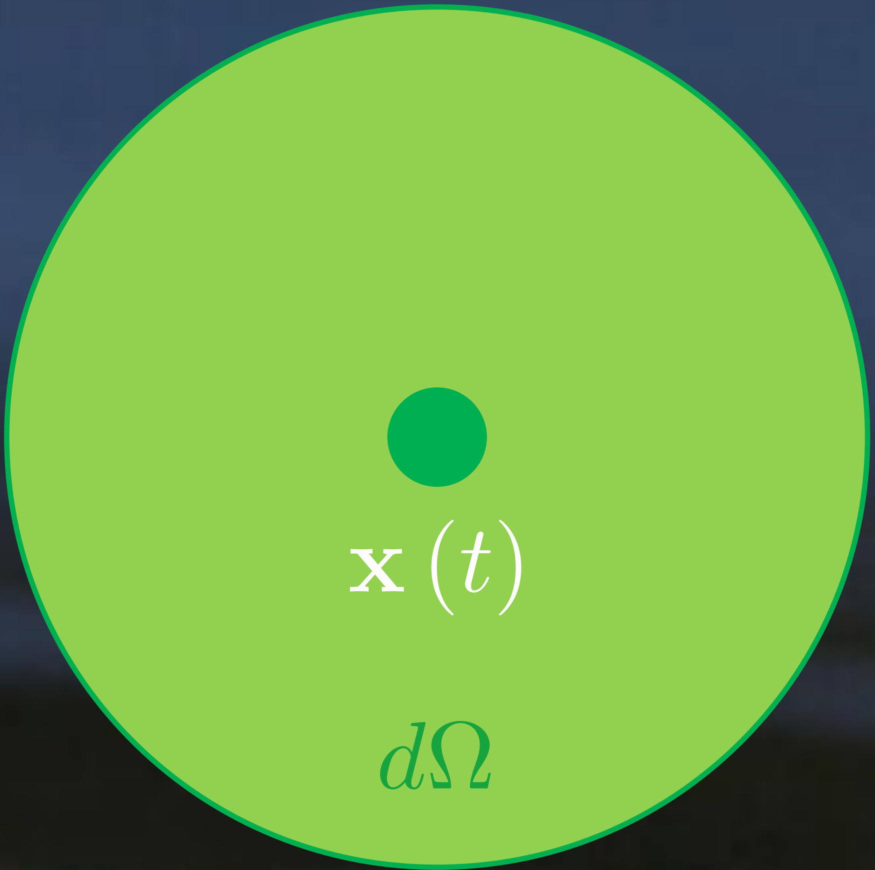
The Navier-Stokes Equations



$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \nabla \mathbf{v} \cdot \mathbf{v} = -\nabla p + \cancel{\nabla \cdot \tau} + \rho \mathbf{g}$$

$$\nabla \cdot \mathbf{v} = 0$$

Equations for Incompressible, Inviscid Flow



$$\underbrace{\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\nabla\mathbf{v}\cdot\mathbf{v}}_{\text{Advection}} = \underbrace{-\nabla p}_{\text{Pressure}} + \overbrace{\rho\mathbf{g}}^{\text{External Force}}$$

$$\nabla\cdot\mathbf{v} = 0$$

Fluid Simulation via Splitting

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$

Advection

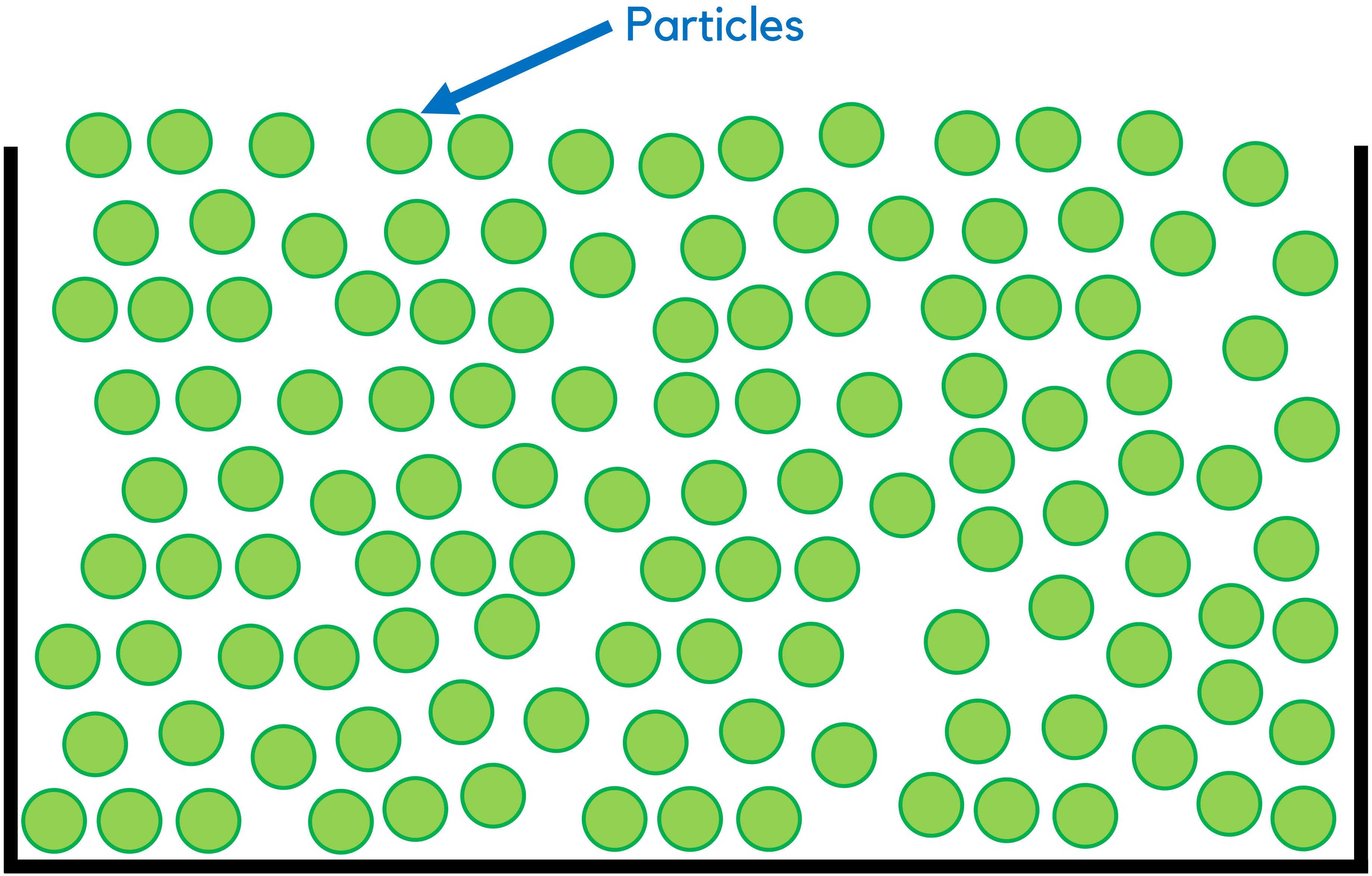
$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$

External Forces

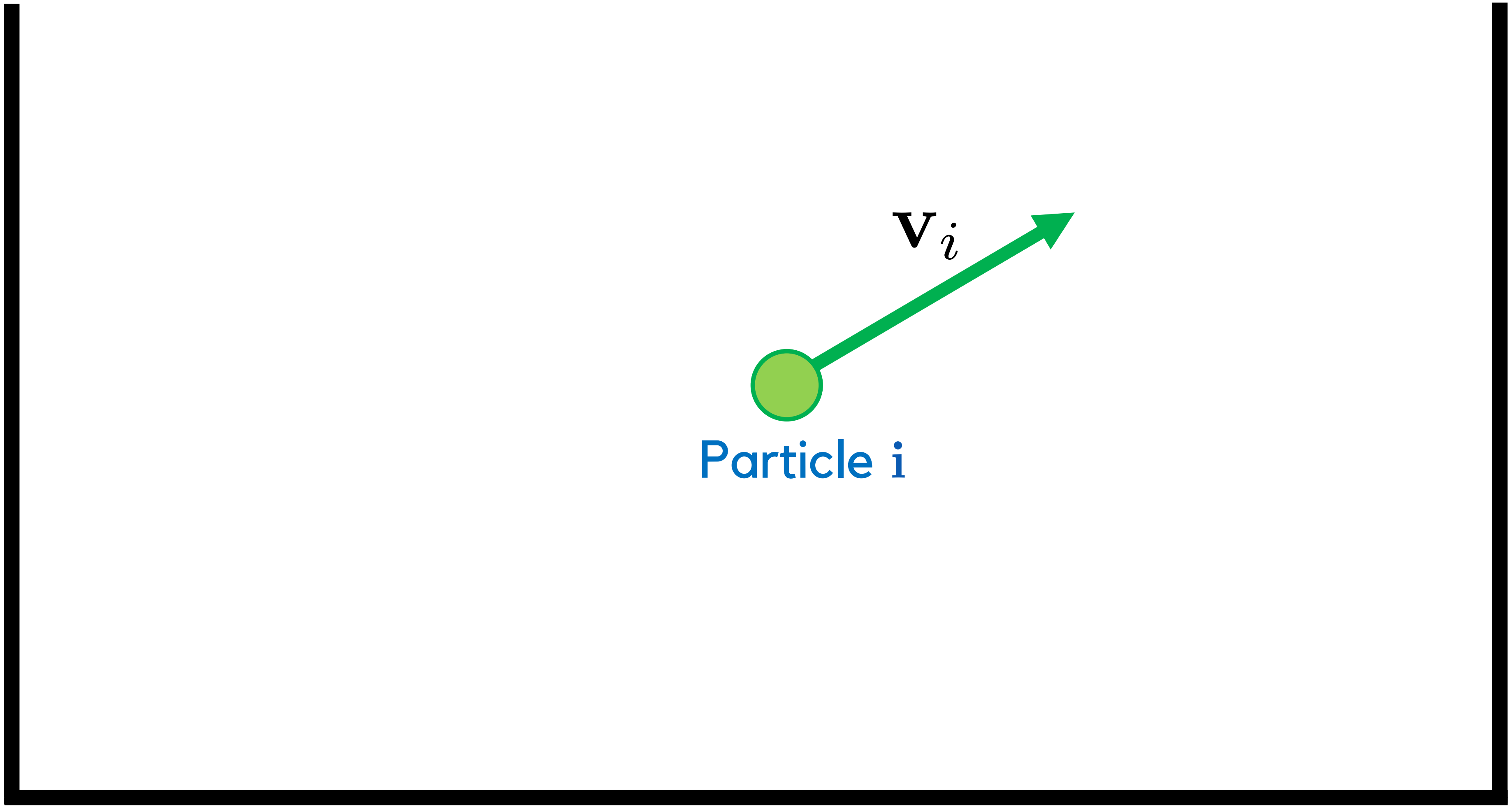
$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$

Pressure Projection

Fluid Discretization



Fluid Discretization



Fluid Simulation - Advection

Input: $\mathbf{v}^t$ (Divergence Free)

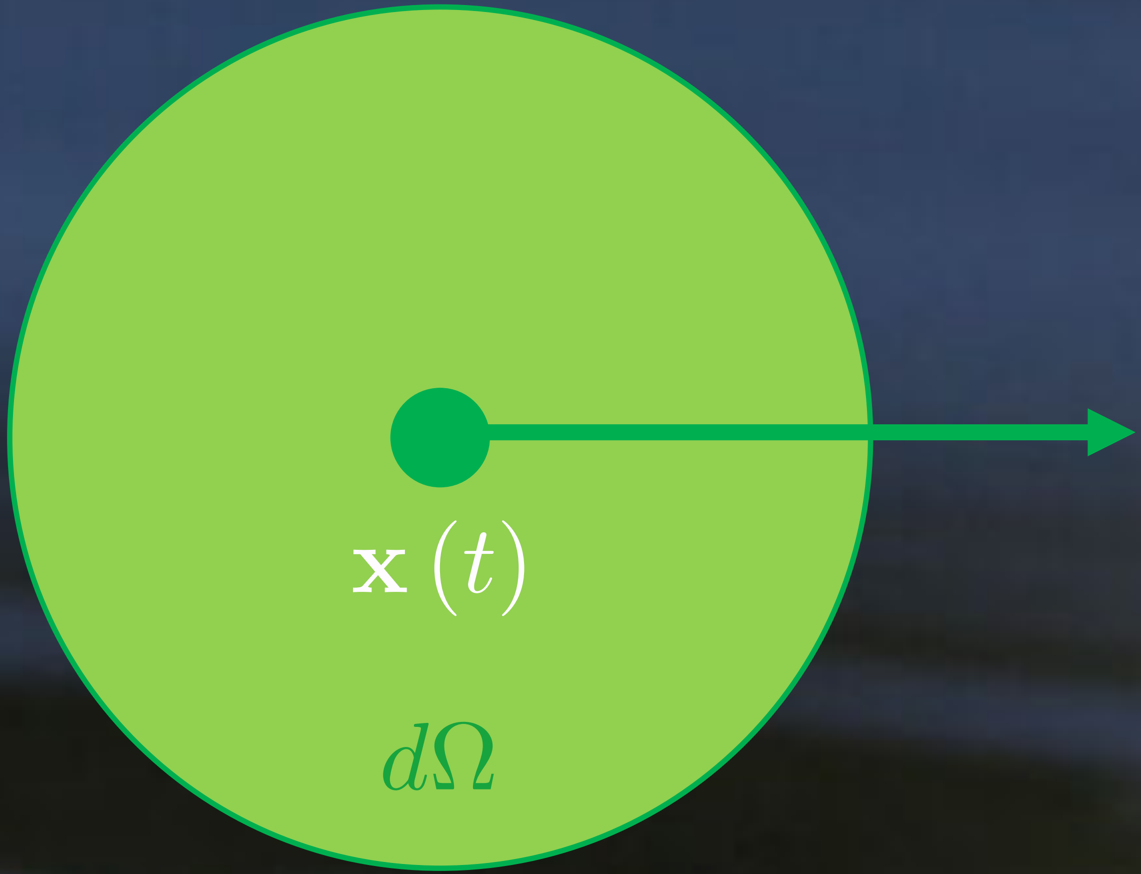
Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$ External Forces

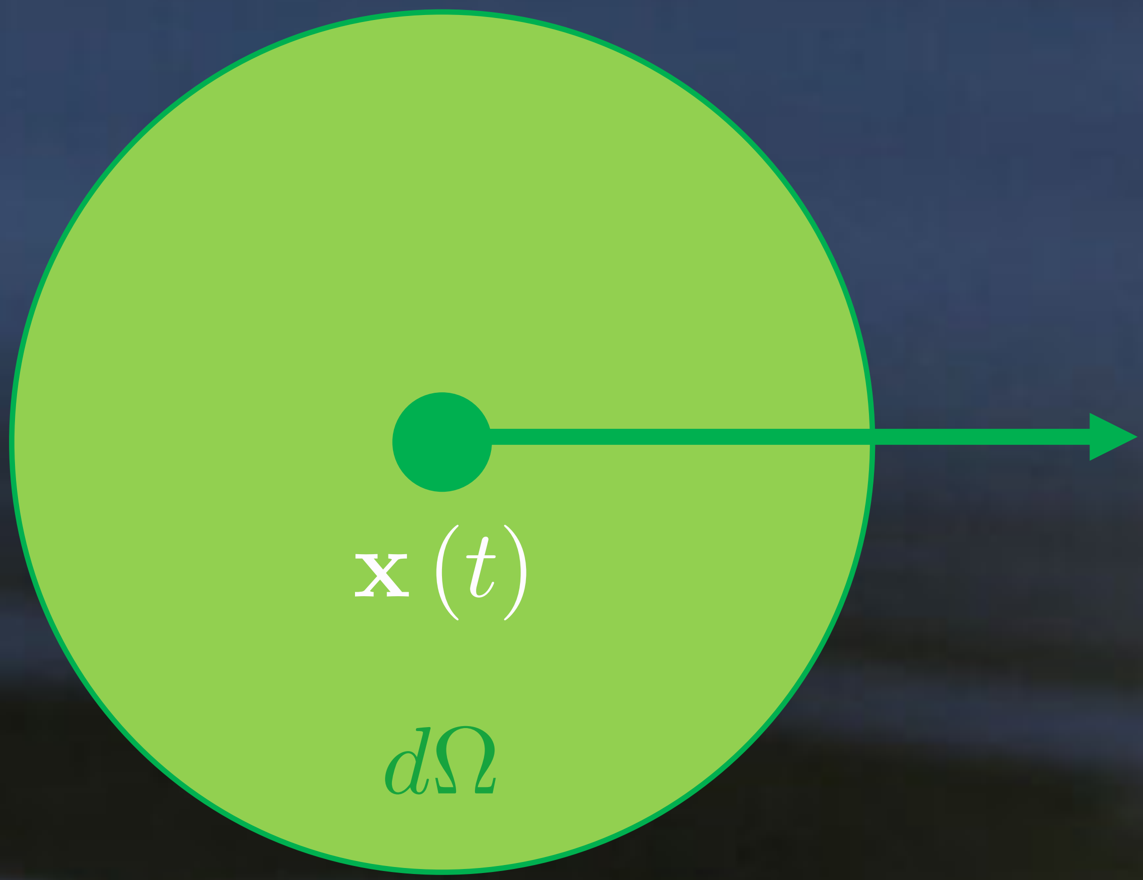
$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$ Pressure Projection

Material Derivative



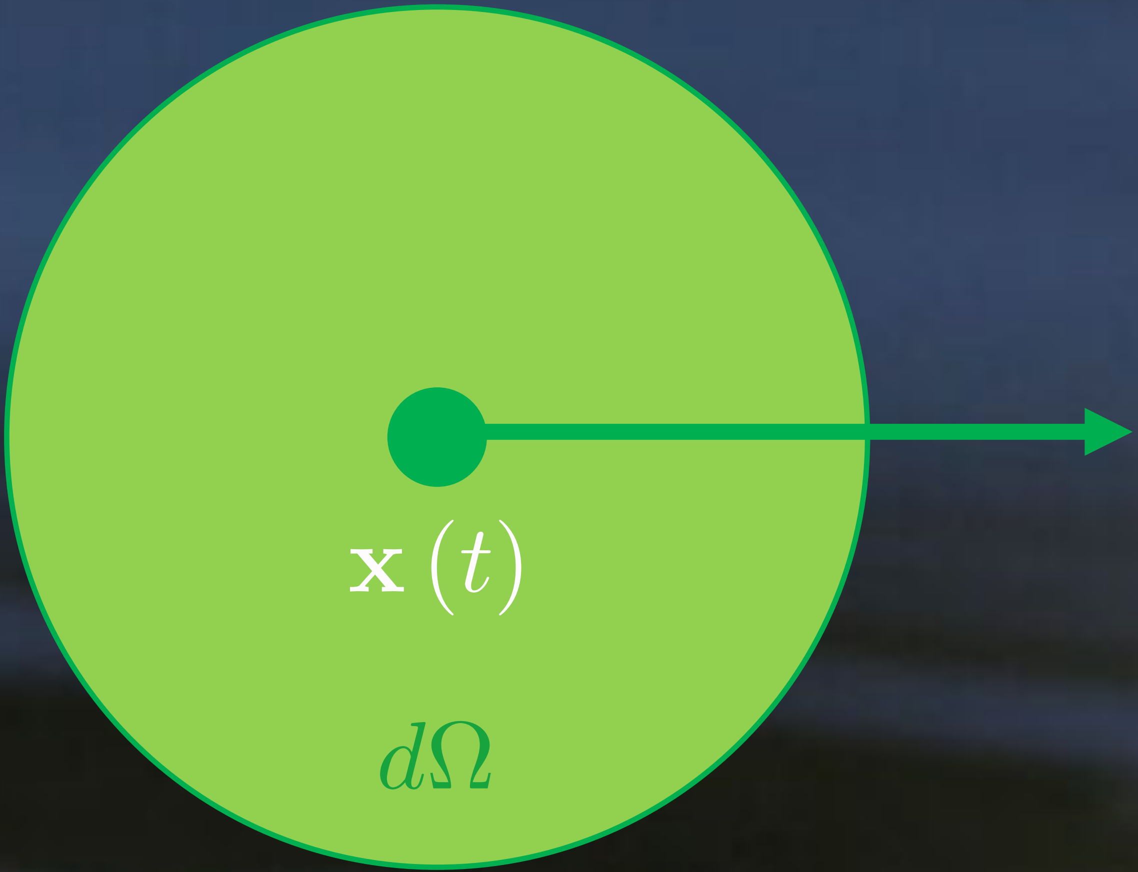
$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v}$$

Fluid Simulation - Advection



$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\mathbf{v}\left(\mathbf{x}\left(t\right), t\right) = \mathbf{0}$$

Fluid Simulation - Advection



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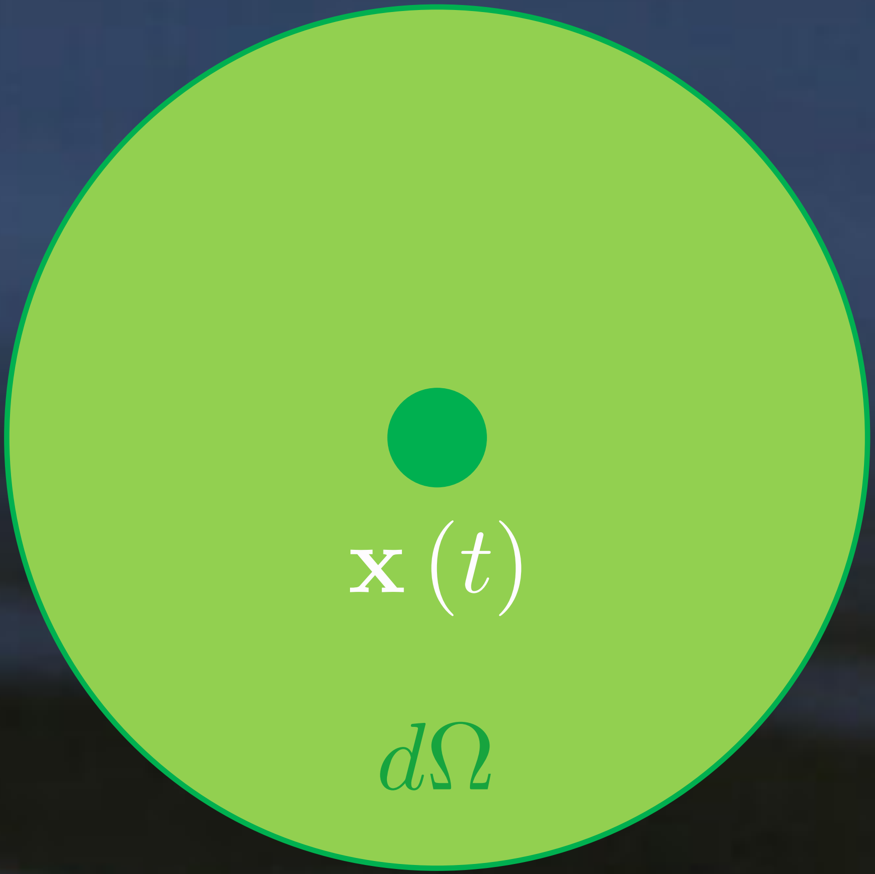
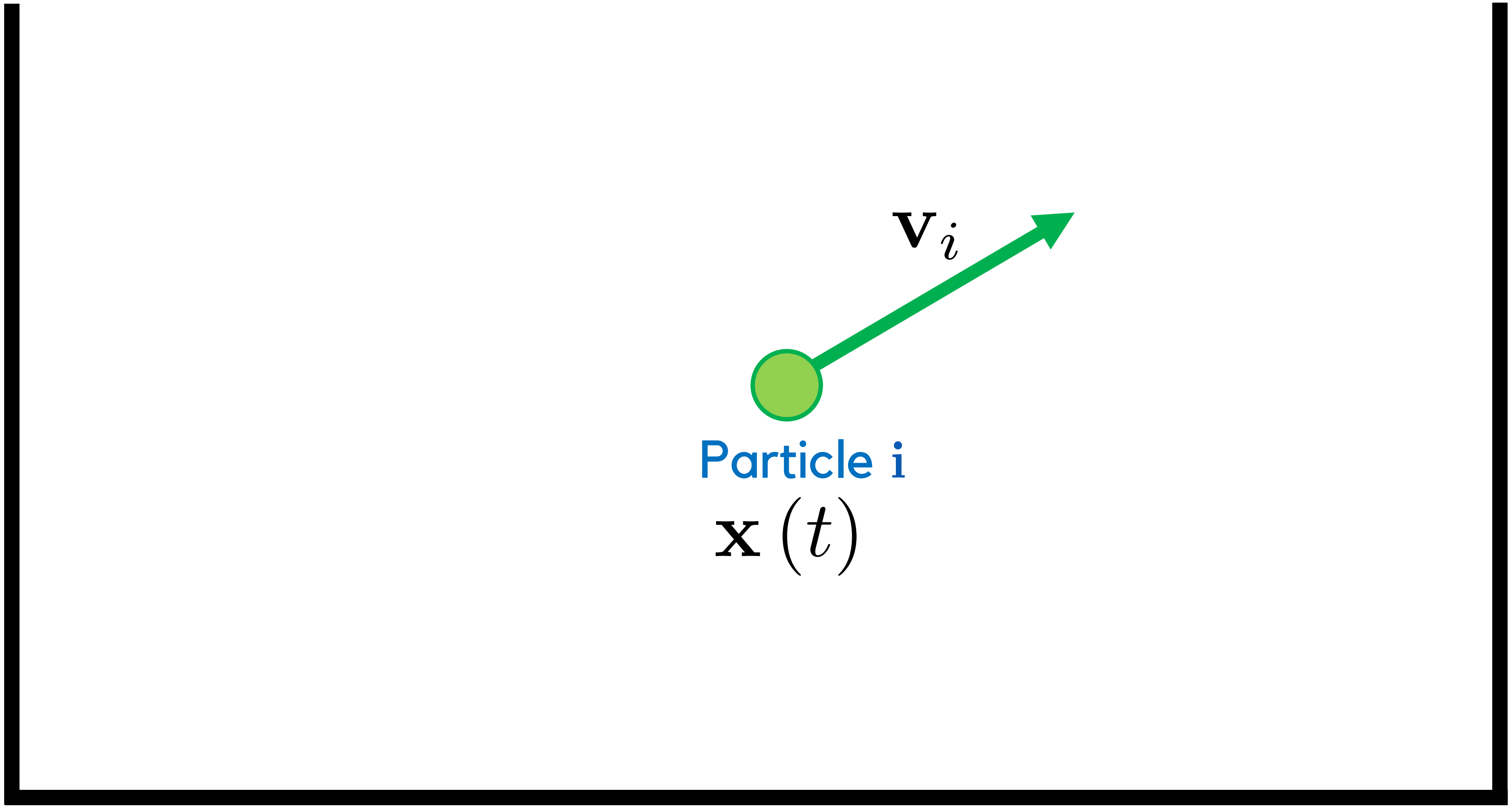
Fluid Simulation - Advection



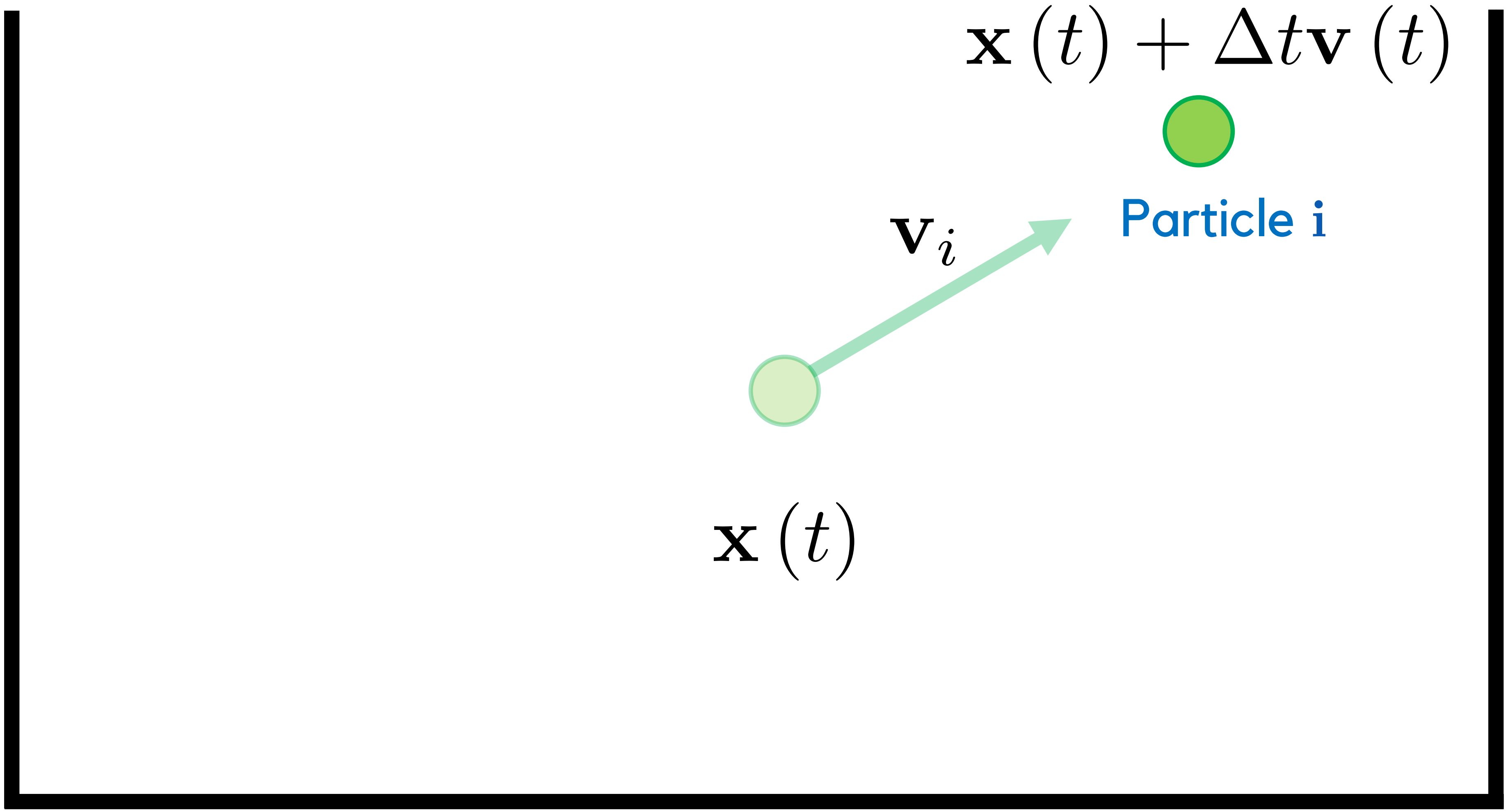
$$\mathbf{v}\left(\underbrace{\mathbf{x}\left(t+\Delta t\right)}_{\text{Updated Position}}, t+\Delta t\right) = \mathbf{v}\left(\underbrace{\mathbf{x}\left(t\right)}_{\text{Current Position}}, t\right)$$

$$\mathbf{x}\left(t+\Delta t\right) \approx \mathbf{x}\left(t\right) + \Delta t \mathbf{v}\left(t\right)$$

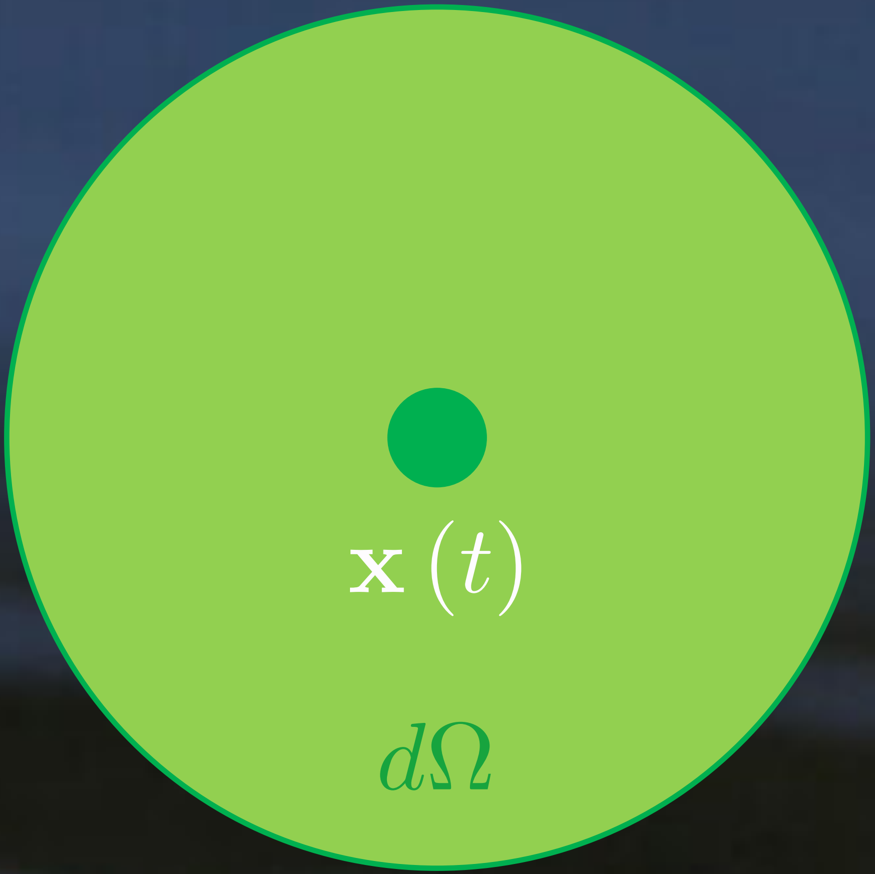
Fluid Simulation - Advection



Fluid Simulation - Advection



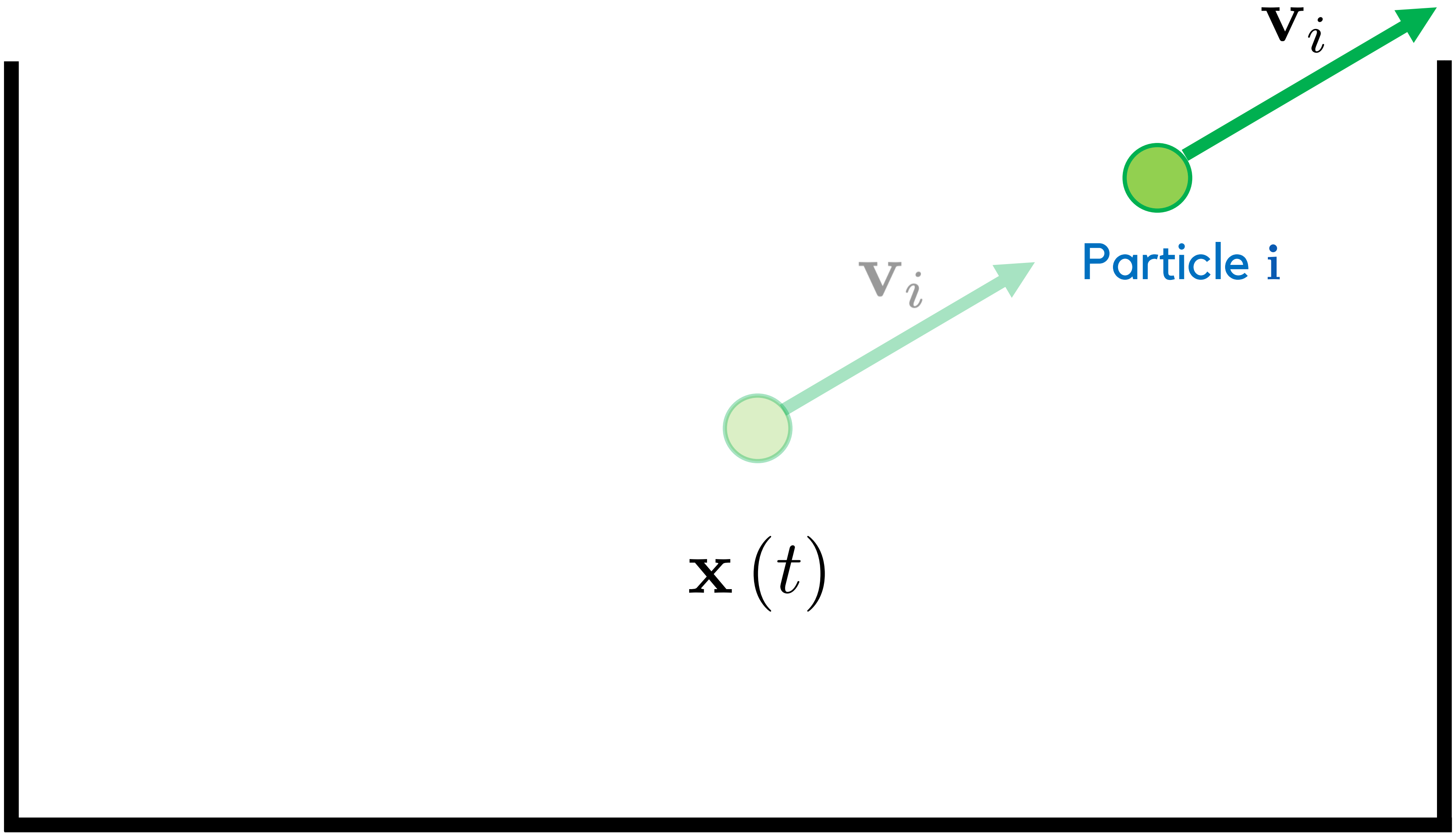
Fluid Simulation - Advection



$$\mathbf{v}\left(\underbrace{\mathbf{x}\left(t+\Delta t\right)}_{\text{Updated Position}}, t+\Delta t\right) = \mathbf{v}\left(\underbrace{\mathbf{x}\left(t\right)}_{\text{Current Position}}, t\right)$$

$$\mathbf{x}\left(t+\Delta t\right) \approx \mathbf{x}\left(t\right) + \Delta t \mathbf{v}\left(t\right)$$

Fluid Simulation - Advection



Fluid Simulation - Advection

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$ External Forces

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$ Pressure Projection

Fluid Simulation – External Forces

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

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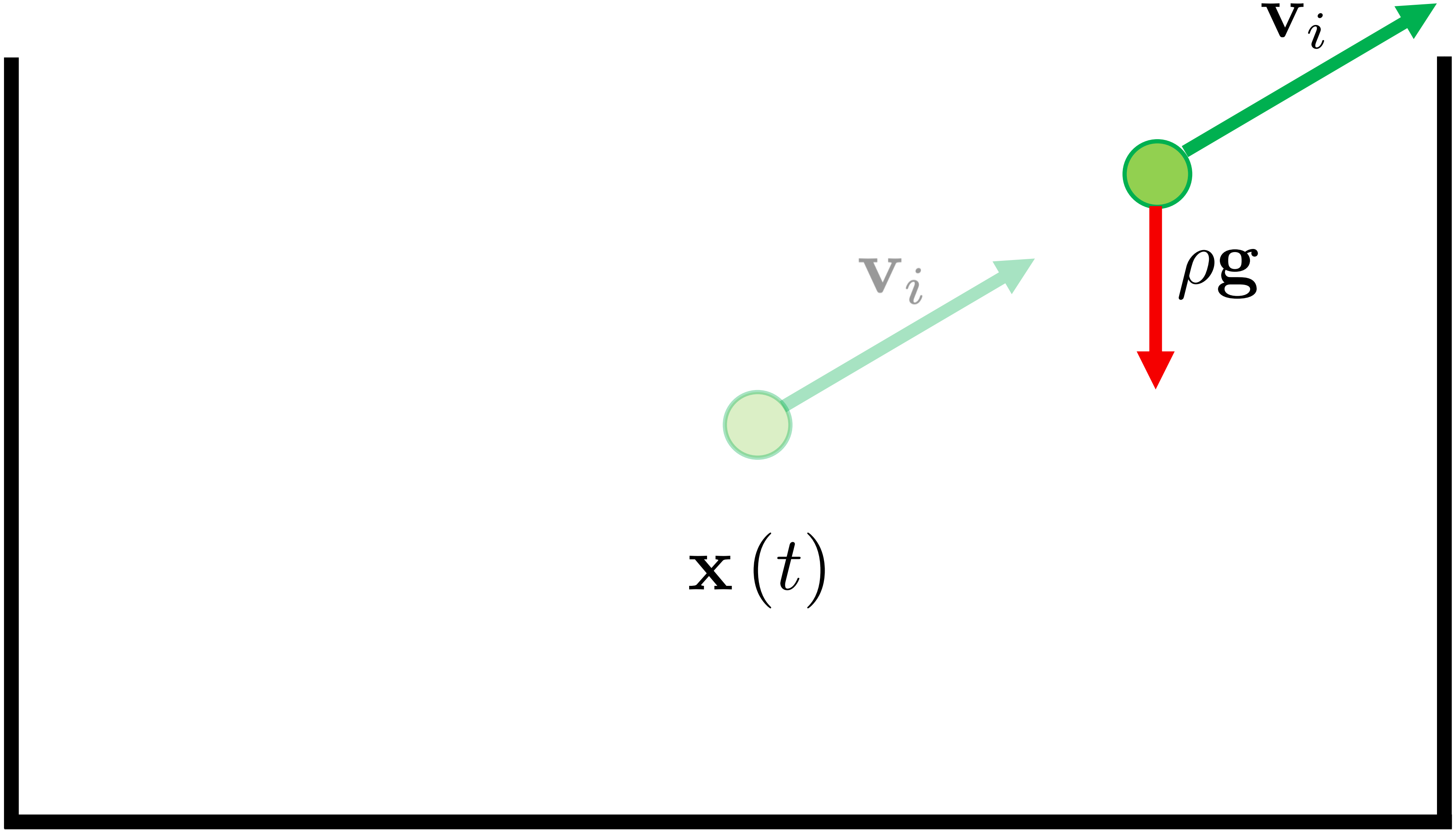
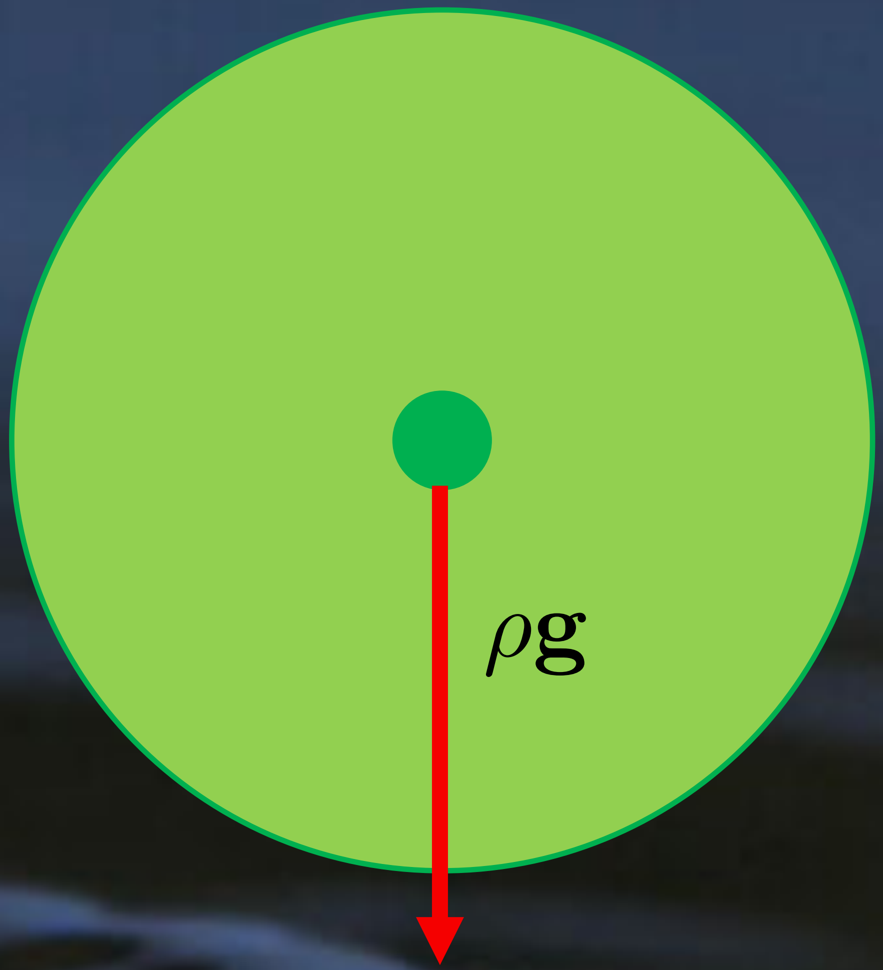
Fluid Simulation – External Forces

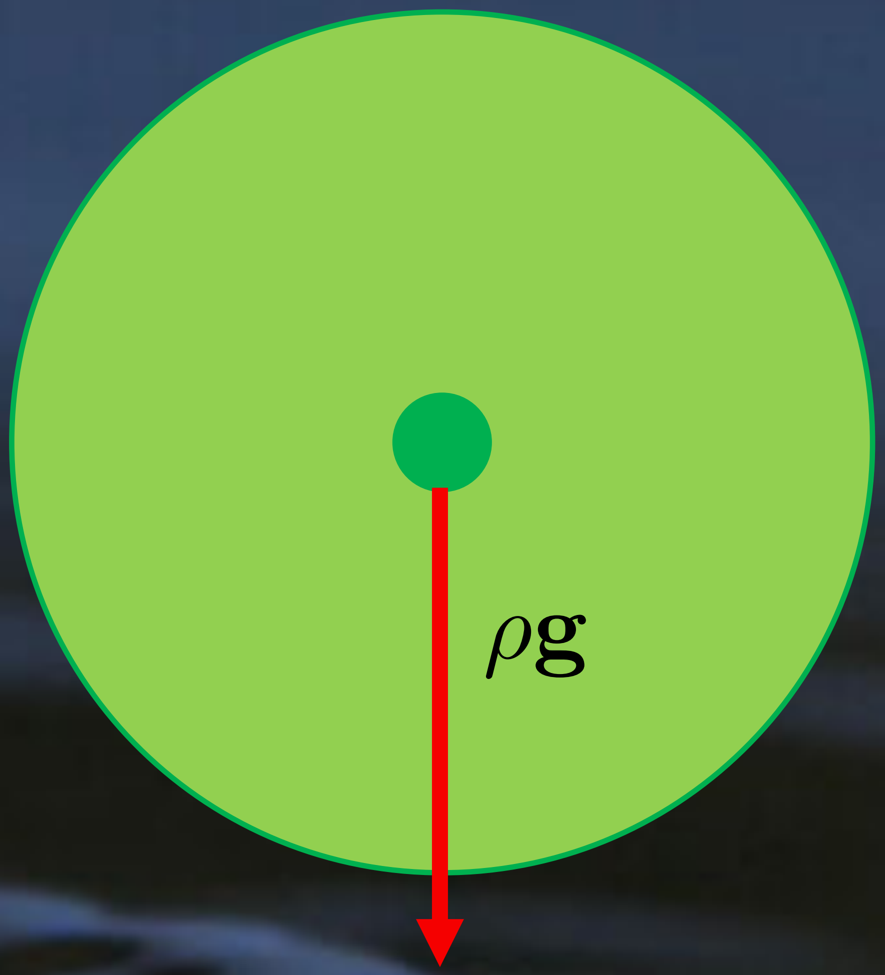


Fluid Simulation – External Forces



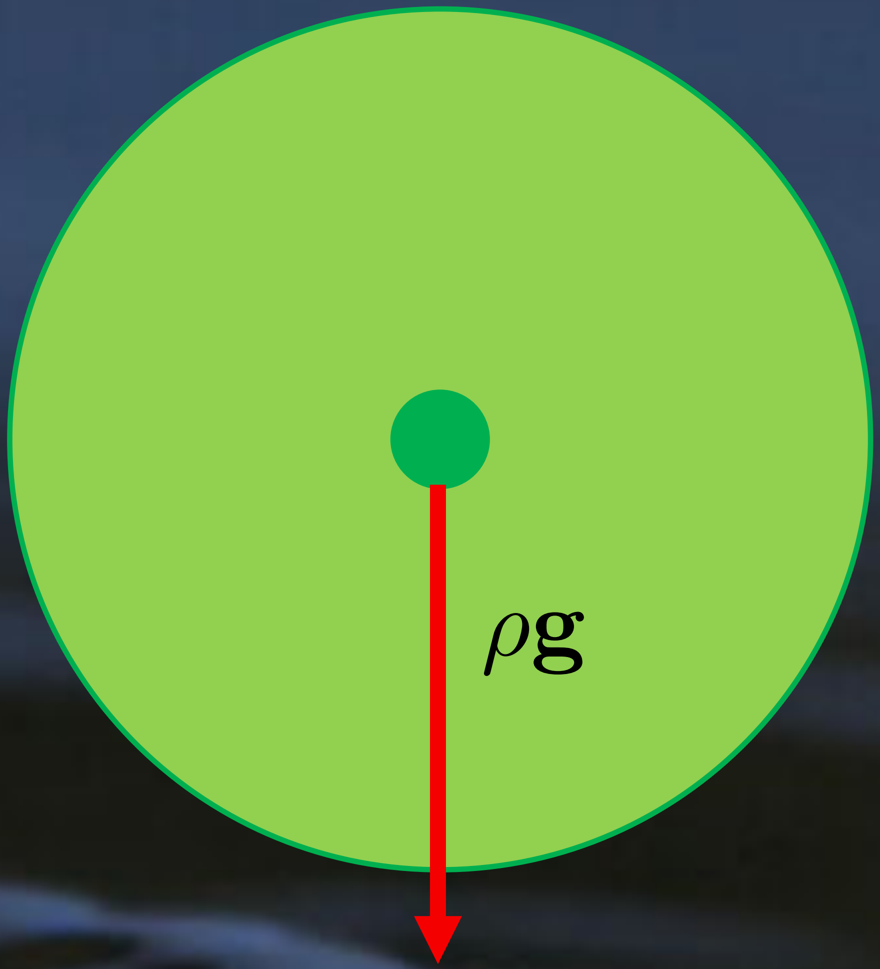
$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$

Fluid Simulation – External Forces



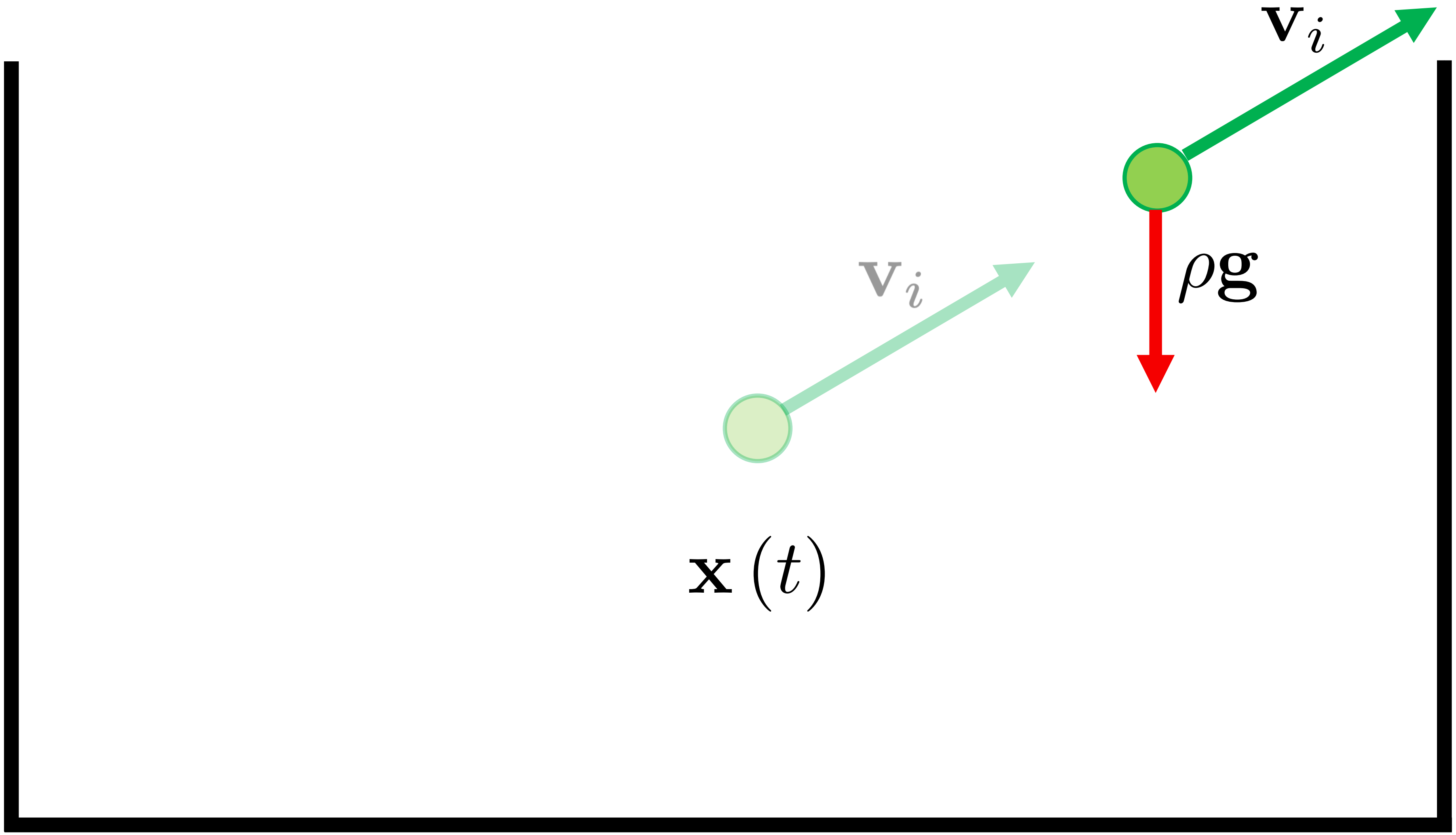
$$\cancel{\rho}\frac{\partial \mathbf{v}}{\partial t} = \cancel{\rho}\mathbf{g}$$

Fluid Simulation – External Forces

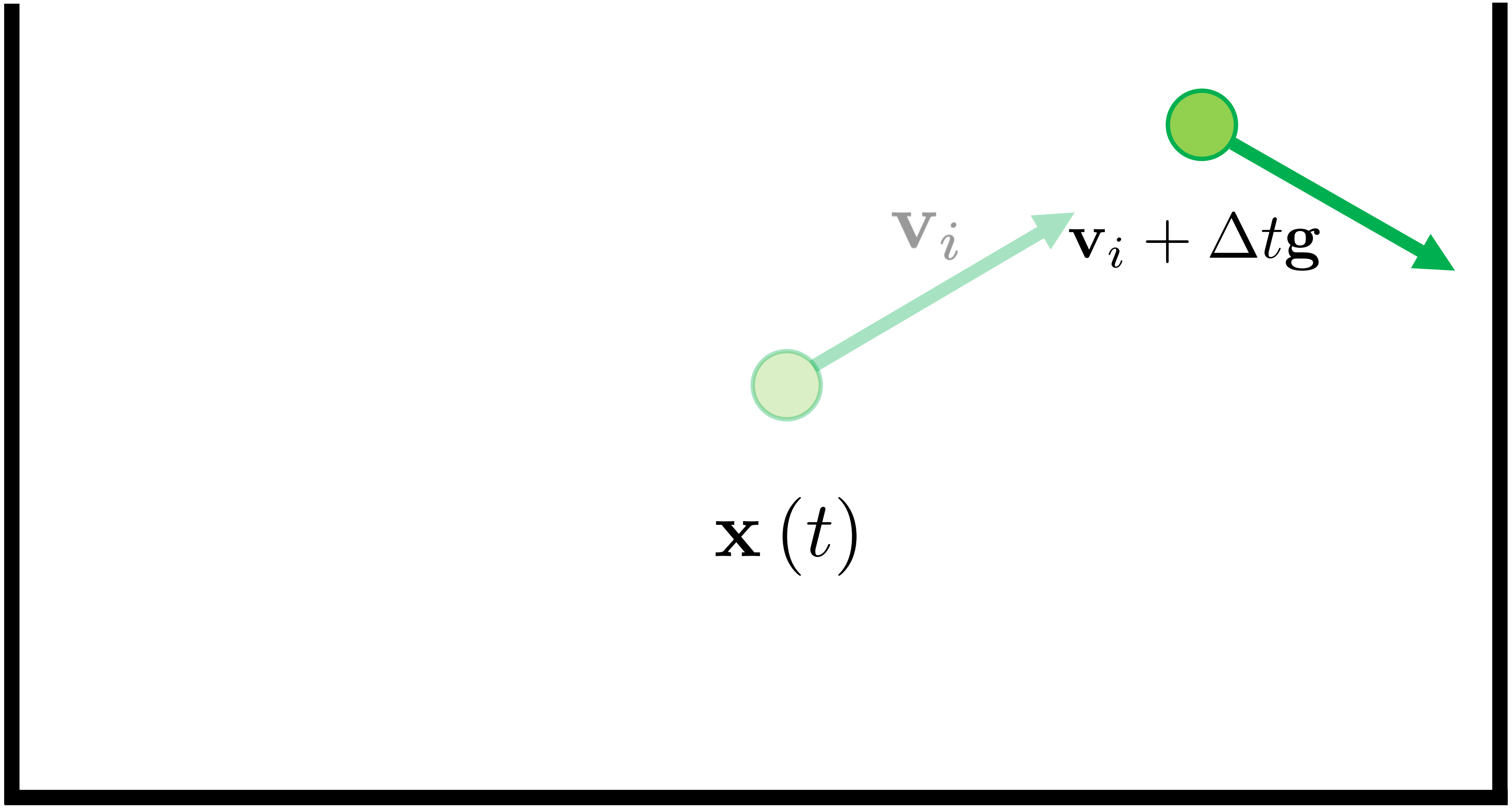


$$\overbrace{\mathbf{v}\left(\mathbf{x}\left(t\right), \underbrace{t+\Delta t}_{\text{Partial Derivative}}\right) \approx \mathbf{v}\left(\mathbf{x}\left(t\right), t\right) + \Delta t\mathbf{g}}^{\text{First—Order Update}}$$

Fluid Simulation – External Forces



Fluid Simulation – External Forces



Fluid Simulation – External Forces

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$

Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$

External Forces

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$

Pressure Projection

Fluid Simulation – Pressure Projection

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$ External Forces

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \text{ subject to } \nabla \cdot \mathbf{v} = 0$$ Pressure Projection

Fluid Simulation – Pressure Projection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p$$ subject to $$\nabla \cdot \mathbf{v} = 0$$

First-Order Update

$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

Divergence

$$\nabla \cdot \mathbf{v}^{t+1} = \nabla \cdot \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla \cdot \nabla p = 0$$

Fluid Simulation – Pressure Projection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$

First-Order Update

$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

Divergence

$$\nabla \cdot \mathbf{v}^{t+1} = \nabla \cdot \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla \cdot \nabla p = 0$$

Fluid Simulation – Pressure Projection

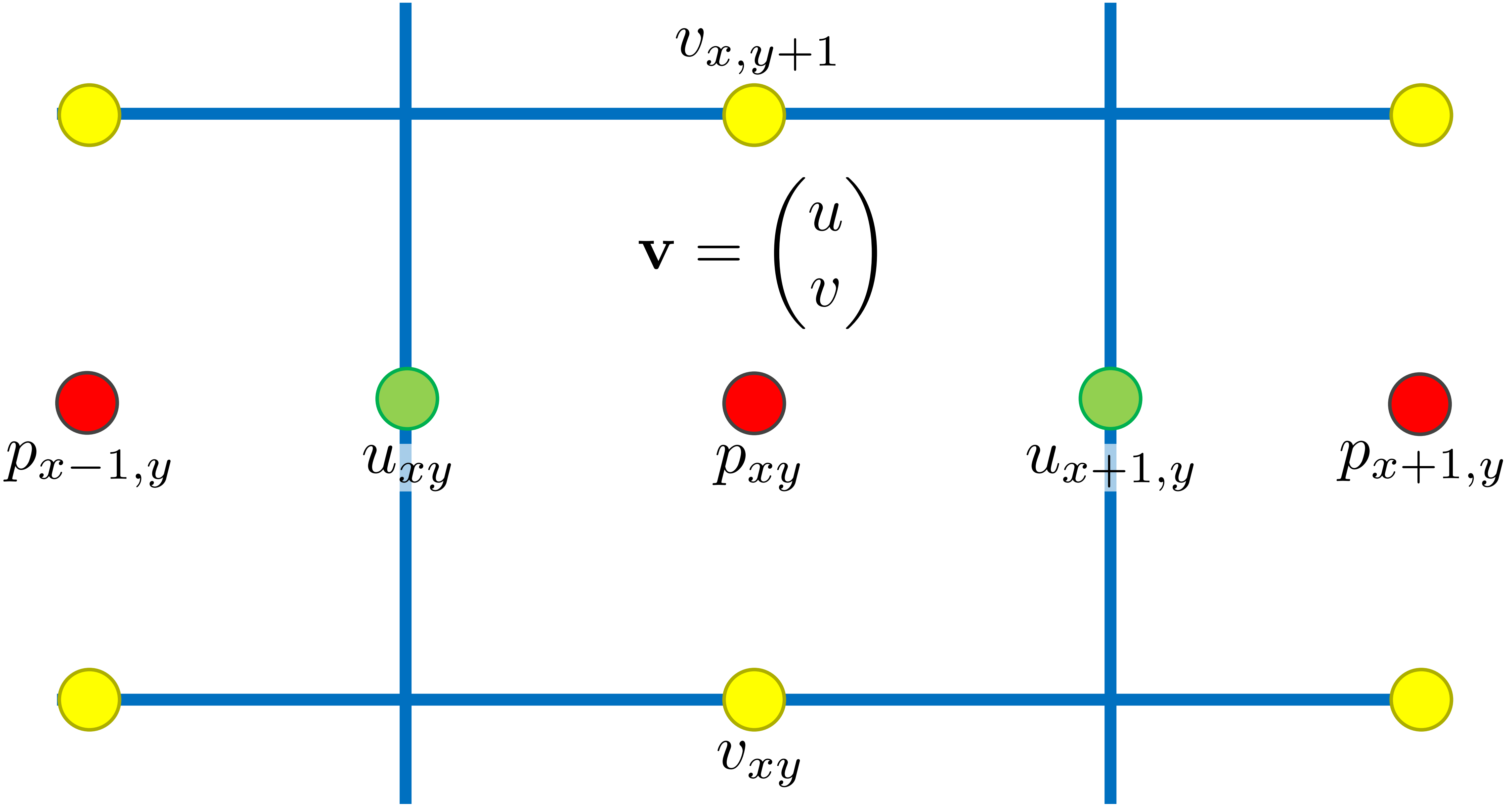
$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$

$$\nabla \cdot \nabla p = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{v}^{\mathbf{t}}$$

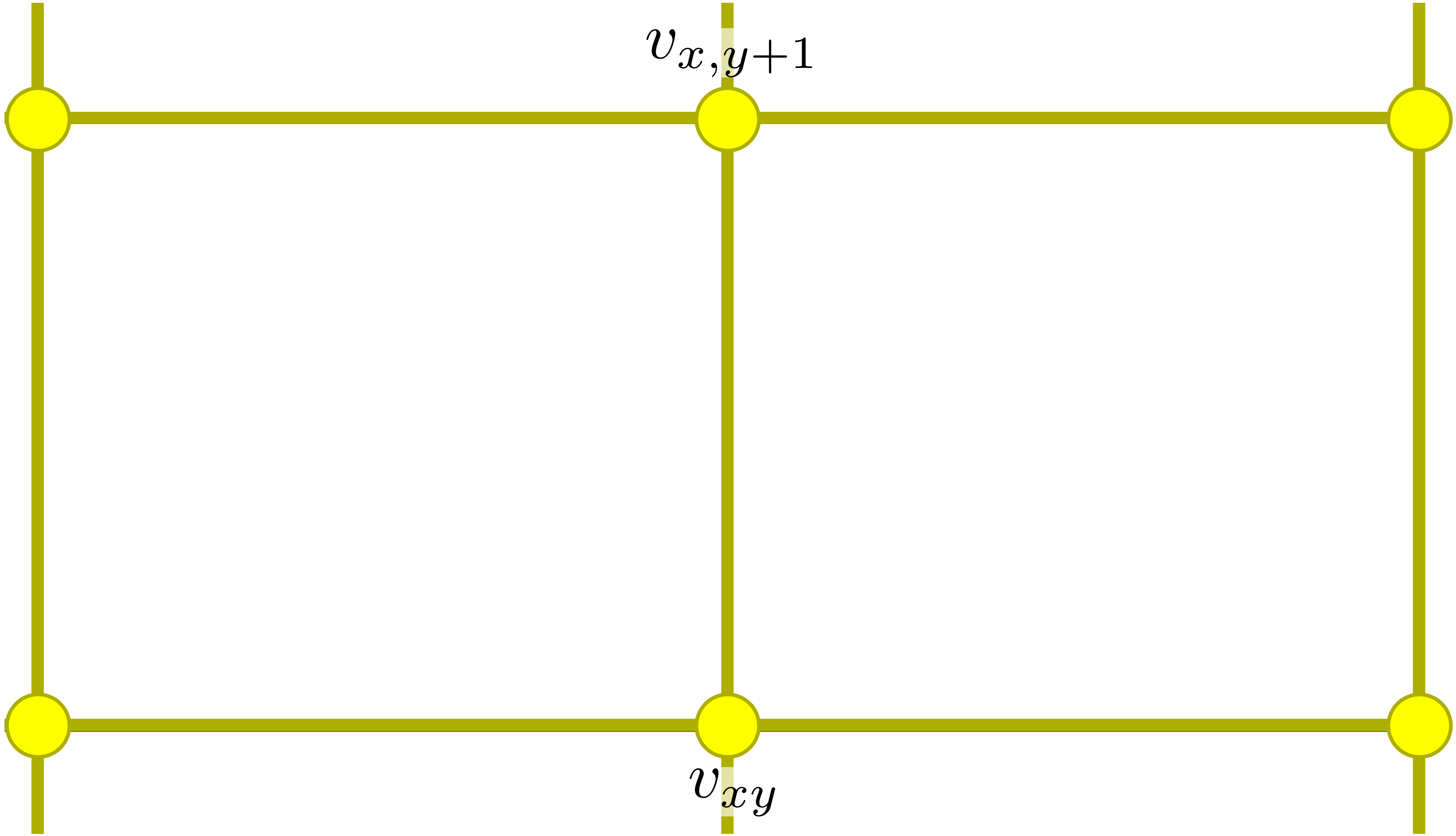
Solve for pressure

Finite Differences

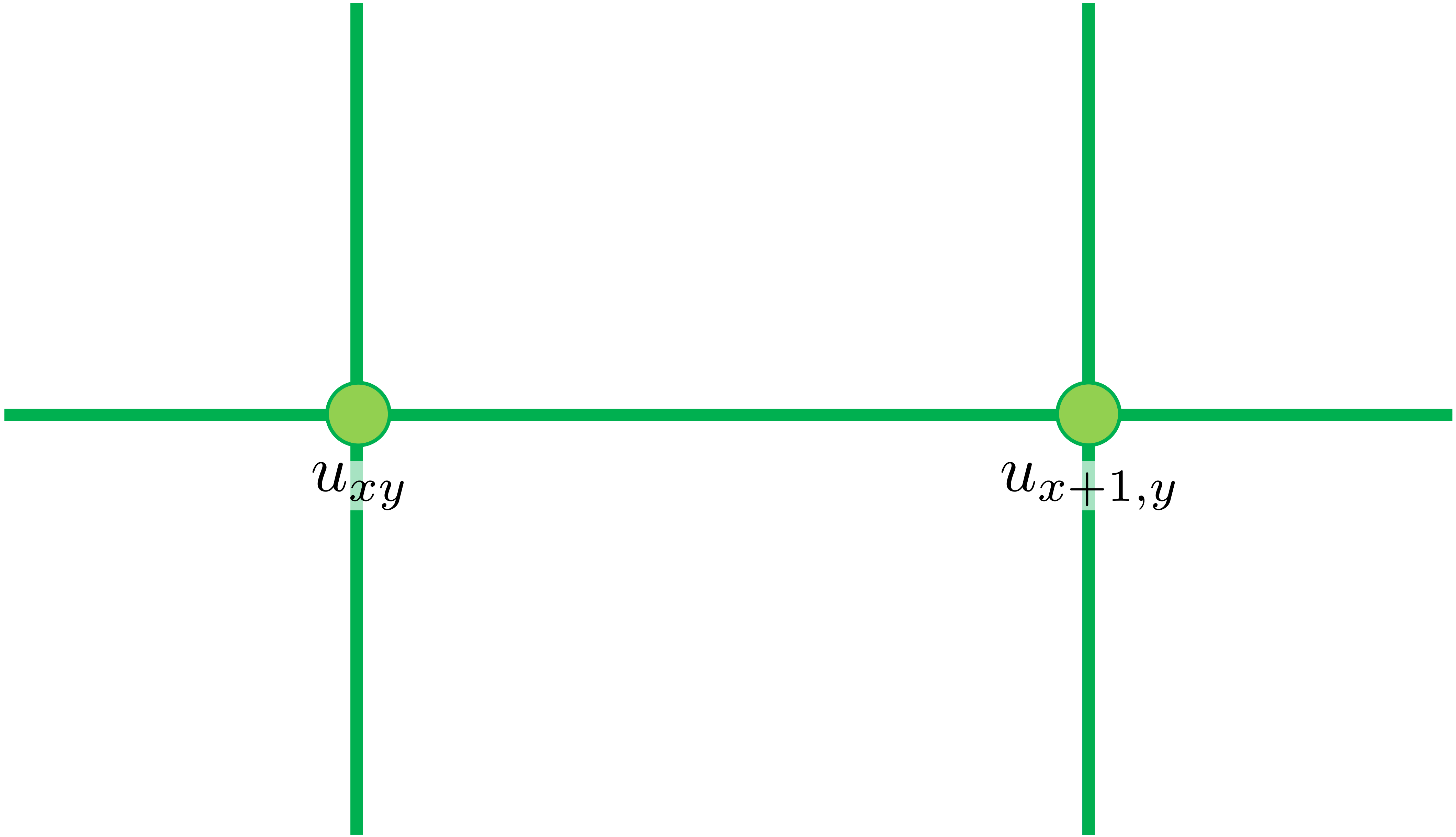
Aside: The Staggered Grid



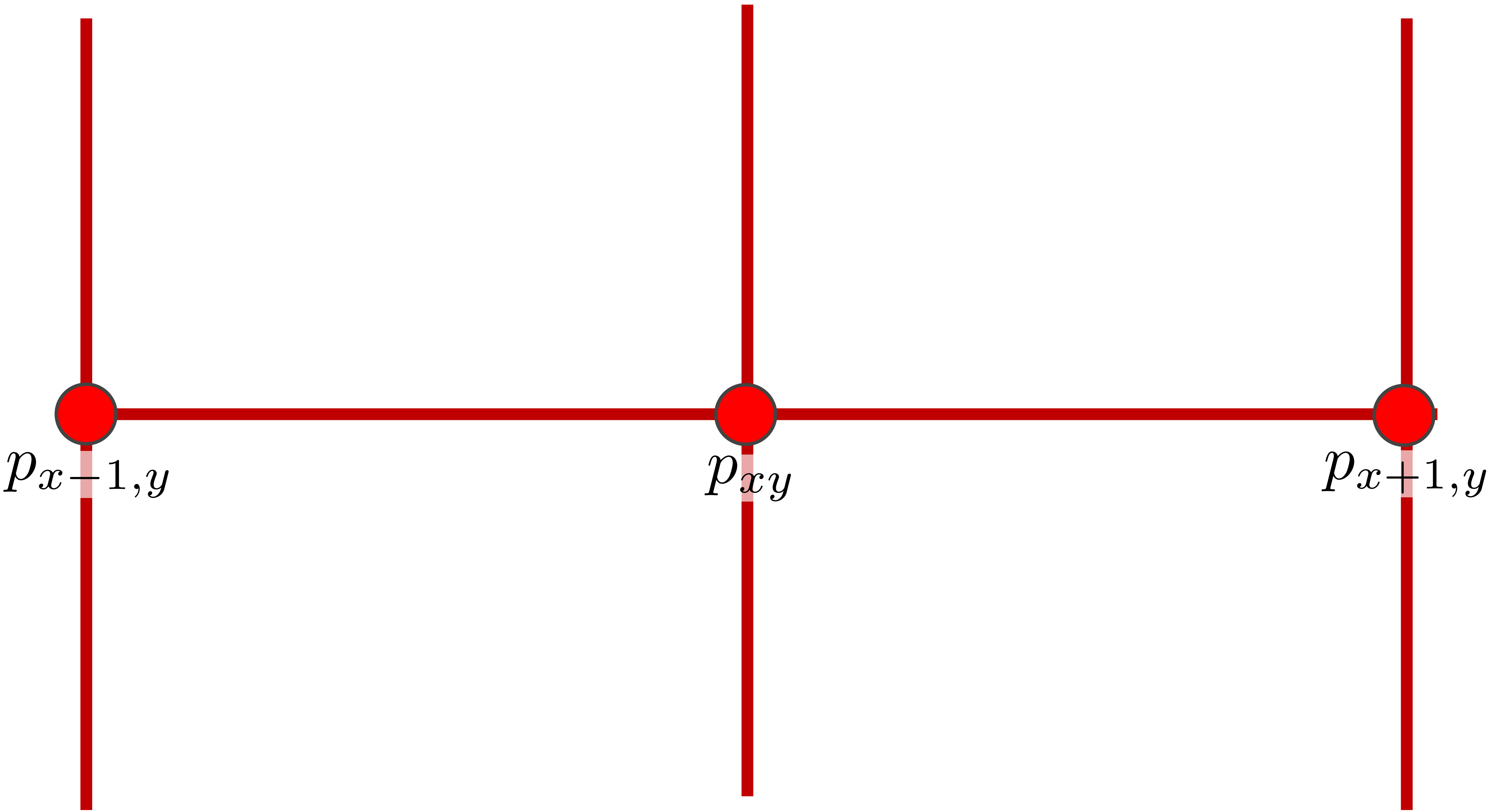
Aside: The Staggered Grid



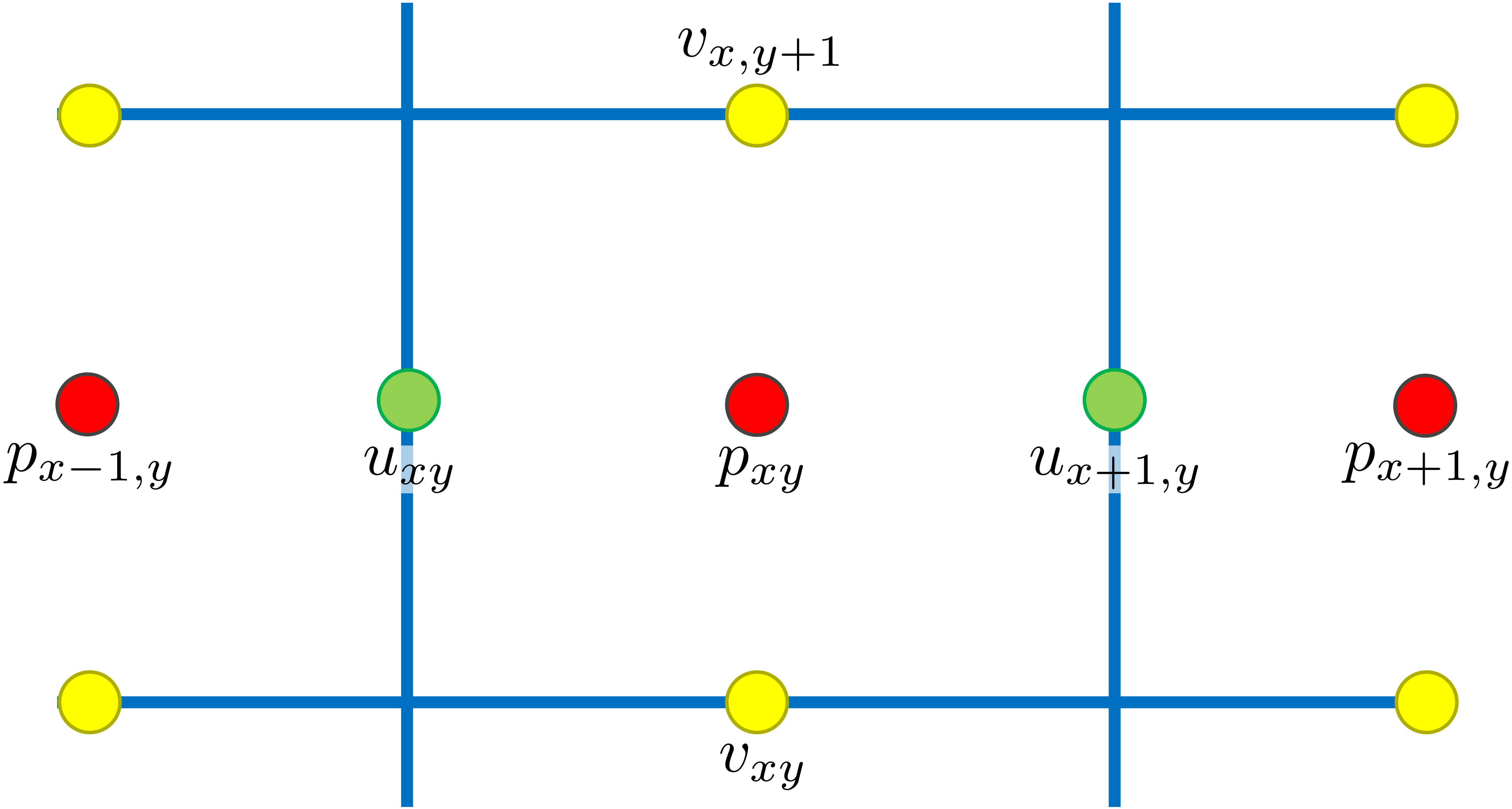
Aside: The Staggered Grid



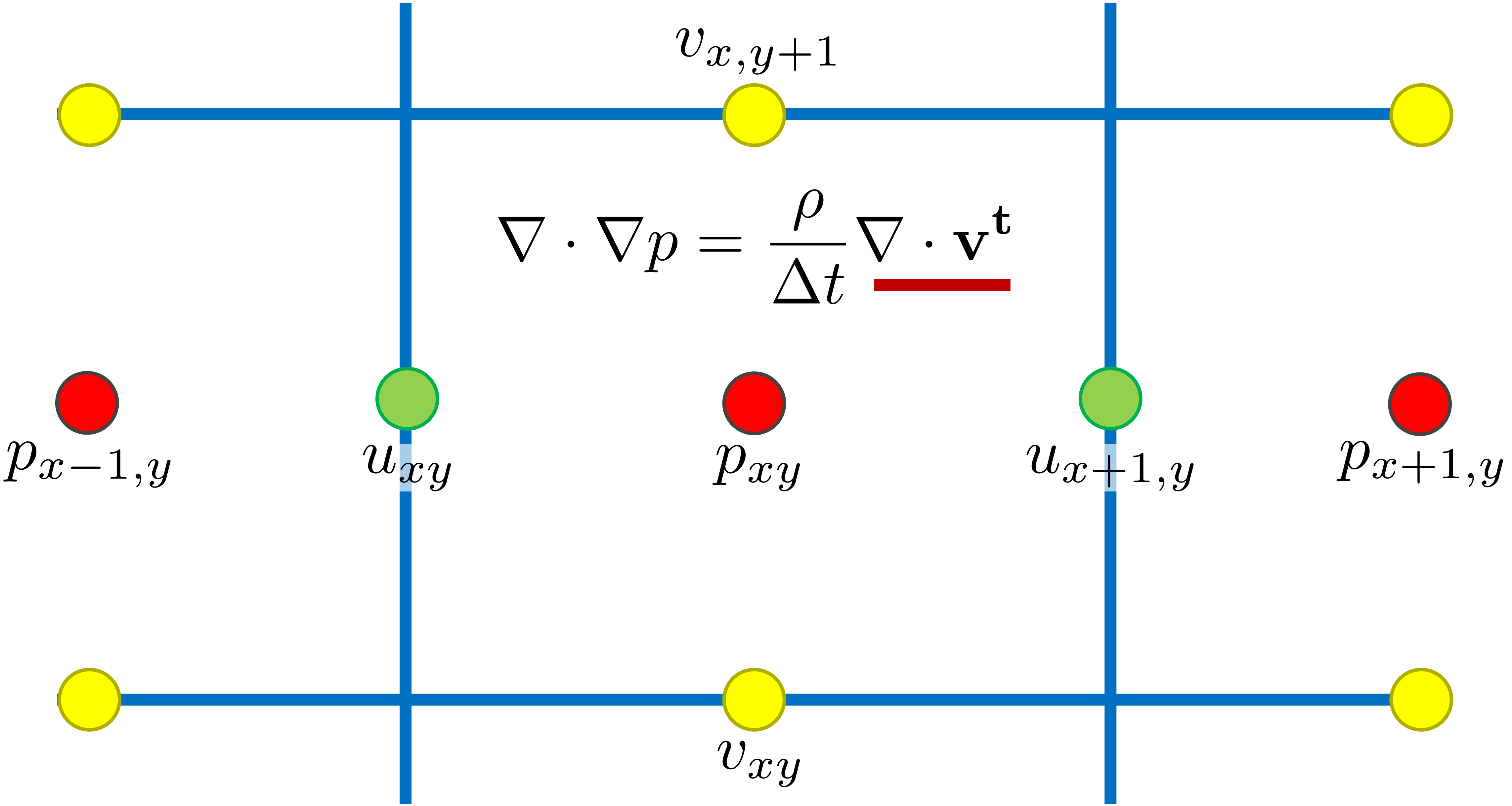
Aside: The Staggered Grid



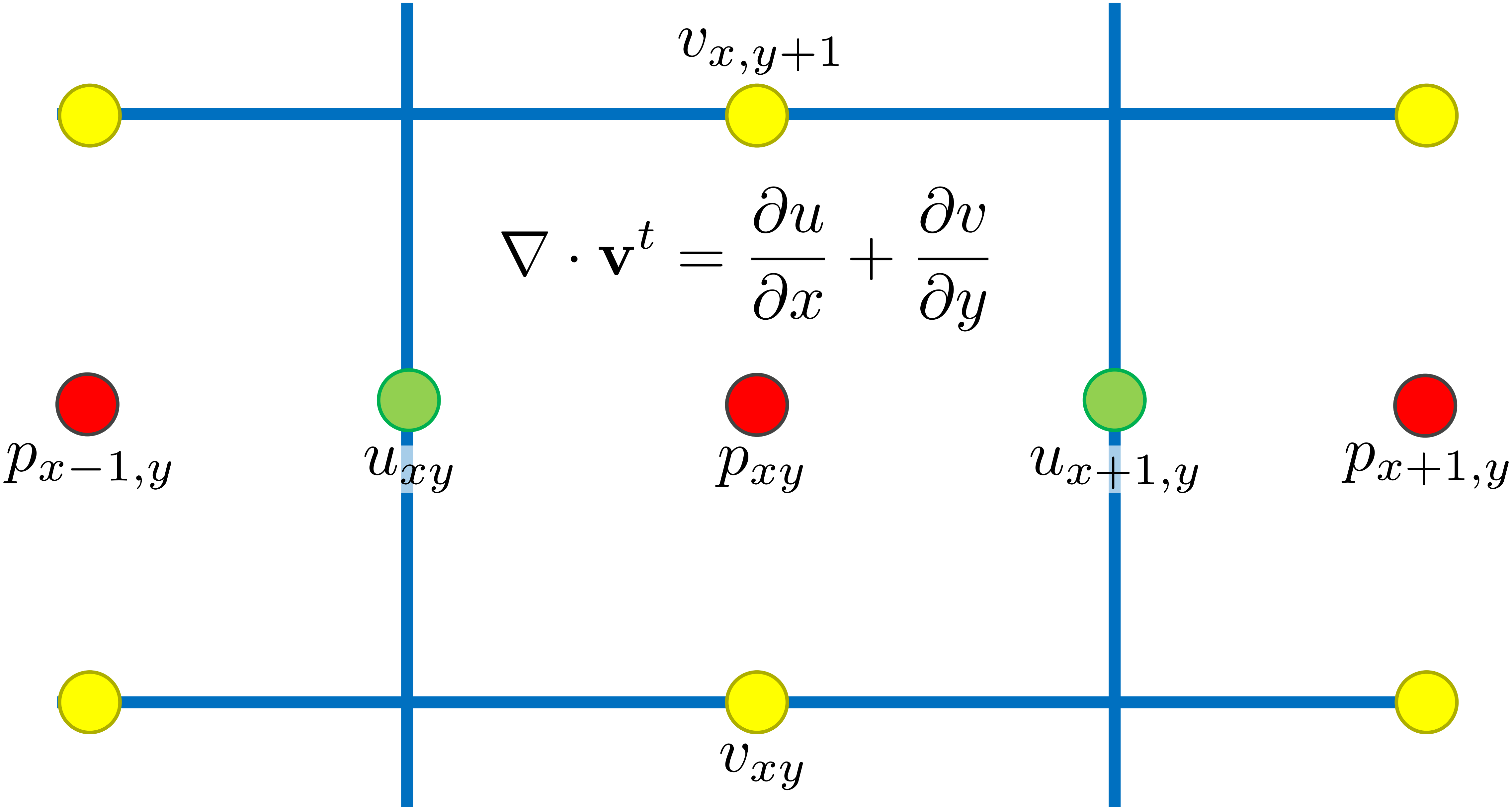
Aside: The Staggered Grid



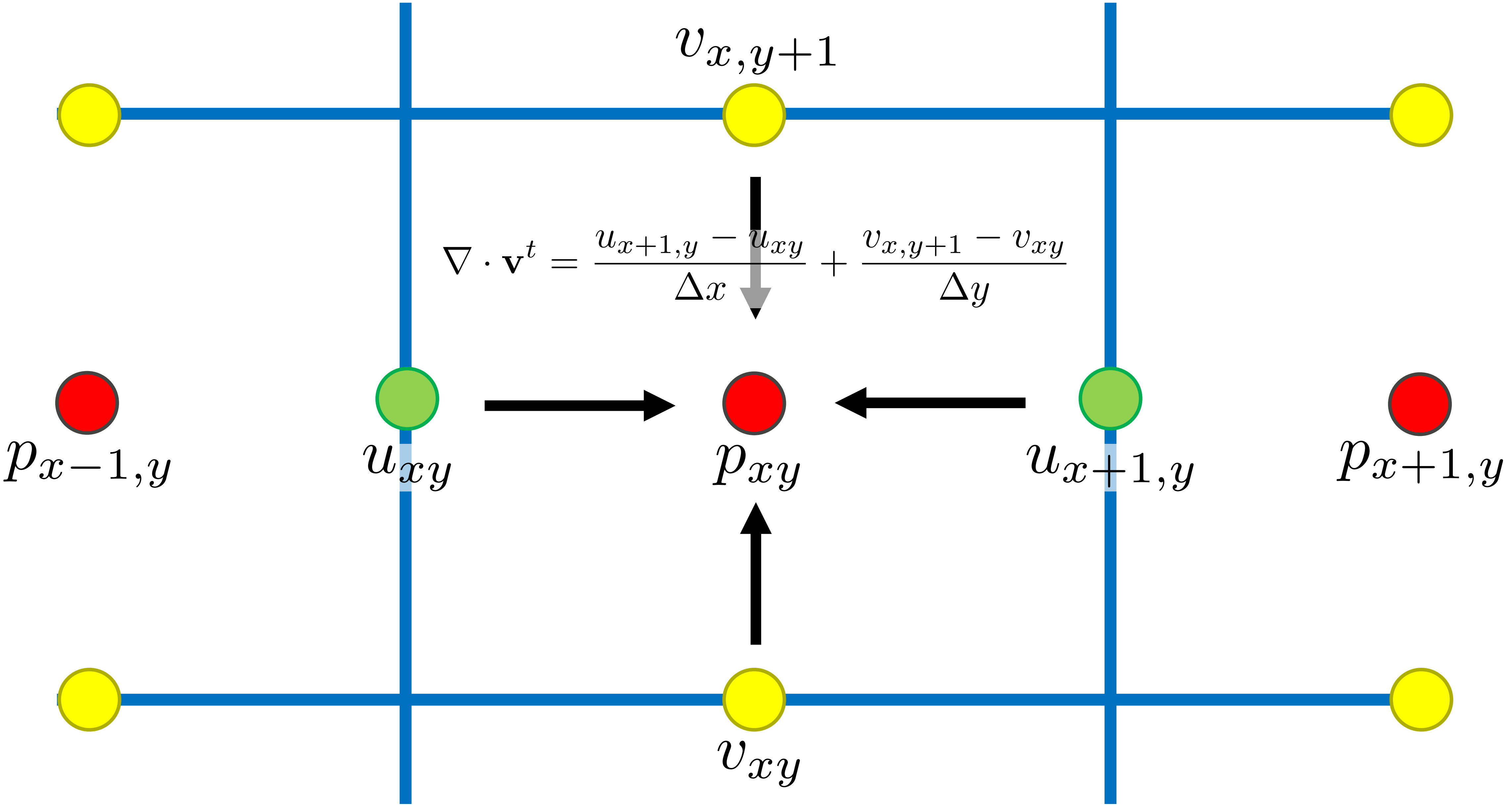
Fluid Simulation – Pressure Projection



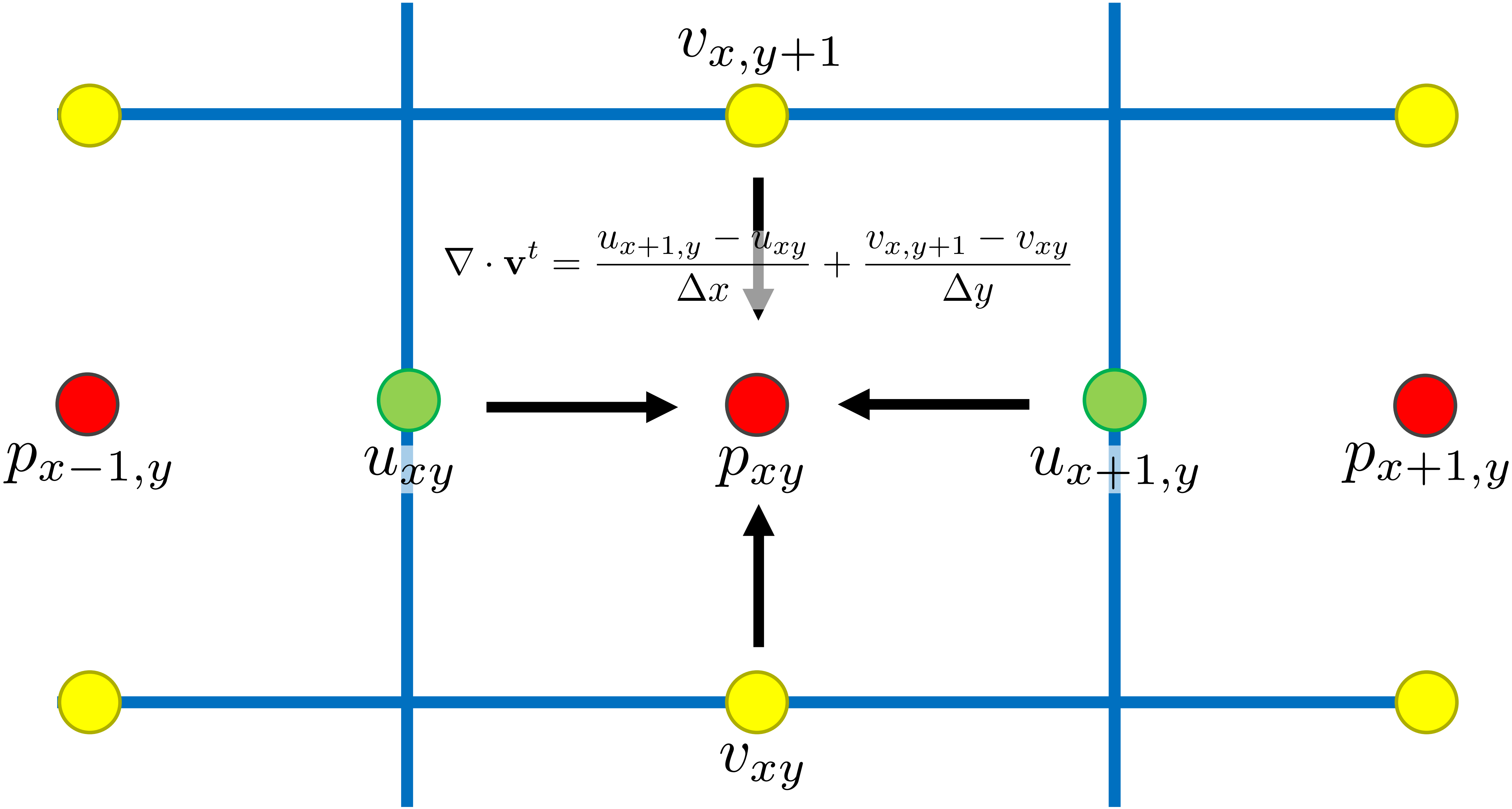
Fluid Simulation – Pressure Projection



Fluid Simulation – Pressure Projection



Fluid Simulation – Pressure Projection



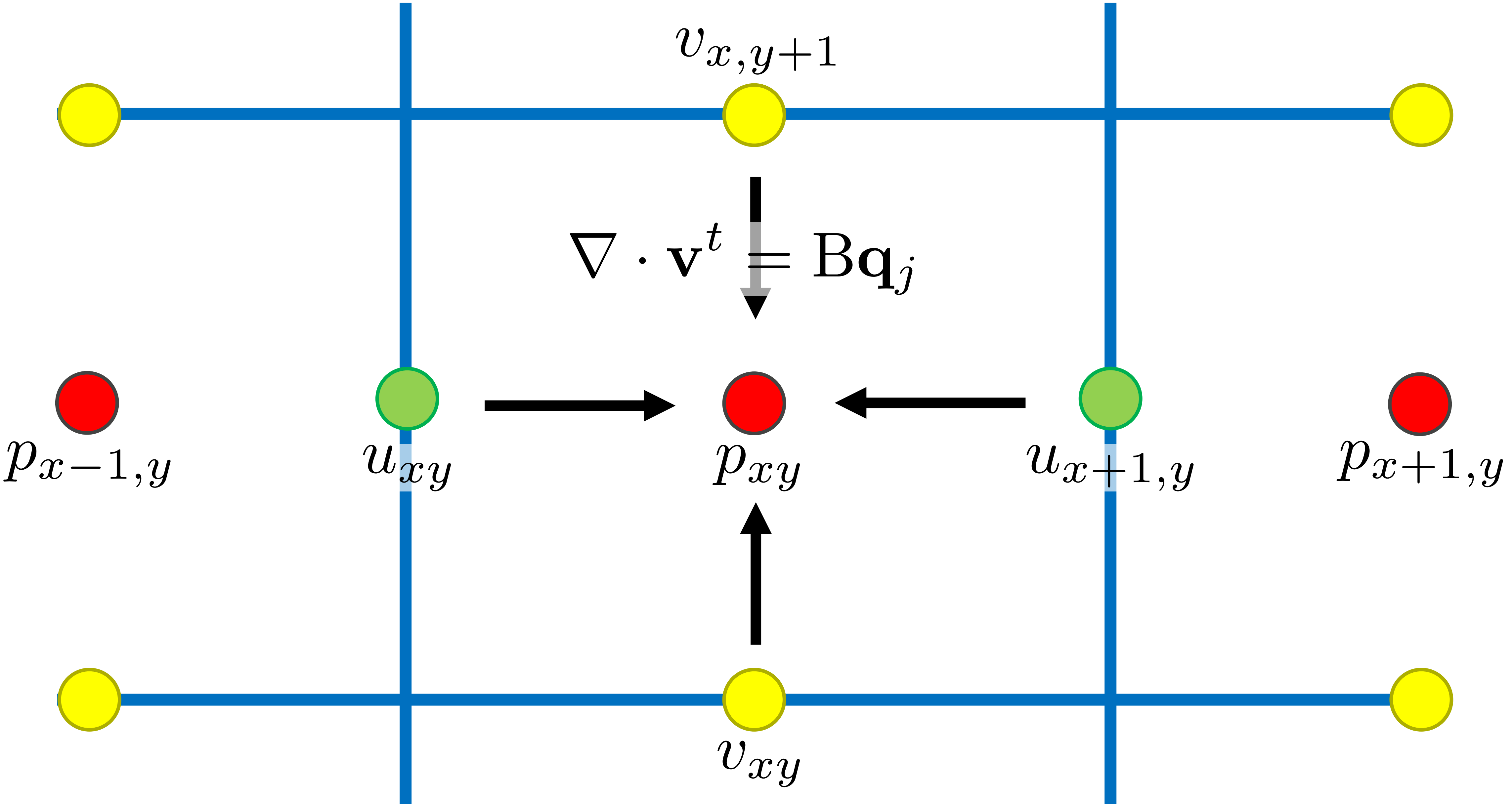
$$\nabla \cdot \mathbf{v}^t = \frac{u_{x+1,y} - u_{xy}}{\Delta x} + \frac{v_{x,y+1} - v_{xy}}{\Delta y}$$

Fluid Simulation – Pressure Projection

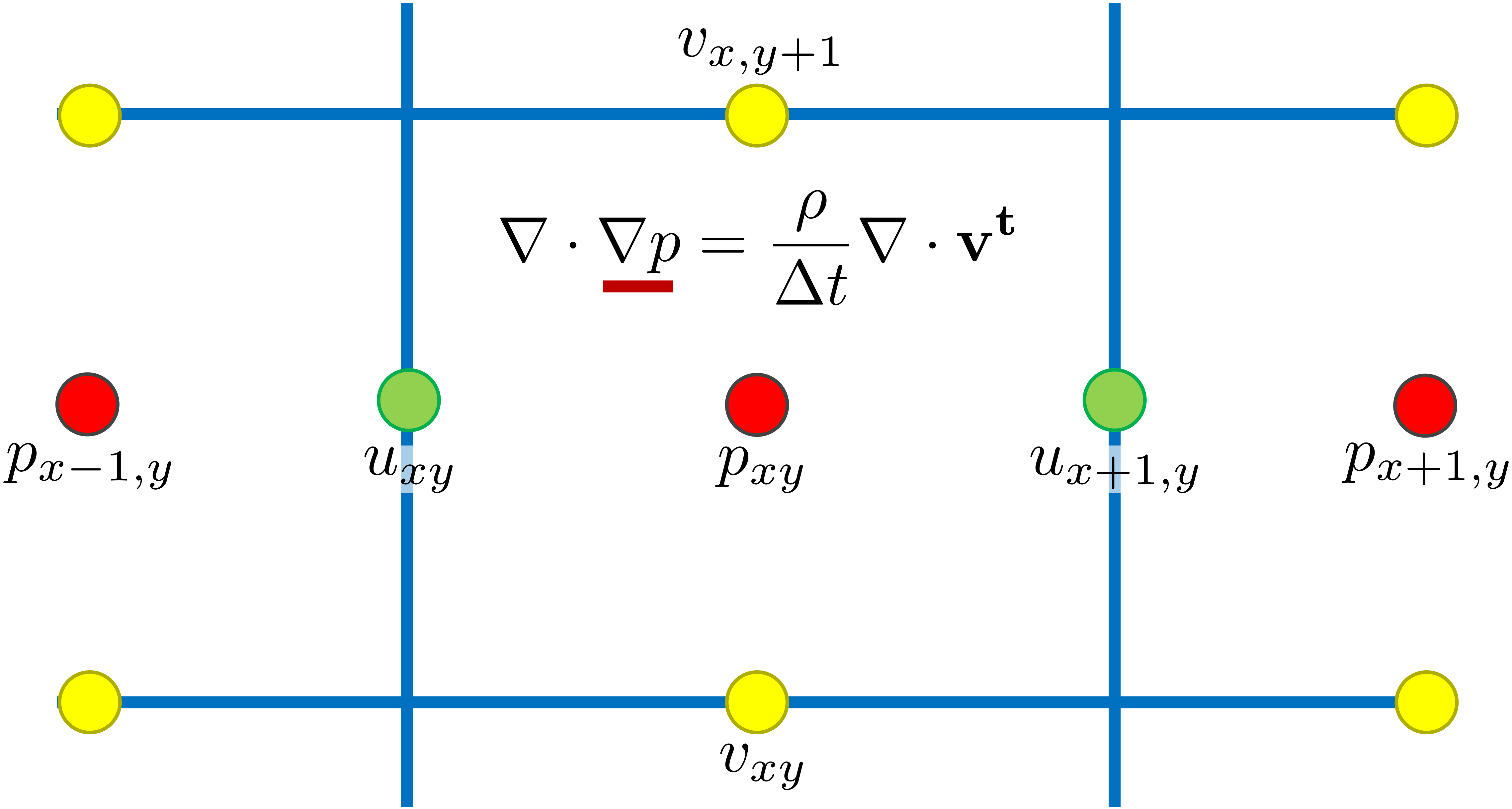
$$\nabla \cdot \mathbf{v}^t = \frac{u_{x+1,y} - u_{xy}}{\Delta x} + \frac{v_{x,y+1} - v_{xy}}{\Delta y}$$

$$\nabla \cdot \mathbf{v}^t = \underbrace{\left(-\frac{1}{\Delta x} \quad \frac{1}{\Delta x} \quad -\frac{1}{\Delta y} \quad \frac{1}{\Delta y}\right)}_{\mathrm{B}} \underbrace{\begin{pmatrix} u_{xy} \\ u_{x+1,y} \\ v_{xy} \\ v_{x,y+1} \end{pmatrix}}_{\mathbf{q}_j}$$

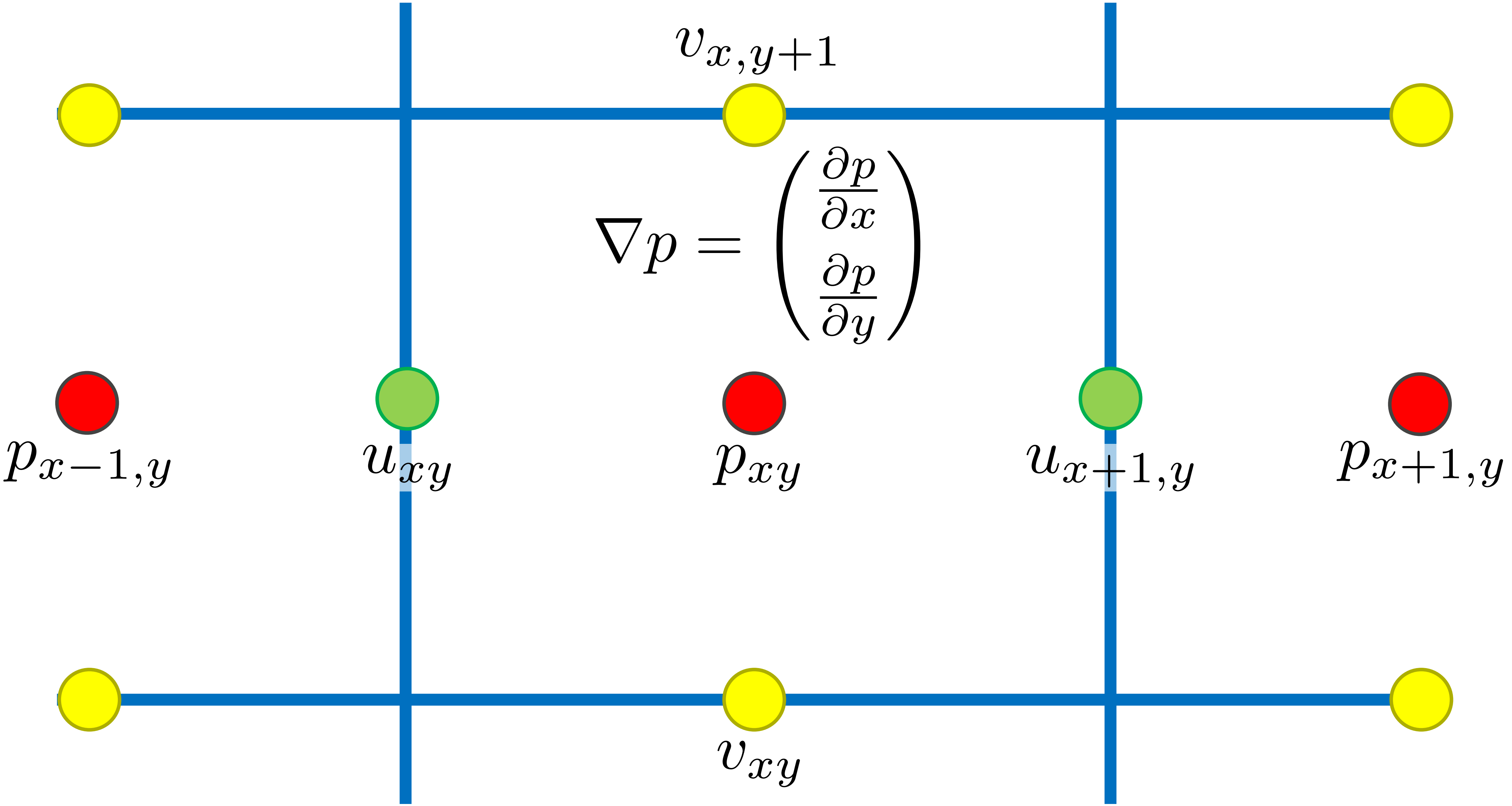
Fluid Simulation – Pressure Projection



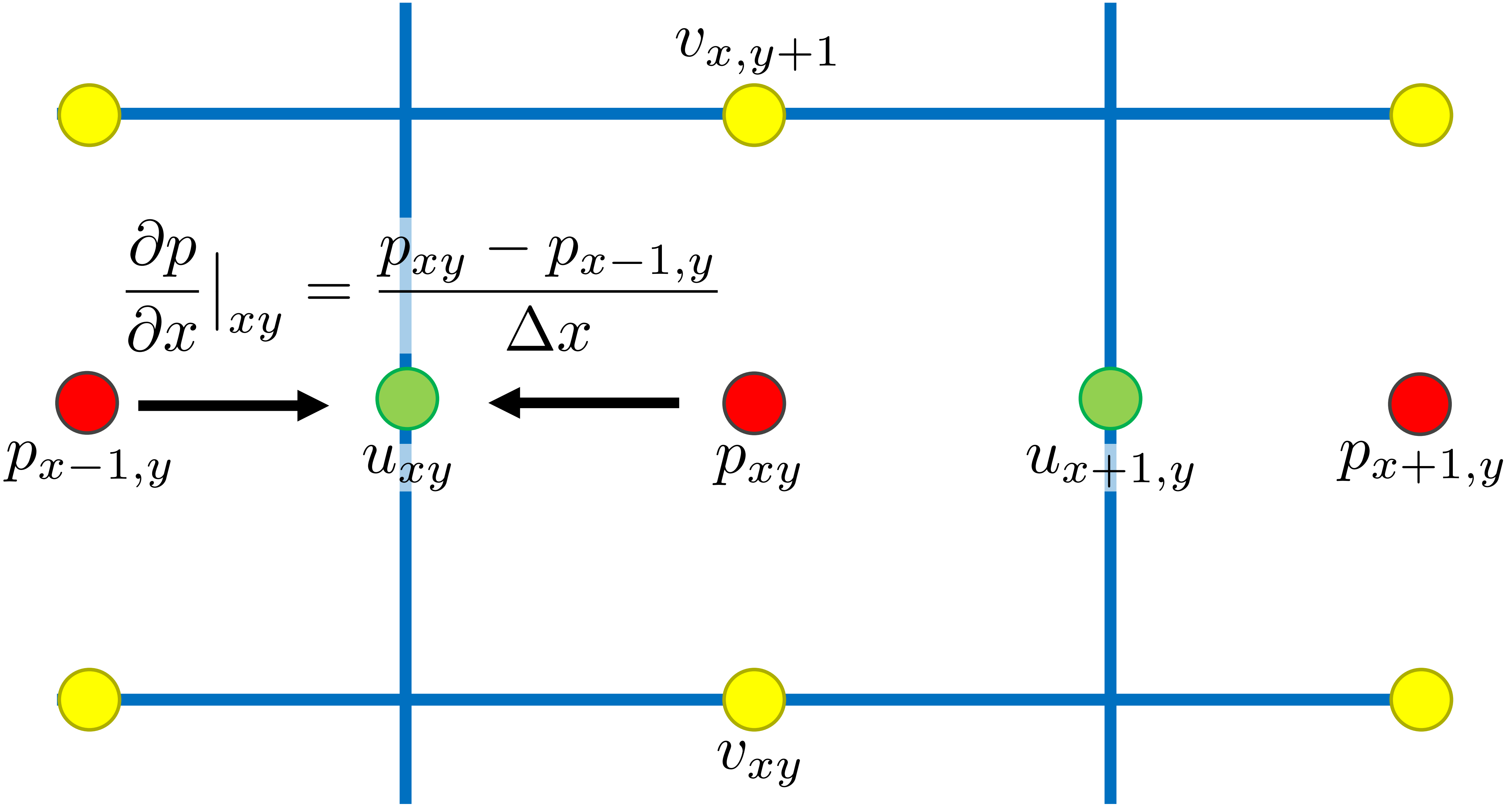
Fluid Simulation – Pressure Projection



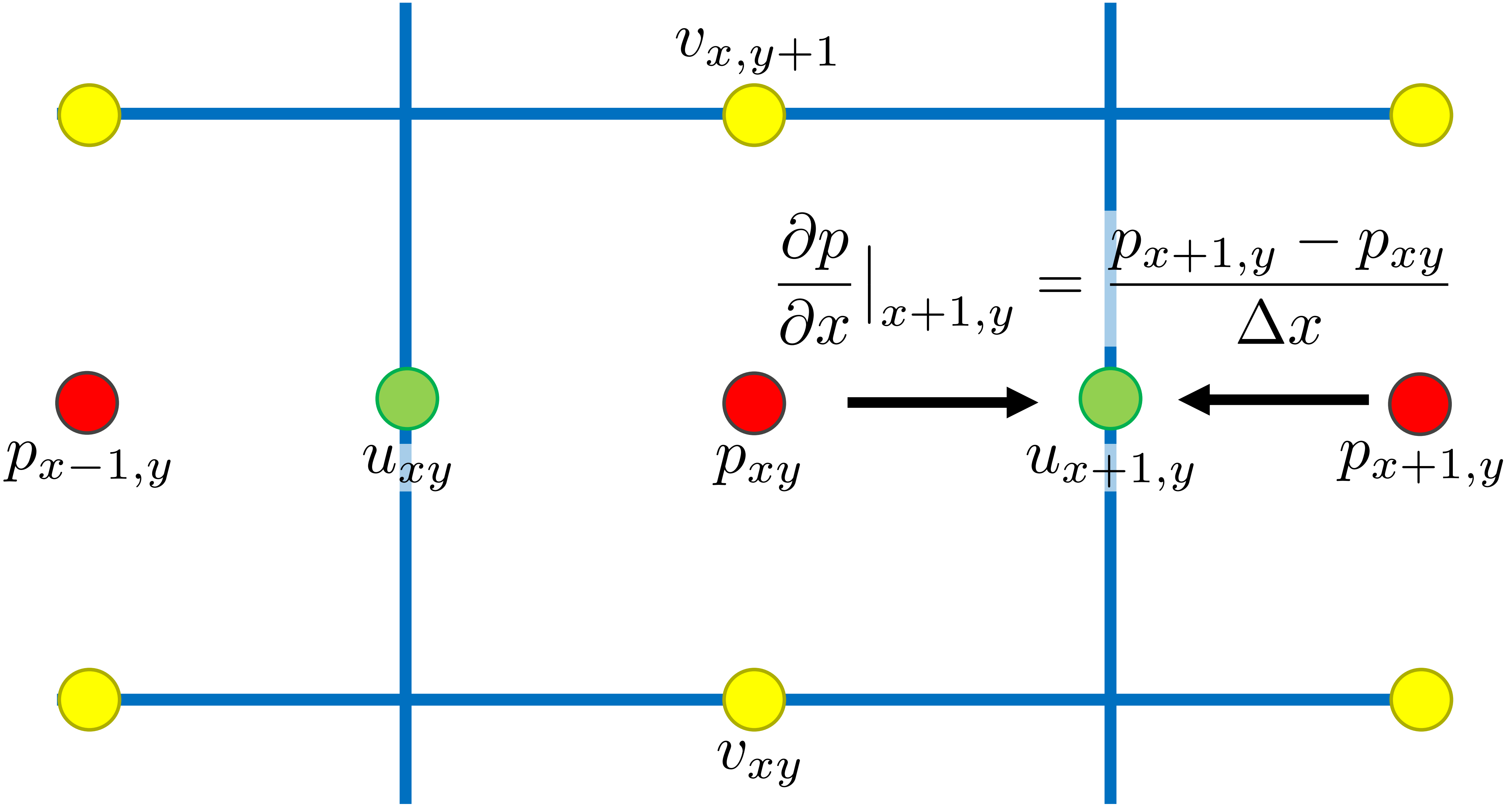
Fluid Simulation – Pressure Projection



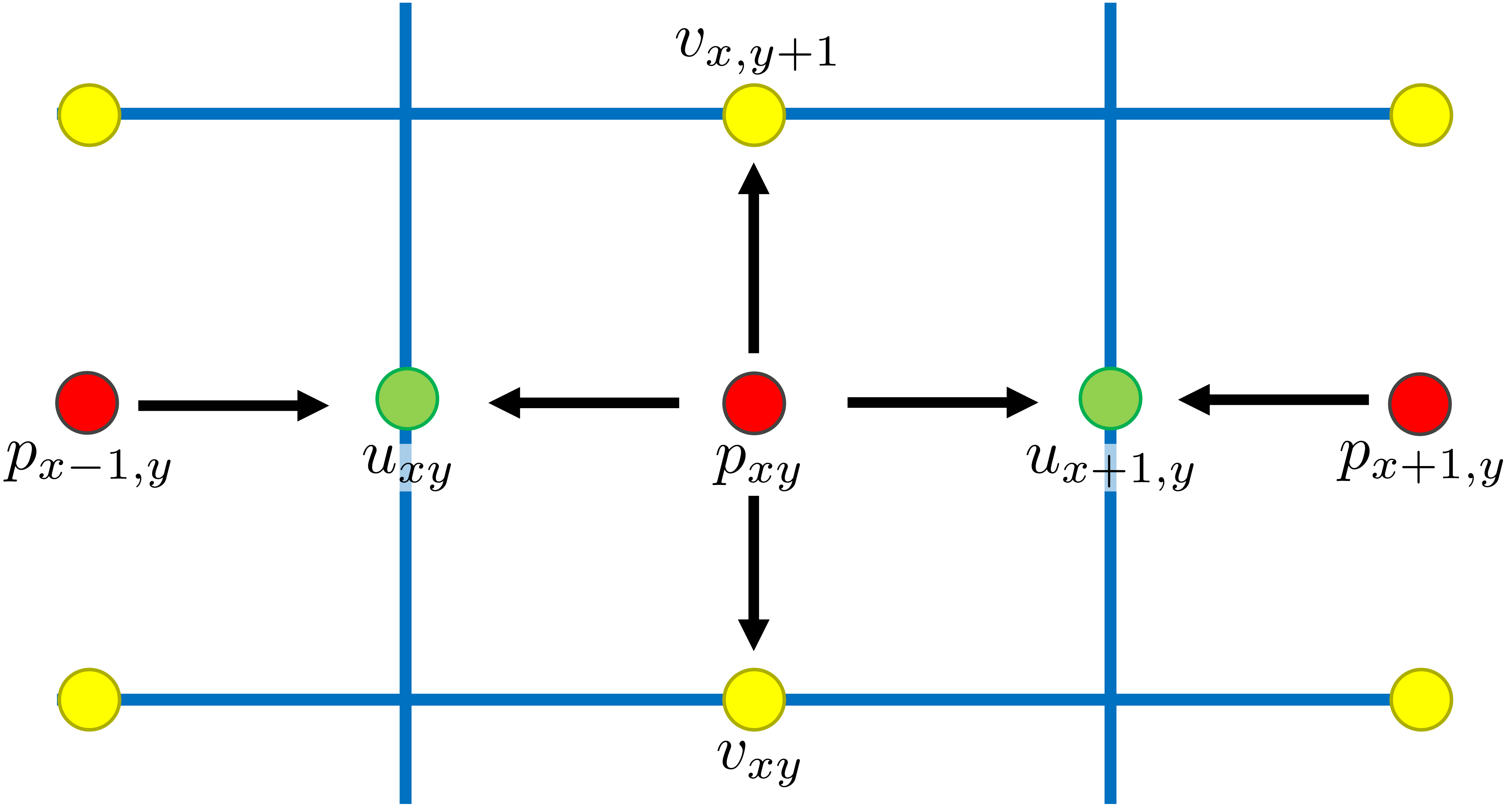
Fluid Simulation – Pressure Projection



Fluid Simulation – Pressure Projection



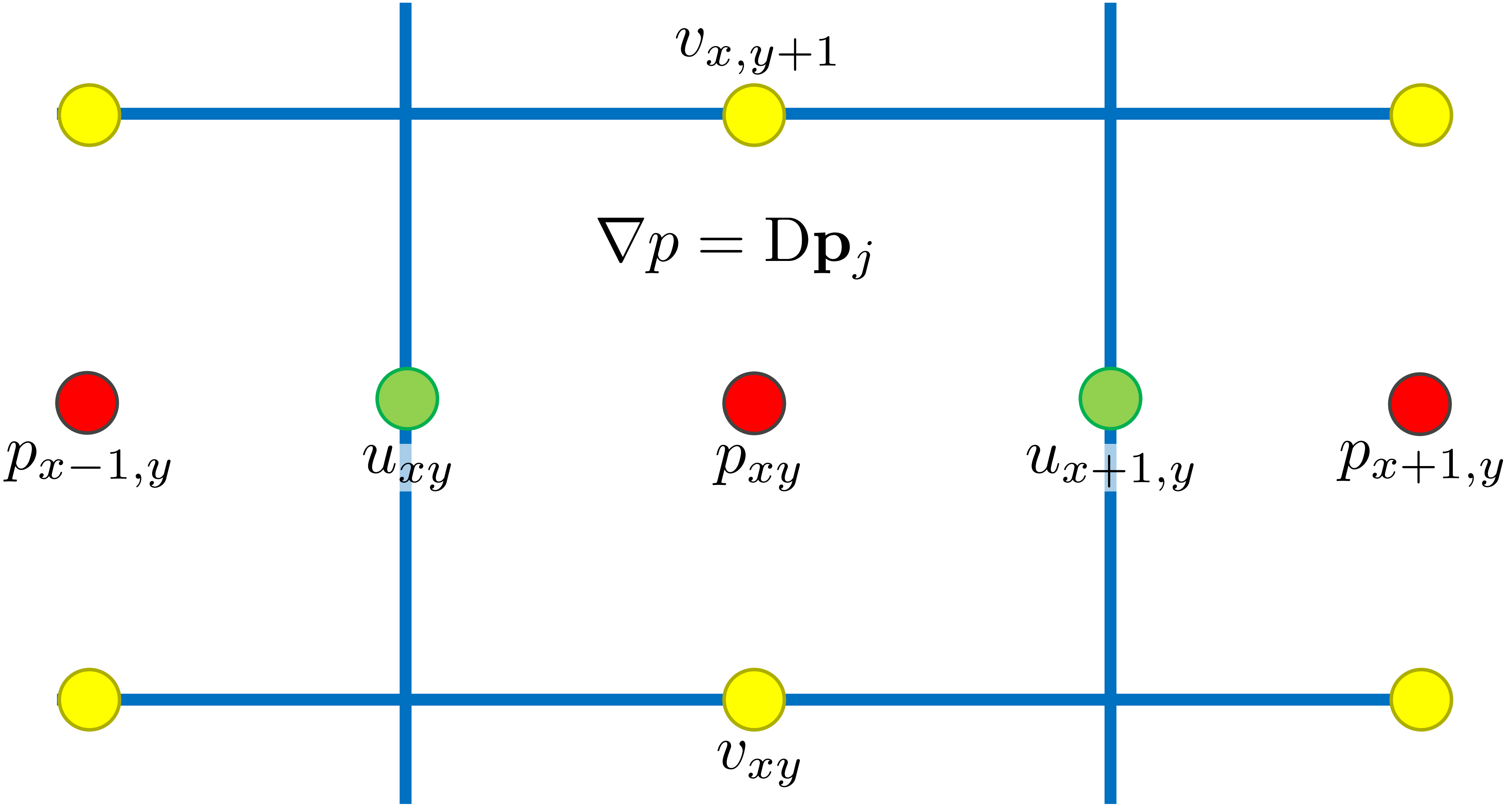
Fluid Simulation – Pressure Projection



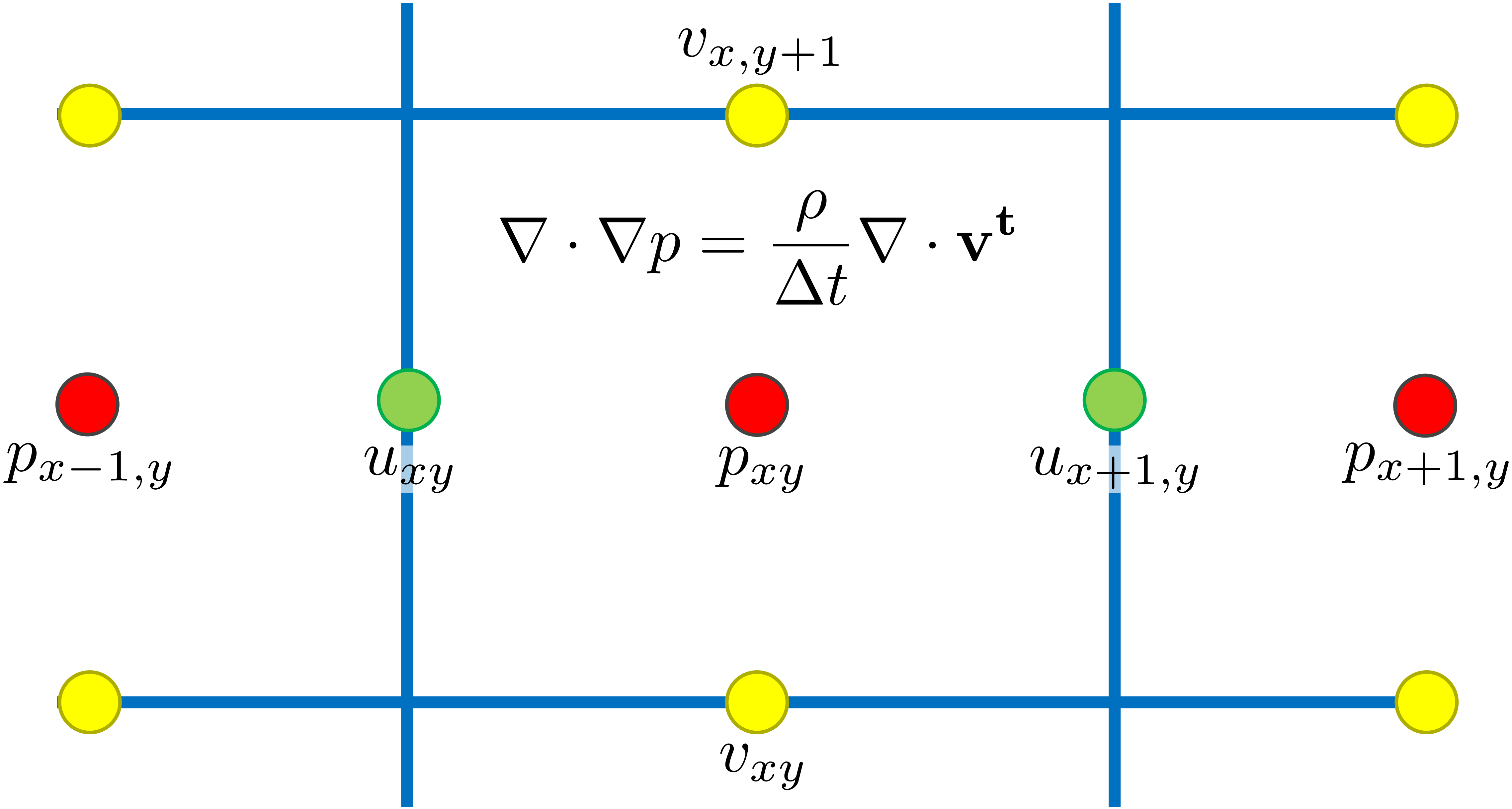
Fluid Simulation – Pressure Projection

$$\nabla p = \underbrace{\begin{pmatrix} -\frac{1}{\Delta x} & 0 & \frac{1}{\Delta x} & 0 & 0 \\ 0 & \frac{1}{\Delta x} & -\frac{1}{\Delta x} & 0 & 0 \\ 0 & 0 & \frac{1}{\Delta y} & -\frac{1}{\Delta y} & 0 \\ 0 & 0 & -\frac{1}{\Delta y} & 0 & \frac{1}{\Delta y} \end{pmatrix}}_{\mathbf{D}} \underbrace{\begin{pmatrix} p_{x-1,y} \\ p_{x+1,y} \\ p_{xy} \\ p_{xy-1} \\ p_{x,y+1} \end{pmatrix}}_{\mathbf{p}_j}$$

Fluid Simulation – Pressure Projection

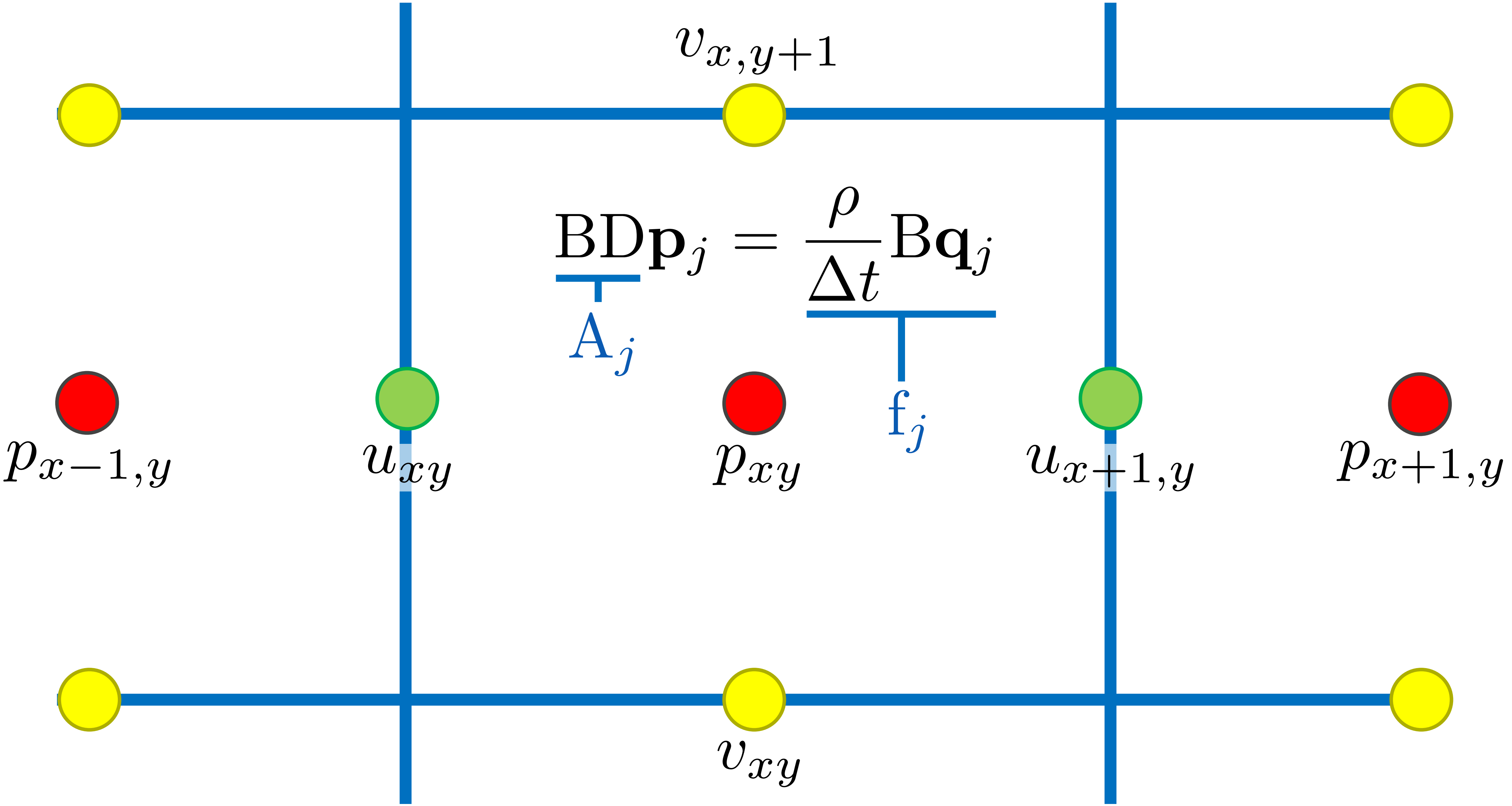


Fluid Simulation – Pressure Projection

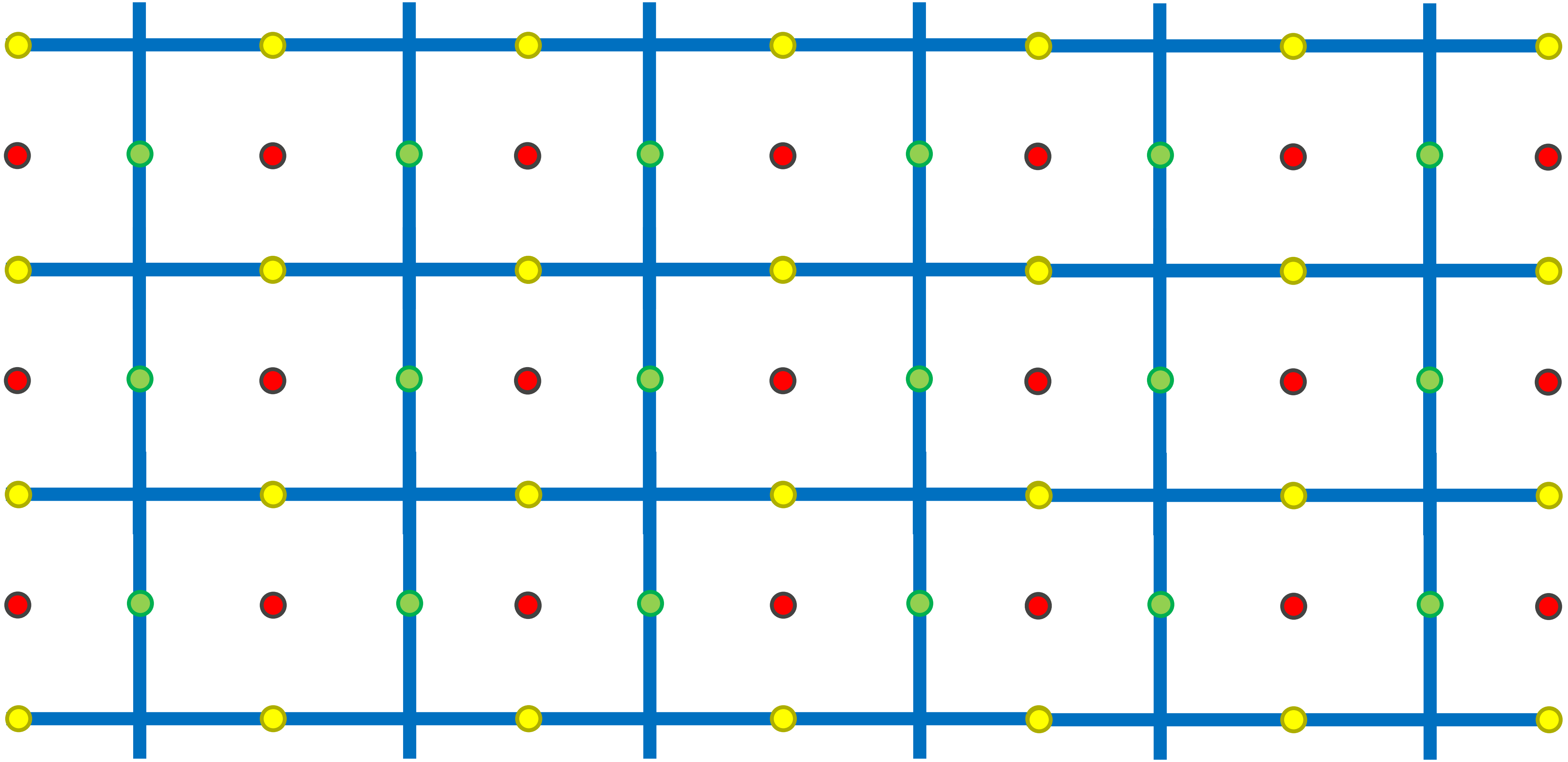


$$\nabla \cdot \nabla p = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{v^t}$$

Fluid Simulation – Pressure Projection



Fluid Simulation – Pressure Projection



Assemble: $\mathrm{A}\mathbf{p} = \mathrm{d}$

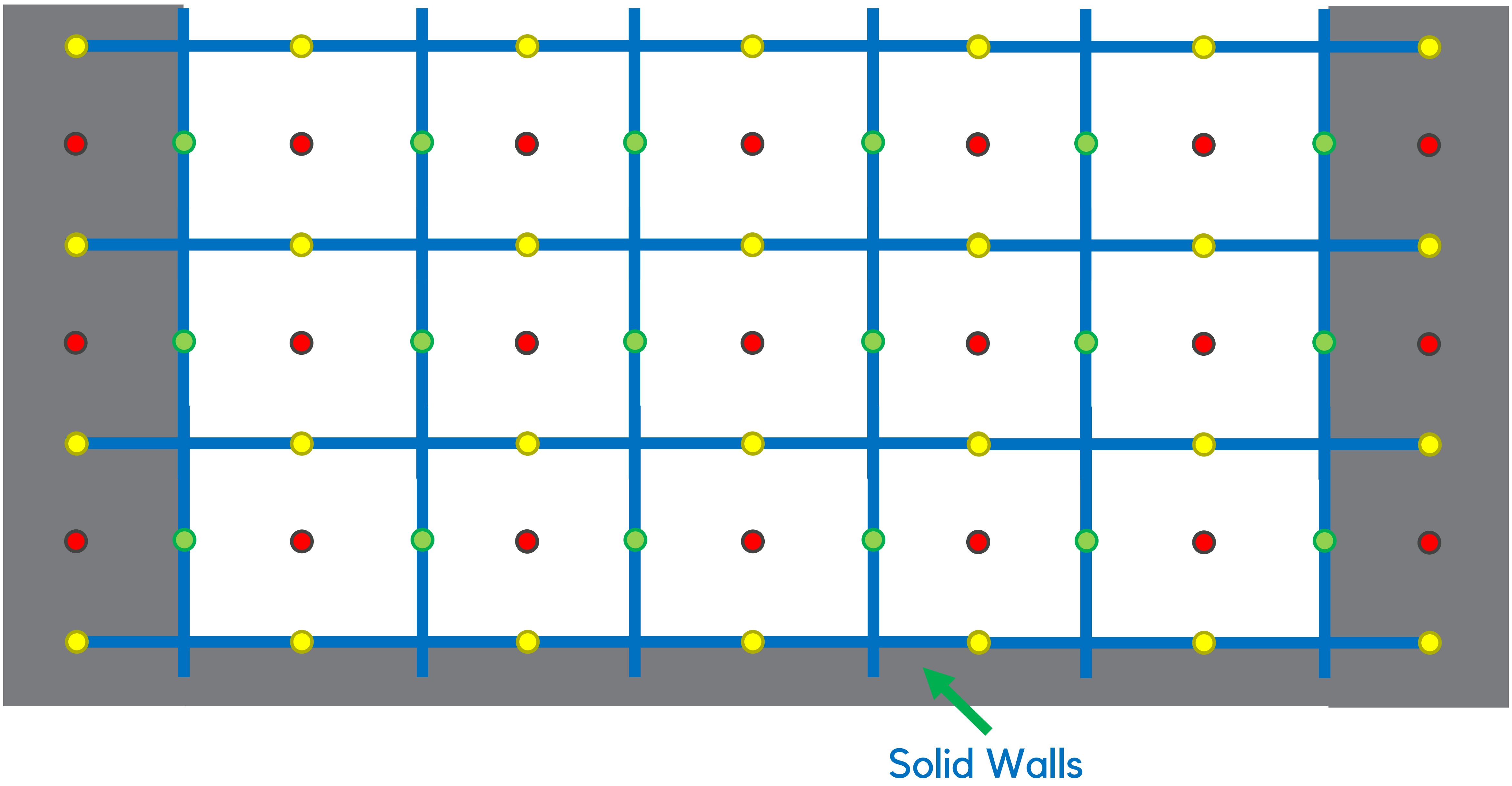
Solve using Conjugate Gradient Method

Fluid Simulation – Pressure Projection

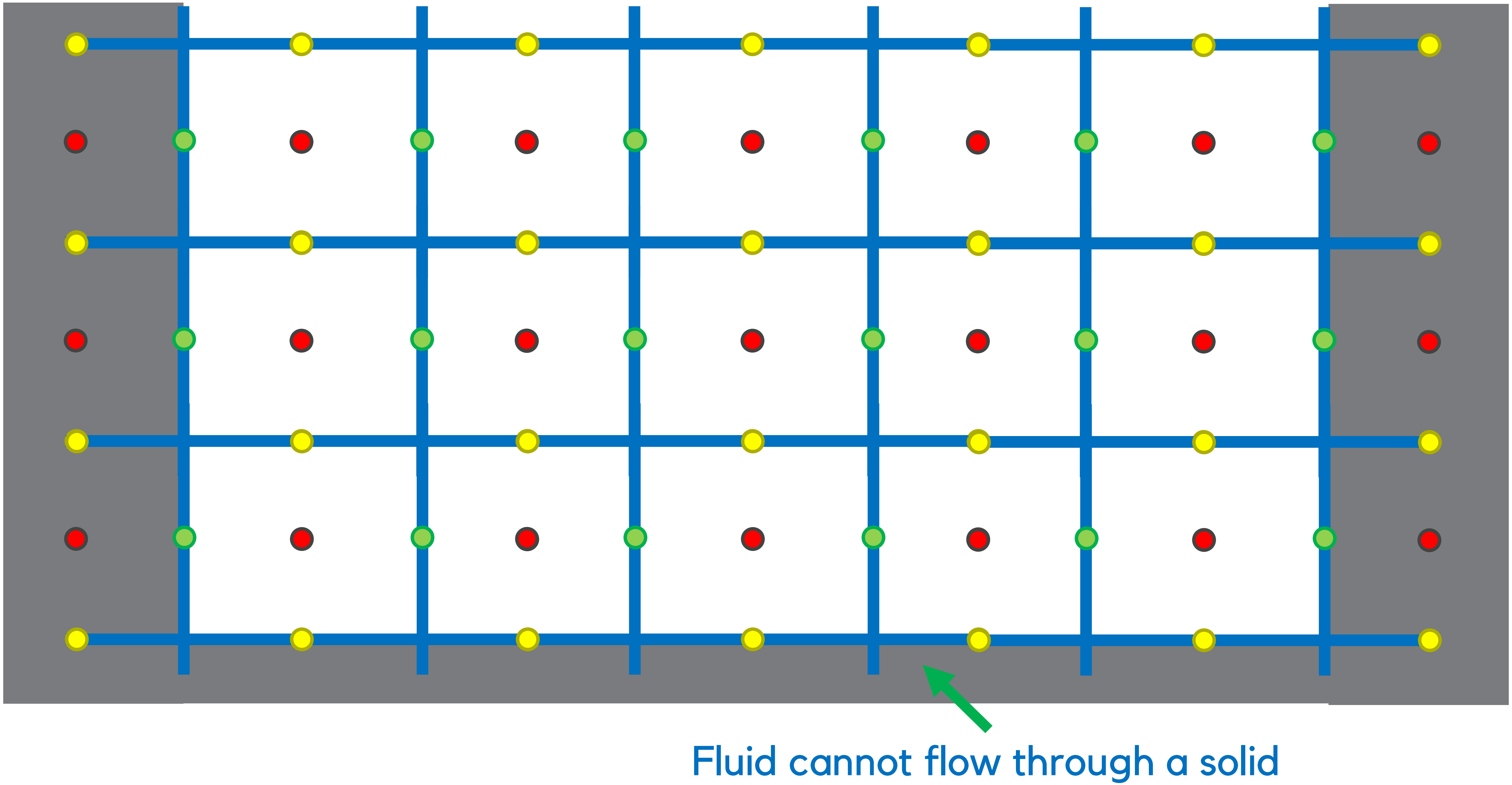
Assemble: $\mathrm{A}\mathbf{p} = \mathrm{d}$

Solve using Conjugate Gradient Method

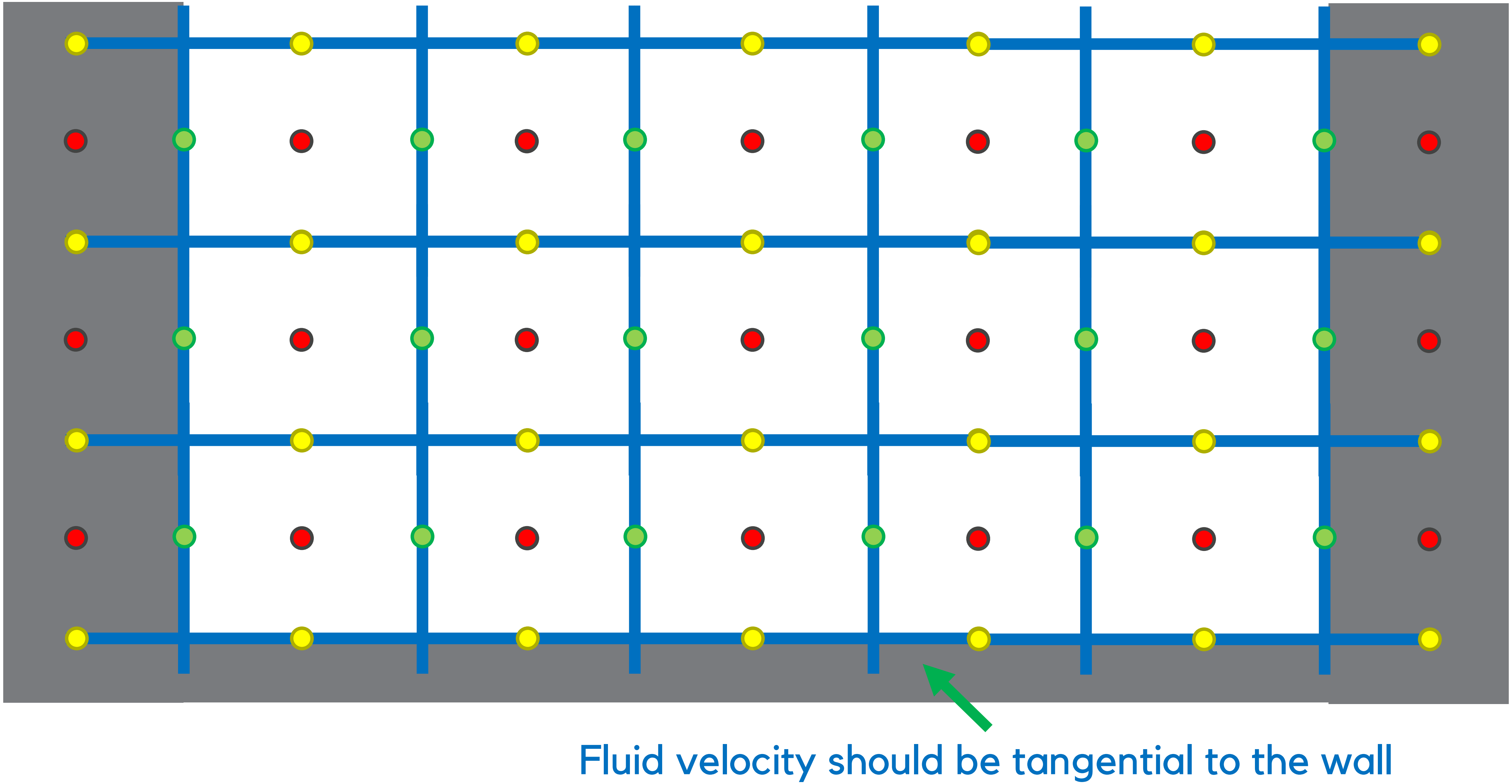
Fluid Simulation – Solid Boundary Conditions



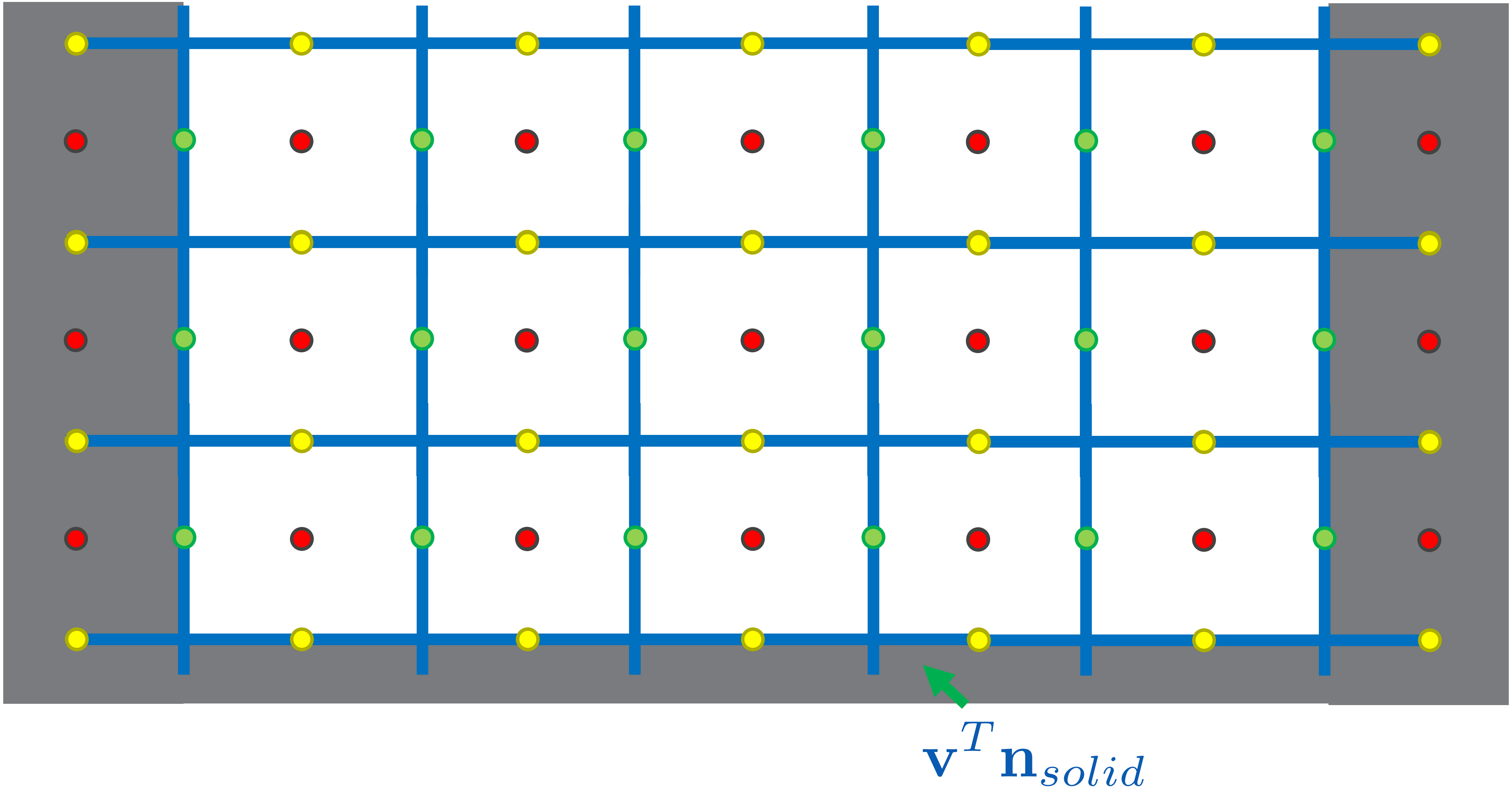
Fluid Simulation – Solid Boundary Conditions



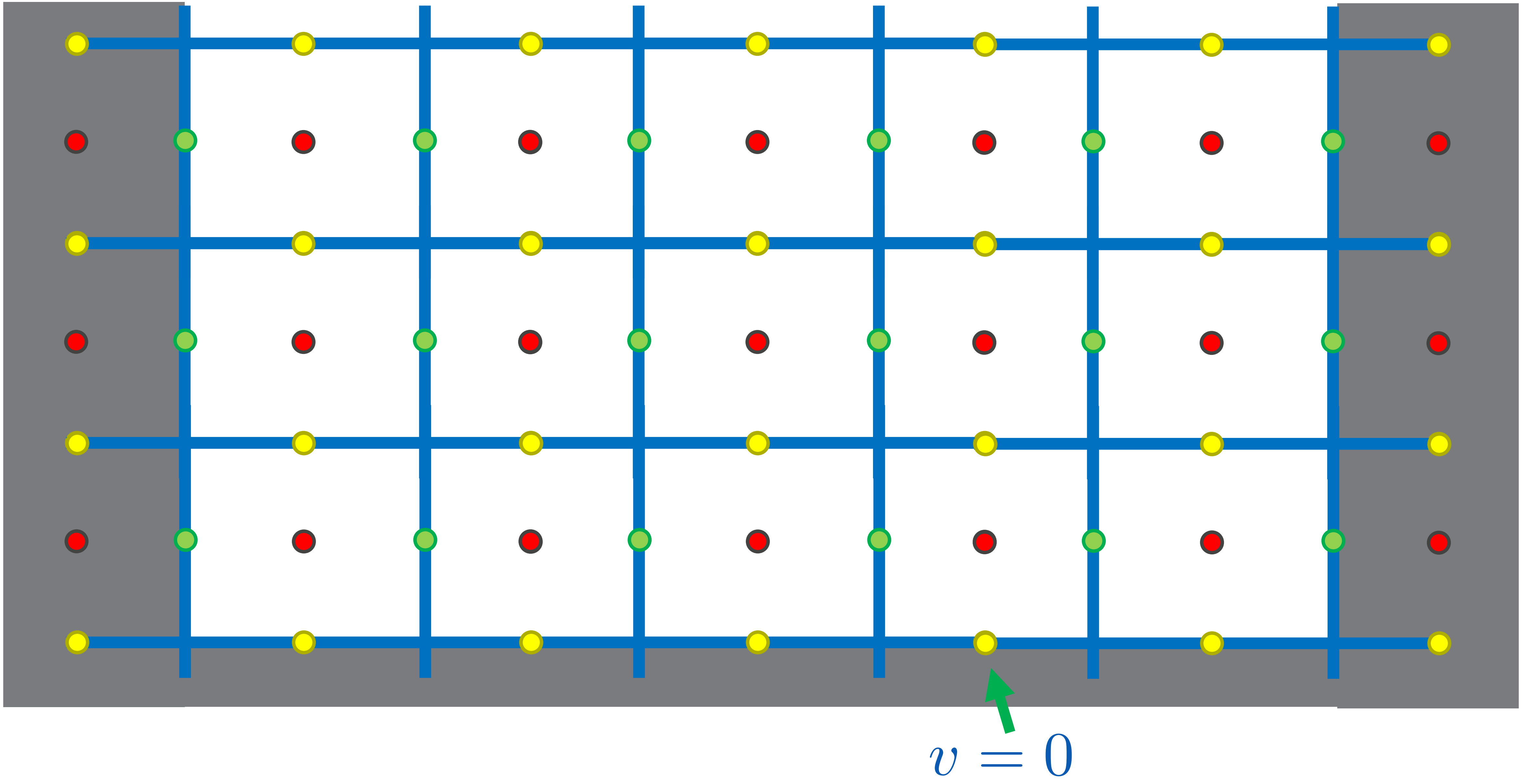
Fluid Simulation – Solid Boundary Conditions



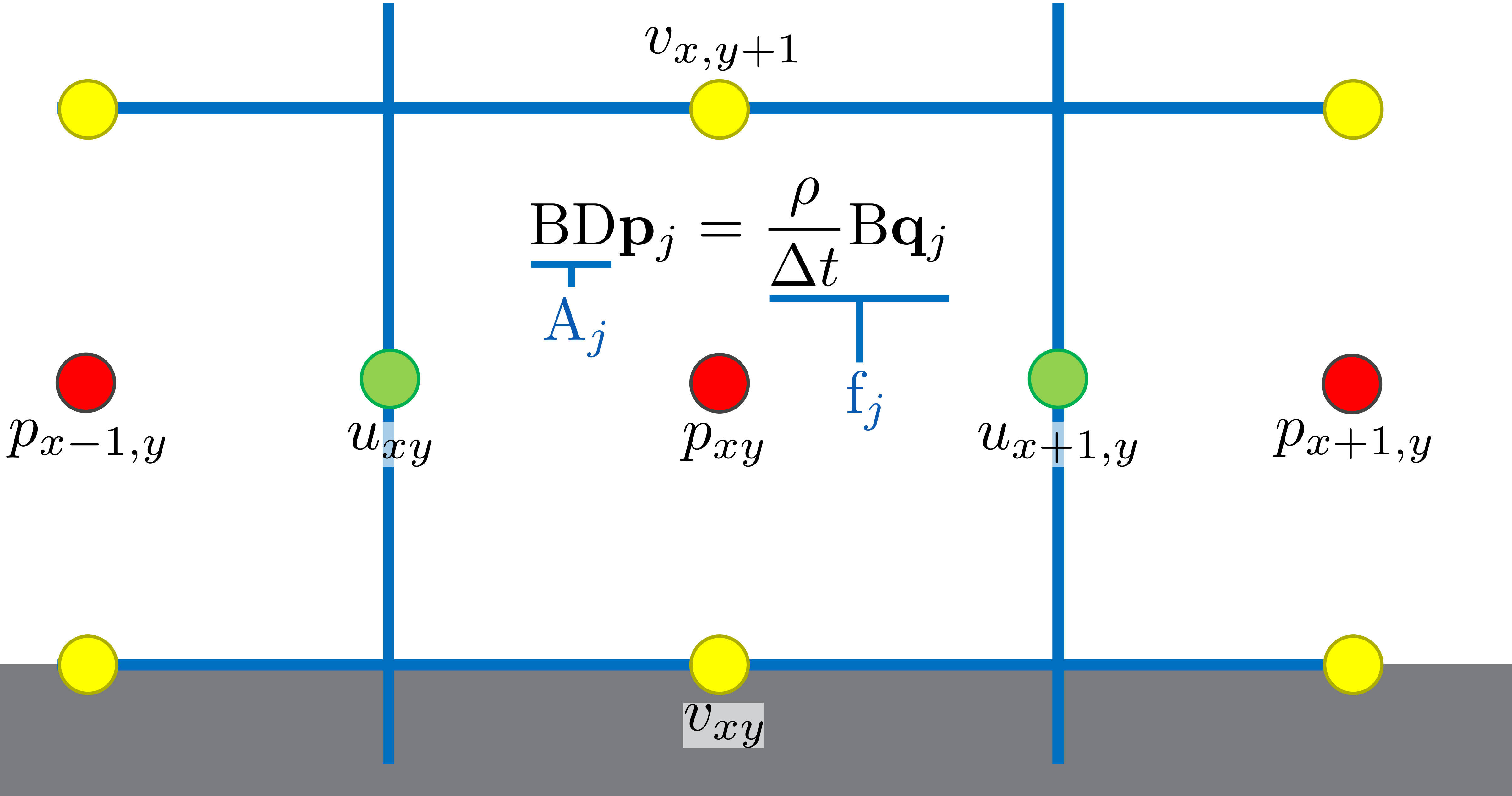
Fluid Simulation – Solid Boundary Conditions



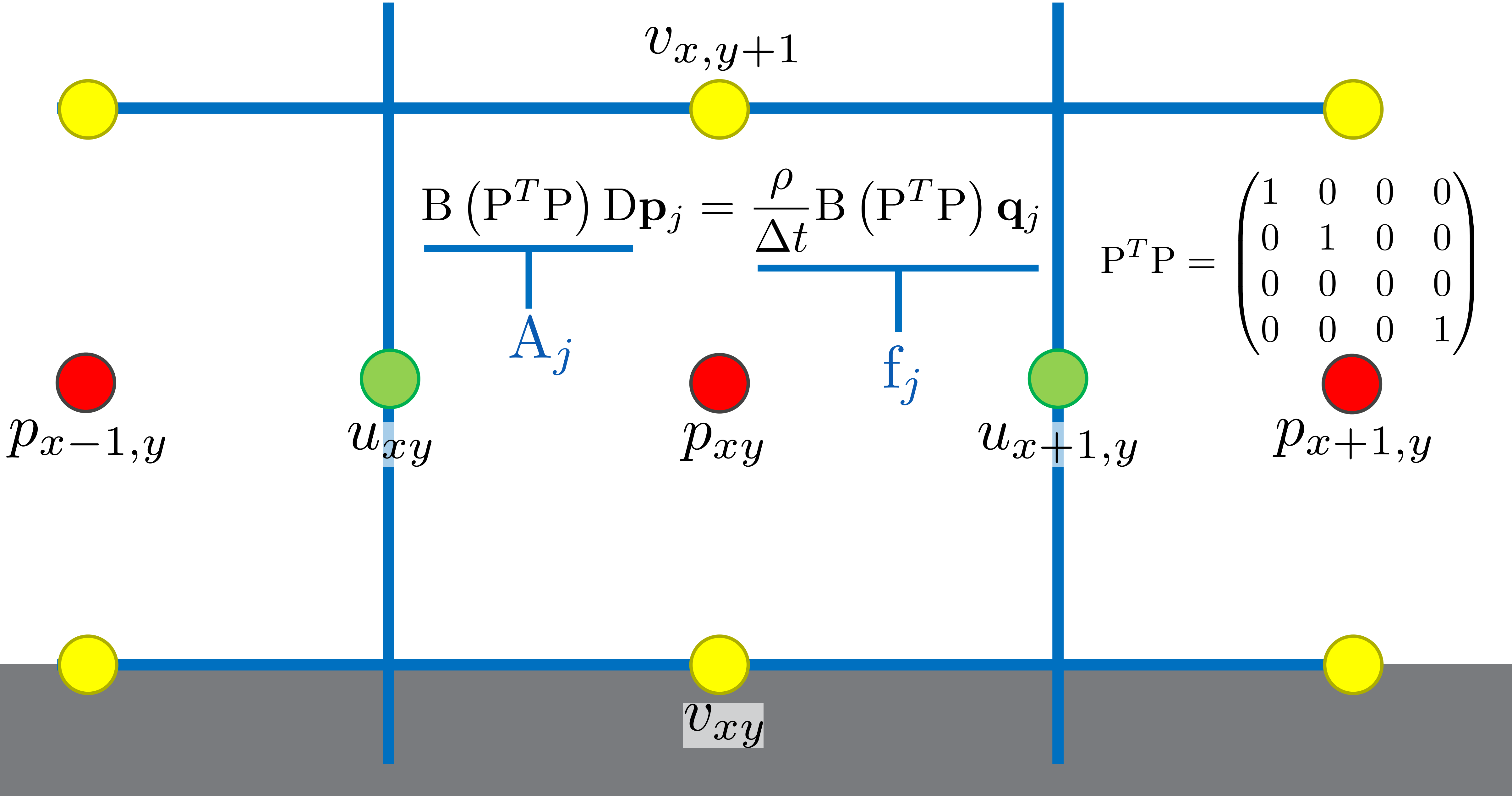
Fluid Simulation – Solid Boundary Conditions



Fluid Simulation – Solid Boundary Conditions



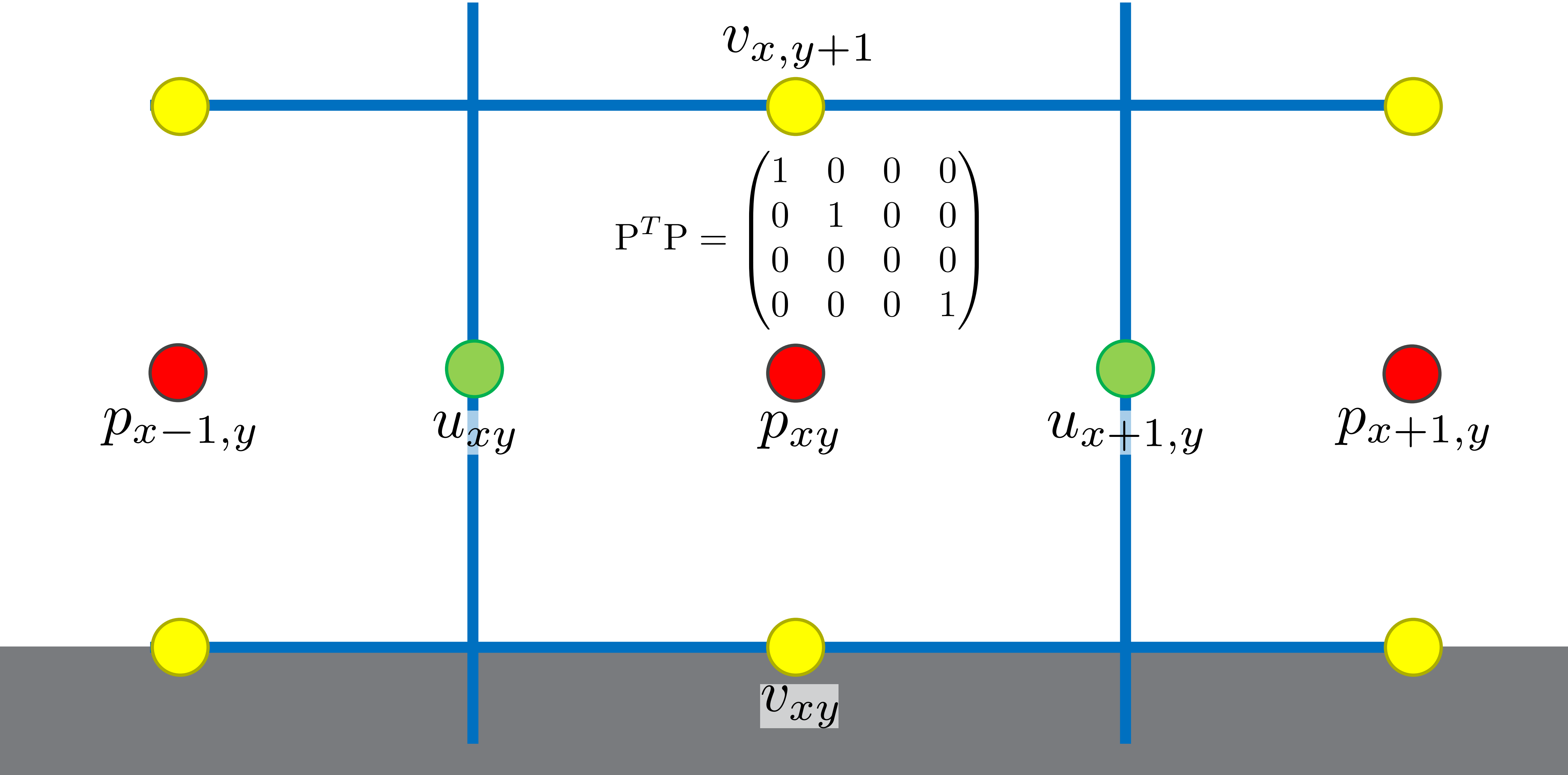
Fluid Simulation – Solid Boundary Conditions



$$\underbrace{\mathrm{B}\left(\mathrm{P}^T\mathrm{P}\right)\mathrm{D}}_{\mathrm{A}_j}\mathbf{p}_j = \underbrace{\frac{\rho}{\Delta t}\mathrm{B}\left(\mathrm{P}^T\mathrm{P}\right)\mathbf{q}_j}_{\mathrm{f}_j}$$

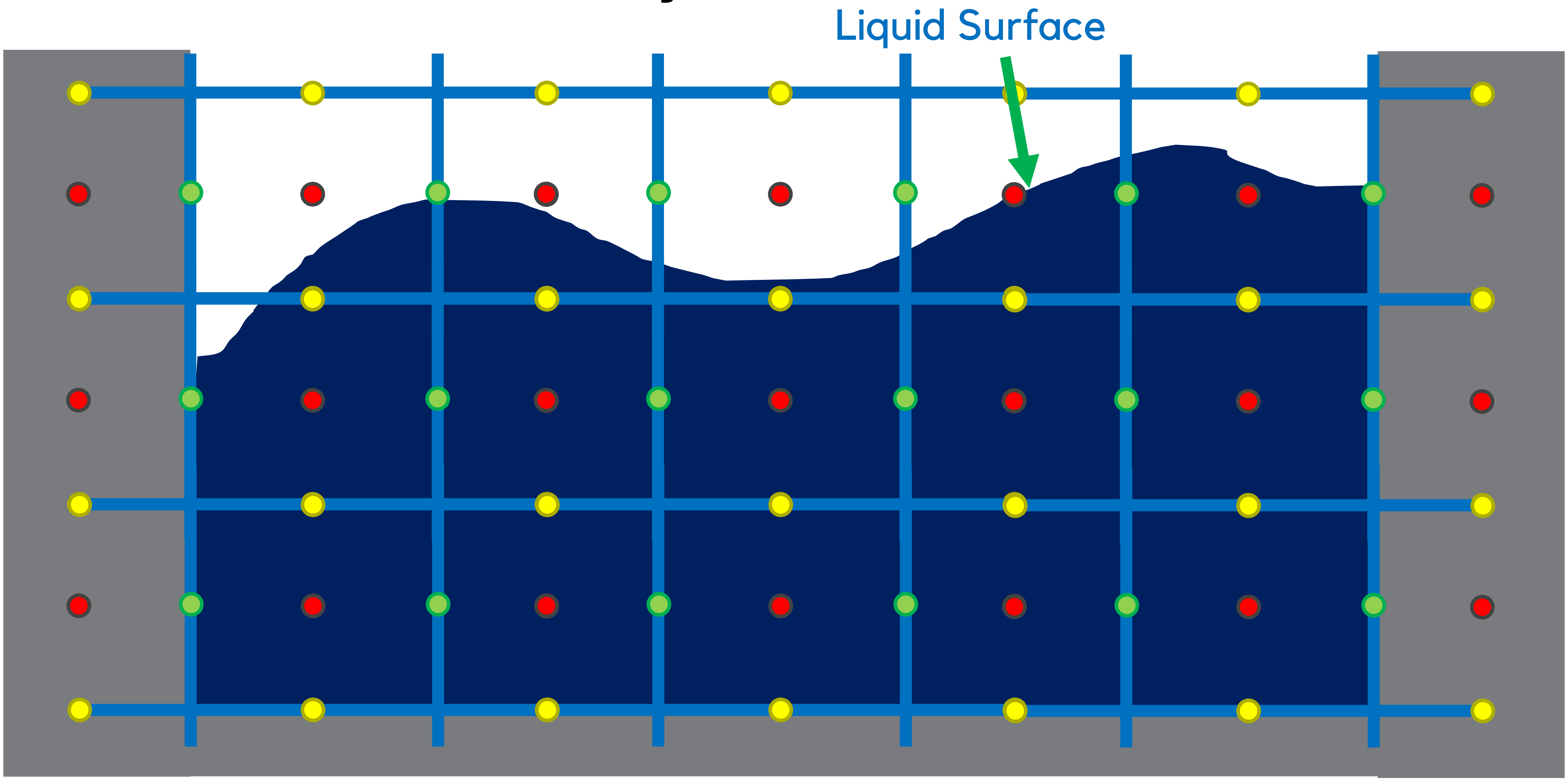
$$\mathrm{P}^T\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fluid Simulation – Solid Boundary Conditions

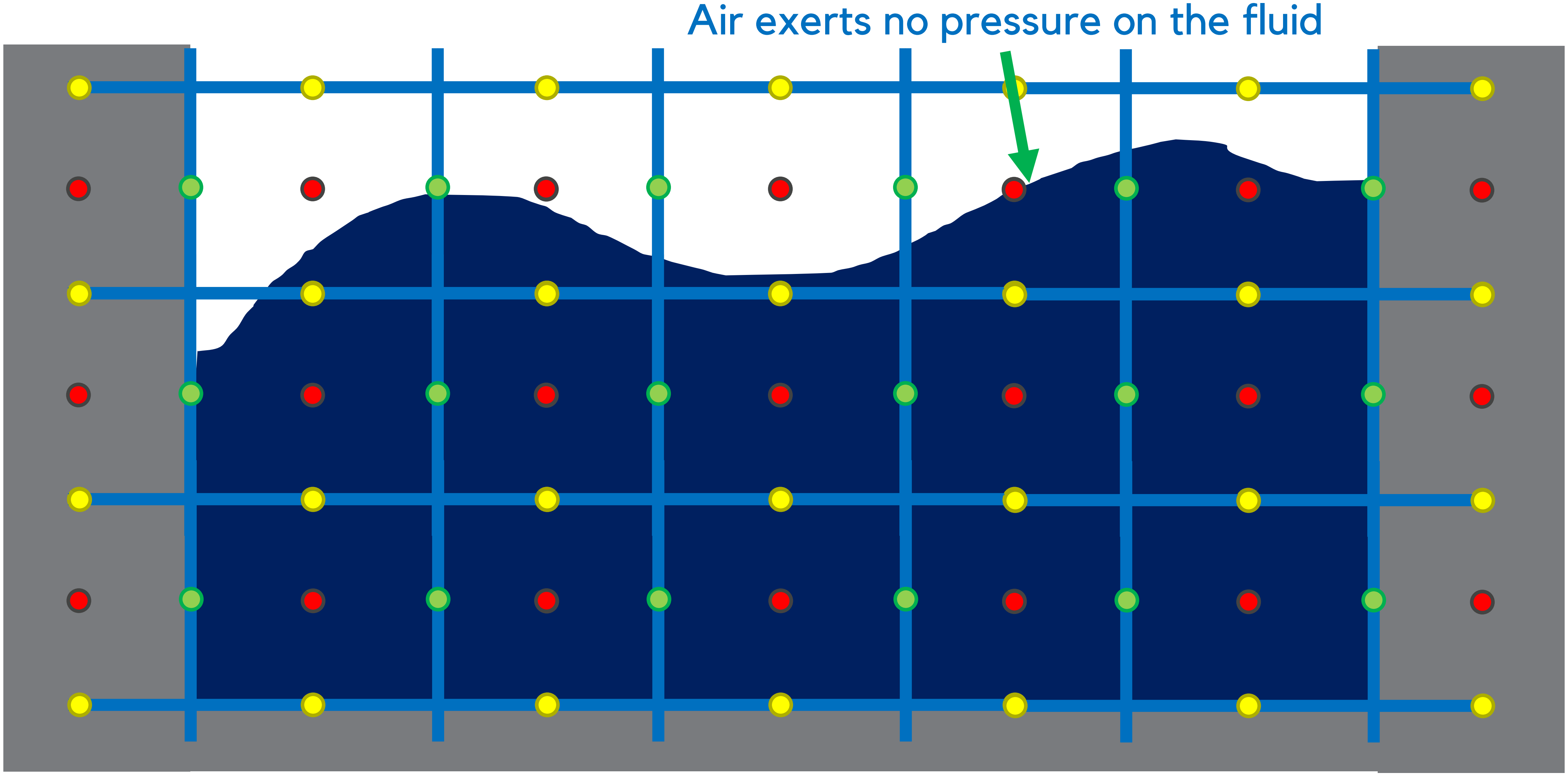


$$\mathrm{P}^T\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

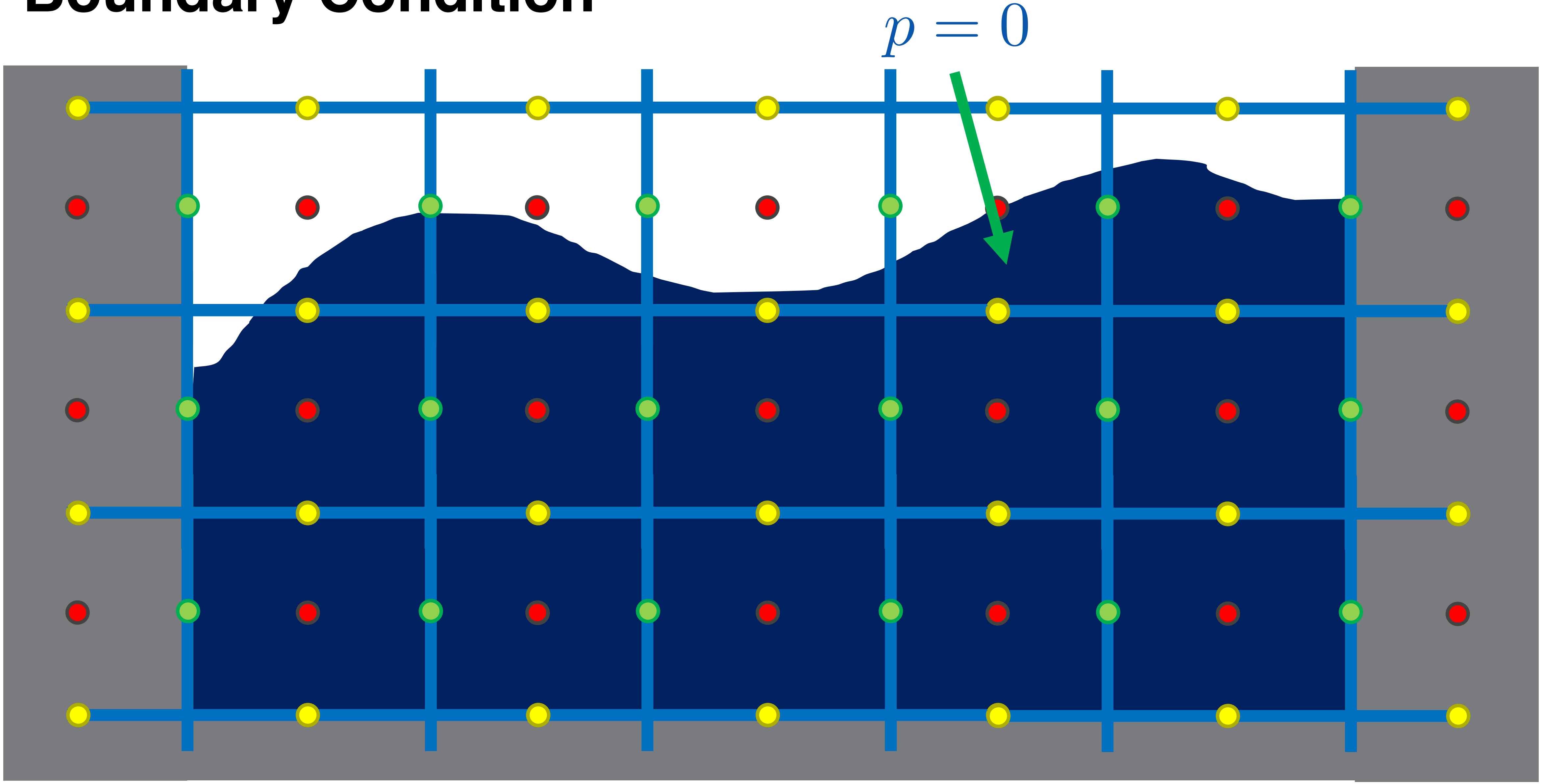
Free Surface Boundary Conditions



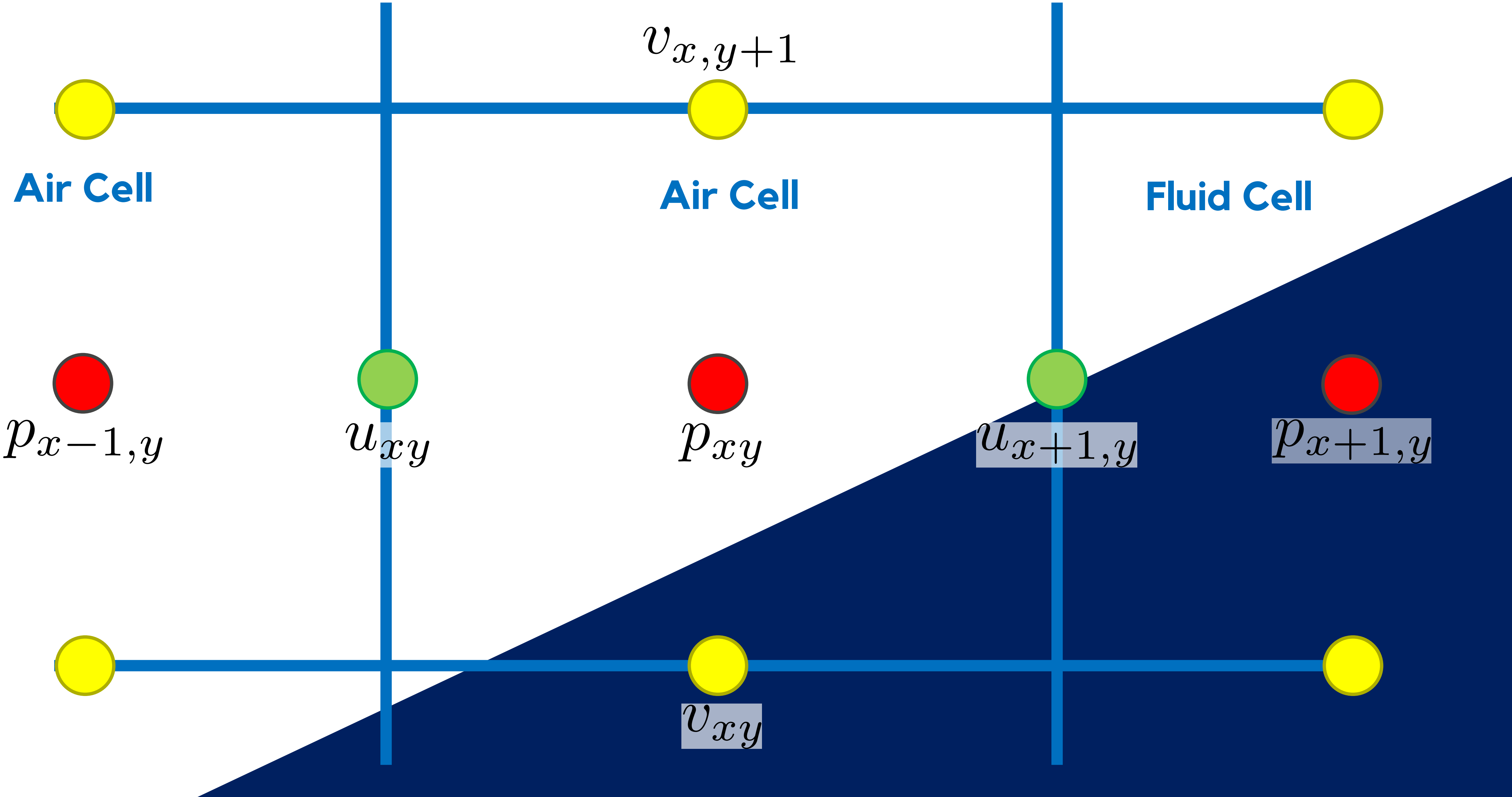
Free Surface Boundary Conditions



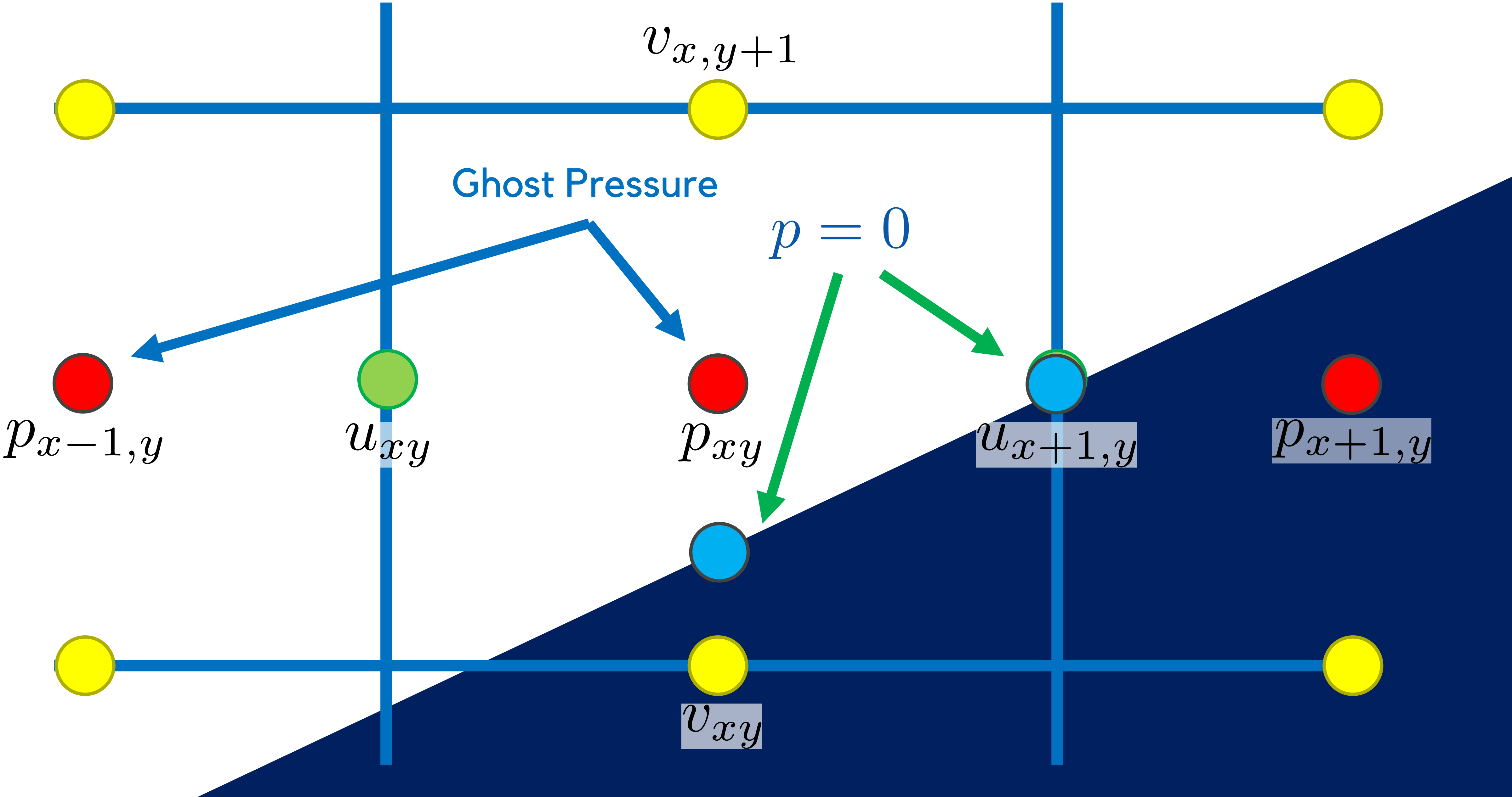
Boundary Condition



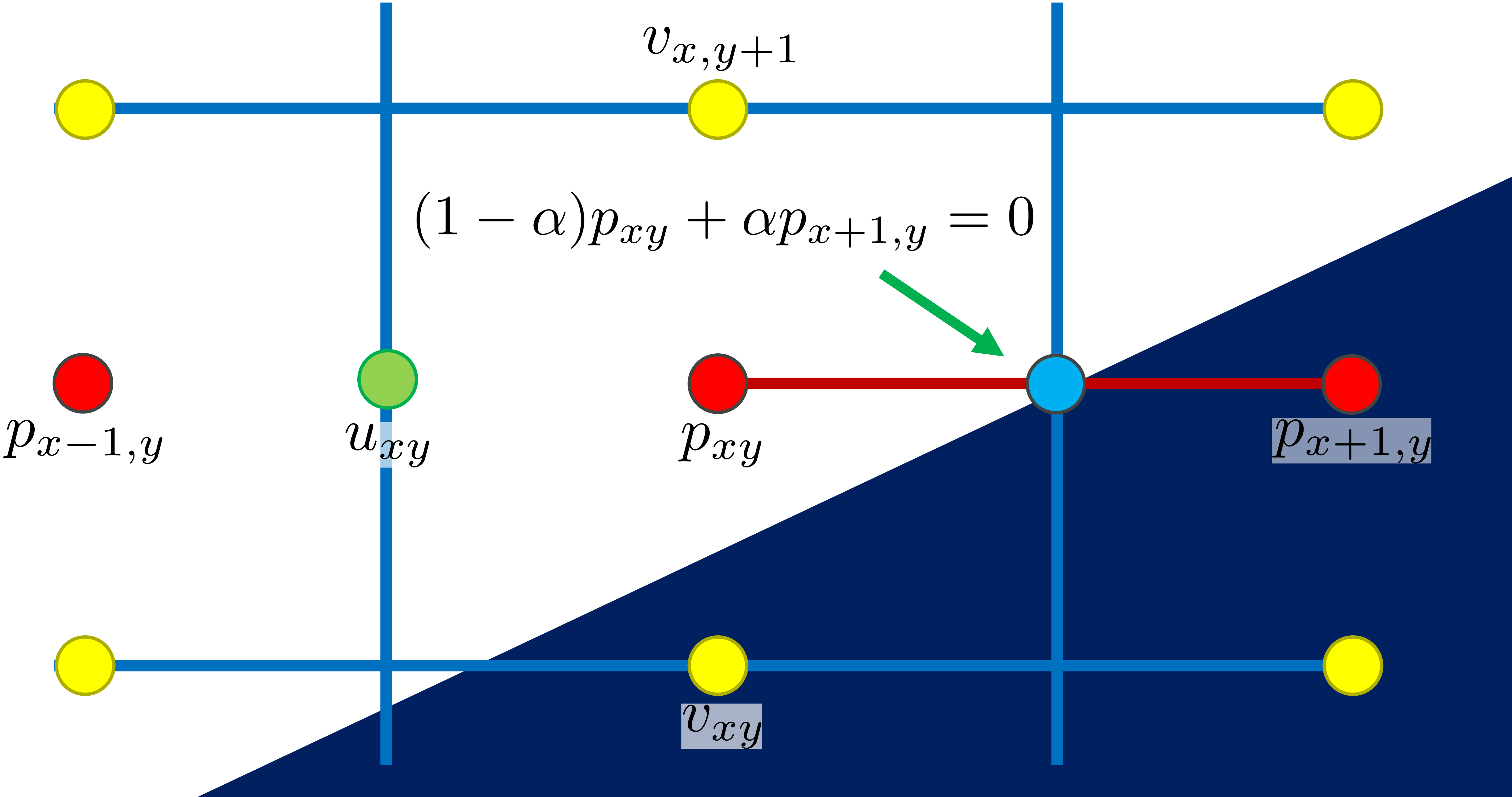
Free Surface Boundary Conditions



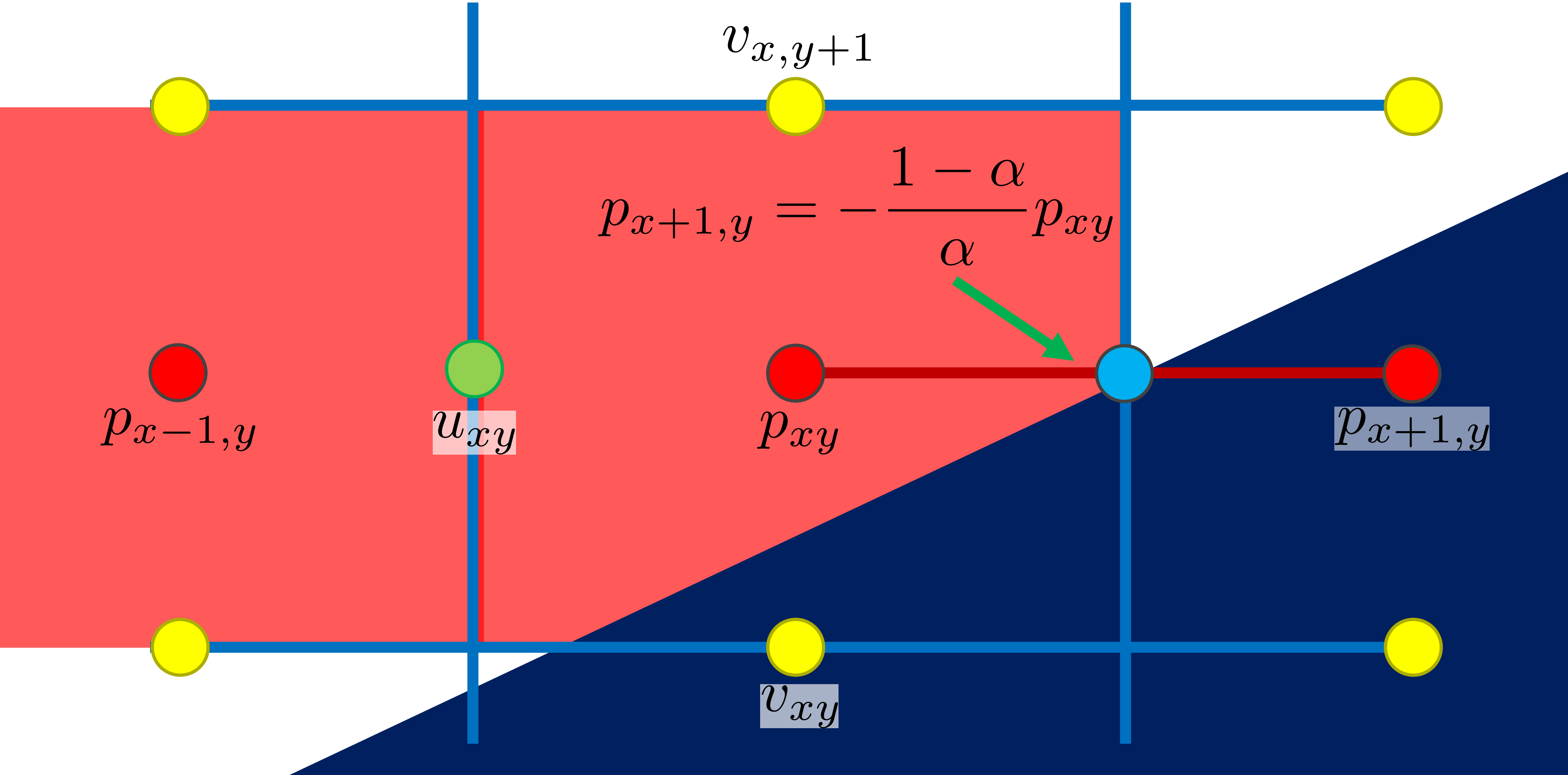
Free Surface Boundary Conditions



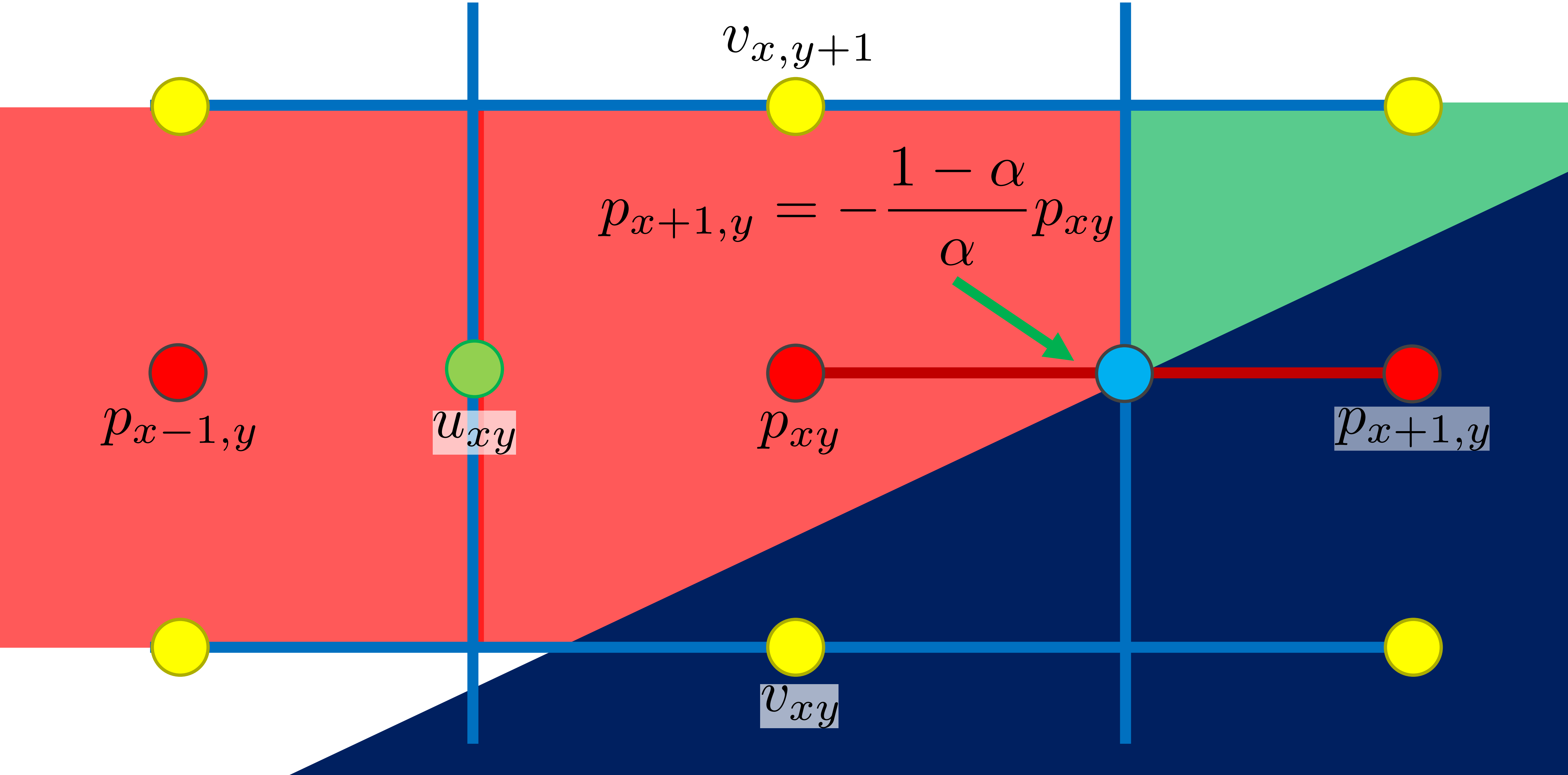
Free Surface Boundary Conditions



Free Surface Boundary Conditions



Free Surface Boundary Conditions

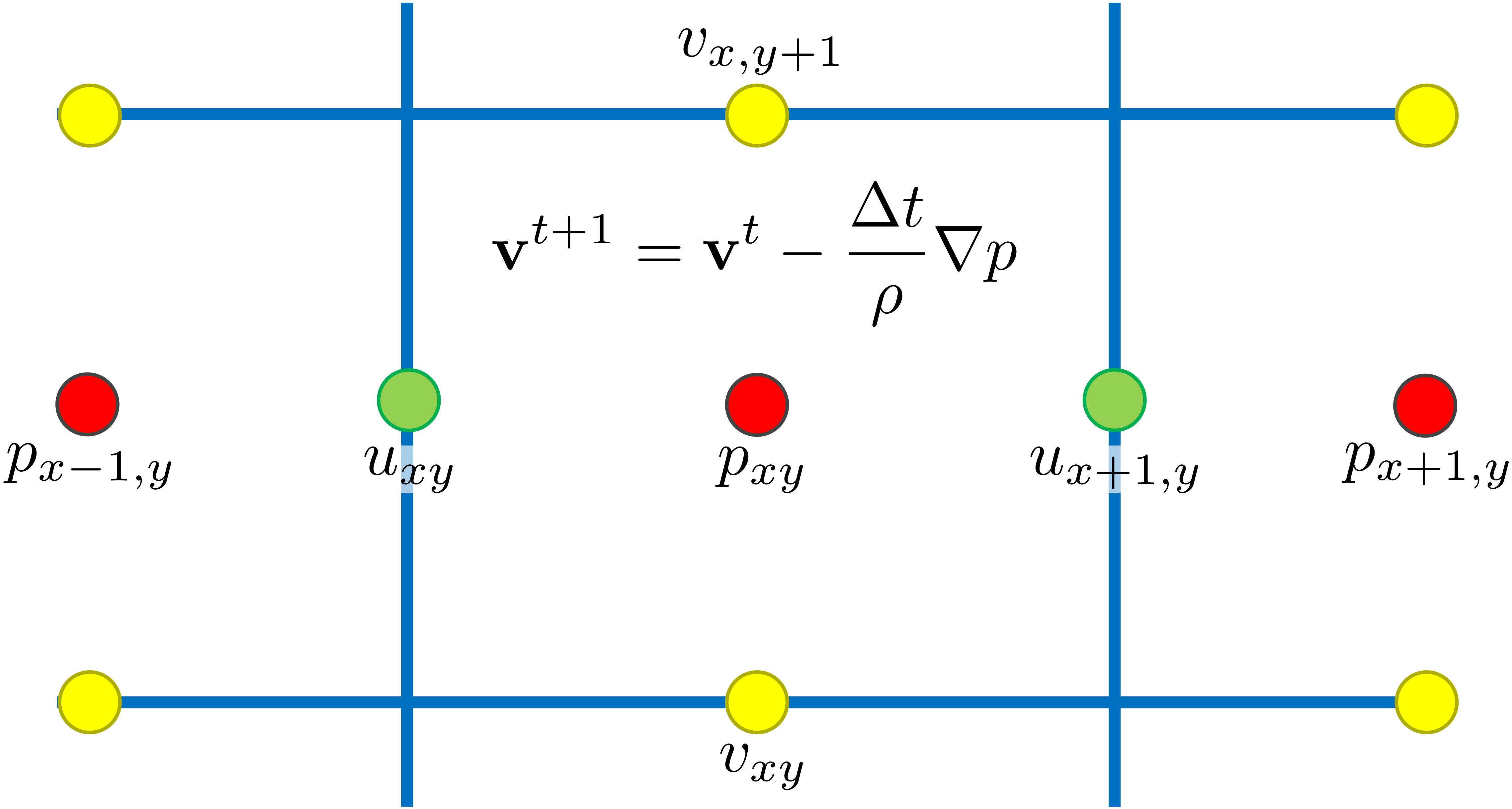


Fluid Simulation – Pressure Projection

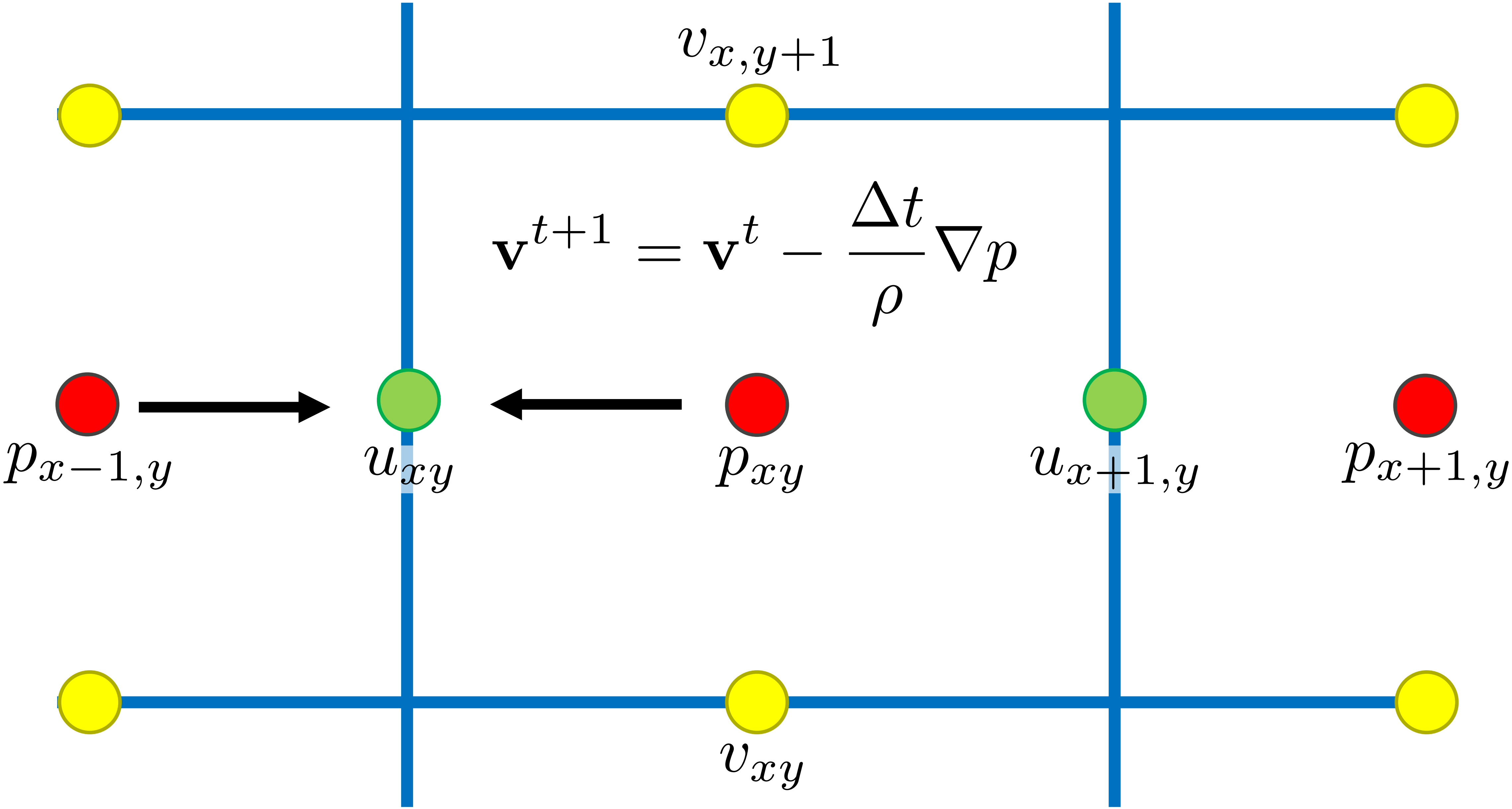
Assemble: $\mathrm{A}\mathbf{p} = \mathrm{d}$

Solve using Conjugate Gradient Method

Fluid Simulation – Solid Boundary Conditions



Fluid Simulation – Solid Boundary Conditions



$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

Fluid Simulation – Pressure Projection

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$ External Forces

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$ Pressure Projection

Fluid Simulation

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} = 0$$ Advection

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \rho \mathbf{g}$$ External Forces

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p \quad \text{subject to} \quad \nabla \cdot \mathbf{v} = 0$$ Pressure Projection

Fluid Simulation Algorithm

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\mathbf{v}^{\mathbf{A}} = \text{Move particles using } \mathbf{v}^t$$

Advection

$$\mathbf{v}^{\mathbf{B}} = \mathbf{v}^{A} + \Delta t \mathbf{g}$$

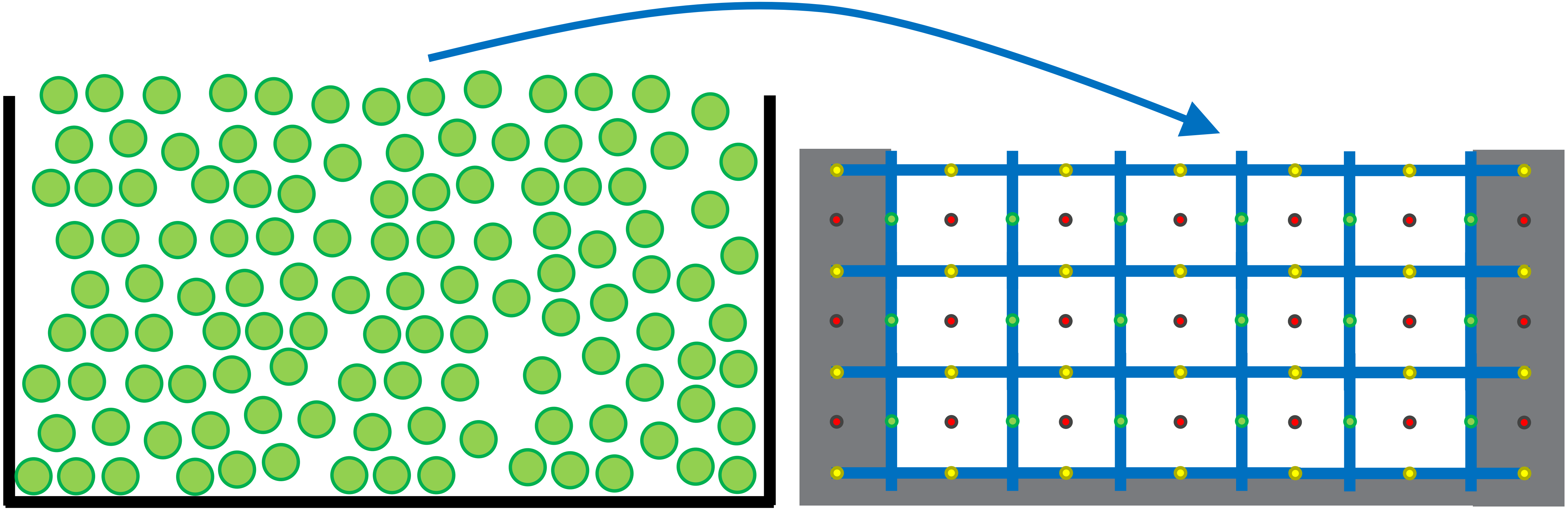
External Forces

$$\text{Solve } \mathrm{A}\mathbf{p} = \mathbf{d}$$

Pressure Projection

$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

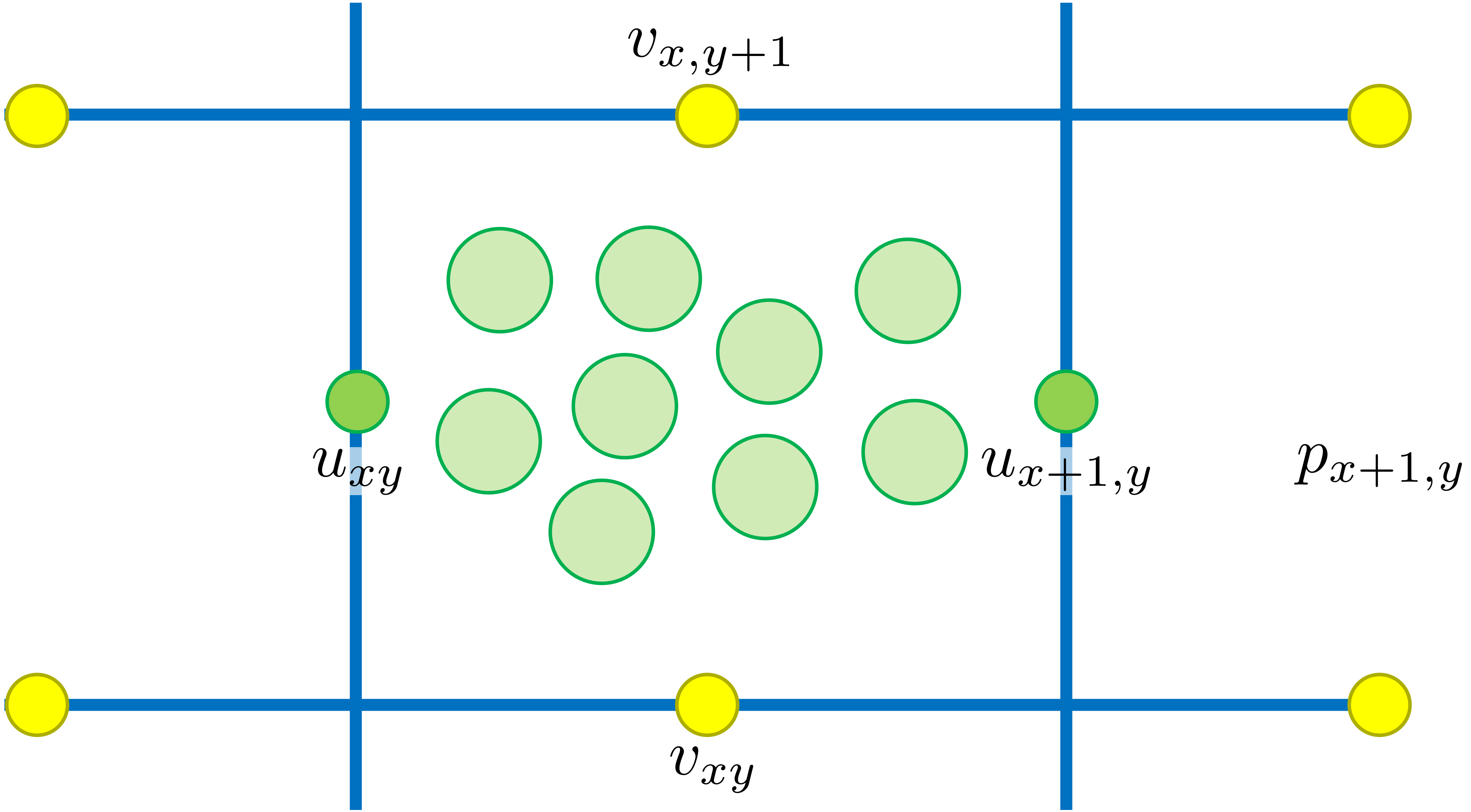
From Particles to the Grid



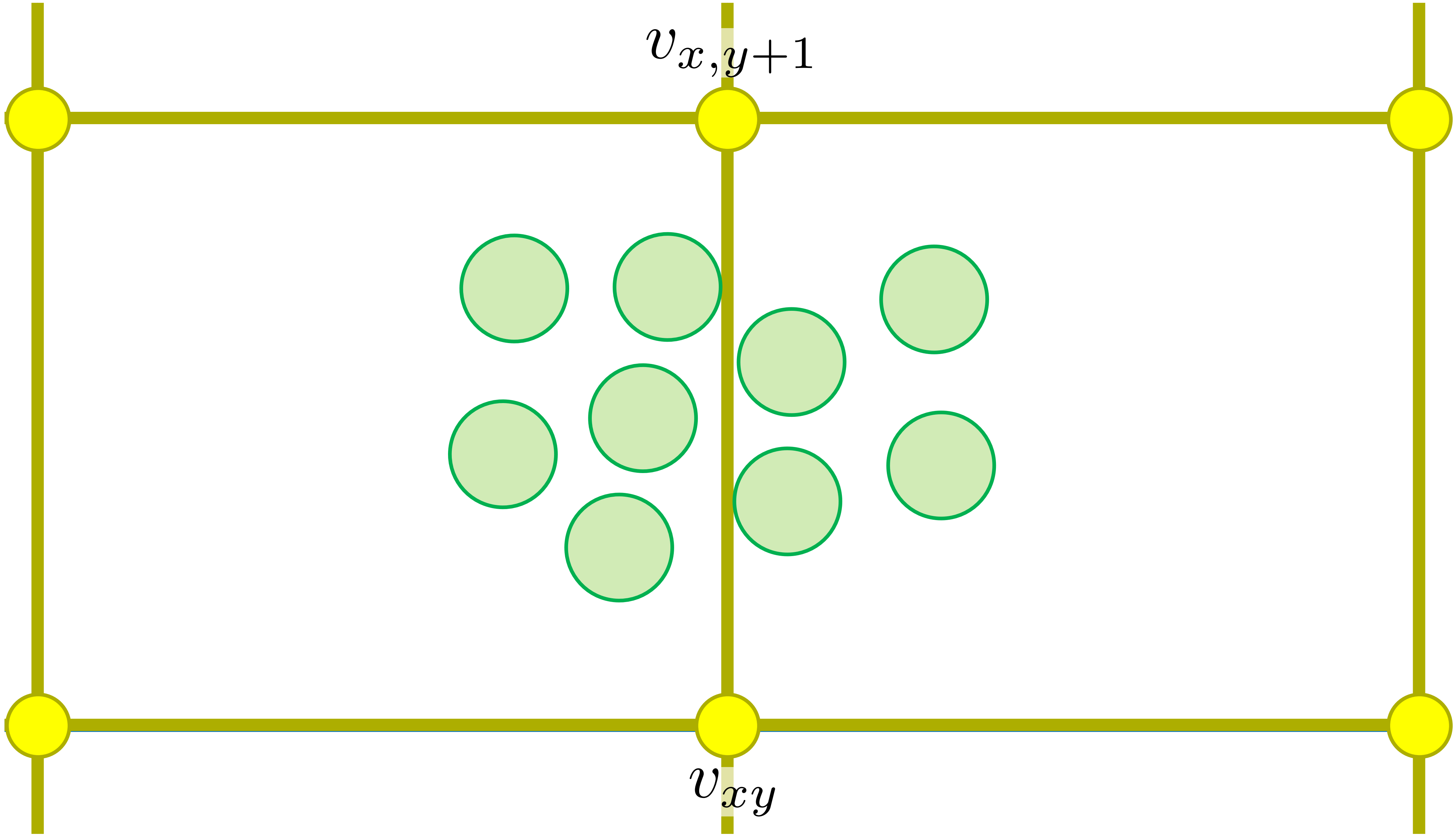
Velocities stored on particles

Pressure projection on grid

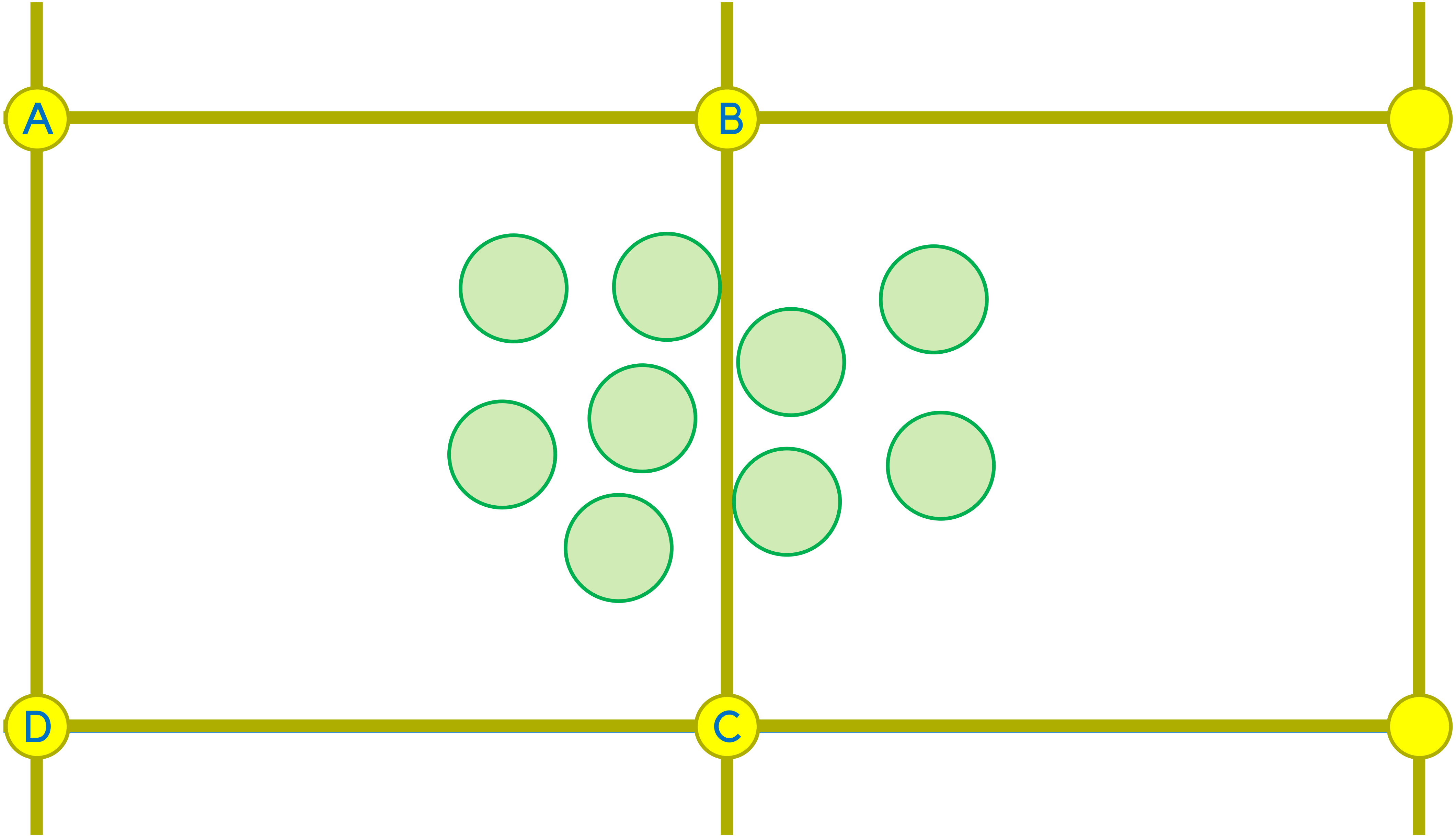
From Particles to the Grid



From Particles to the Grid



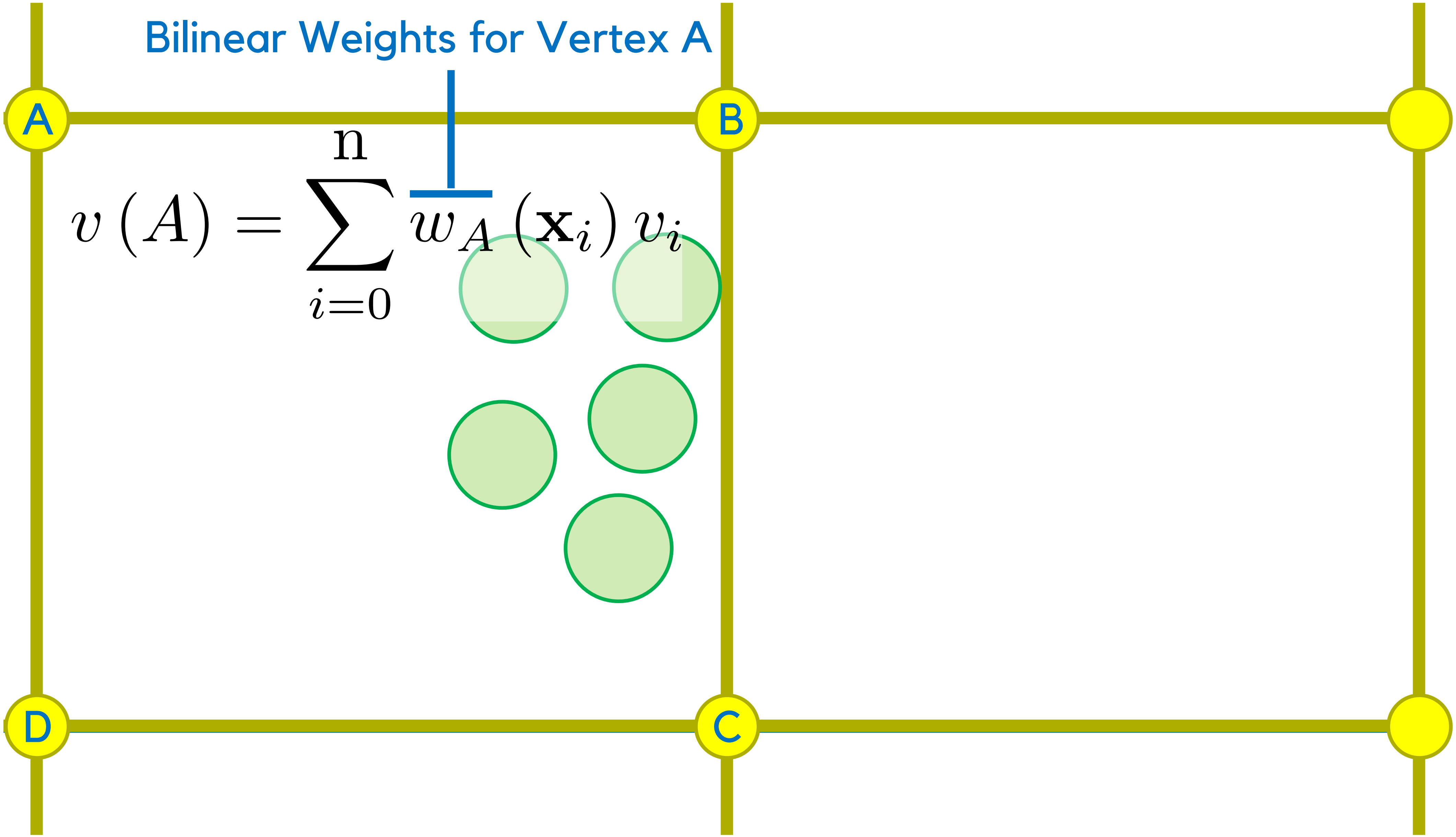
From Particles to the Grid



From Particles to the Grid

Bilinear Weights for Vertex A

$$v\left(A\right) = \sum_{i=0}^{n} w_A\left(\mathbf{x}_i\right) v_i$$



Fluid Simulation Algorithm

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$$\mathbf{v}^{\mathbf{A}} = \text{Move particles using } \mathbf{v}^t$$

Advection

$$\mathbf{v}^{\mathbf{B}} = \mathbf{v}^A + \Delta t \mathbf{g}$$

External Forces

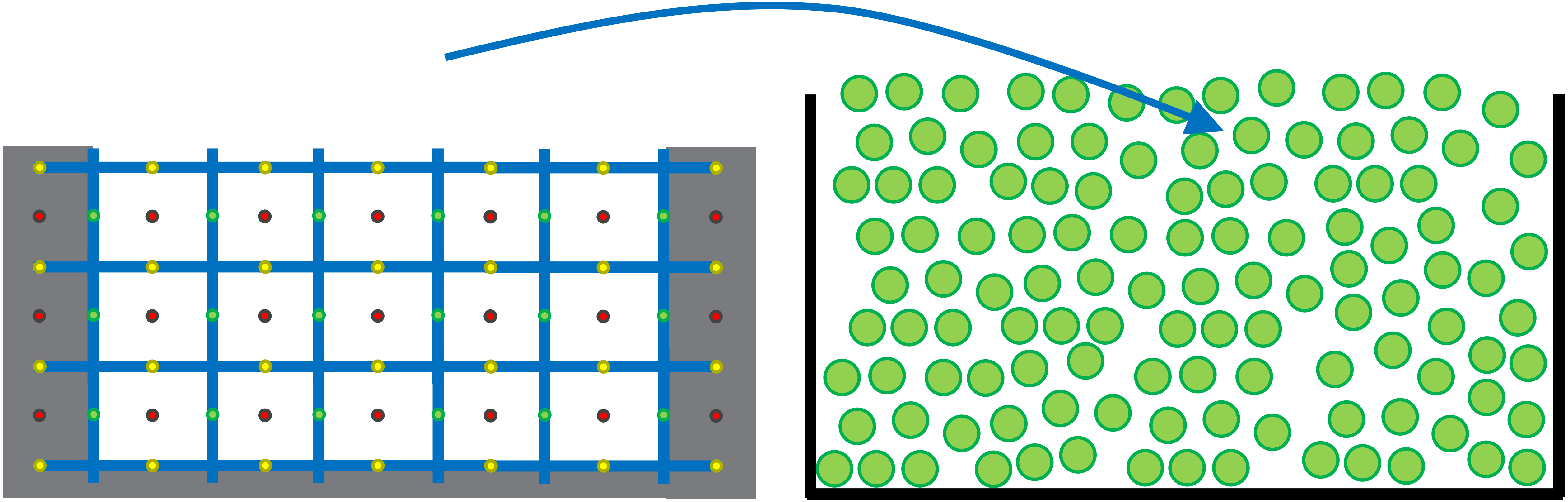
Transfer $\mathbf{v}^B$ to the grid

Solve $\mathbf{A}\mathbf{p} = \mathbf{d}$

Pressure Projection

$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

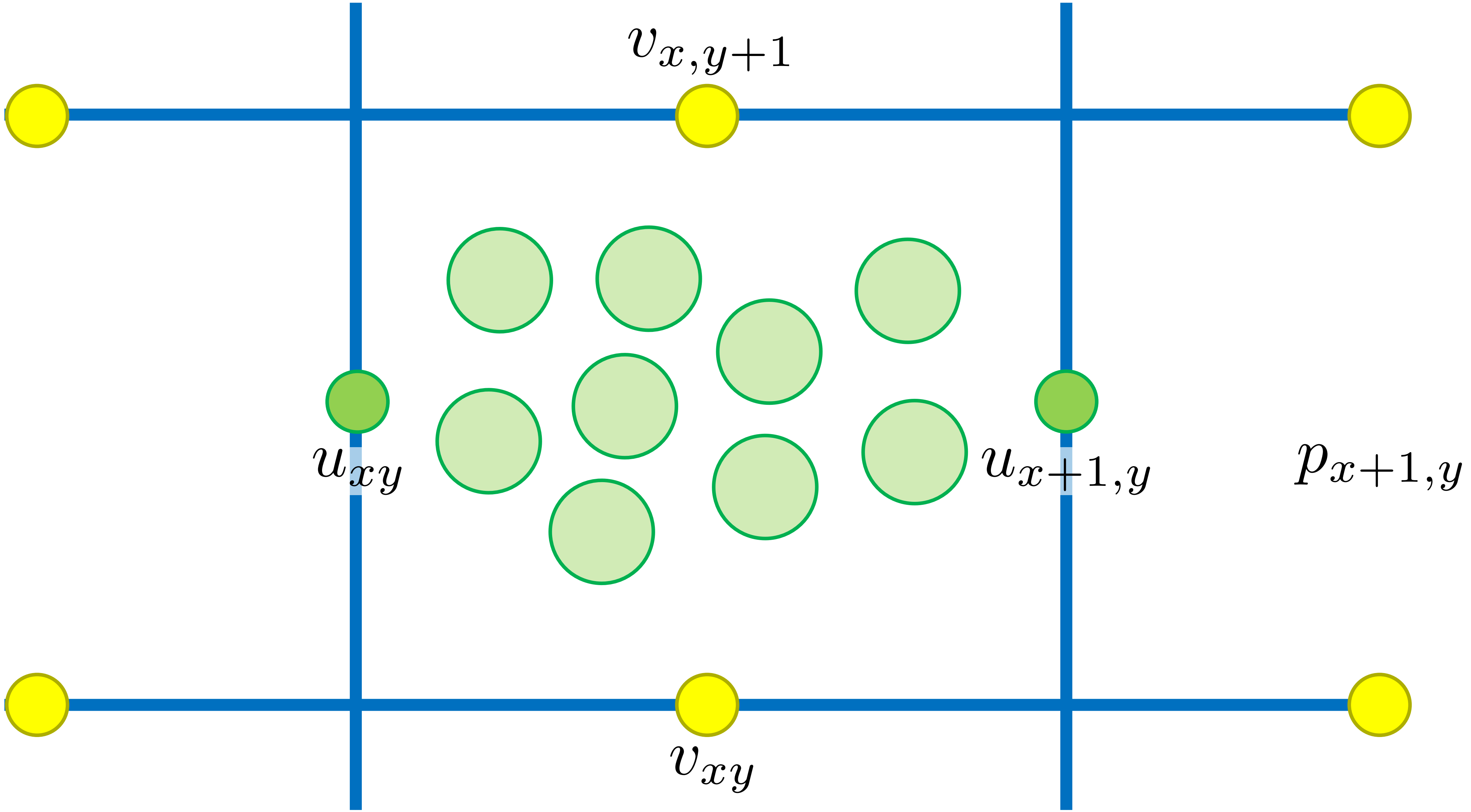
From Grid to the Particles



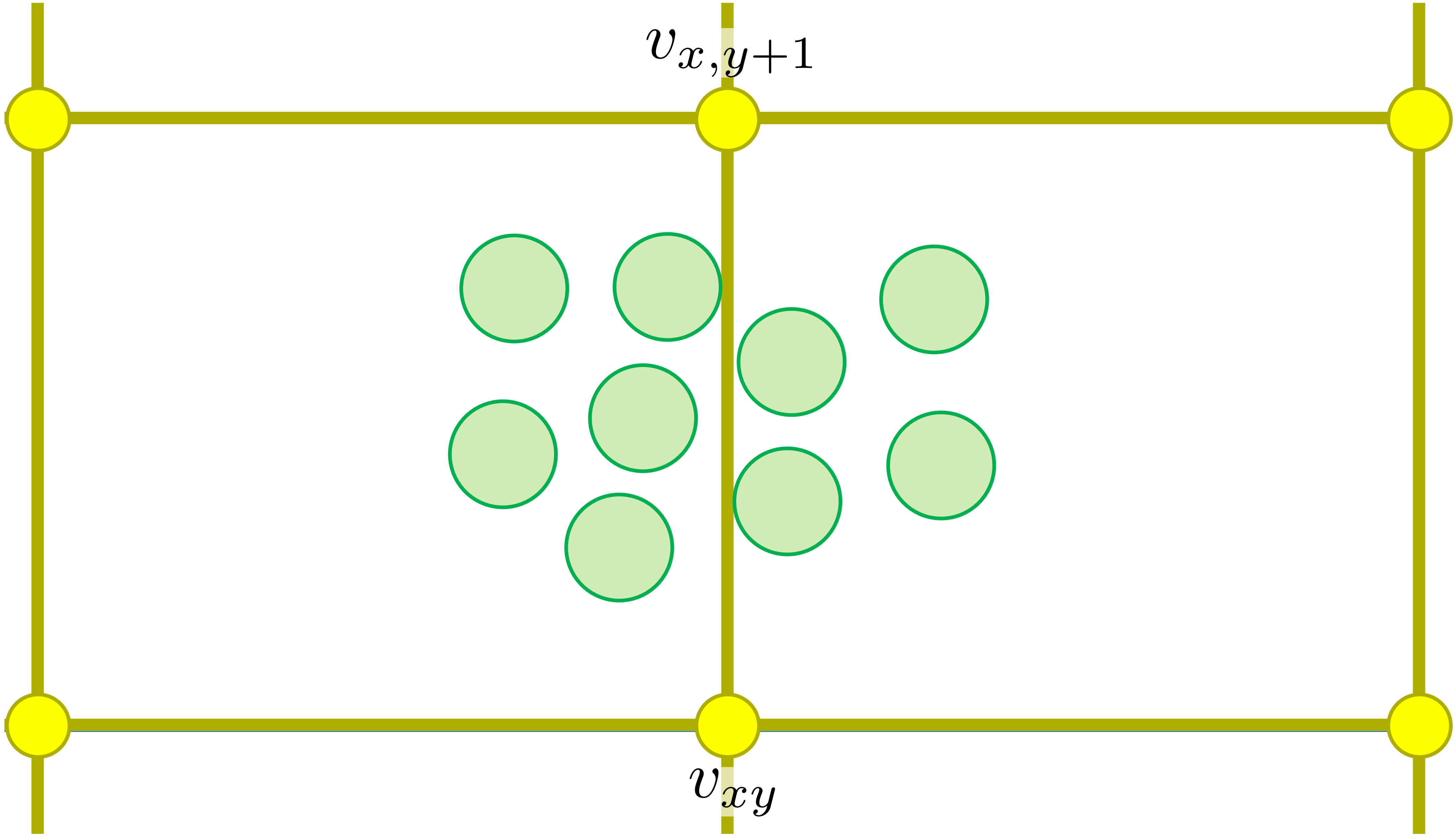
Updated Velocities on Grid

Old Velocities on Particles

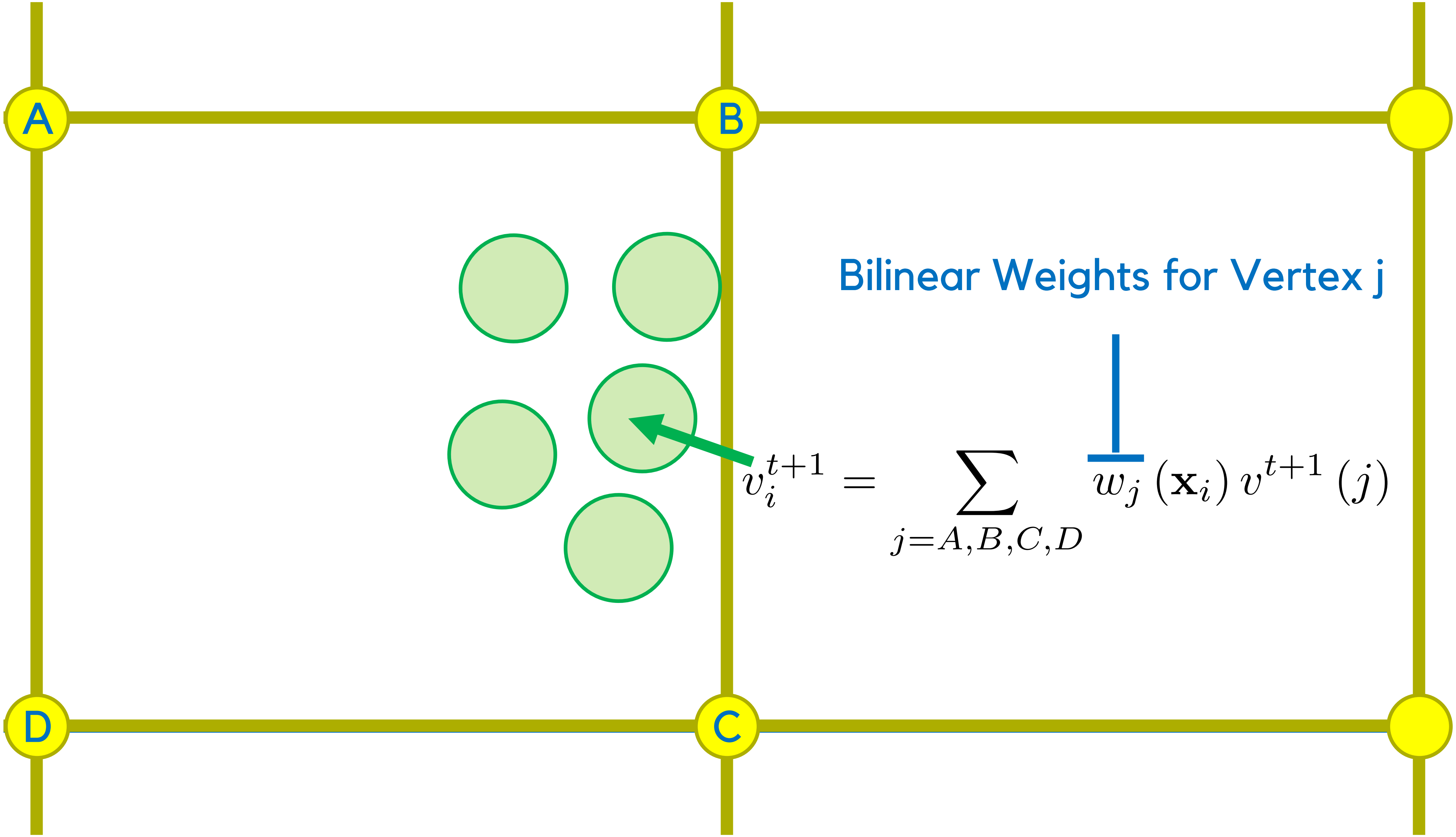
Particle-in-Cell (PIC) Transfer



Particle-in-Cell (PIC) Transfer

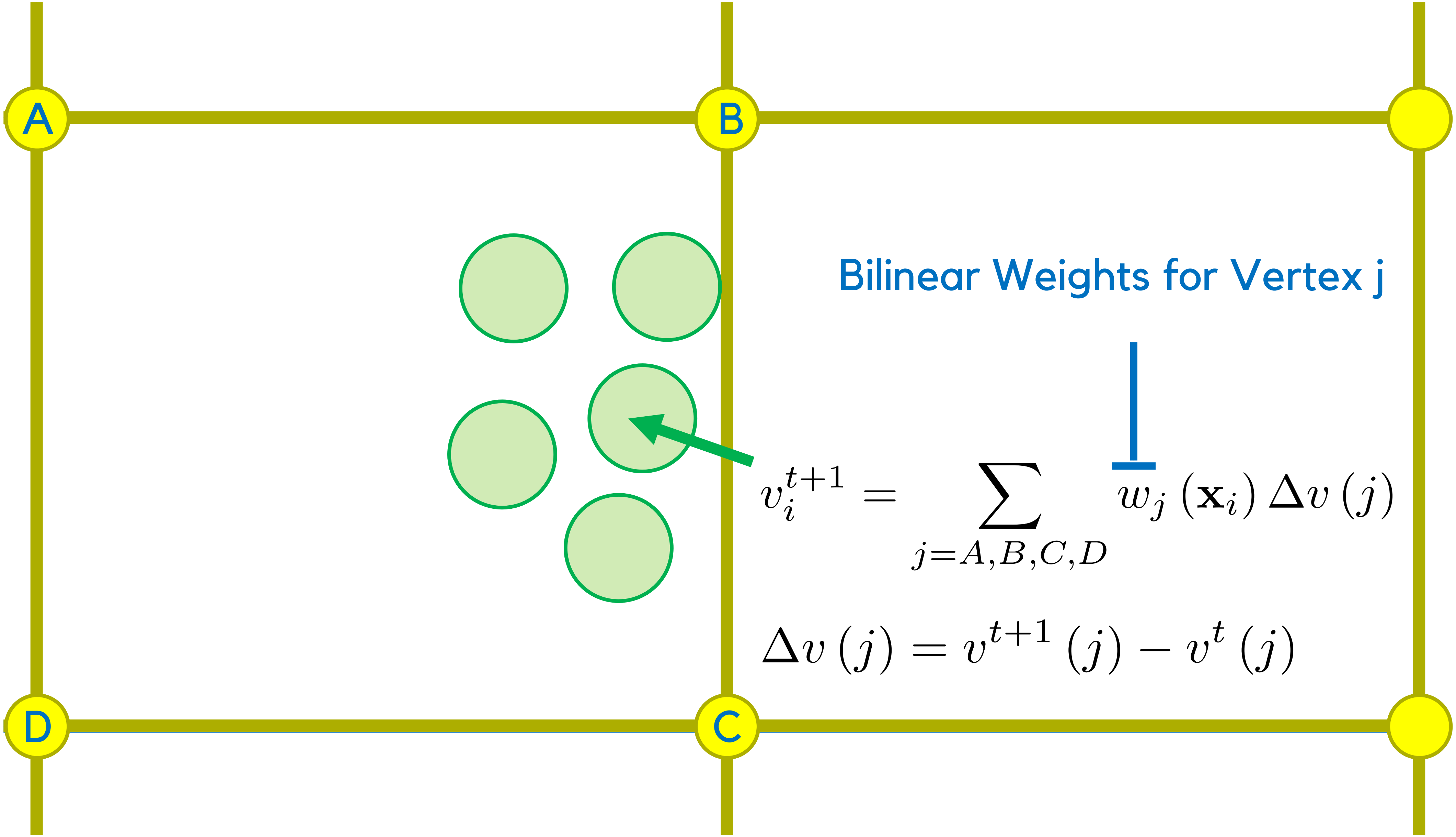


Particle-in-Cell (PIC) Transfer



$$v_i^{t+1} = \sum_{j=A,B,C,D} w_j\left(\mathbf{x}_i\right) v^{t+1}\left(j\right)$$

Fluid-Implicit-Particle (FLIP) Transfer



$$v_i^{t+1} = \sum_{j=A,B,C,D} w_j\left(\mathbf{x}_i\right) \Delta v\left(j\right)$$

$$\Delta v\left(j\right) = v^{t+1}\left(j\right) - v^{t}\left(j\right)$$

Fluid Simulation Algorithm

Input: $\mathbf{v}^t$ (Divergence Free)

Output: $\mathbf{v}^{t+1}$

$\mathbf{v}^{\mathbf{A}} = \text{Move particles using } \mathbf{v}^t$ Advection

$\mathbf{v}^{\mathbf{B}} = \mathbf{v}^A + \Delta t \mathbf{g}$ External Forces

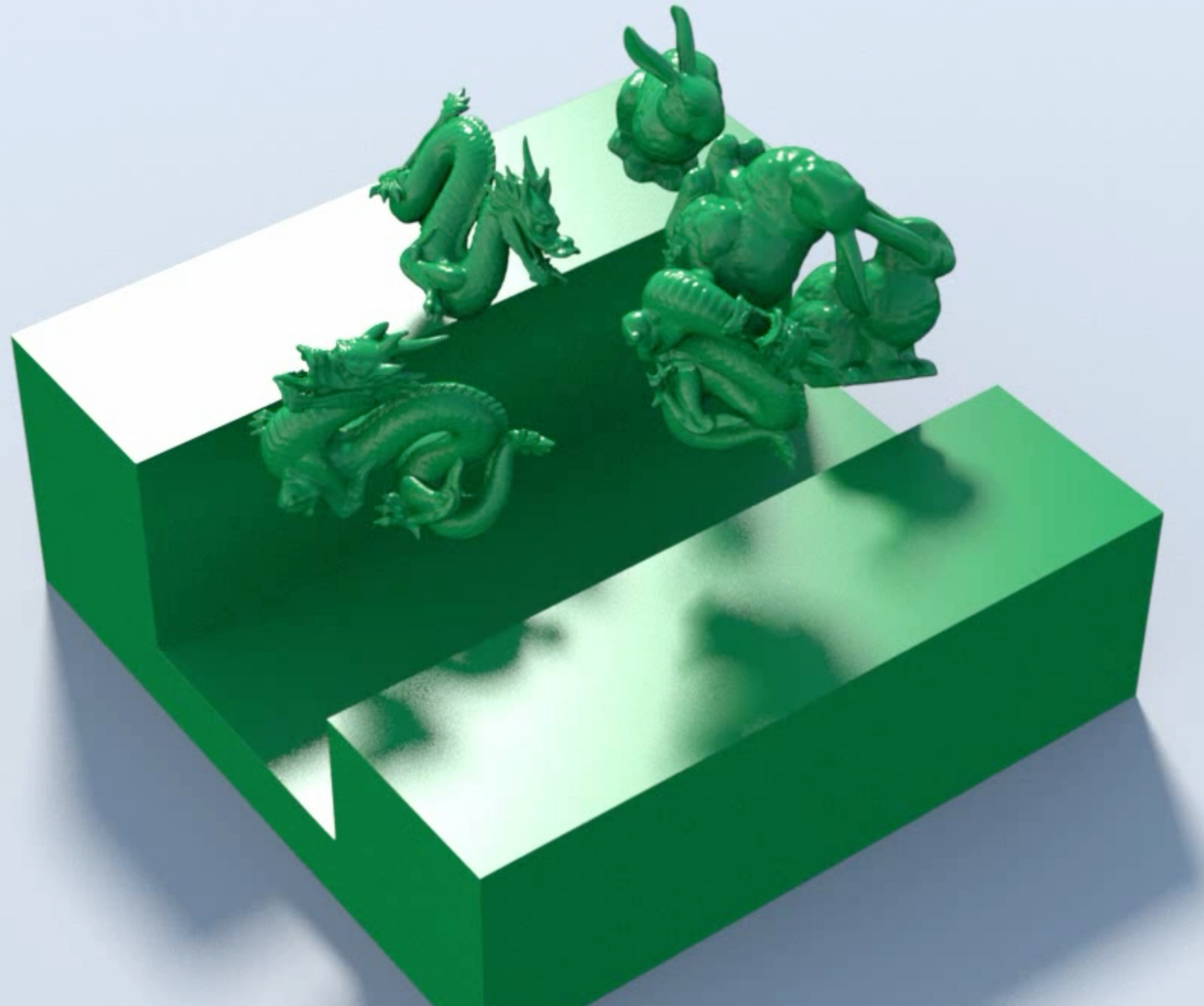
Transfer $\mathbf{v}^B$ to the grid

Solve $\mathbf{A}\mathbf{p} = \mathbf{d}$

Pressure Projection

$$\mathbf{v}^{t+1} = \mathbf{v}^t - \frac{\Delta t}{\rho} \nabla p$$

Transfer $\mathbf{v}^{t+1}$ to particles (PIC or FLIP)



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The End …
(Or is it ?)
OK Go | This Too Shall Pass