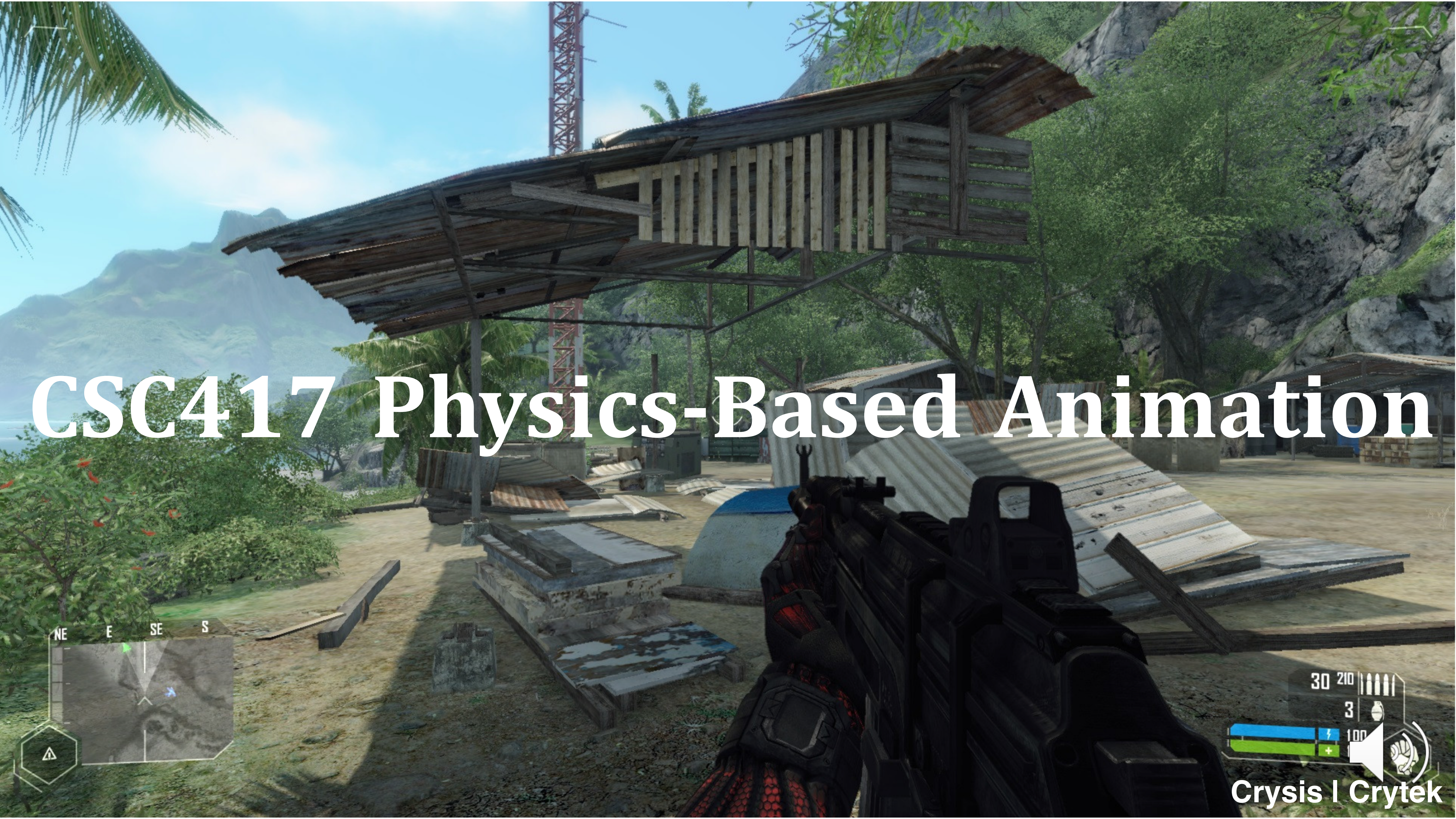
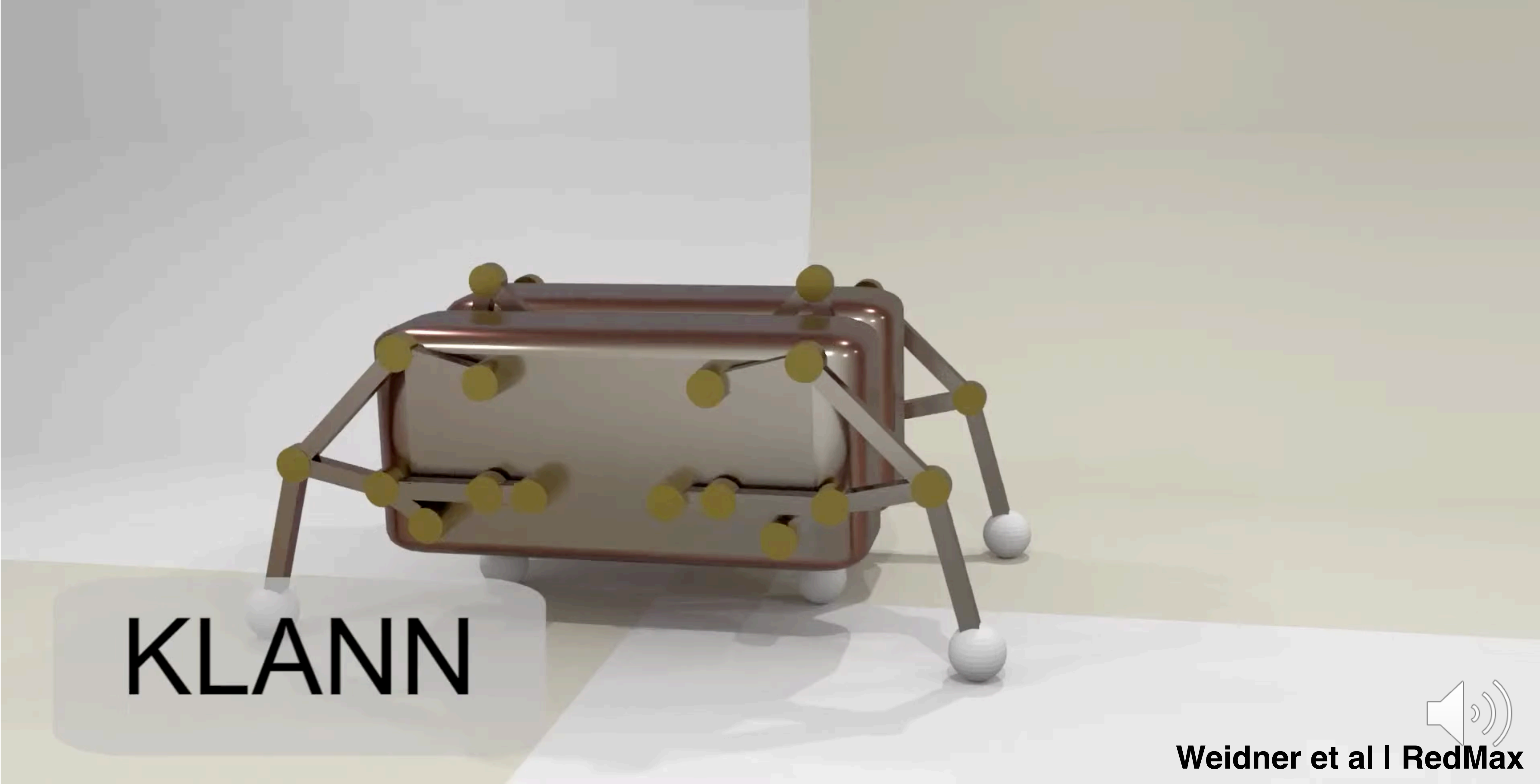
CSC417 Physics-Based Animation



Crysis | Crytek

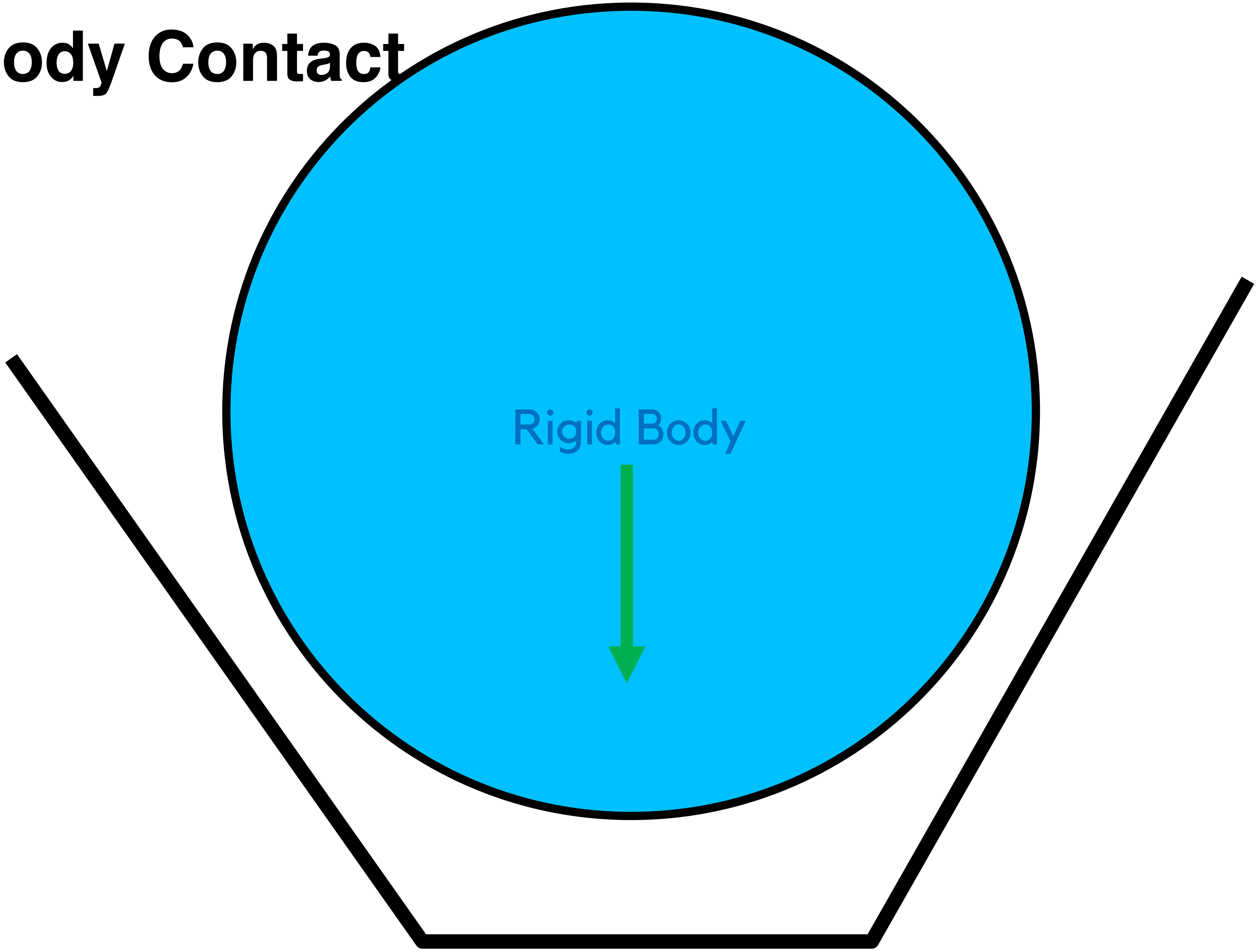
Last Video: Jointed Rigid Body Systems



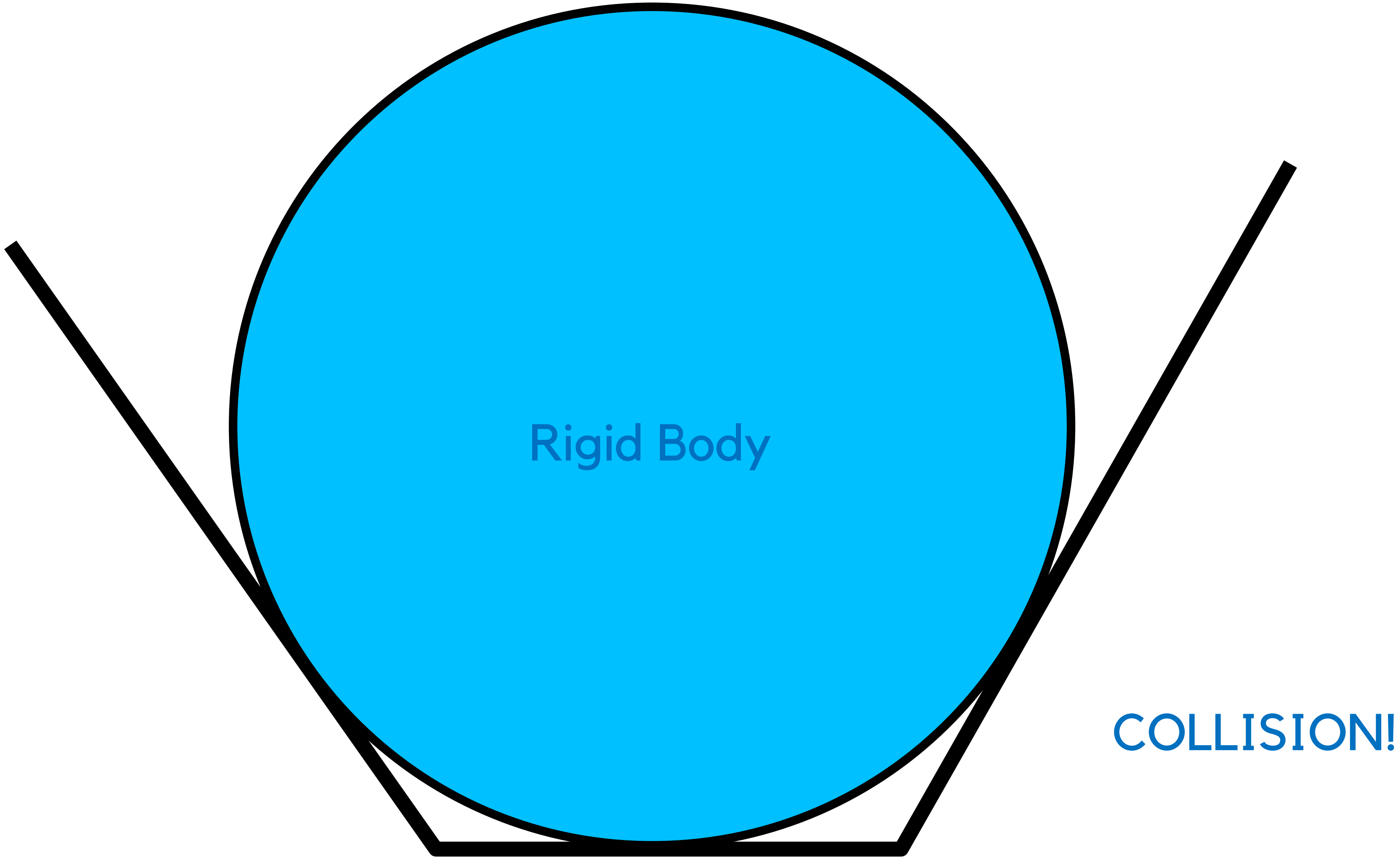
Weidner et al | RedMax

This Video: Rigid Body Simulation with Contact

Rigid Body Contact



Rigid Body Contact



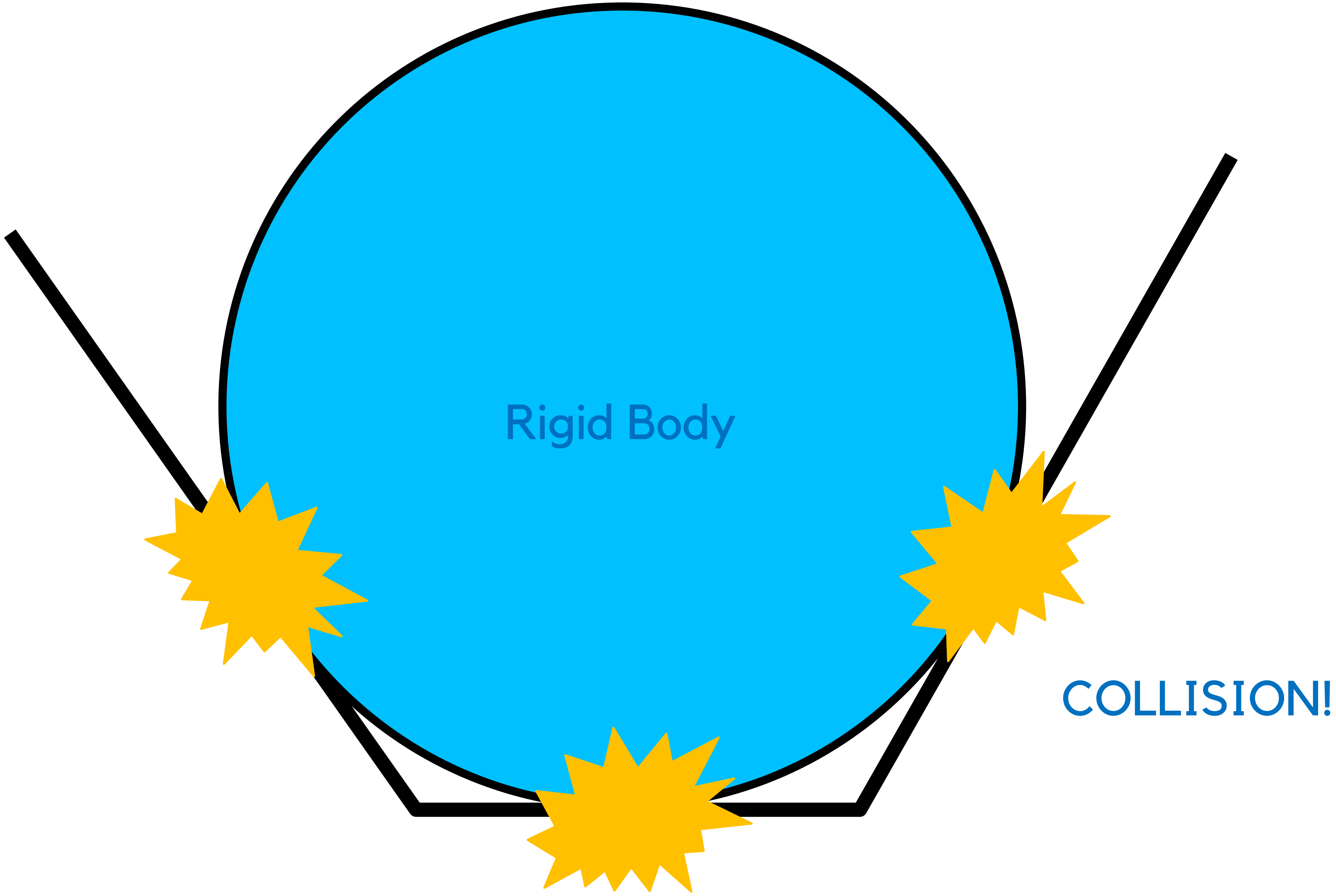
Collisions in Simulation

Two phases detection and response

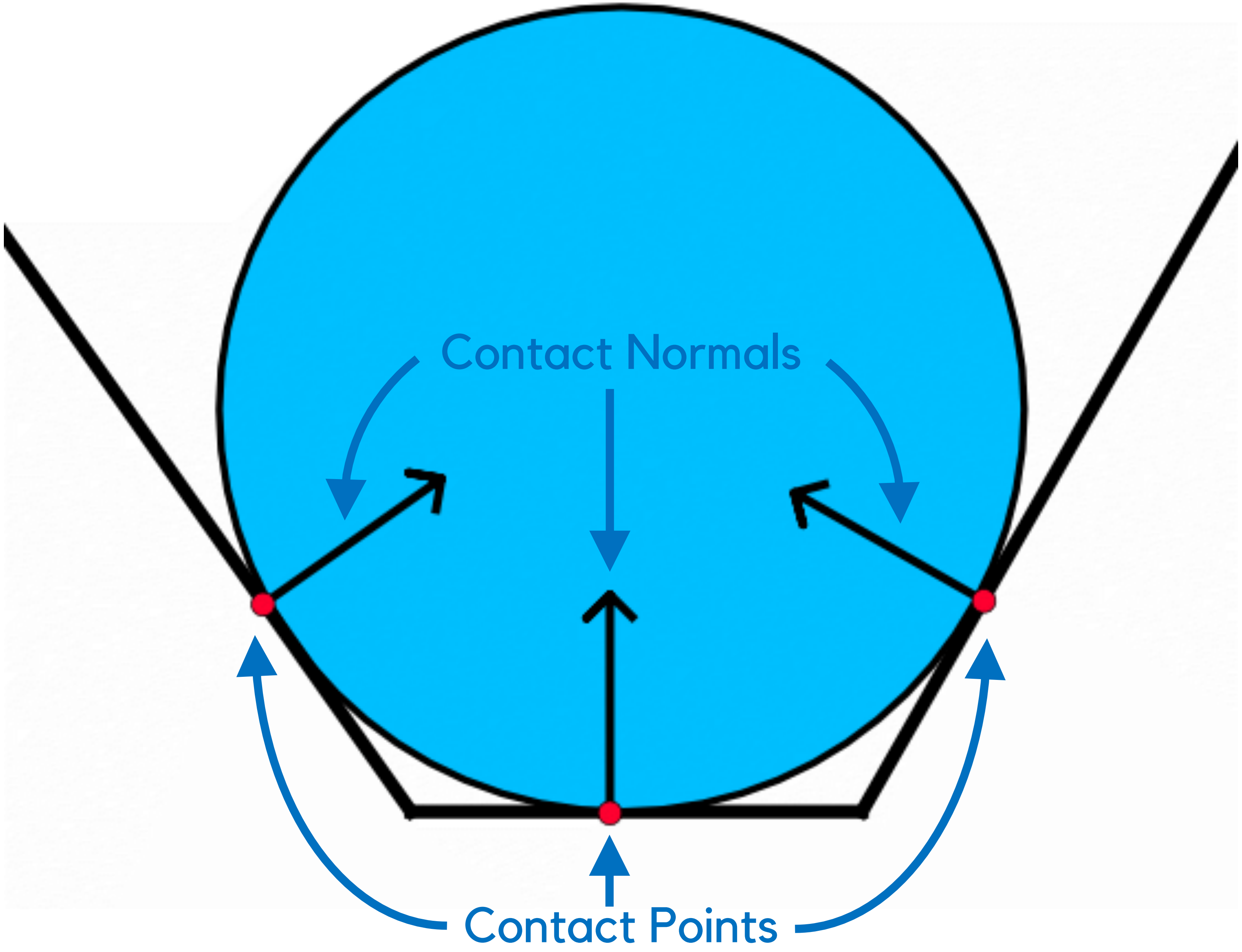
Detection: Did I hit anything ?

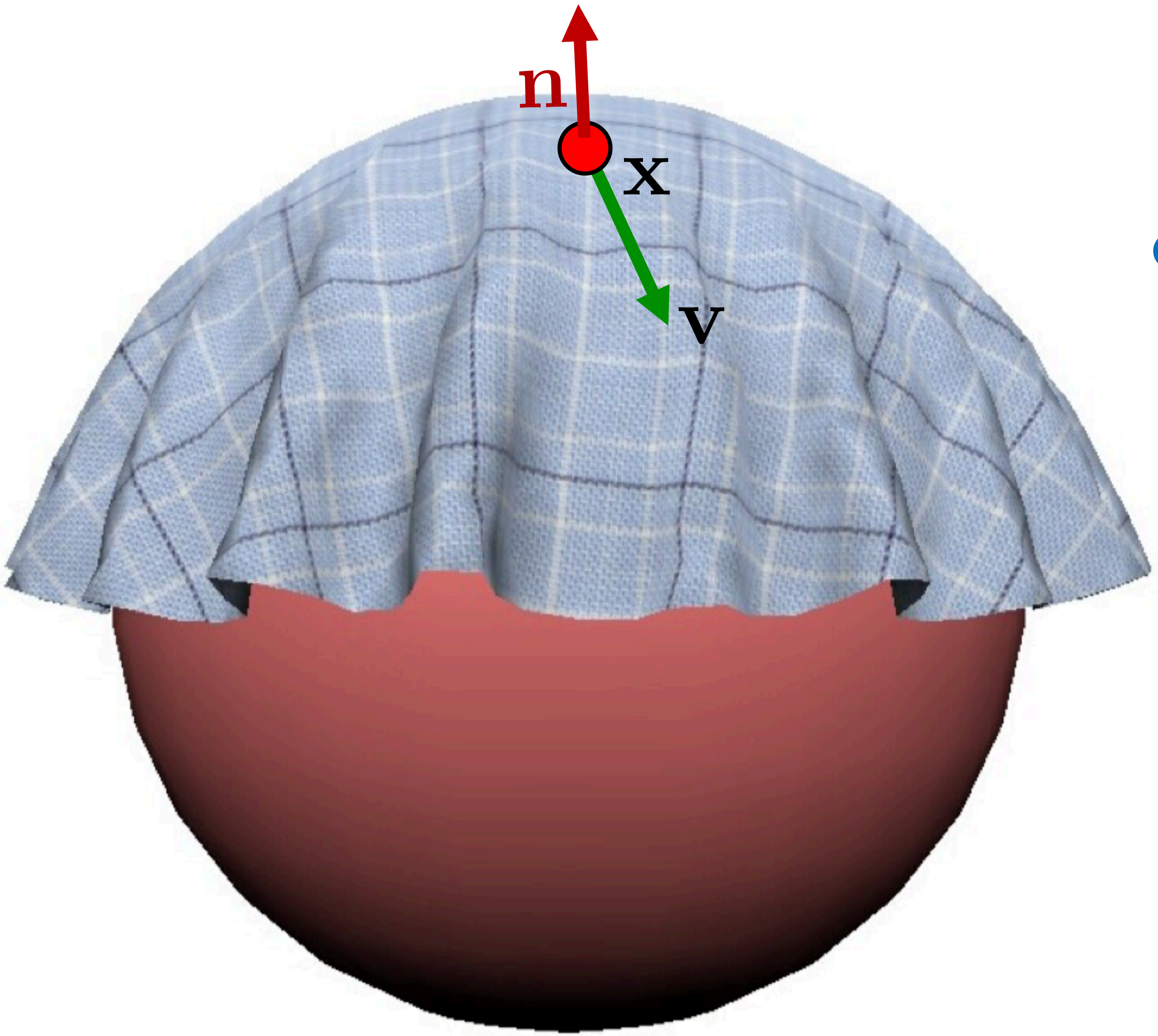
Response: I hit something ! What do I do ?

Rigid Body Contact



Rigid Body Contact





Check if $||\mathbf{x} - \mathbf{c}||_2^2 \leq r^2$

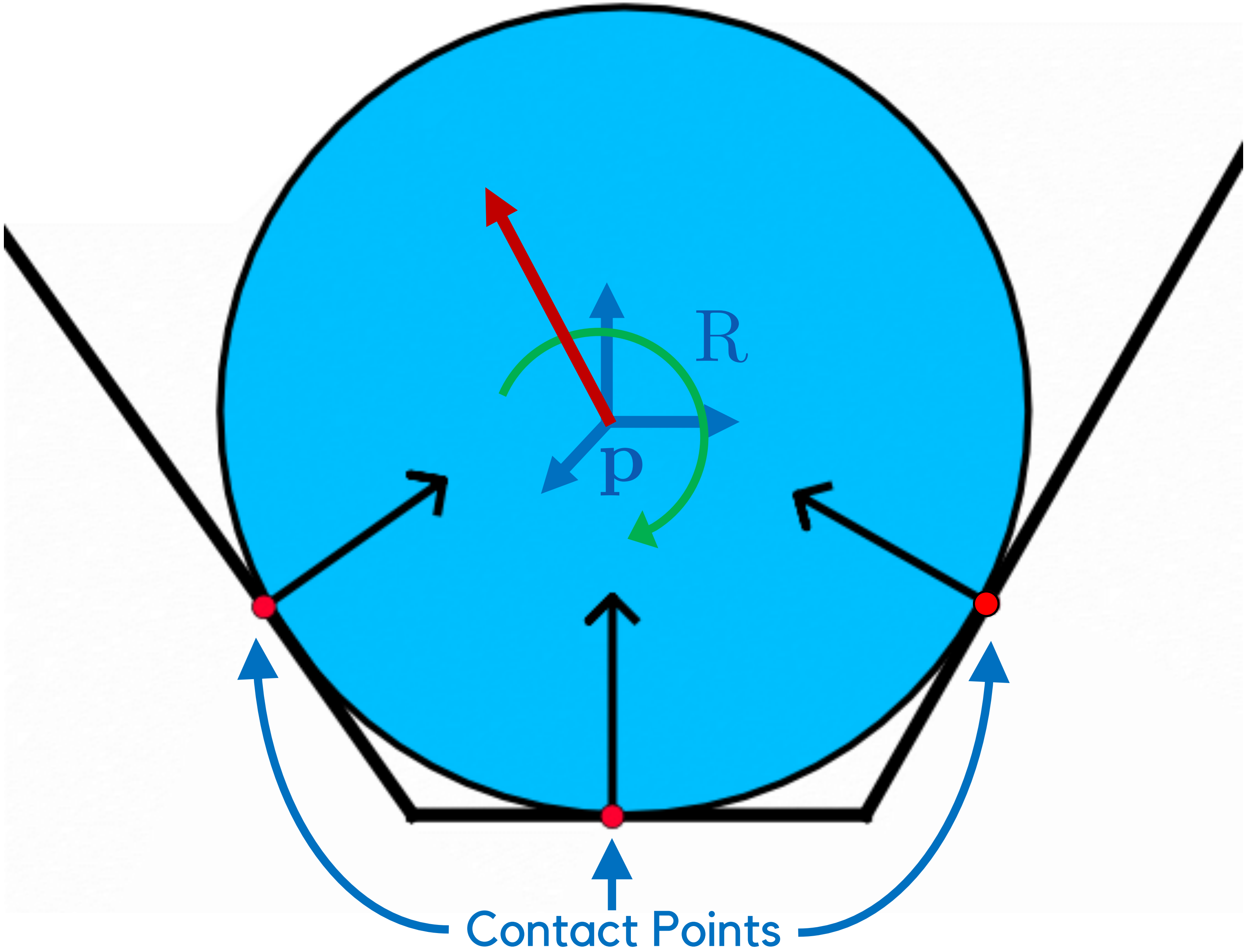
center radius

If colliding ...

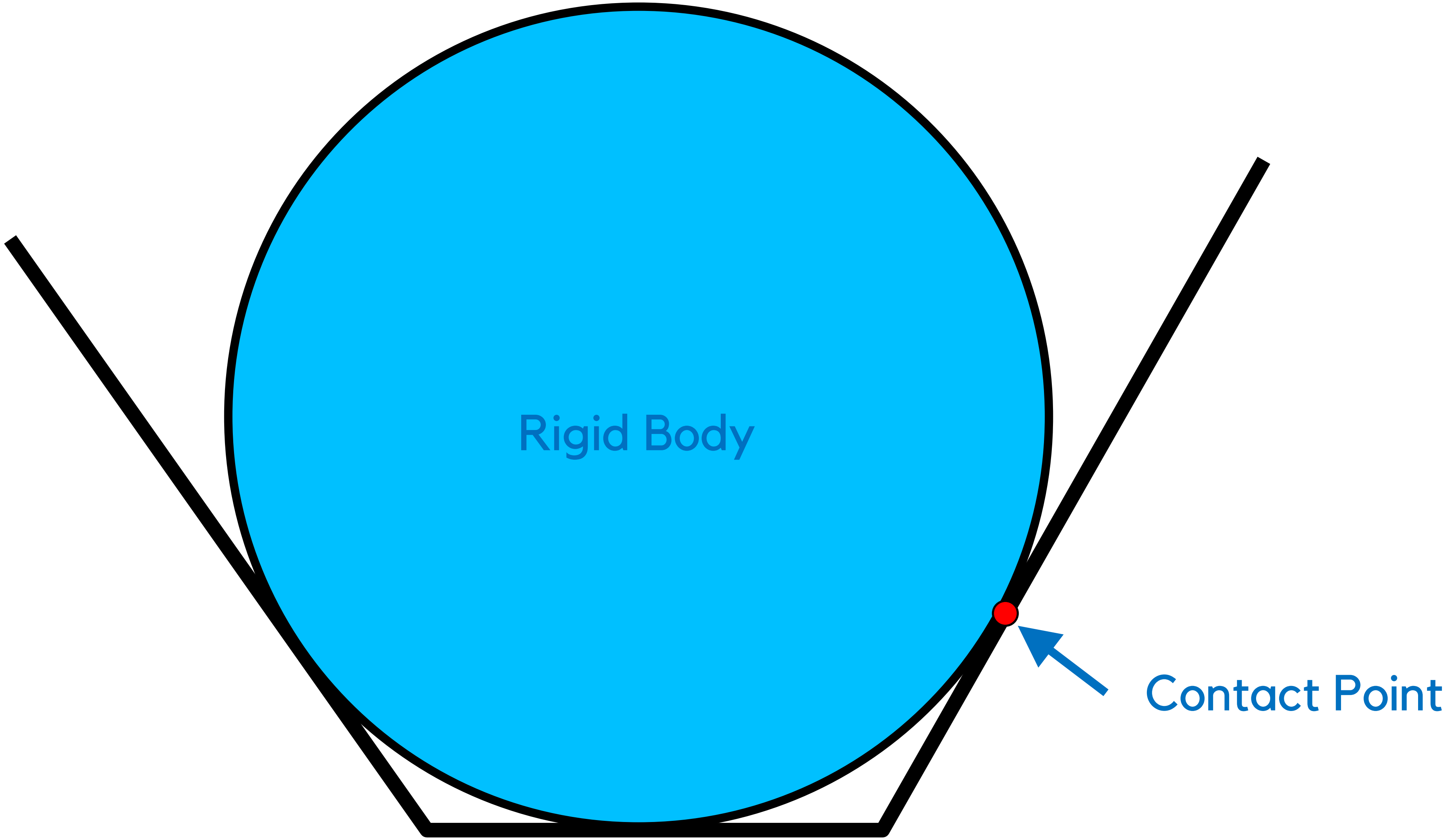
$$\alpha = -\min\left(0, \mathbf{n}^T \mathbf{v}\right)$$

$$\mathbf{v}_{\text{filtered}} = \mathbf{v} + \alpha \mathbf{n}$$

Rigid Body Contact

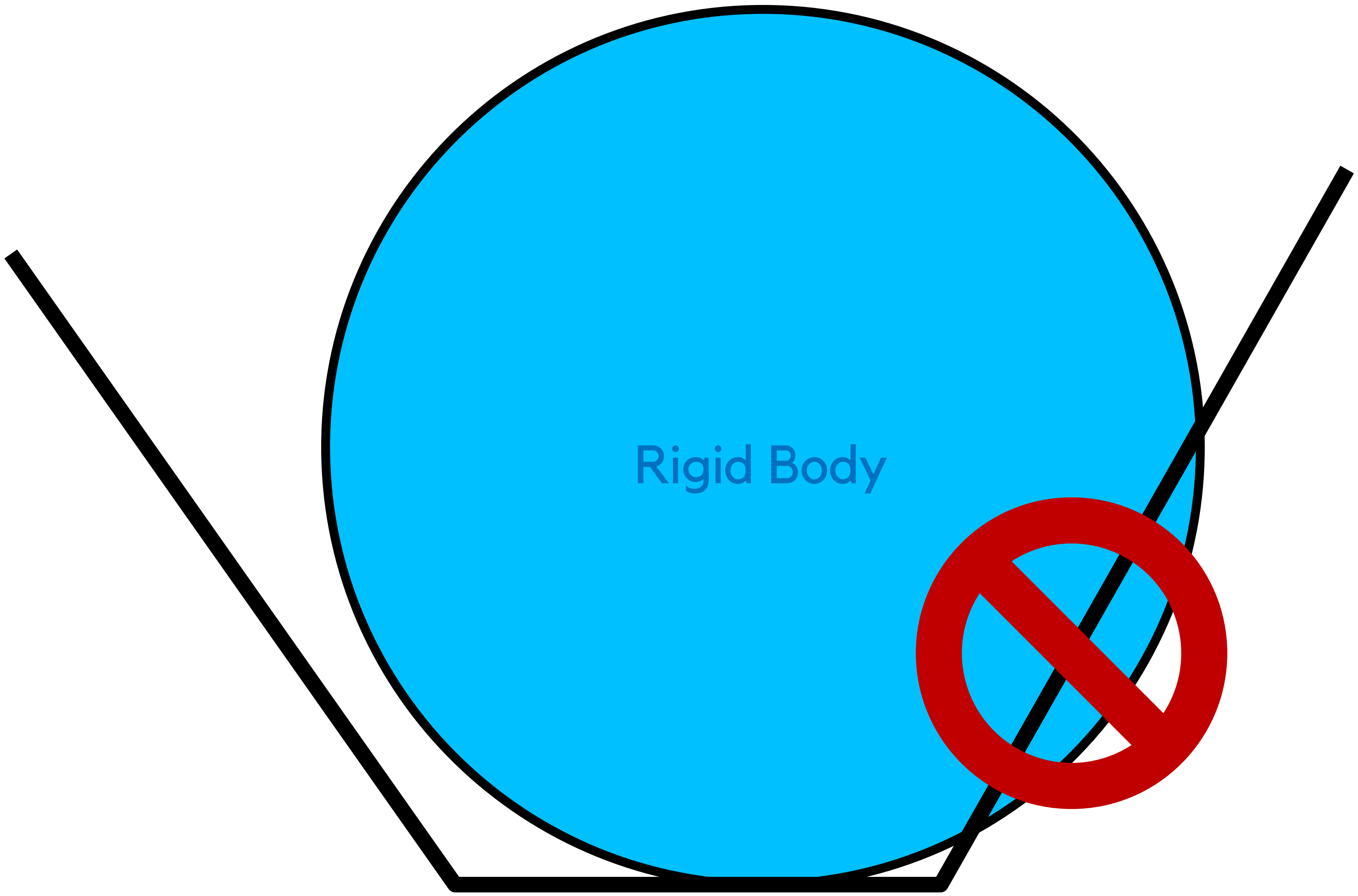


Three Rules of Contact Mechanics



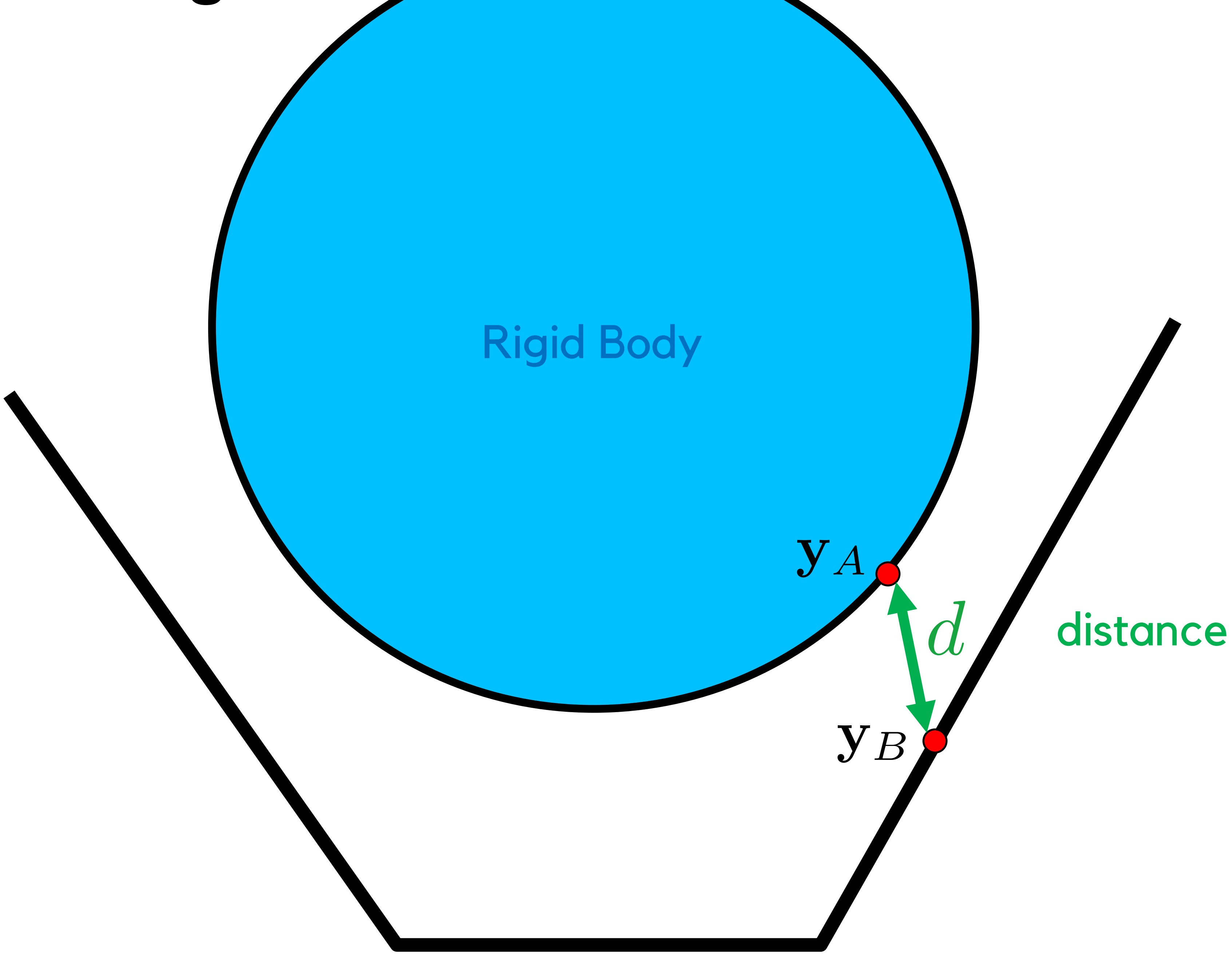
No interpenetration at contact point

Three Rules of Contact Mechanics

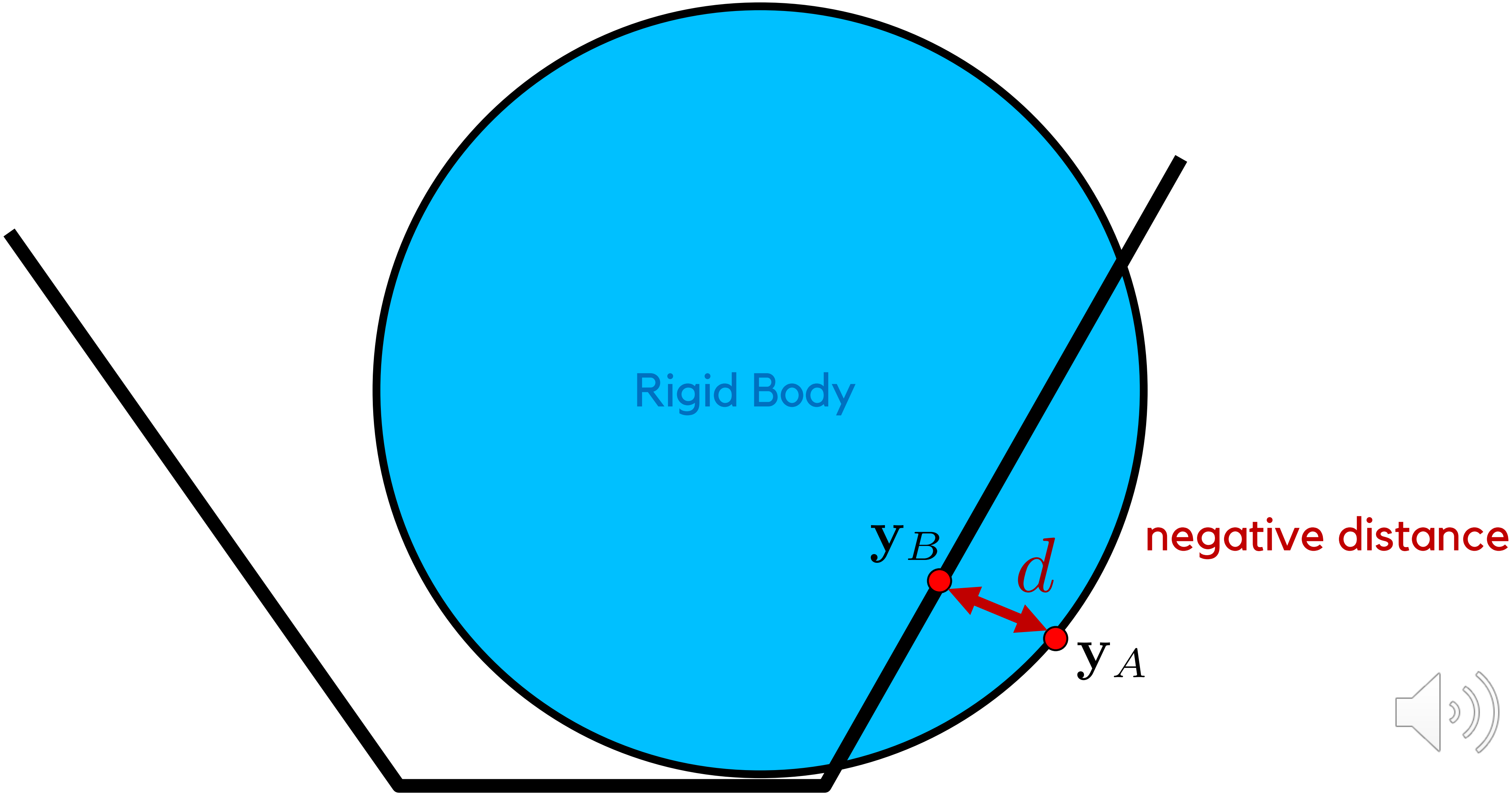


No interpenetration at contact point

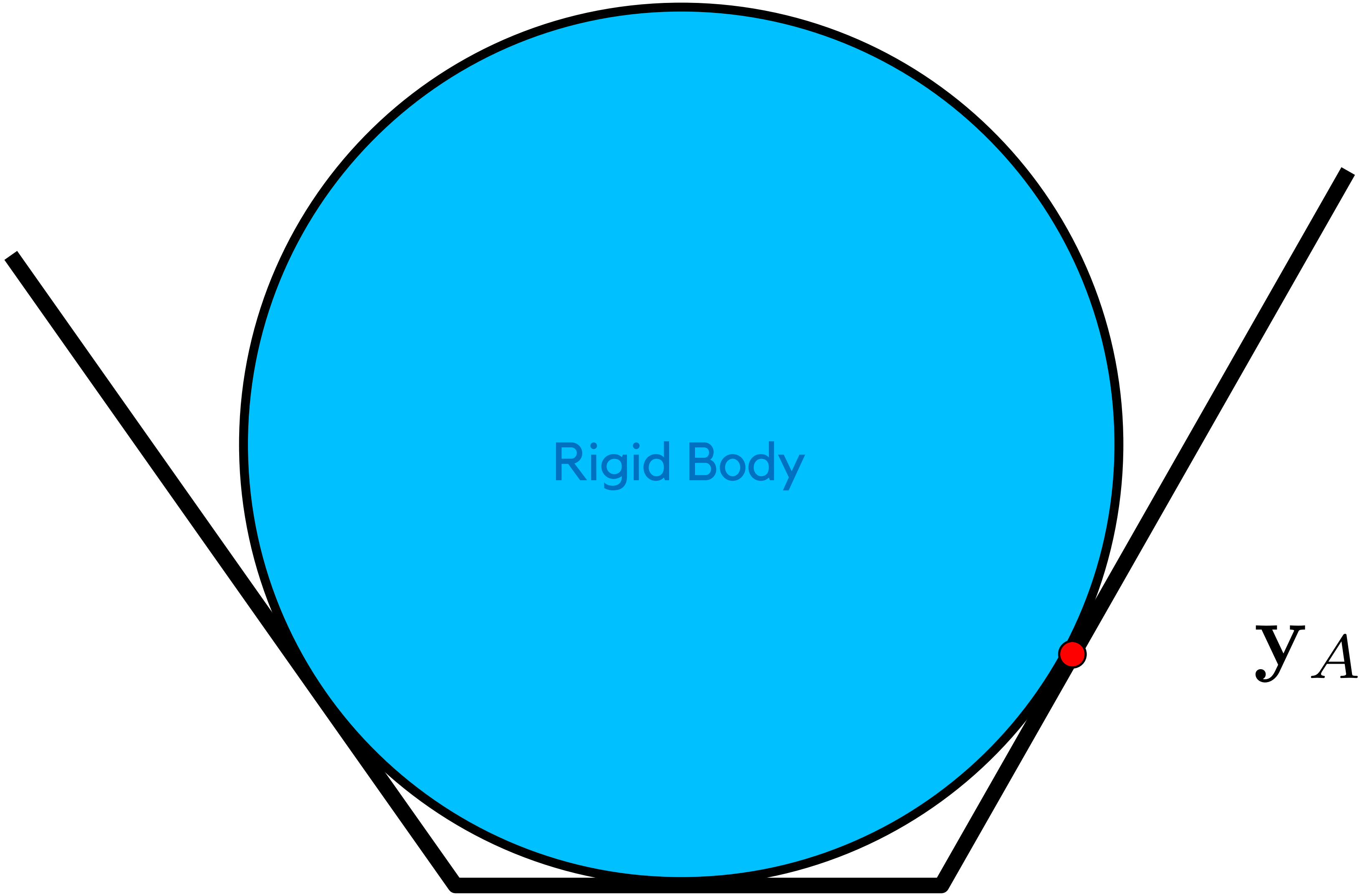
Aside: The Signed Distance



Aside: The Signed Distance



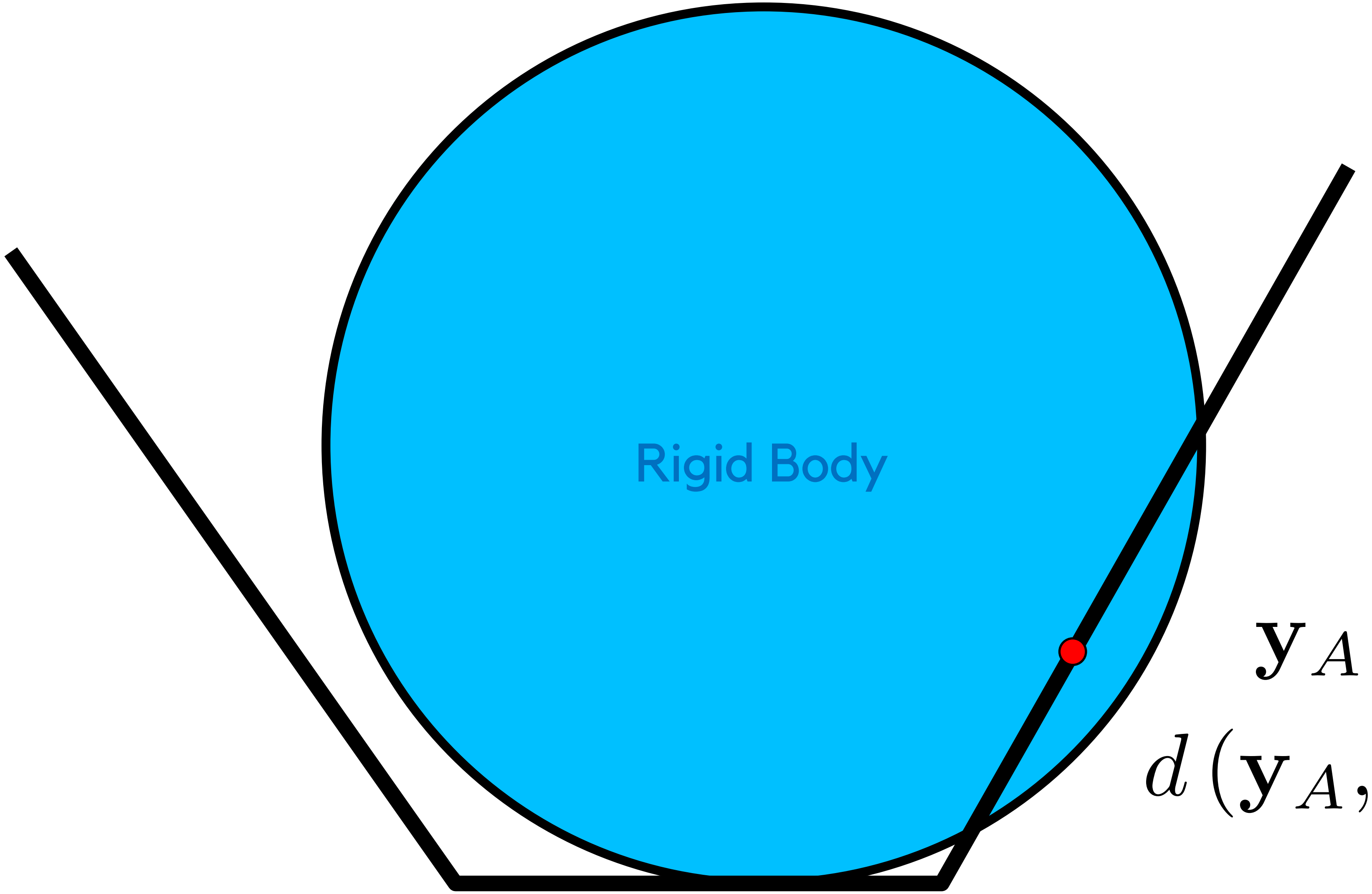
Three Rules of Contact Mechanics



$$\mathbf{y}_A = \mathbf{y}_B$$

No interpenetration at contact point

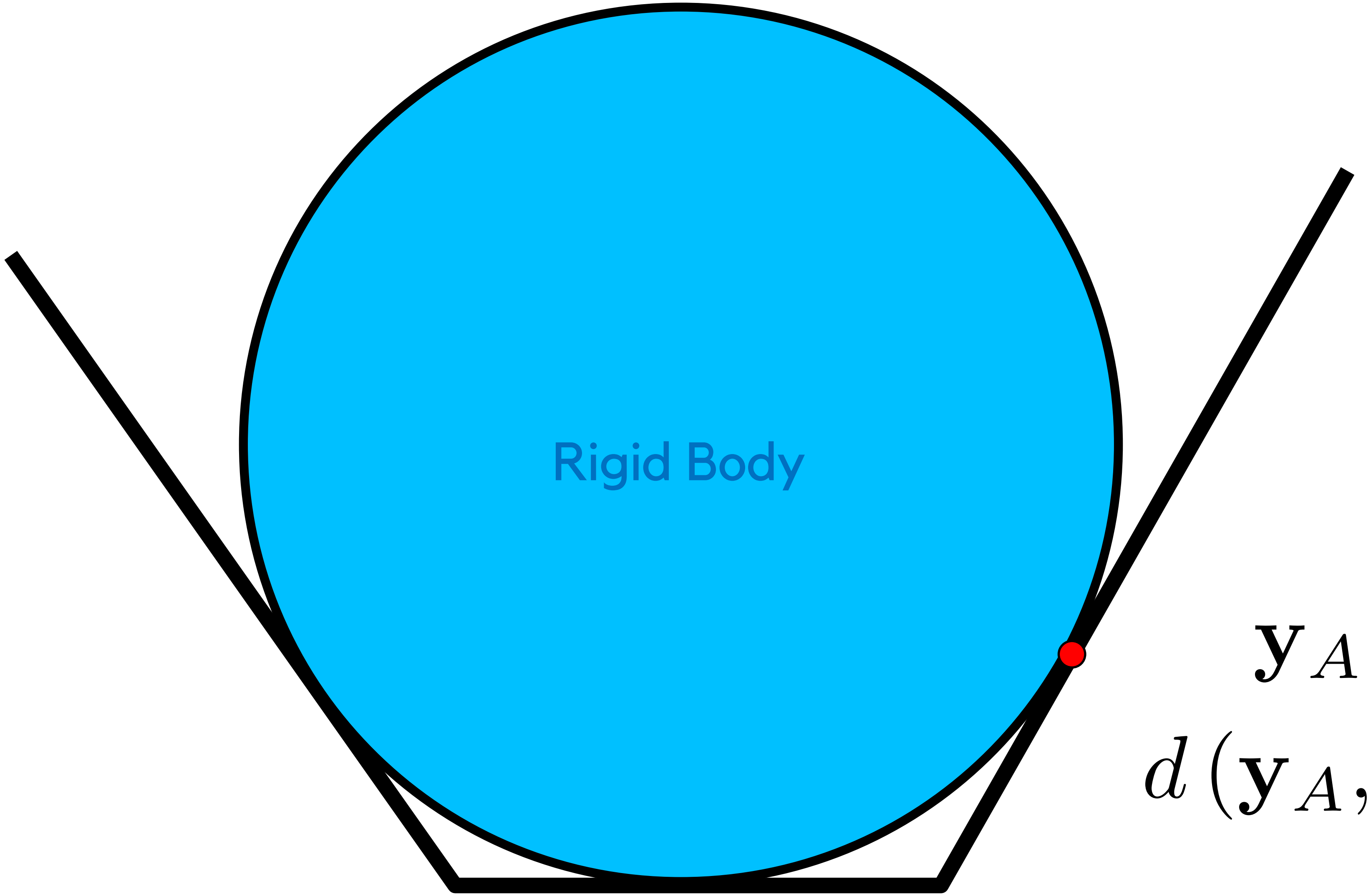
Three Rules of Contact Mechanics



$$\mathbf{y}_A = \mathbf{y}_B$$

$$d(\mathbf{y}_A, \mathbf{y}_B) < 0$$

Three Rules of Contact Mechanics

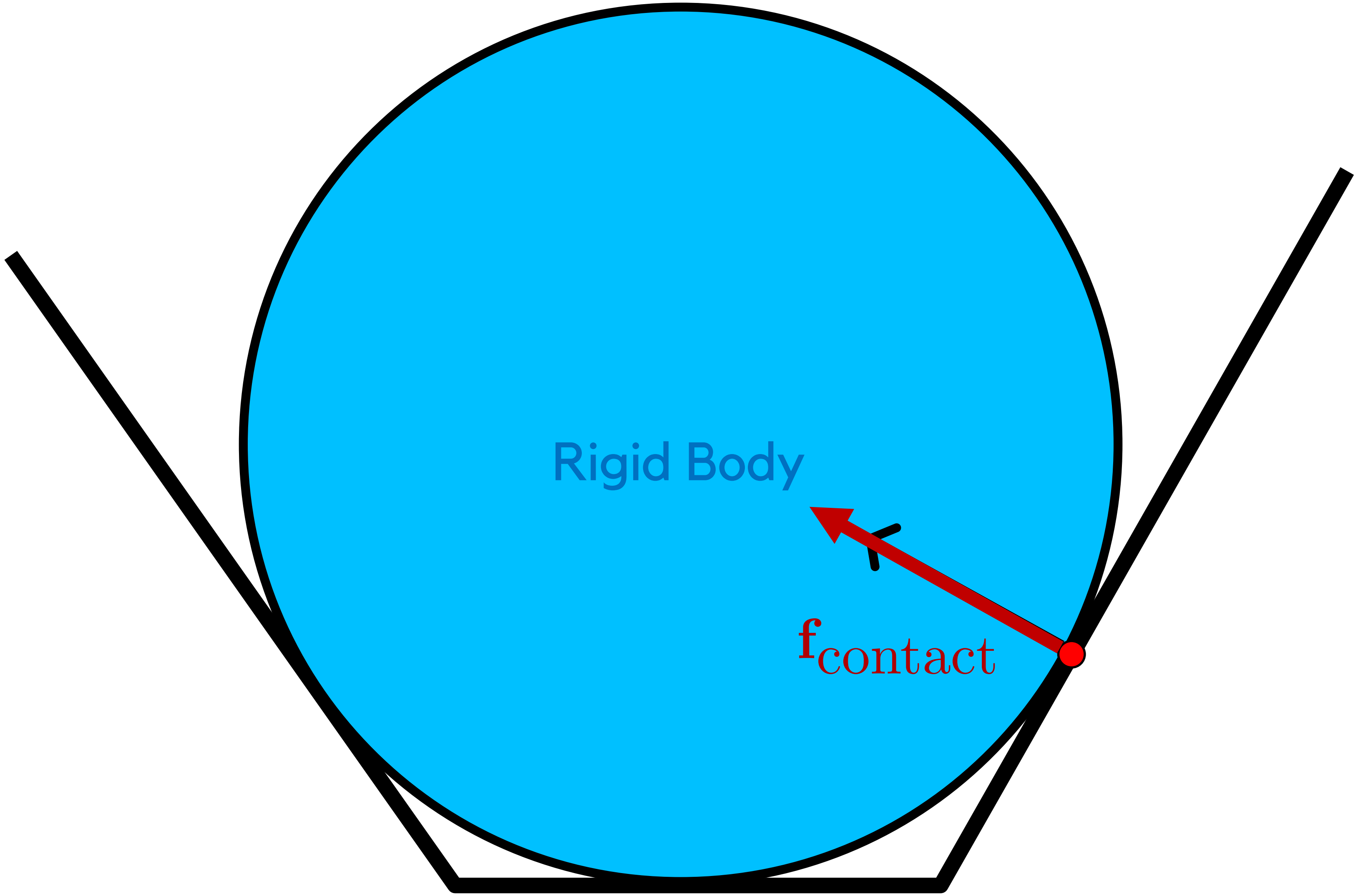


$$\mathbf{y}_A = \mathbf{y}_B$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

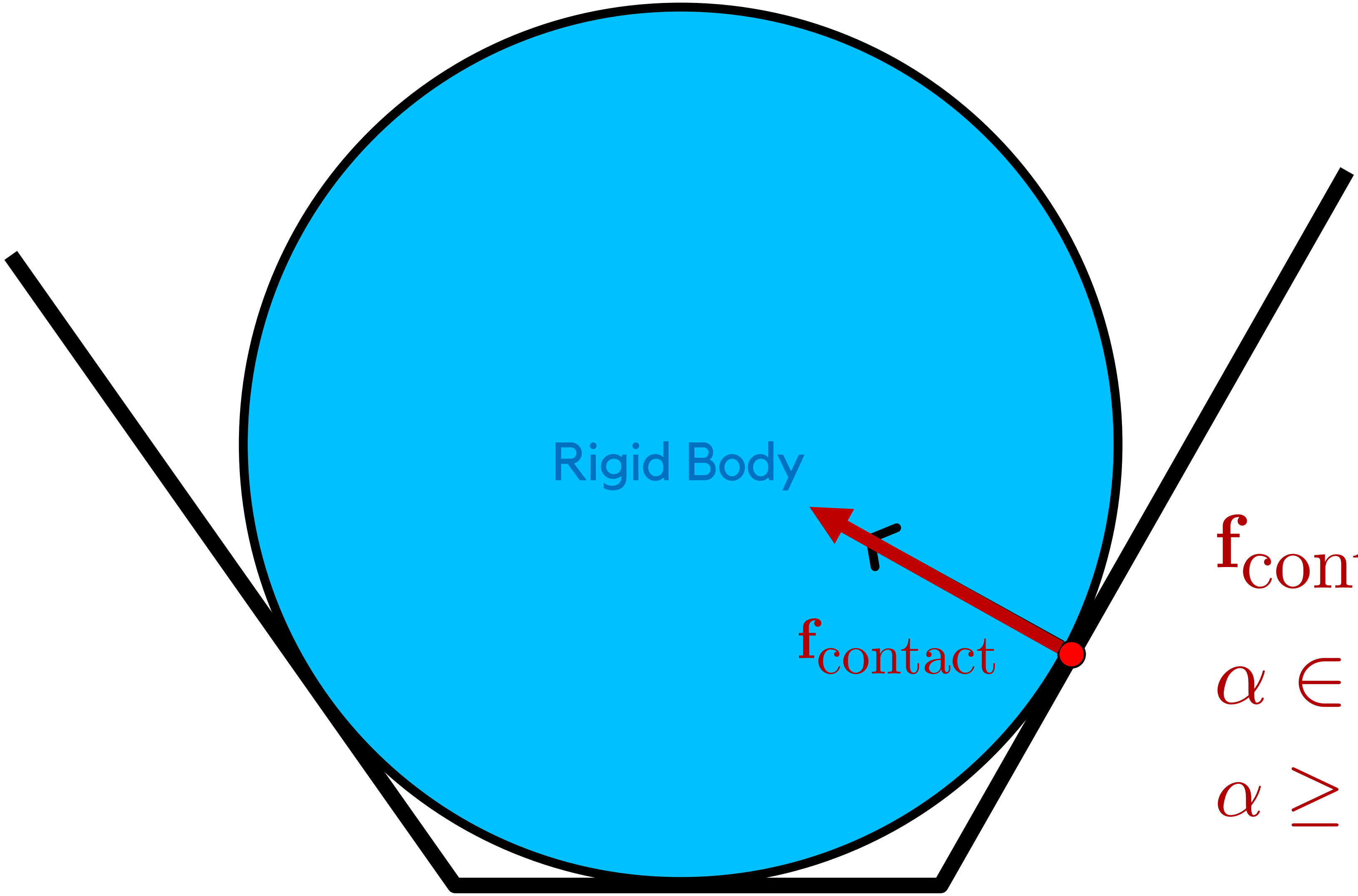
No interpenetration at contact point

Three Rules of Contact Mechanics



Contact forces "push" objects apart

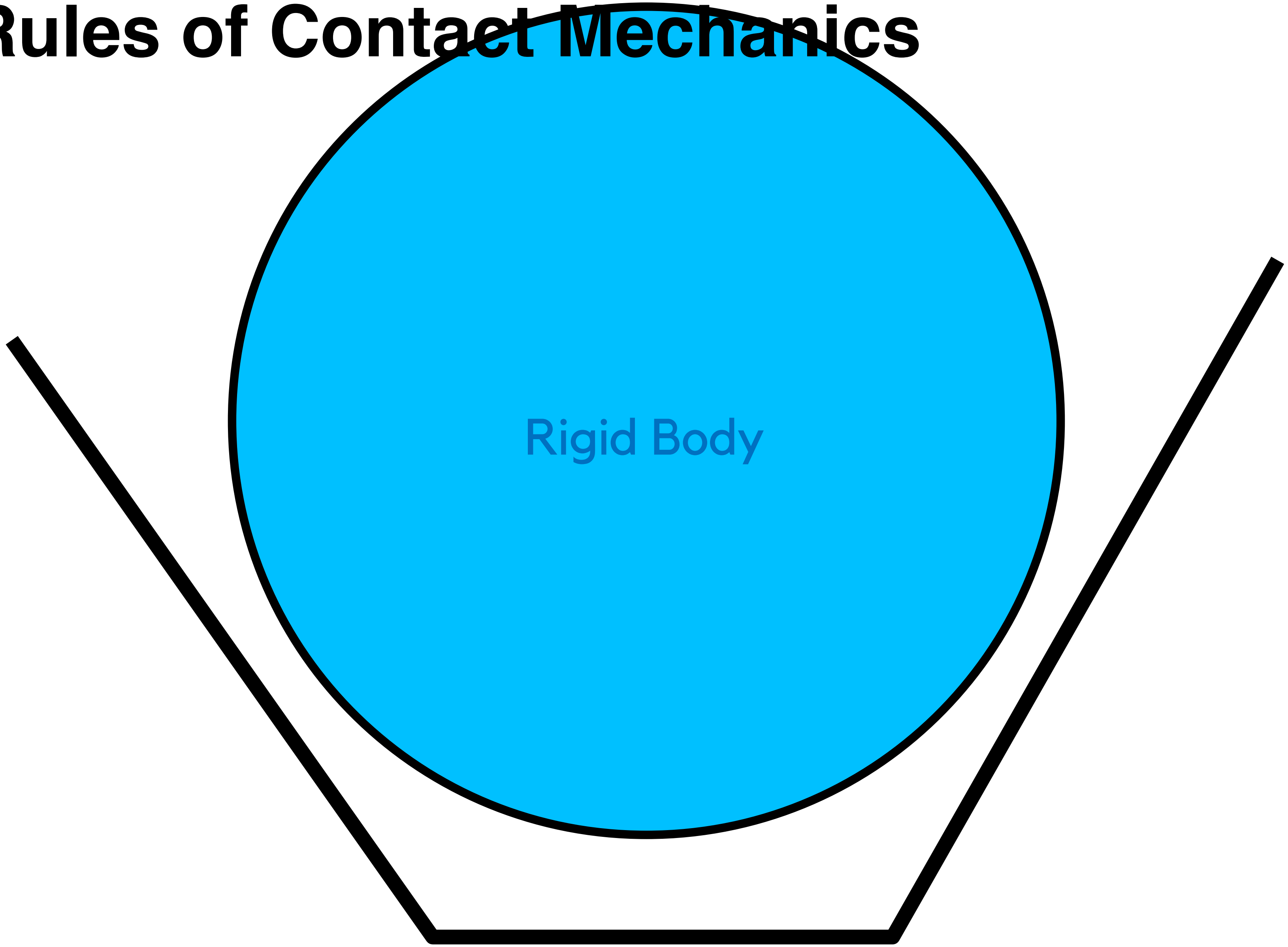
Three Rules of Contact Mechanics



$$\mathbf{f}_{\text{contact}} = \alpha \mathbf{n}$$
$$\alpha \in \mathbb{R}$$
$$\alpha \geq 0$$

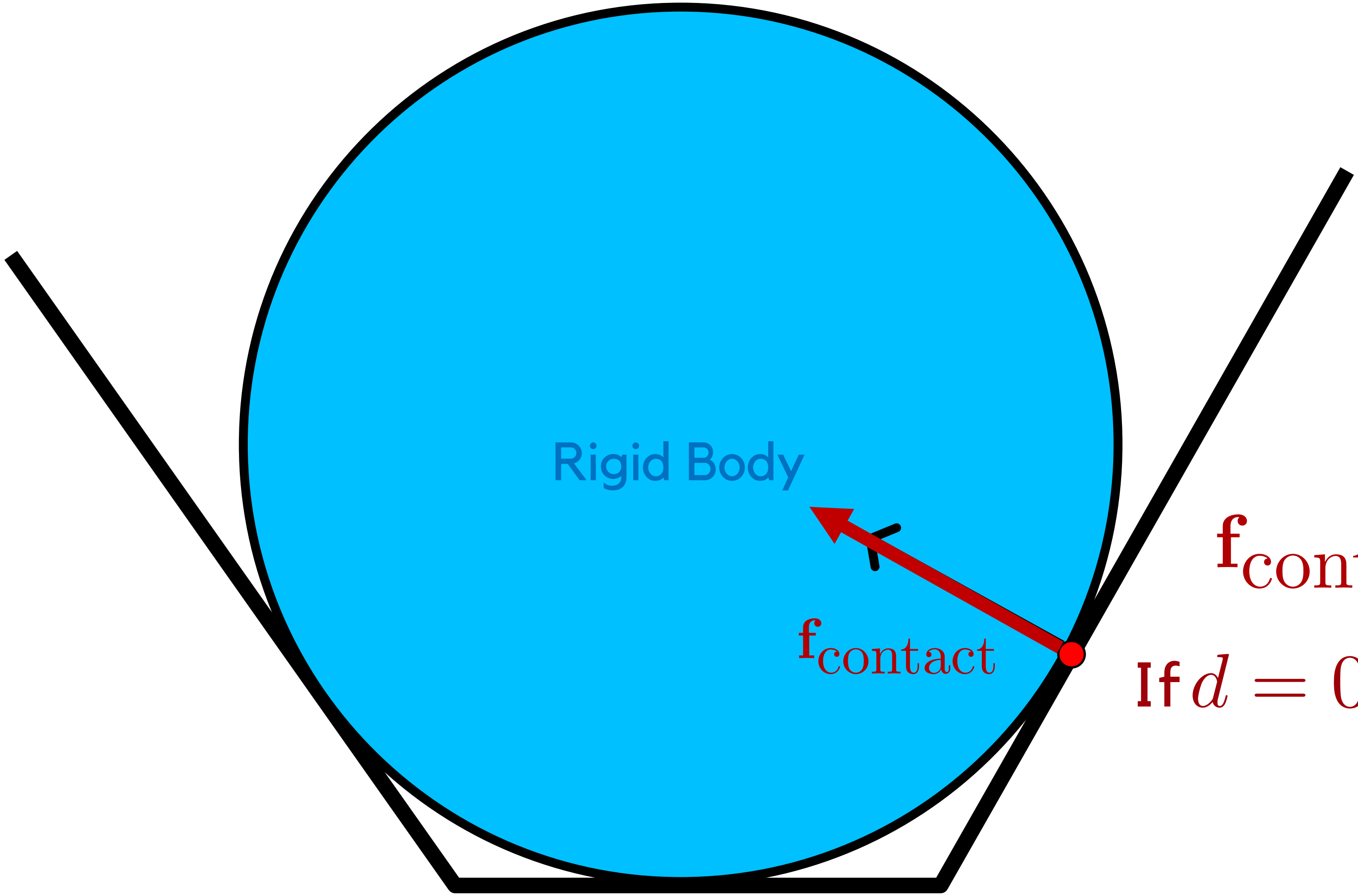
Contact forces “push” objects apart

Three Rules of Contact Mechanics



Contact forces can only be applied when objects are in contact

Three Rules of Contact Mechanics

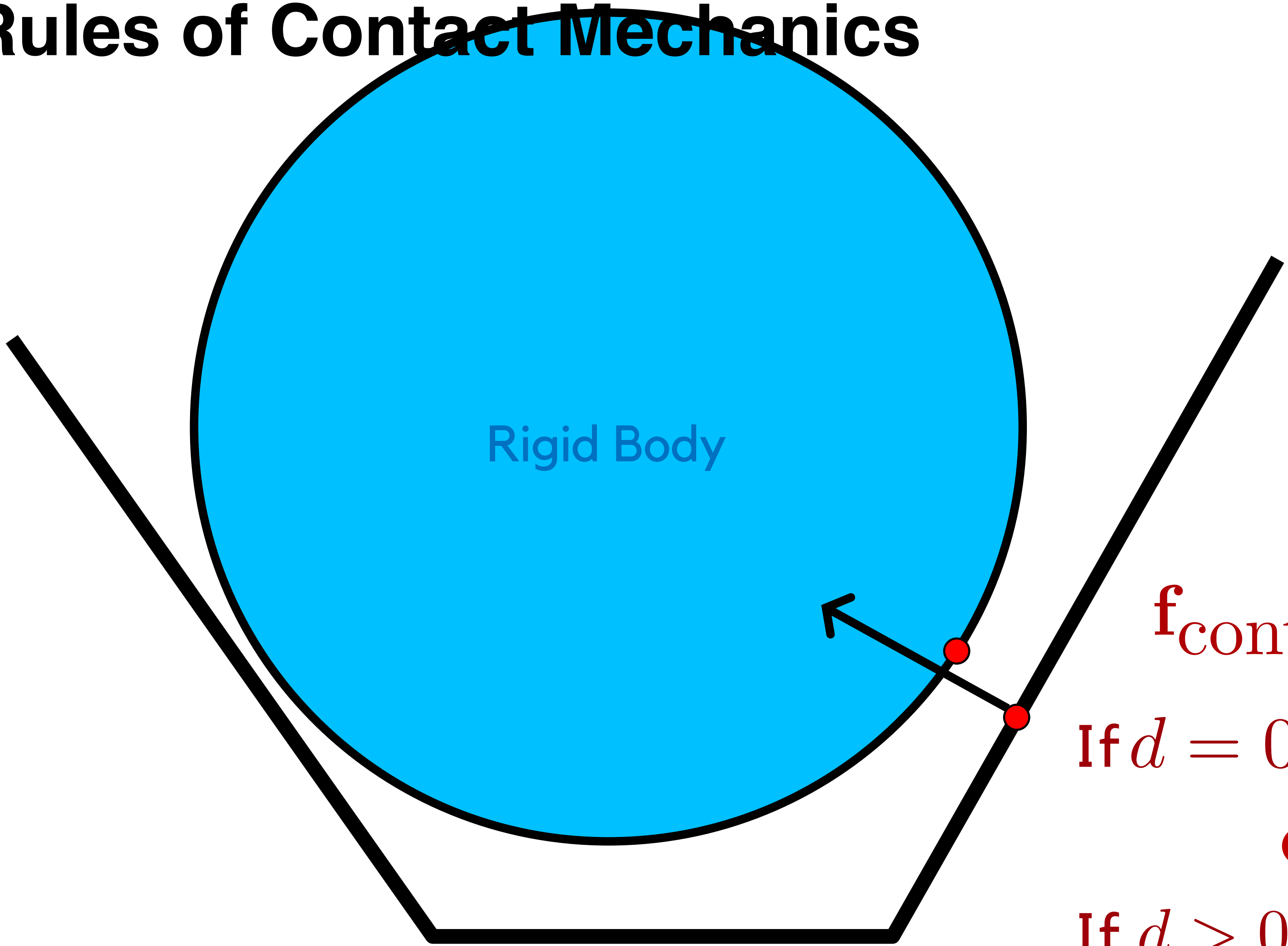


$\mathbf{f}_{\text{contact}} = \alpha \mathbf{n}$

If $d = 0$ then $\alpha \geq 0$

Contact forces can only be applied when objects are in contact

Three Rules of Contact Mechanics



$$\mathbf{f}_{\text{contact}} = \alpha \mathbf{n}$$

If $d = 0$ then $\alpha \geq 0$

OR

If $d > 0$ then $\alpha = 0$

Contact forces can only be applied when objects are in contact

Signorini Conditions

$$\mathbf{f}_{\text{contact}} = \alpha\mathbf{n} \quad \alpha >= 0 \quad \alpha \in \mathbb{R}$$

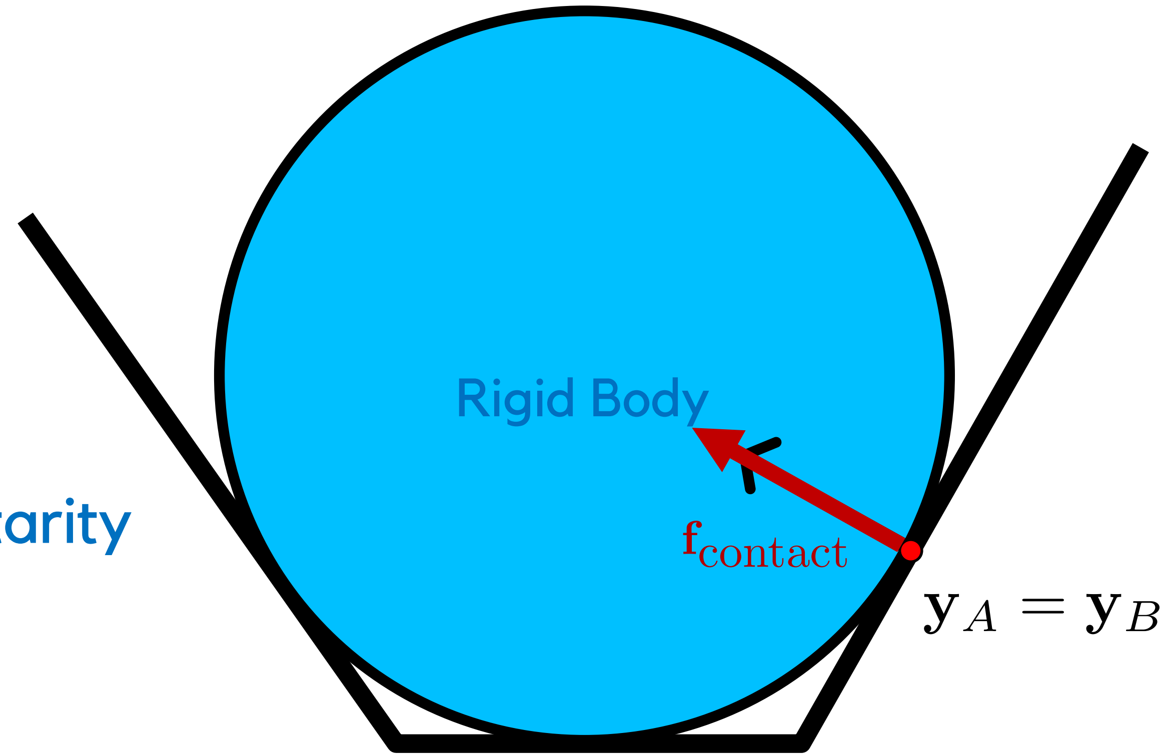
$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

If $d = 0$ then $\alpha > 0$

OR

If $d > 0$ then $\alpha = 0$

Complementarity



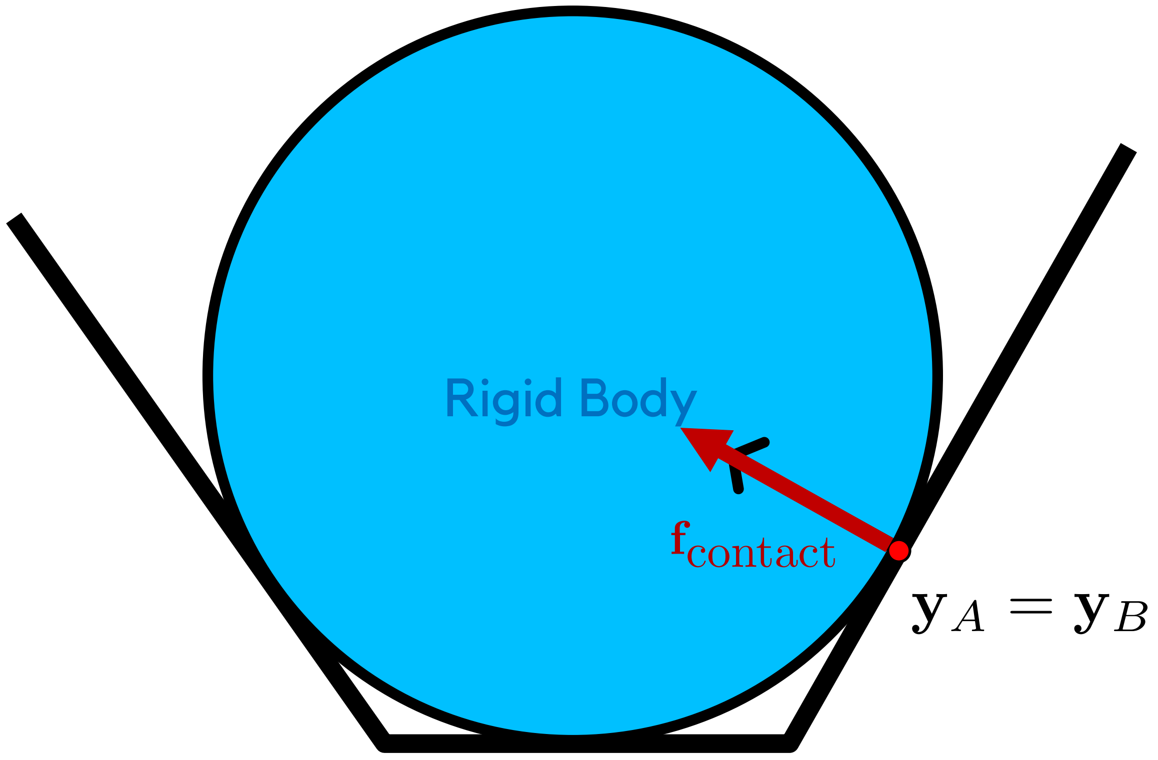
Signorini Conditions

$$\mathbf{f}_{\text{contact}} = \alpha \mathbf{n} \quad \alpha >= 0 \quad \alpha \in \mathbb{R}$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \perp \alpha$$

Complementarity



Newton-Euler Equations

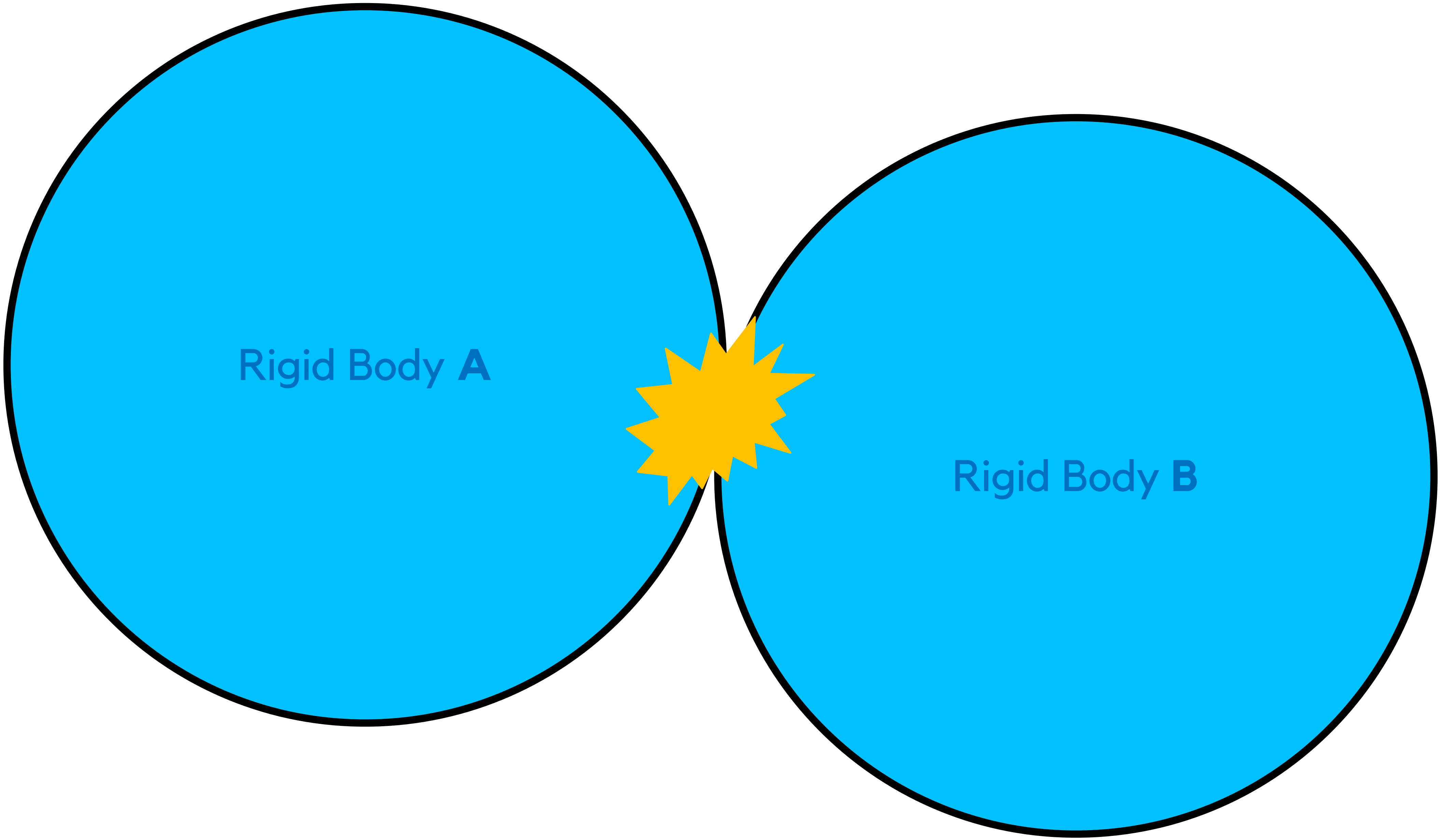
Conservation of Angular Momentum

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^{T}\right)\underbrace{\dot{\omega}}_{\text{angular acceleration}} = \underbrace{\omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^{T}\right)\omega\right)}_{\text{quadratic velocity vector}} + \underbrace{\tau_{ext}}_{\text{external torque}}$$

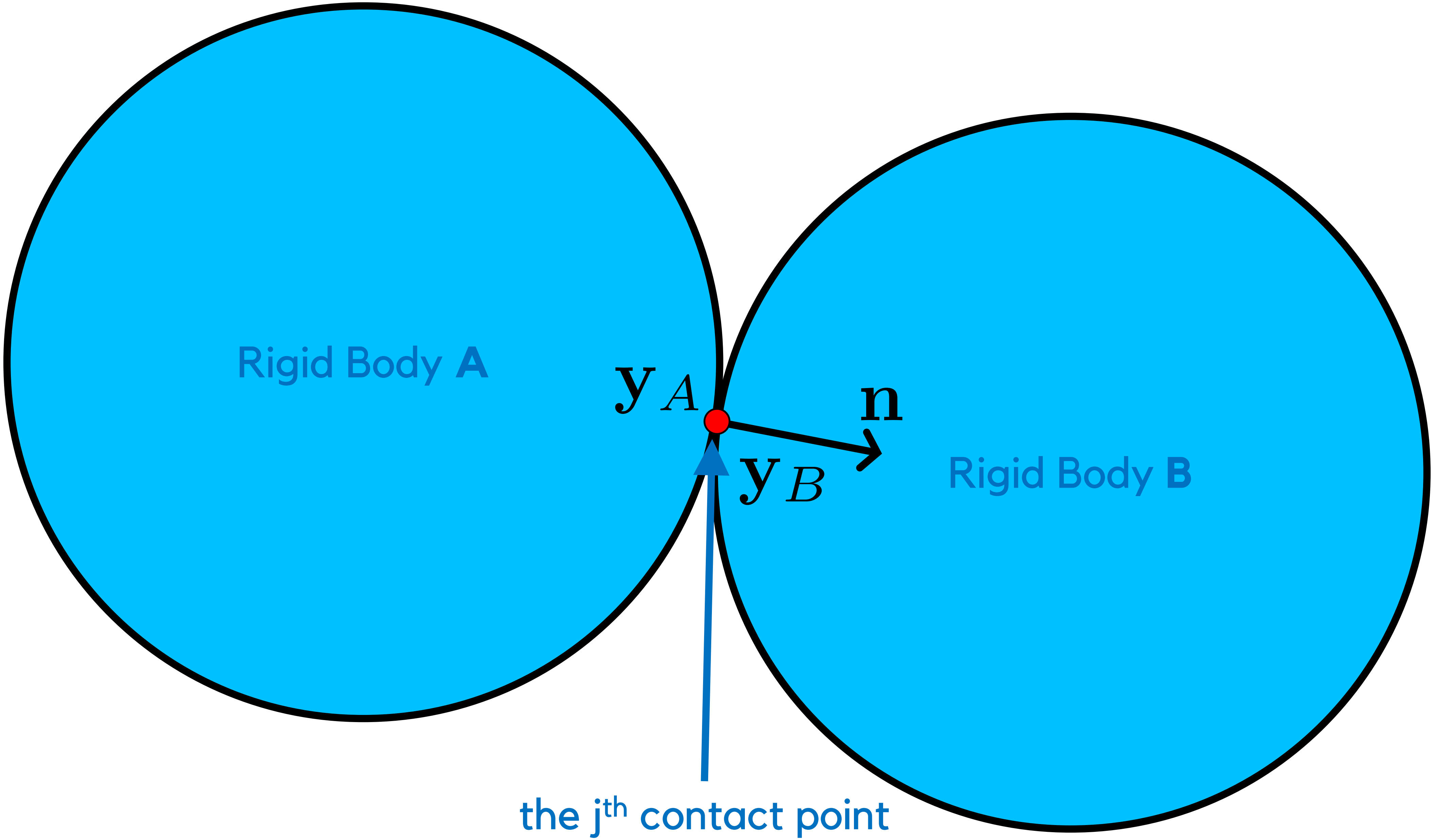
Conservation of Linear Momentum

$$m\mathrm{I}\,\underbrace{\ddot{\mathbf{p}}}_{\text{acceleration}} = \underbrace{\mathbf{f}_{ext}}_{\text{external force}}$$

Two-Body Contact Example

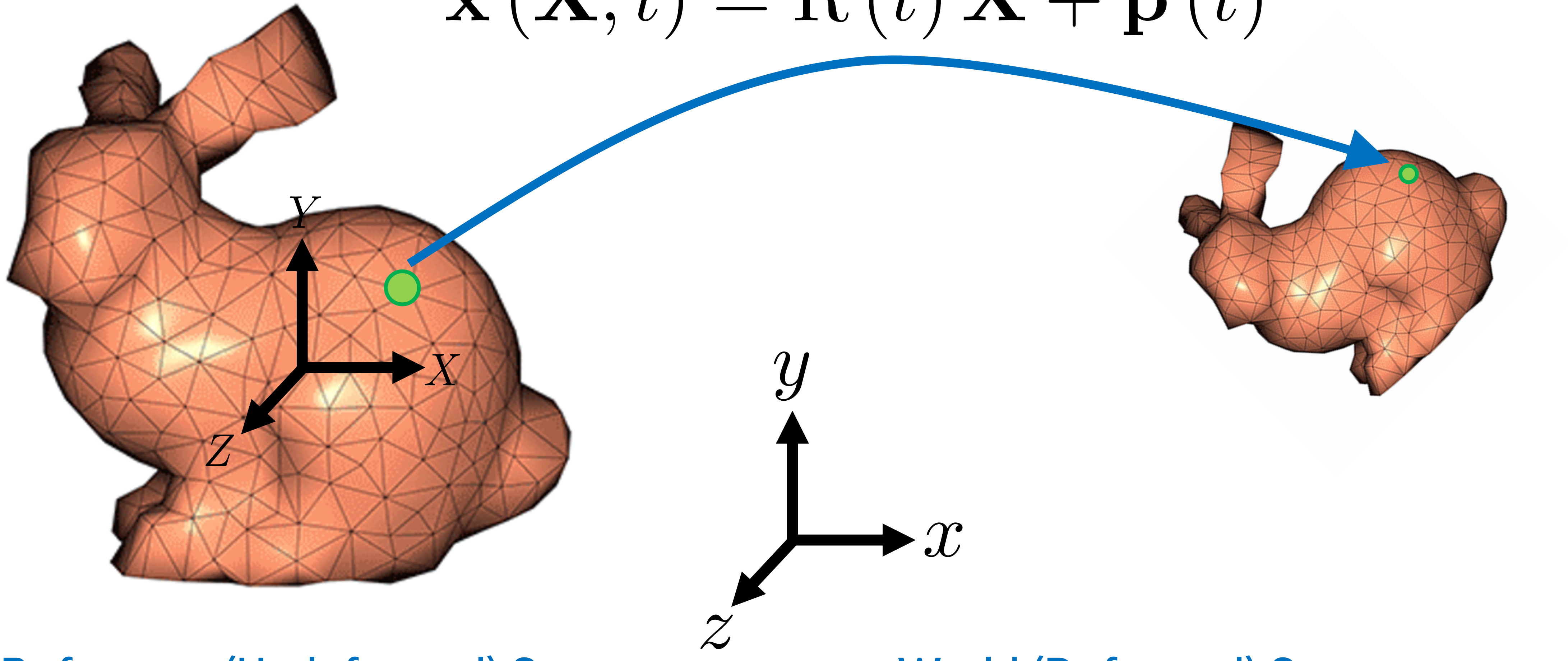


Two-Body Contact Example



The Rigid Body Mapping

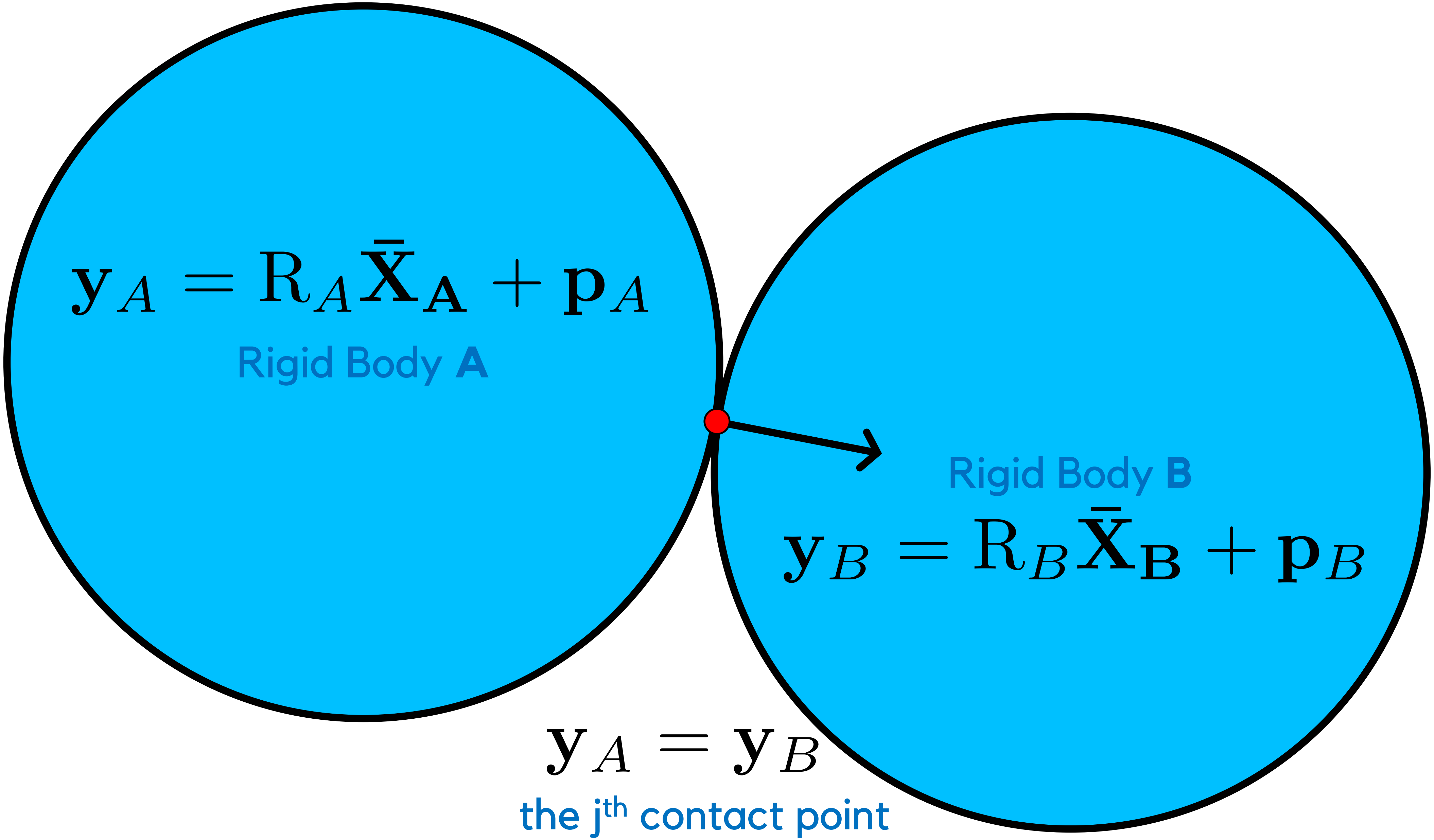
$$\mathbf{x}\left(\bar{\mathbf{X}}, t\right) = \mathrm{R}\left(t\right)\bar{\mathbf{X}} + \mathbf{p}\left(t\right)$$



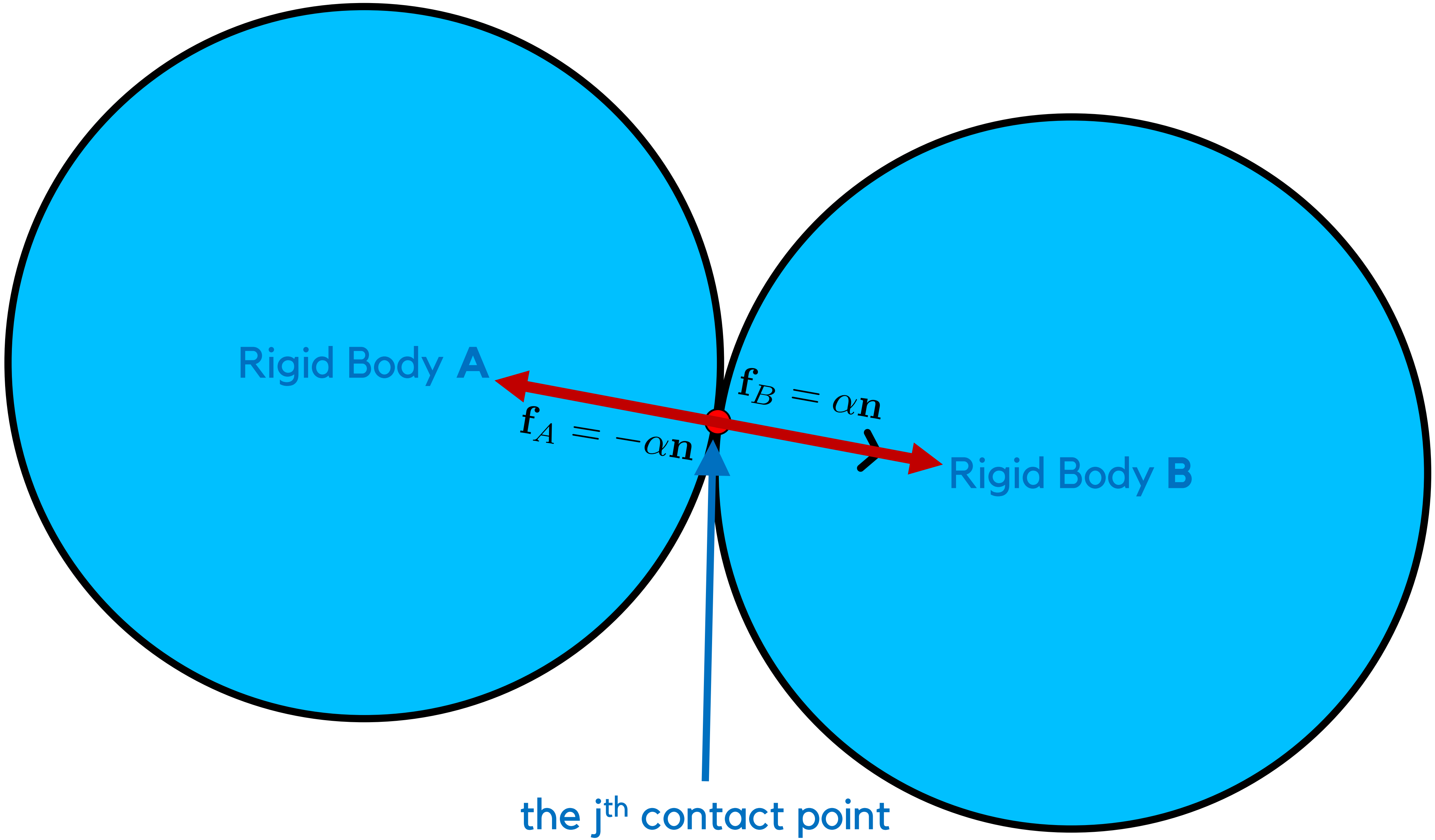
Reference (Undeformed) Space

World (Deformed) Space

Two-Body Contact Example



Two-Body Contact Example



The Whole Picture

For each rigid body

$$\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\dot{\omega} = \omega \times \left(\left(\mathrm{R}\,\mathcal{I}\,\mathrm{R}^T\right)\omega\right) + \tau_{ext}$$

$$m\mathrm{I}\,\ddot{\mathbf{p}} = \mathbf{f}_{ext}$$

For each contact point

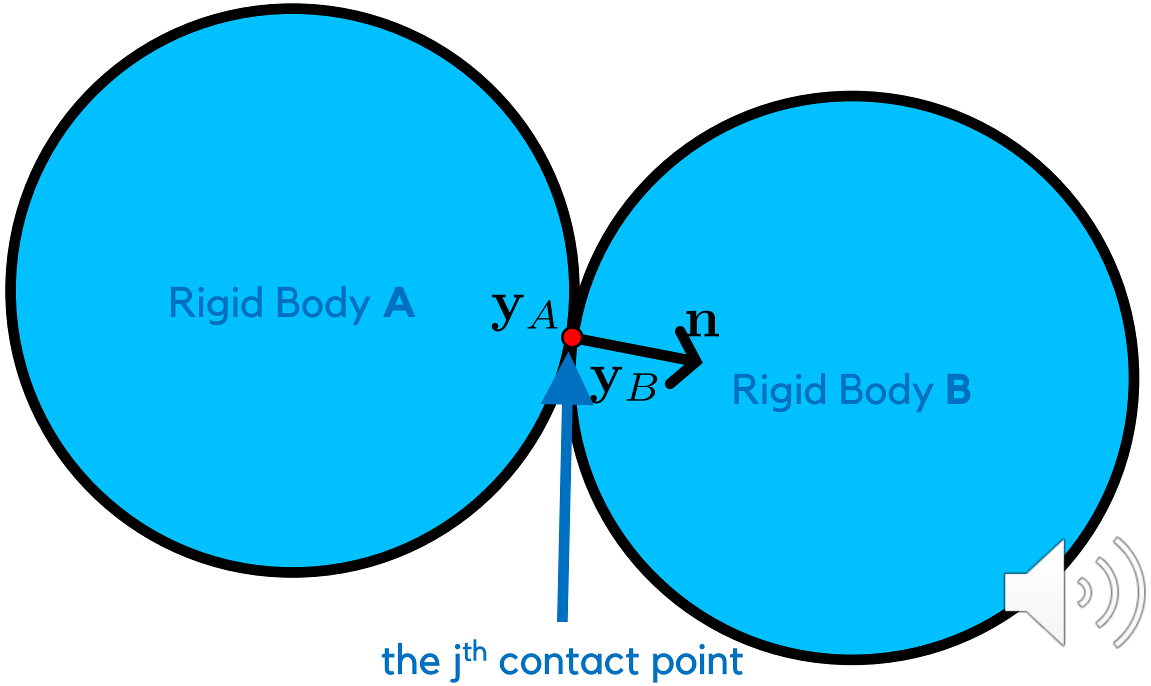
$$\mathbf{f}_A = -\alpha\mathbf{n}$$

$$\mathbf{f}_B = \alpha\mathbf{n}$$

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \perp \alpha$$



Discrete Equations of Motion

$$\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^{t+1} = \left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^t + \Delta t\,\omega^t \times \left(\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega^t\right) + \Delta t\,\tau^t_{ext}$$

$$m\dot{\mathbf{p}}^{t+1} = m\dot{\mathbf{p}}^t + \Delta t\mathbf{f}_{ext}$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right]\Delta t\right)\mathrm{R}^t$$

$$\mathbf{p}^{t+1} = \mathbf{p}^t + \Delta t\dot{\mathbf{p}}^t$$

For each contact point

$$\mathbf{f}_A = -\alpha\mathbf{n}$$

$$\mathbf{f}_B = \alpha\mathbf{n}$$

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

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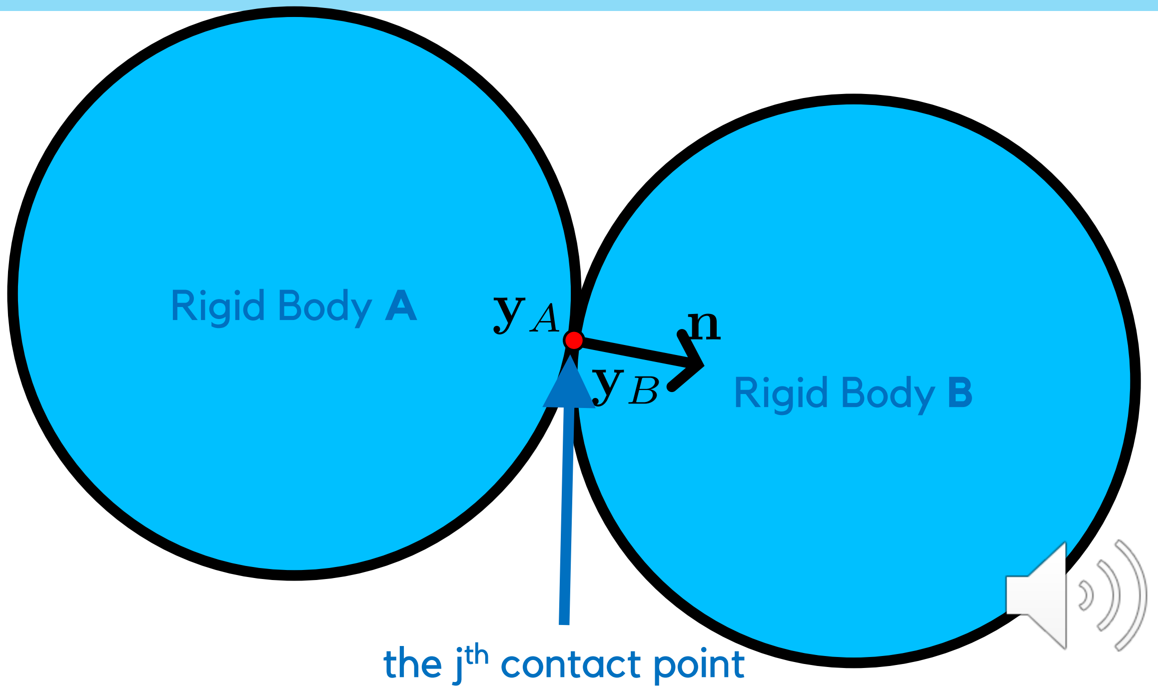


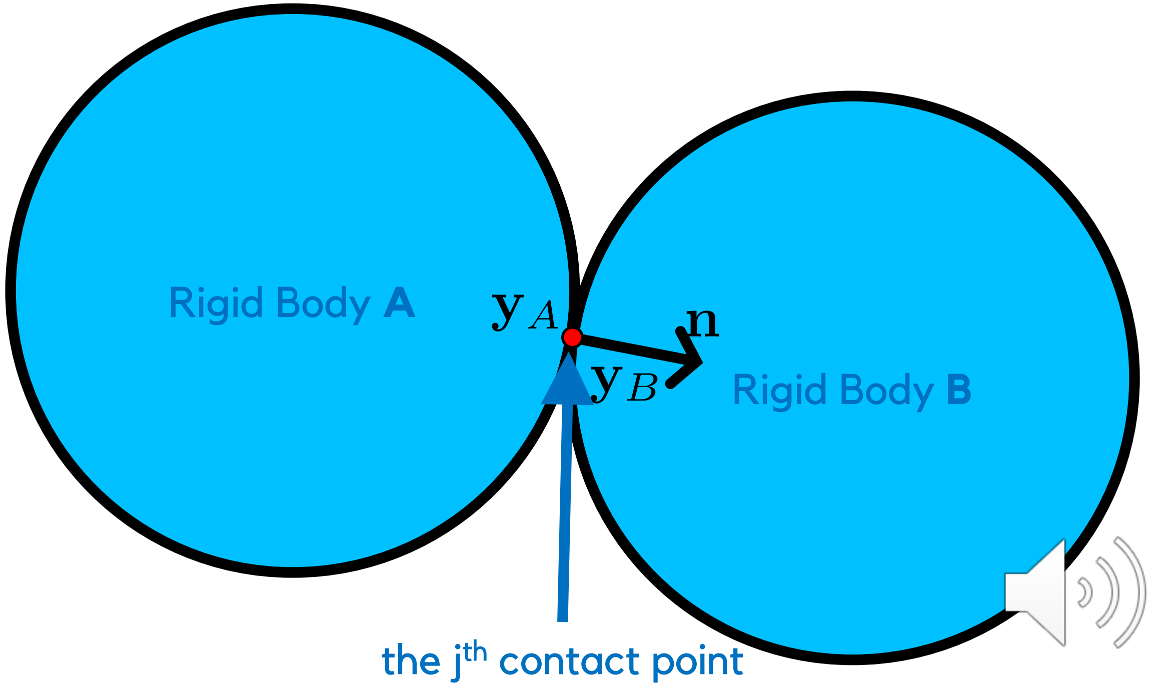
Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t}$$

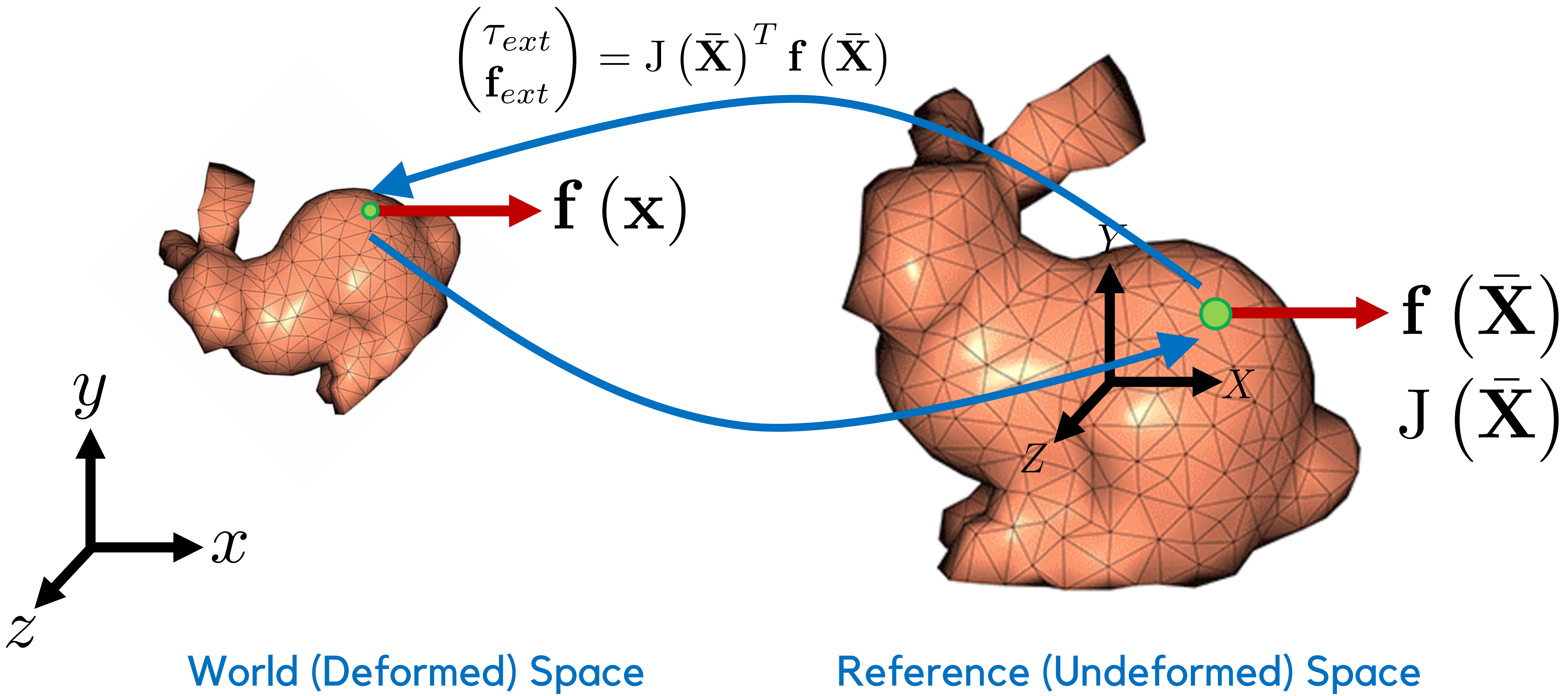
$$\begin{pmatrix} \mathrm{R}\mathcal{I}\mathrm{R}^T & 0 \\ 0 & m\mathrm{I} \end{pmatrix} \quad \dot{\mathbf{q}} = \begin{pmatrix} \omega \\ \dot{\mathbf{p}} \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} \omega \times \left(\left(\mathrm{R}\mathcal{I}\mathrm{R}^T\right)\omega\right) + \tau_{ext} \\ \mathbf{f}_{ext} \end{pmatrix}$$

For each contact point

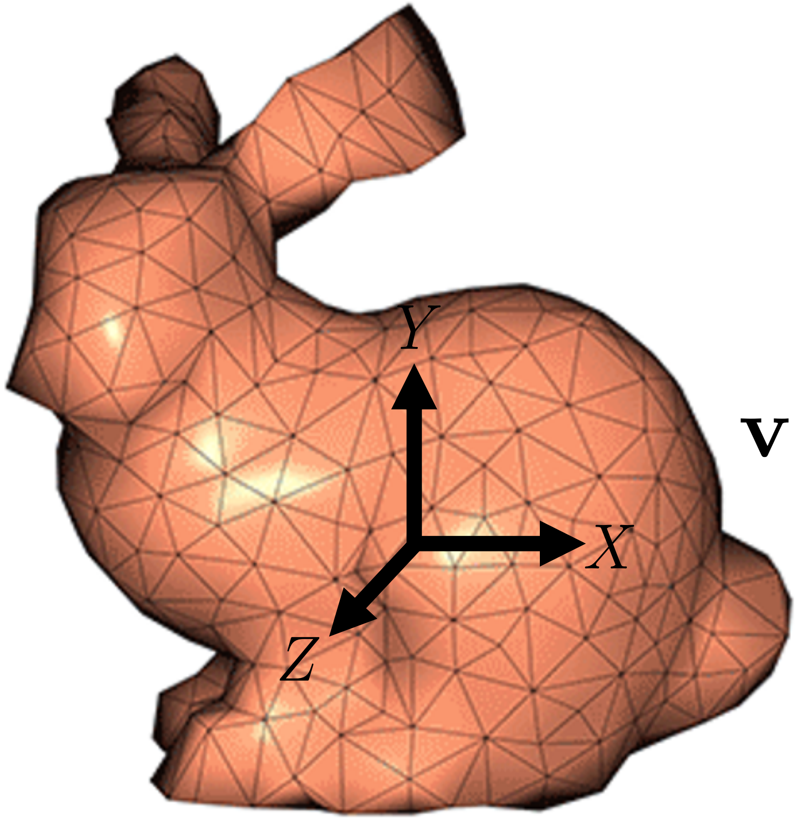
$$\mathbf{f}_A = -\alpha \mathbf{n}$$
$$\mathbf{f}_B = \alpha \mathbf{n}$$
$$\alpha >= 0$$
$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$
$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \perp \alpha$$



External Torques and Forces



The Rigid Body Jacobian



$$\mathbf{v}\left(\bar{\mathbf{X}}, t\right) = \underbrace{\mathrm{R}\left([\bar{\mathbf{X}}]^T \quad \mathrm{I}\right)\begin{pmatrix}\mathrm{R}^T & 0 \\ 0 & \mathrm{R}^T\end{pmatrix}}_{\substack{\text{Jacobian} \\ \mathrm{J} \in \mathbb{R}^{3\times 6}}} \underbrace{\begin{pmatrix}\omega \\ \dot{\mathbf{p}}\end{pmatrix}}_{\dot{\mathbf{q}} \in \mathbb{R}^6}$$

Reference (Undeformed) Space

Discrete Equations of Motion

$$\mathbf{M}\dot{\mathbf{q}}^{t+1} = \mathbf{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t} + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathbf{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\hat{\mathbf{n}} = \begin{cases} -\mathbf{n} & \text{If Object A} \\ \mathbf{n} & \text{If Object B} \end{cases}$$

For each contact point

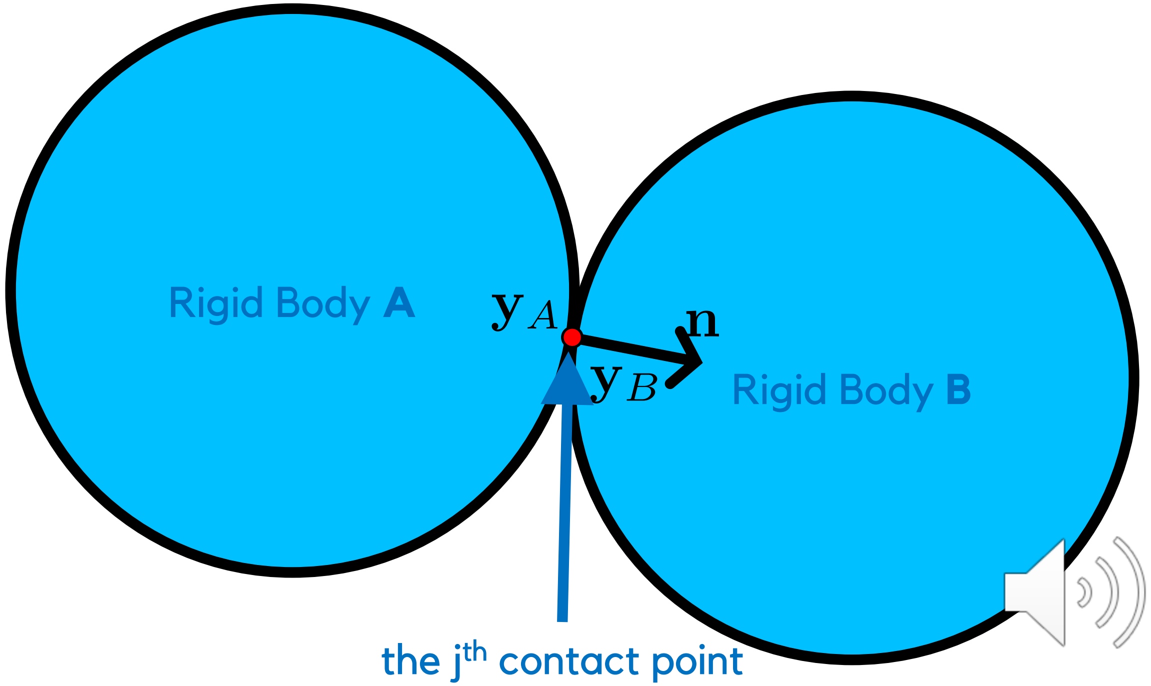
$$\cancel{\mathbf{f}_A = -\alpha \mathbf{n}}$$

$$\cancel{\mathbf{f}_B = \alpha \mathbf{n}}$$

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \perp \alpha$$



Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t} + \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right]\Delta t\right)\mathrm{R}^{t}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t \dot{\mathbf{p}}^{t}$$

For each contact point

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

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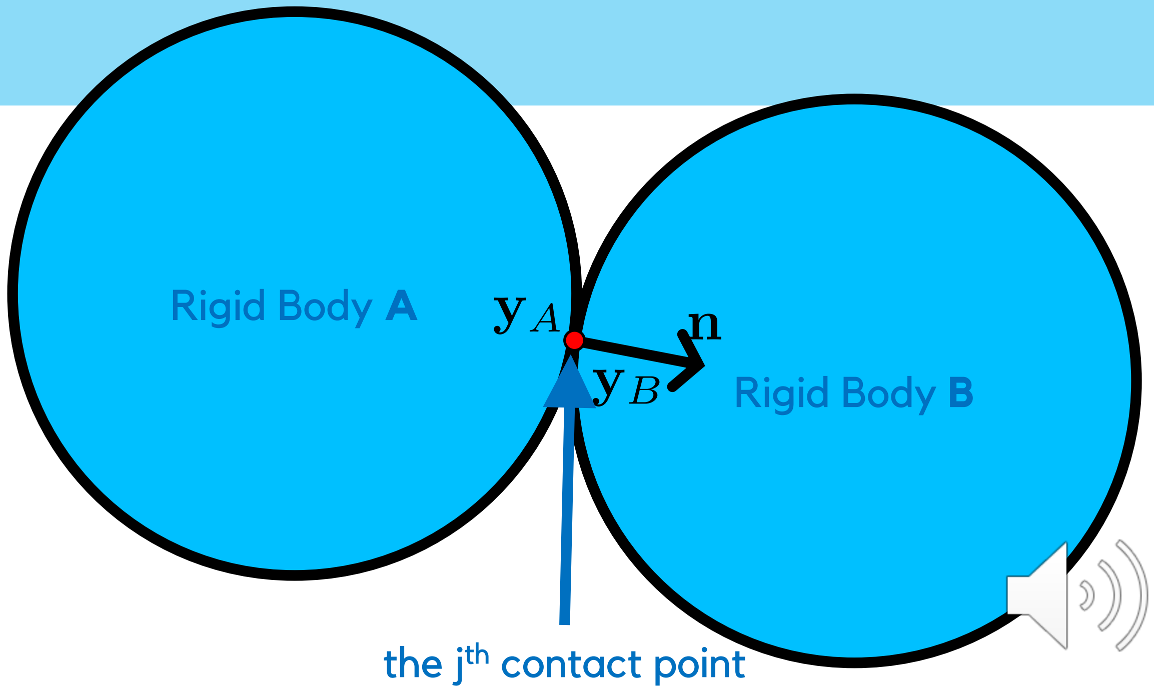

Discrete Equations of Motion

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$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right]\Delta t\right)\mathrm{R}^{t}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t \dot{\mathbf{p}}^{t}$$

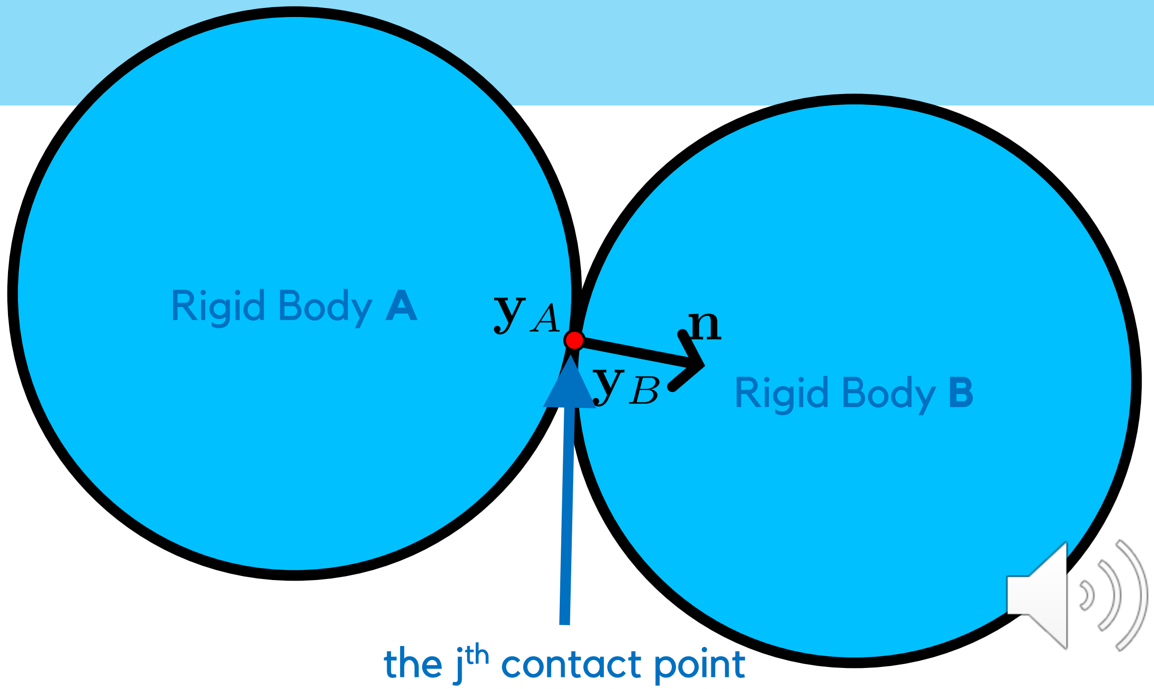
For each contact point

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \geq 0$$

Discretize ?

$$d\left(\mathbf{y}_A, \mathbf{y}_B\right) \perp \alpha$$



Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t} + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right] \Delta t\right) \mathrm{R}^{t}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t \dot{\mathbf{p}}^{t}$$

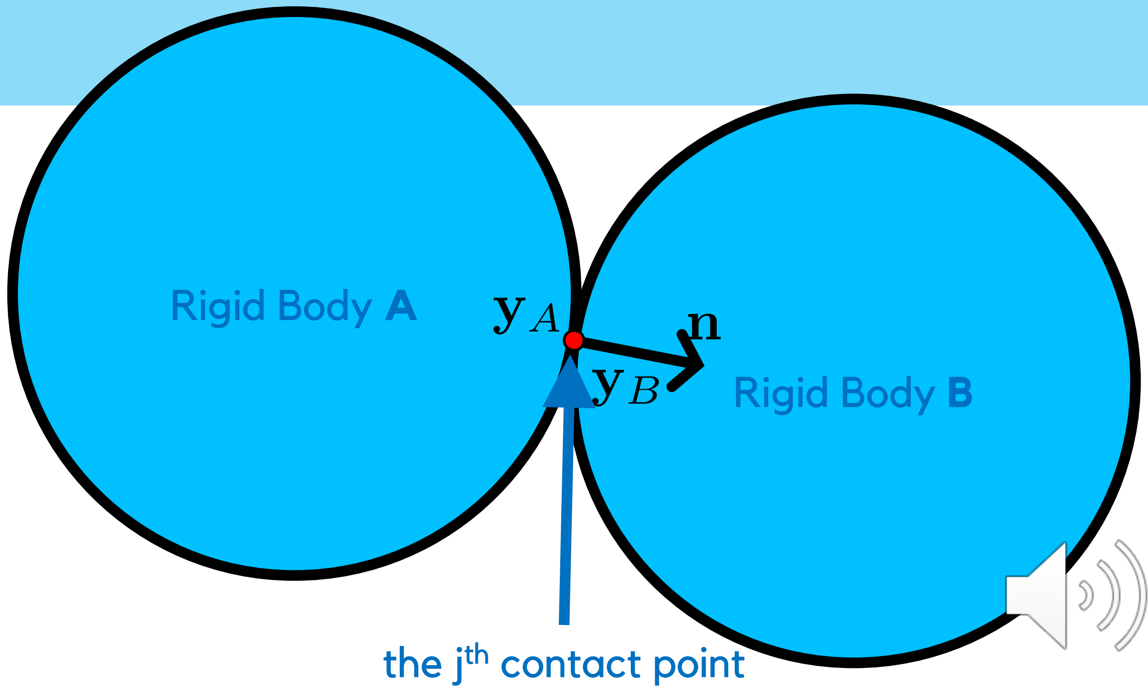
For each contact point

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A^{t+1}, \mathbf{y}_B^{t+1}\right) \geq 0$$

Nonlinear ☹

$$d\left(\mathbf{y}_A^{t+1}, \mathbf{y}_B^{t+1}\right) \perp \alpha$$



Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t} + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right]\Delta t\right)\mathrm{R}^{t}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t \dot{\mathbf{p}}^{t}$$

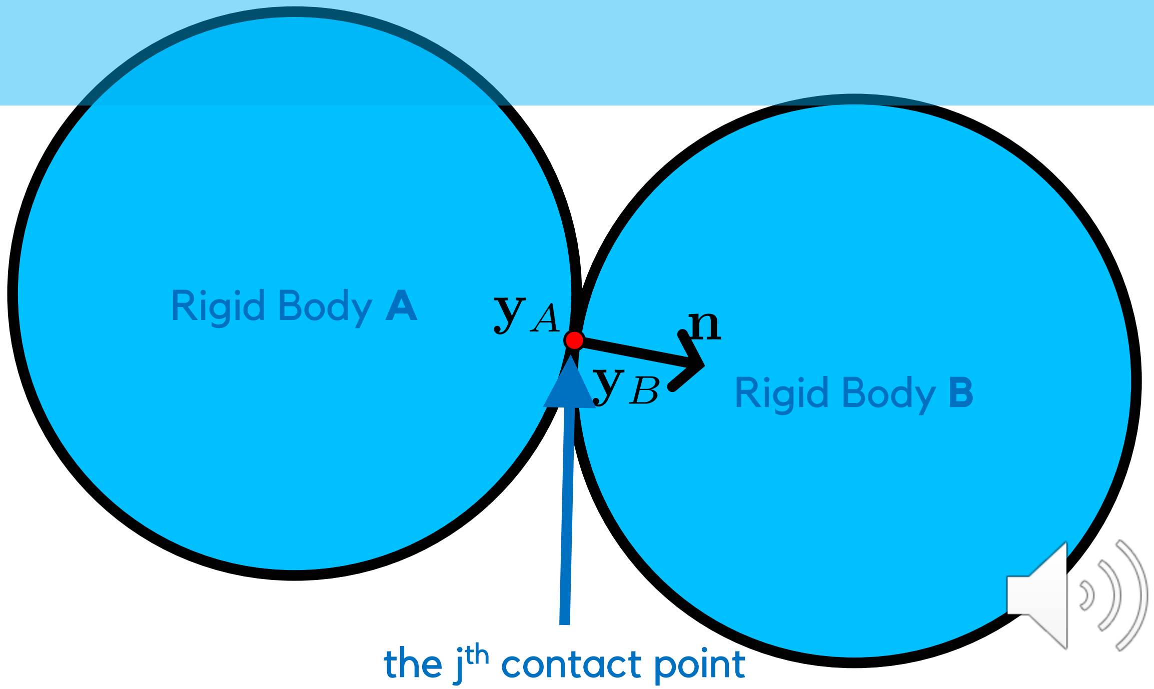
For each contact point

$$\alpha >= 0$$

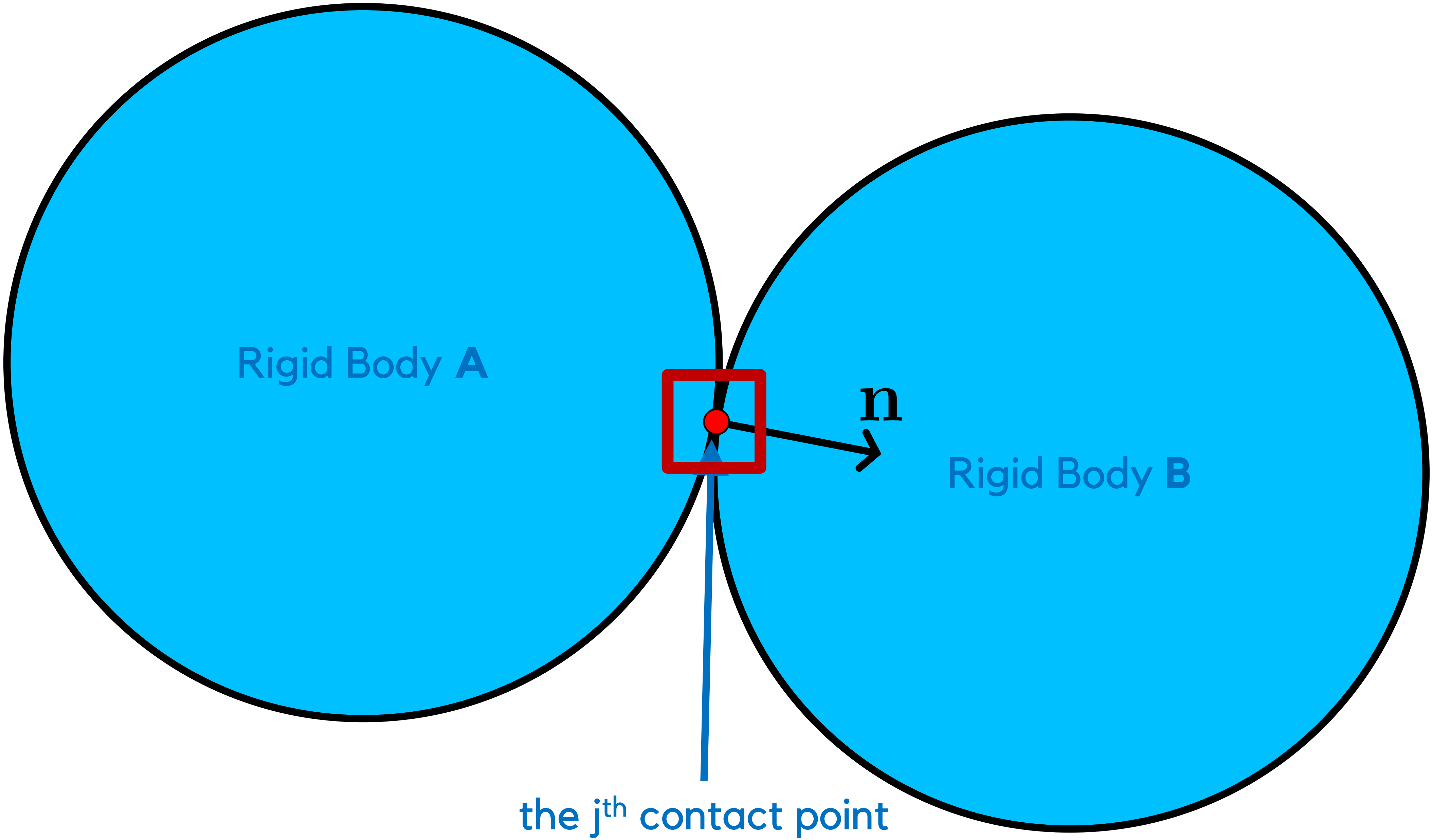
$$d\left(\mathbf{y}_A^t + \Delta t \dot{\mathbf{y}}_A^{t+1}, \mathbf{y}_B^t + \Delta t \dot{\mathbf{y}}_B^{t+1}\right) \geq 0$$

Linearize

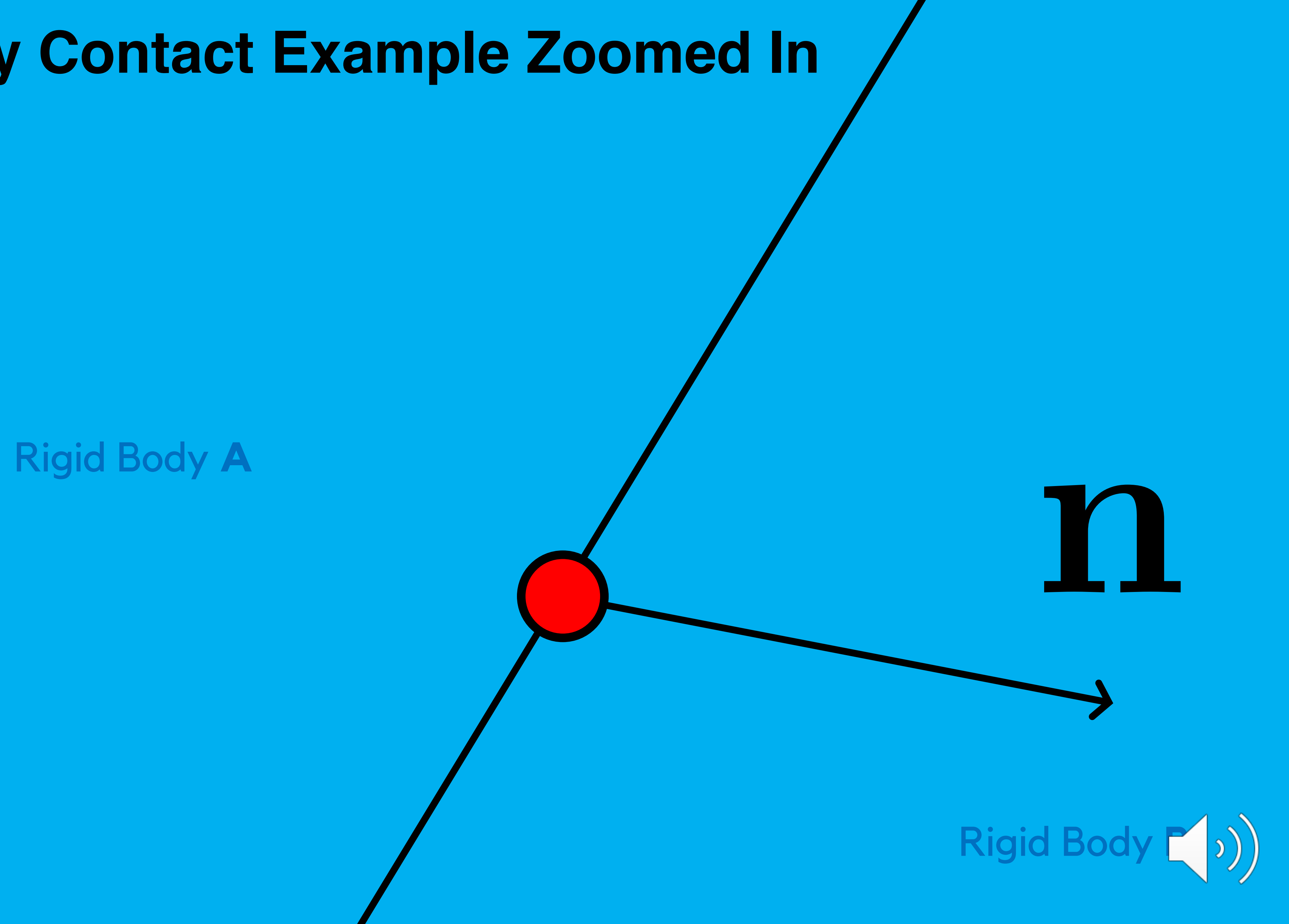
$$d\left(\mathbf{y}_A^t + \Delta t \dot{\mathbf{y}}_A^{t+1}, \mathbf{y}_B^t + \Delta t \dot{\mathbf{y}}_B^{t+1}\right) \perp \alpha$$



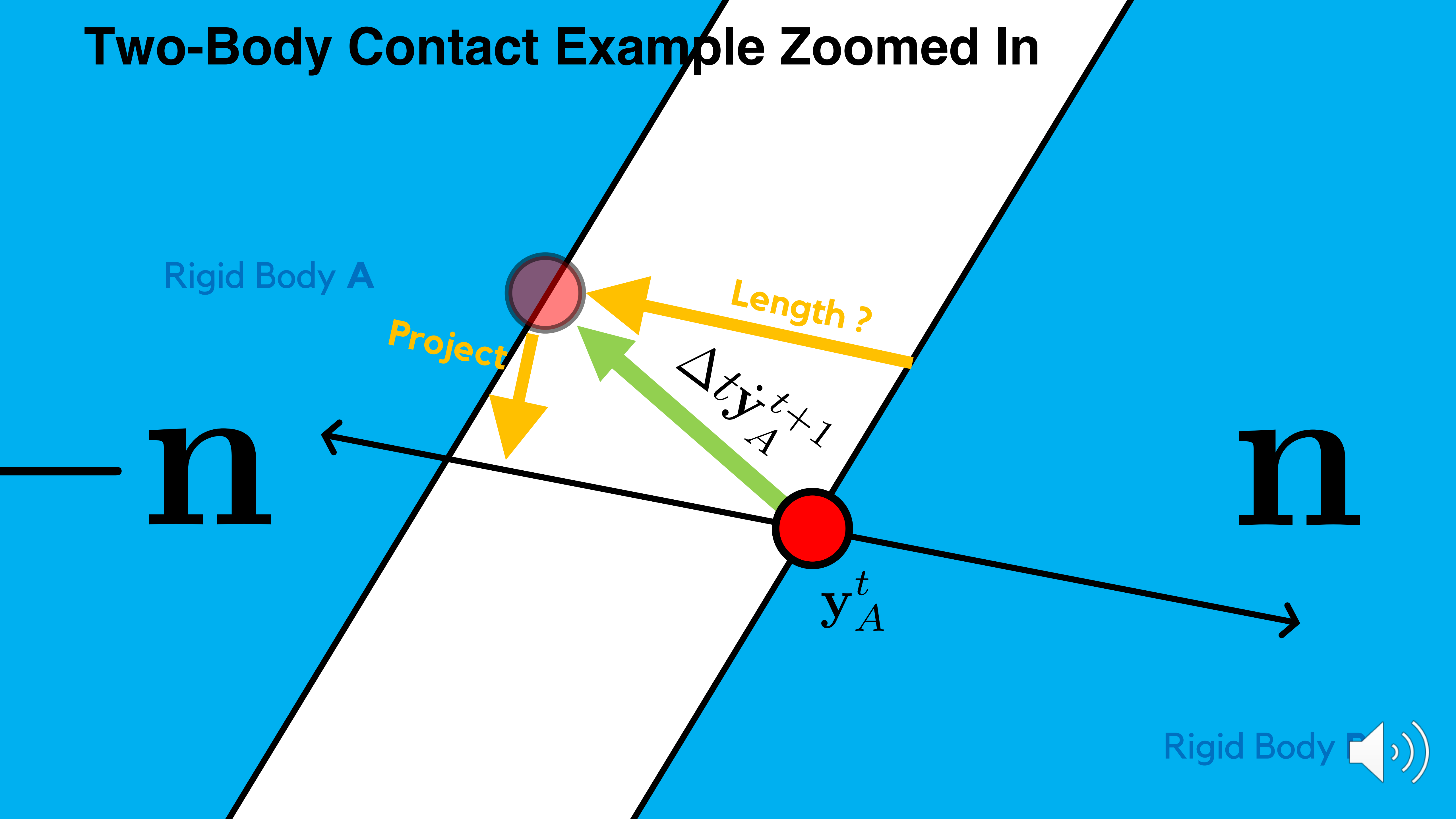
Two-Body Contact Example



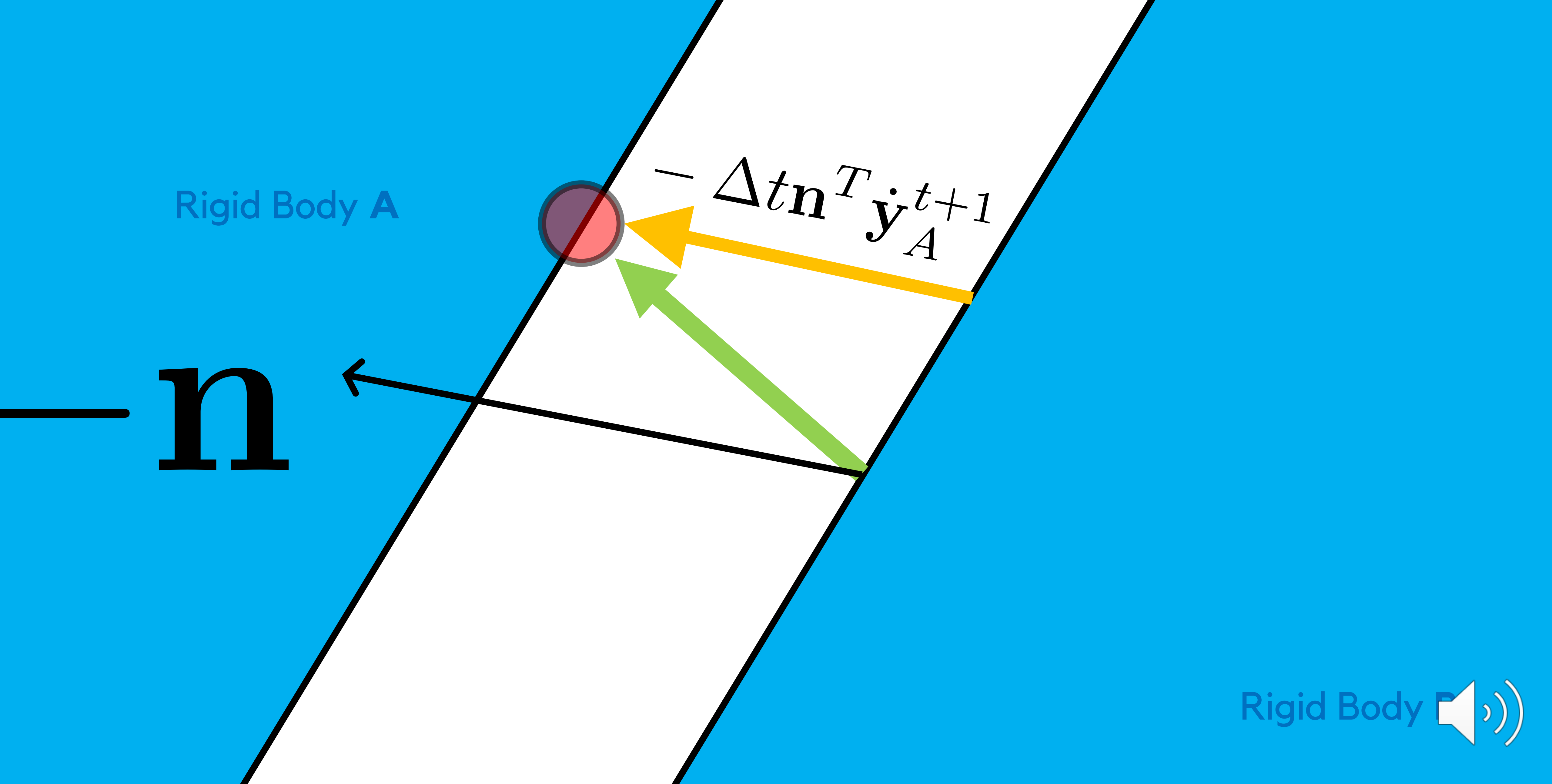
Two-Body Contact Example Zoomed In



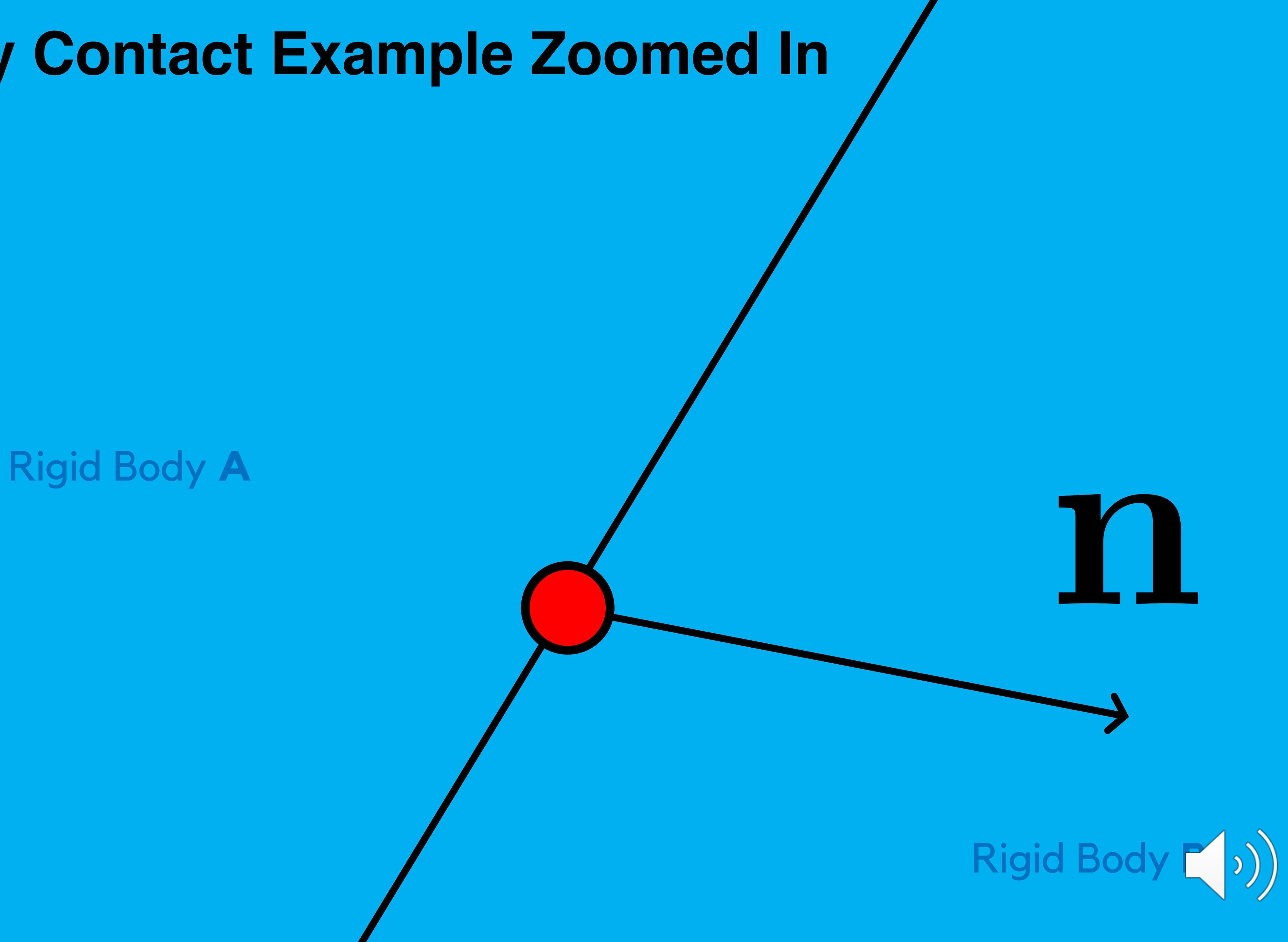
Two-Body Contact Example Zoomed In



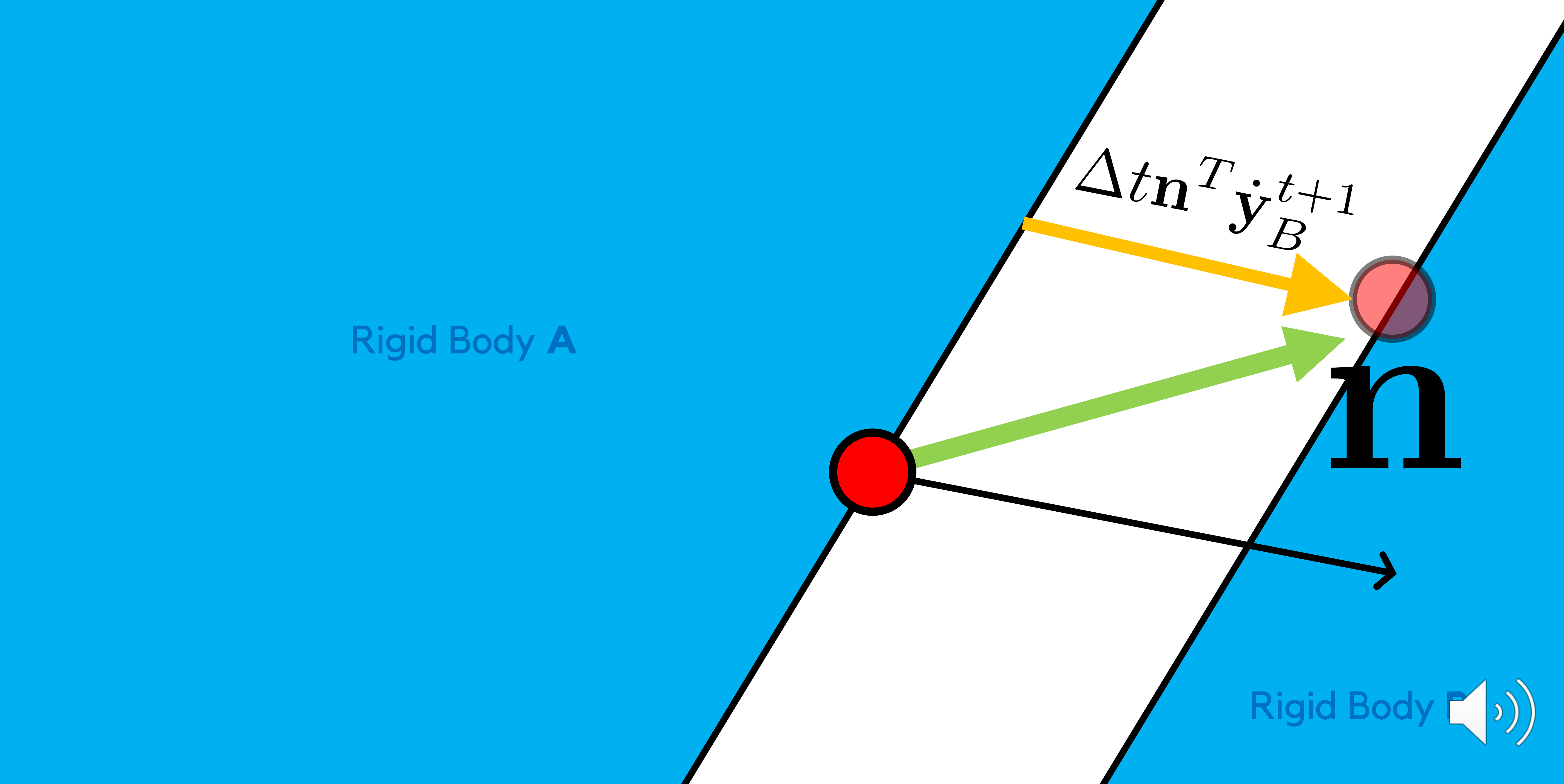
Two-Body Contact Example Zoomed In



Two-Body Contact Example Zoomed In



Two-Body Contact Example Zoomed In



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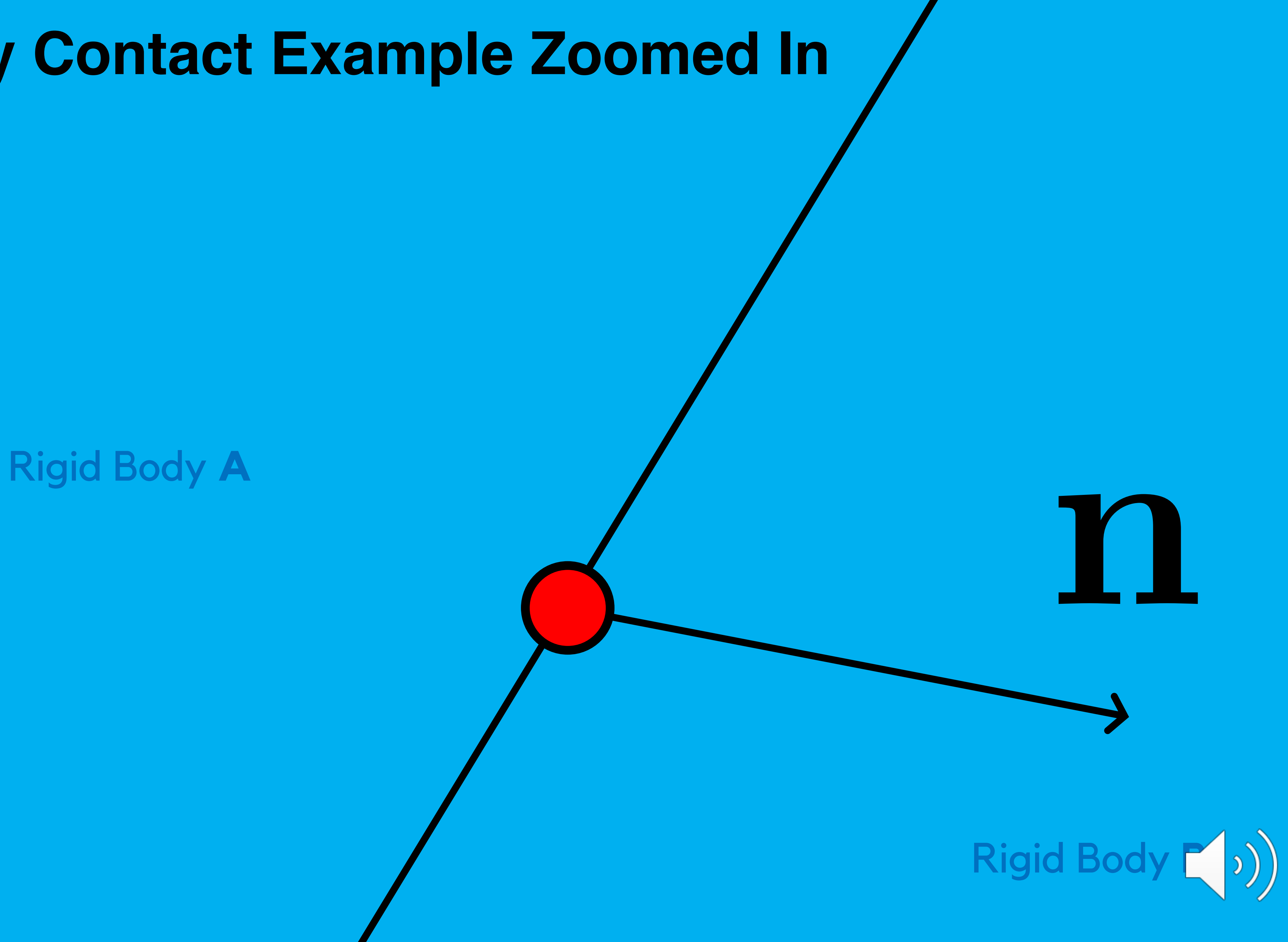

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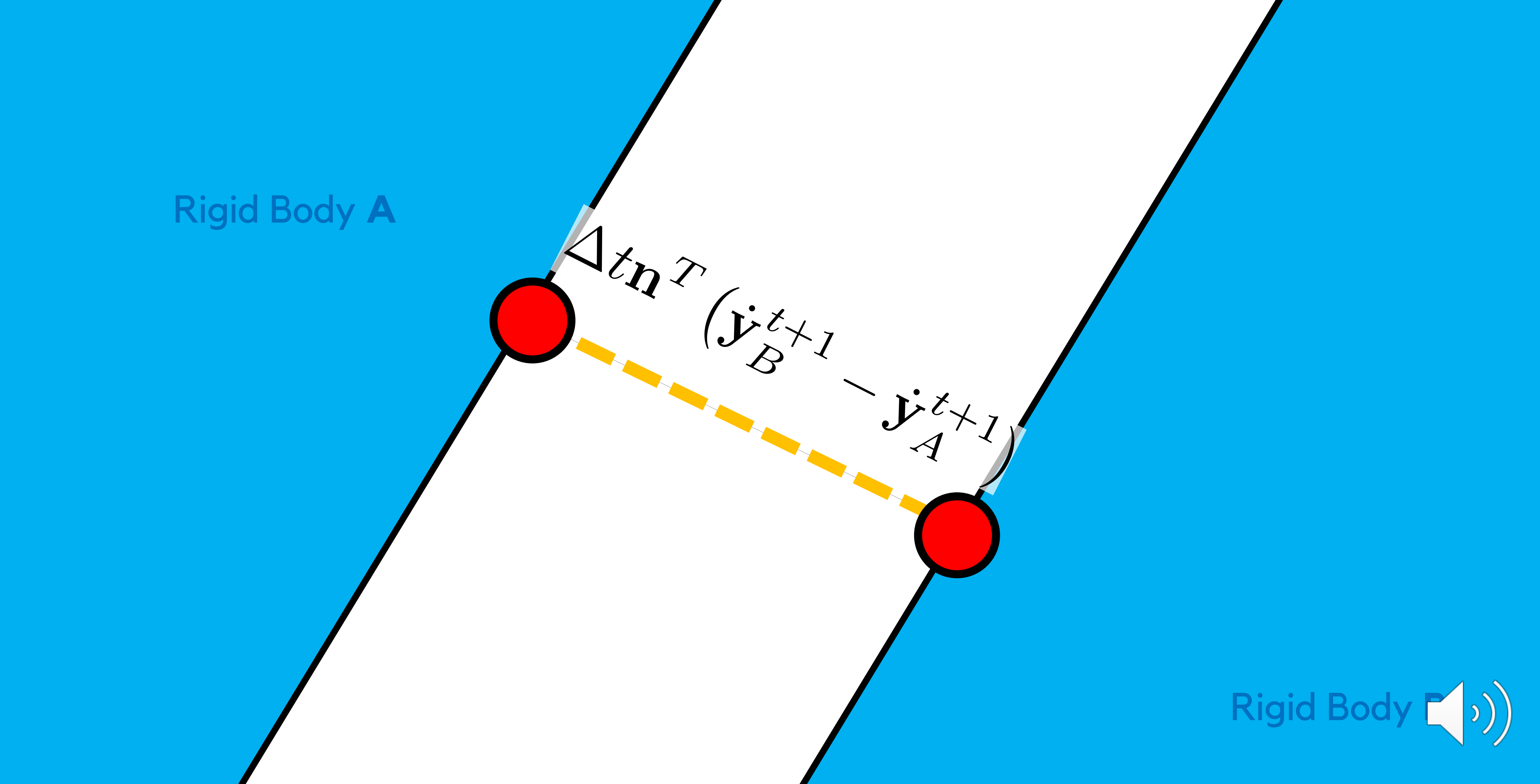
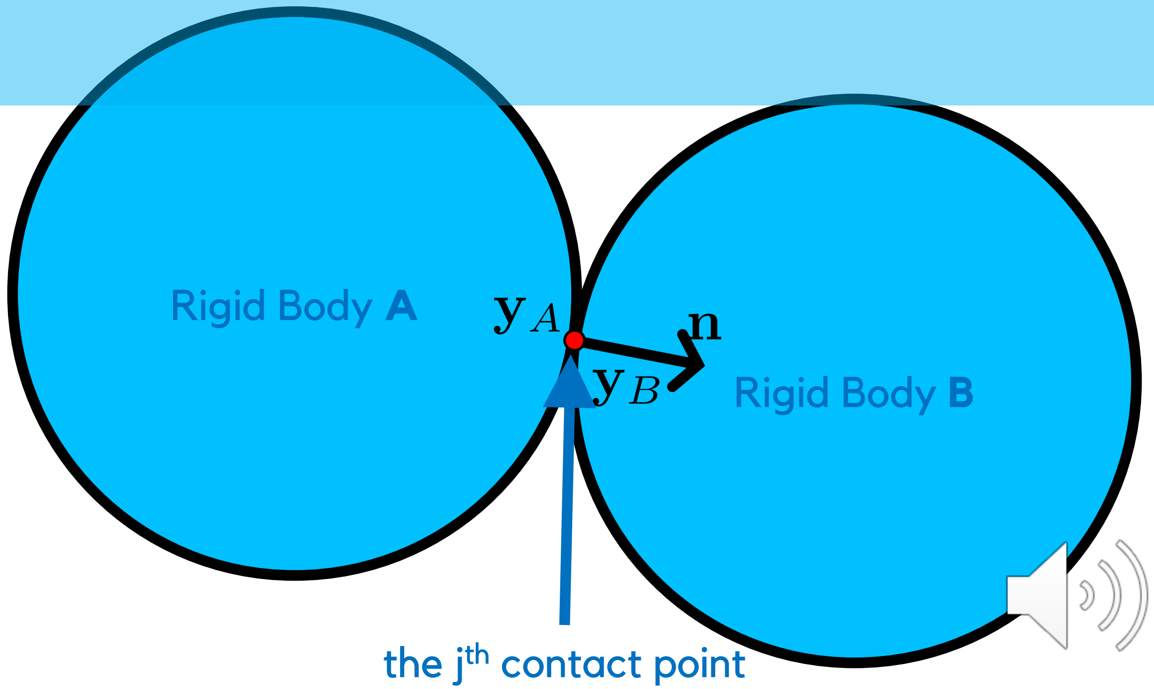


Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t\mathbf{f}^{t} + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right]\Delta t\right)\mathrm{R}^{t}$$

$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t\dot{\mathbf{p}}^{t}$$



For each contact point

$$\alpha >= 0$$

$$d\left(\mathbf{y}_A^t + \Delta t\dot{\mathbf{y}}_A^{t+1}, \mathbf{y}_B^t + \Delta t\dot{\mathbf{y}}_B^{t+1}\right) \geq 0$$

$$d\left(\mathbf{y}_A^t + \Delta t\dot{\mathbf{y}}_A^{t+1}, \mathbf{y}_B^t + \Delta t\dot{\mathbf{y}}_B^{t+1}\right) \perp \alpha$$

Discrete Equations of Motion

$$\mathrm{M}\dot{\mathbf{q}}^{t+1} = \mathrm{M}\dot{\mathbf{q}}^{t} + \Delta t \mathbf{f}^{t} + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}\left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{R}^{t+1} = \exp\left(\left[\omega\right] \Delta t\right) \mathrm{R}^{t}$$

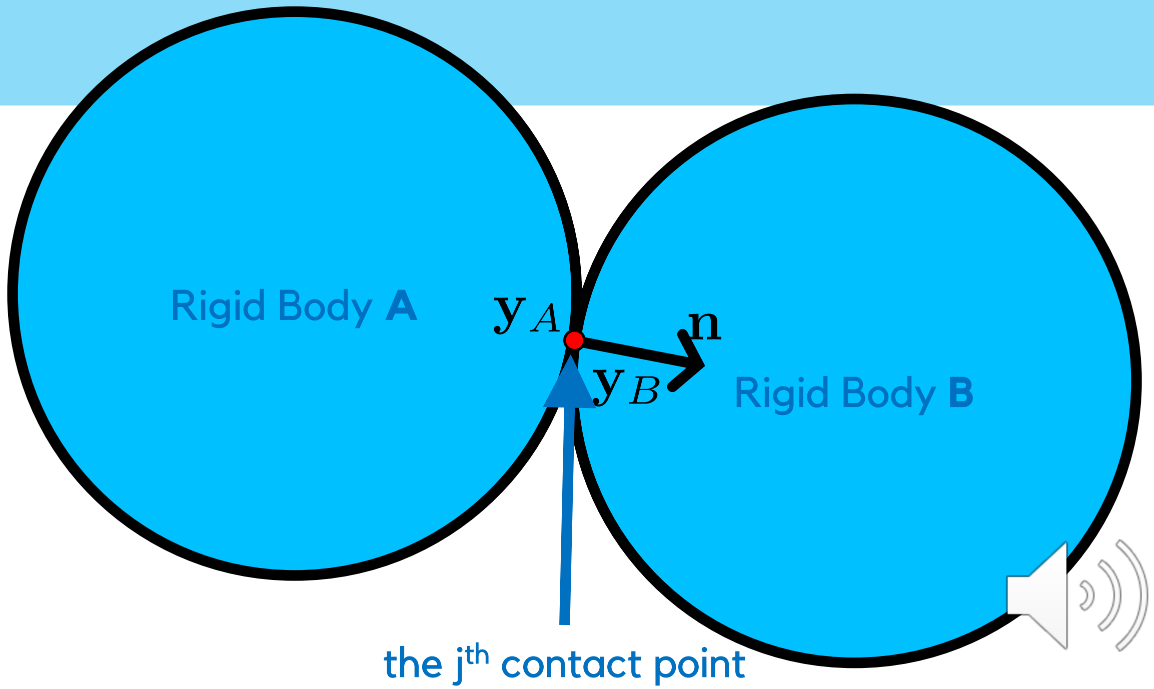
$$\mathbf{p}^{t+1} = \mathbf{p}^{t} + \Delta t \dot{\mathbf{p}}^{t}$$

For each contact point

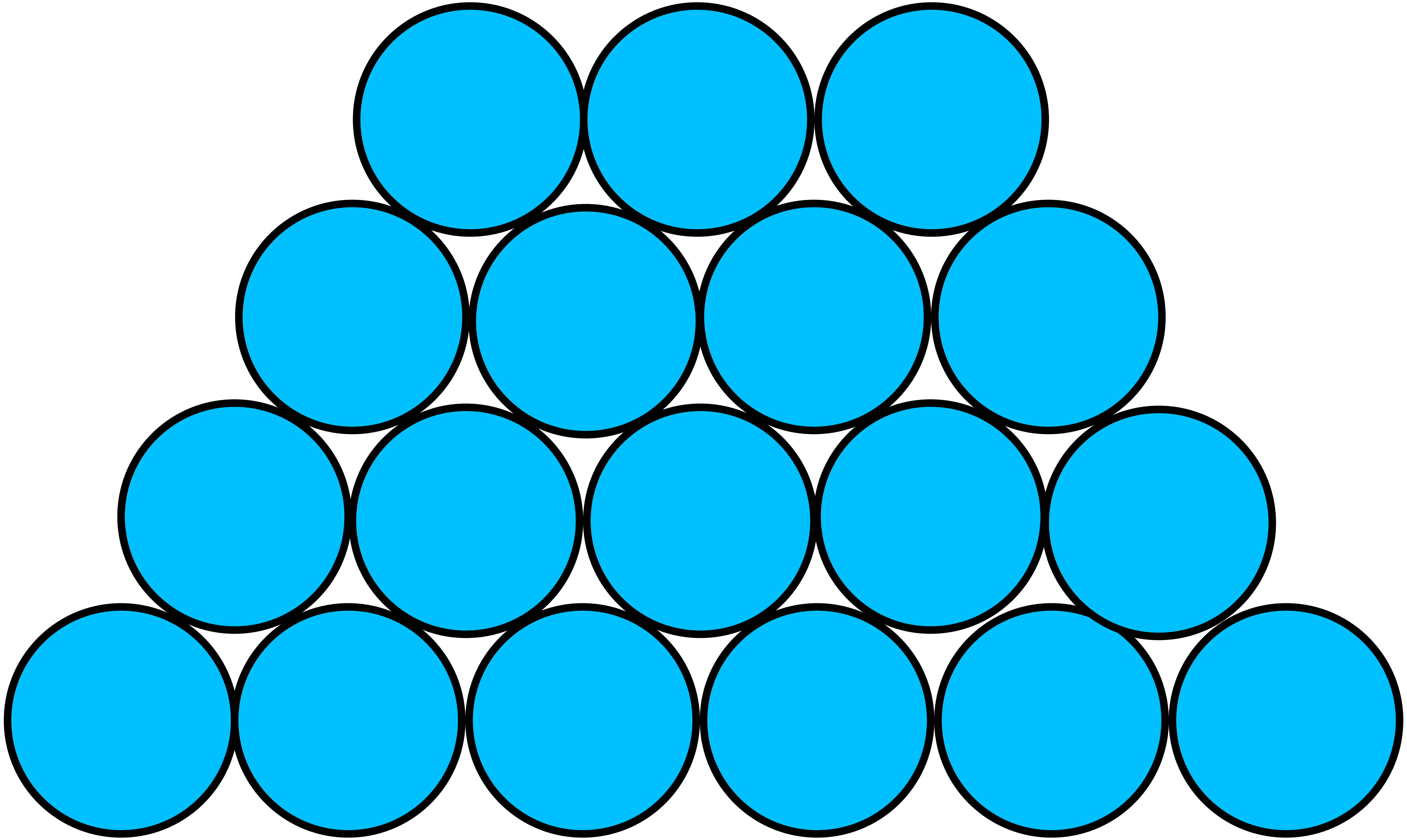
$$\alpha >= 0$$

$$\mathbf{n}^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \geq 0$$

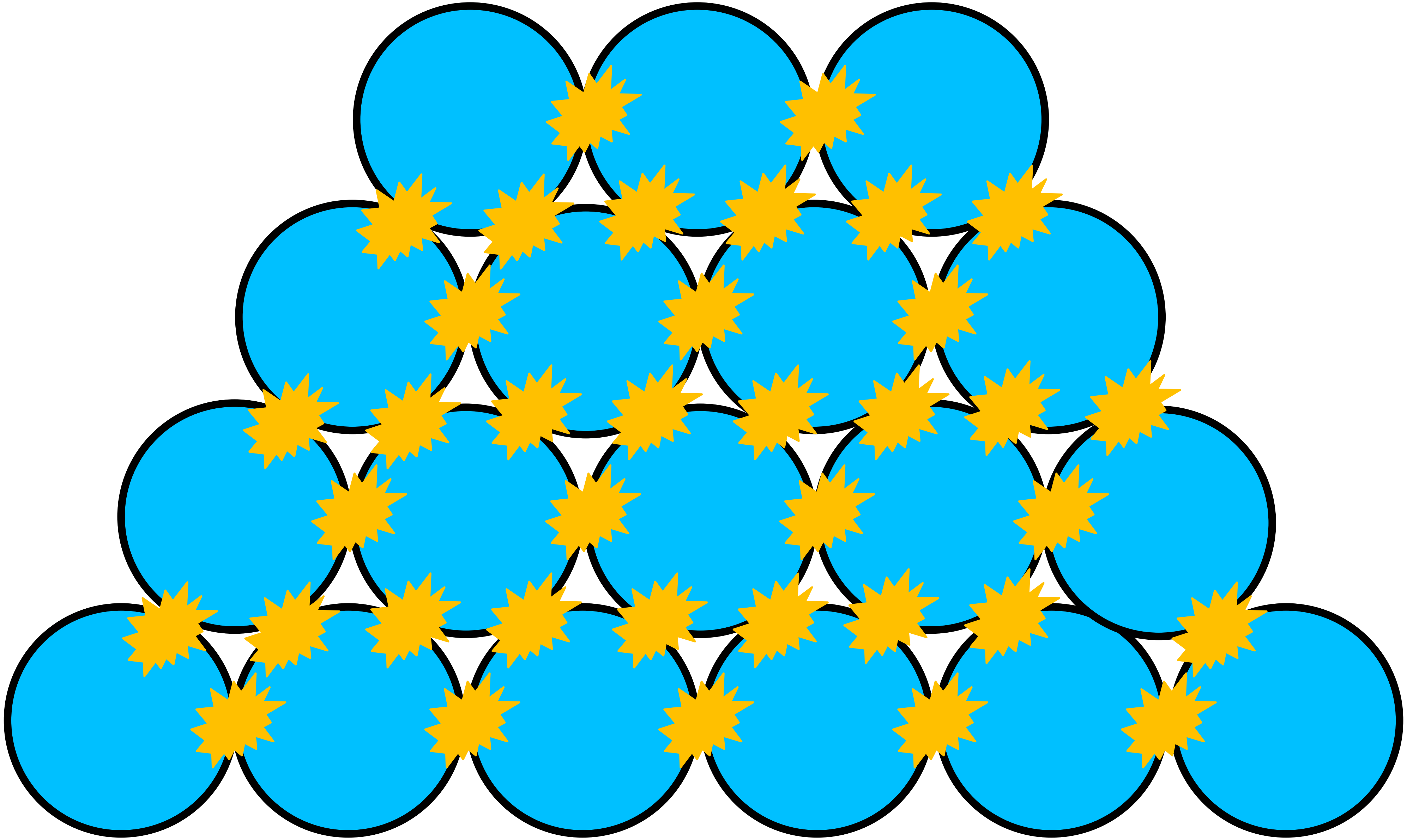
$$\mathbf{n}^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \perp \alpha$$



A More Complicated Example



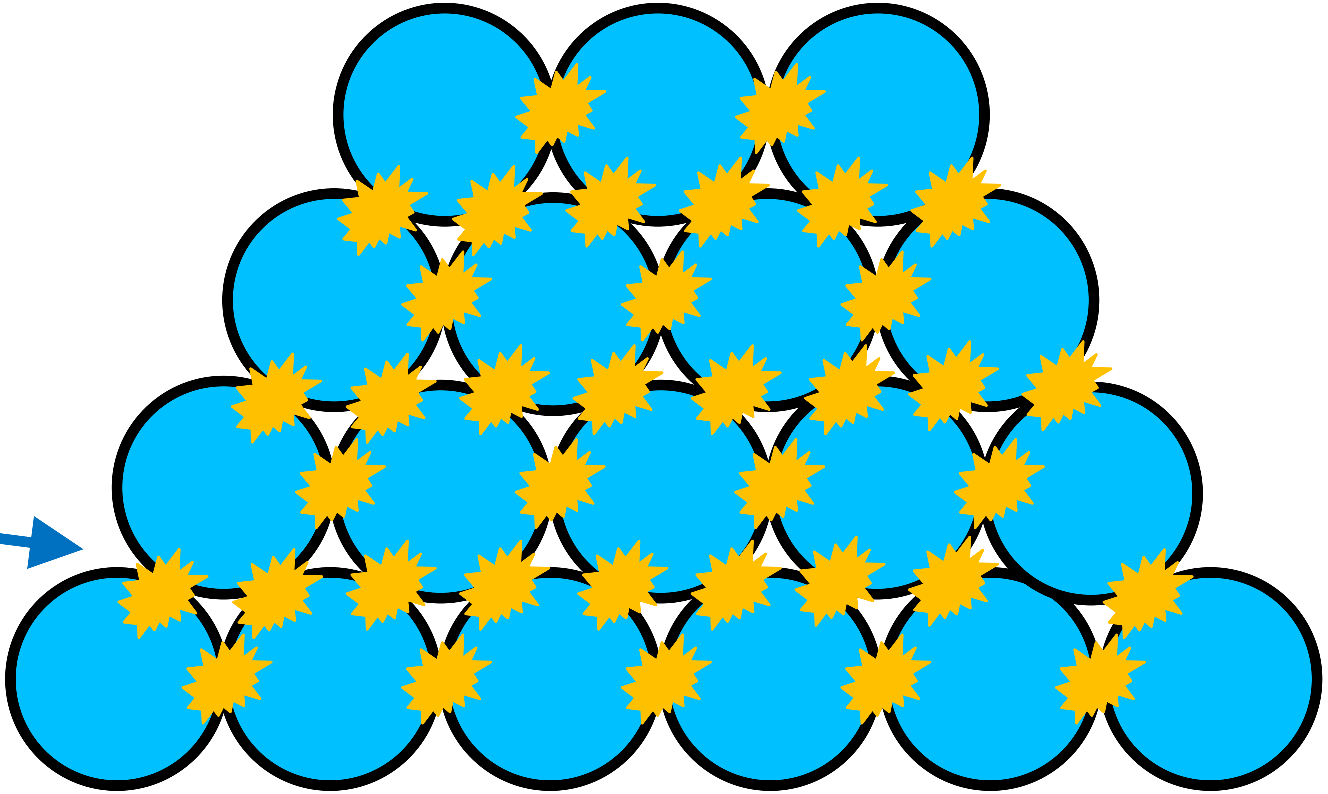
A More Complicated Example



Detect Collisions

A More Complicated Example

For i^{th} contact	
Object A ID	
Object B ID	
Contact Normal	$\mathbf{n}_i$
Contact Point	$\mathbf{y}_i$



Solving the Contact Problem

Dynamics Equations

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_A \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

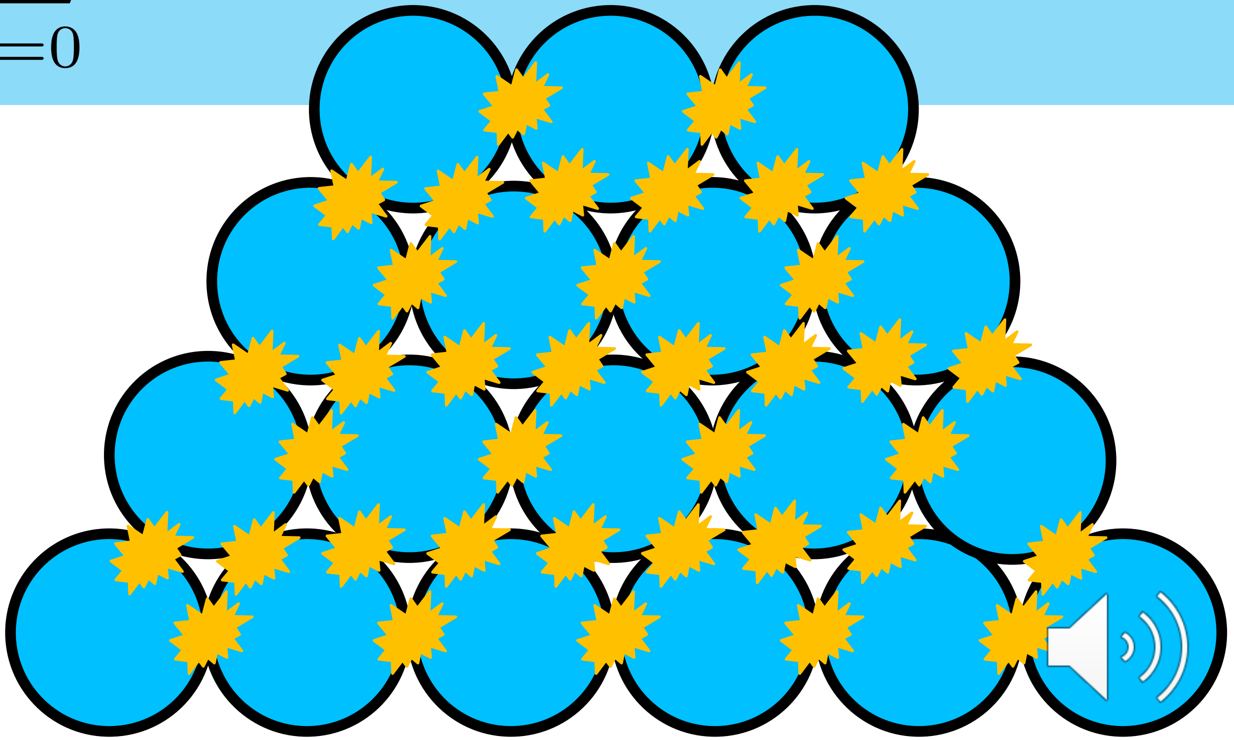
$$\mathrm{M}_B \dot{\mathbf{q}}_B^{t+1} = \mathrm{M}_B \dot{\mathbf{q}}_B^{t} + \Delta t \mathbf{f}_B^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_B \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

For each contact point

$$\alpha >= 0$$

$$\mathbf{n}^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \geq 0$$

$$\mathbf{n}^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \perp \alpha$$



Solving the Contact Problem Iteratively

Dynamics Equations

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_A \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{M}_B \dot{\mathbf{q}}_B^{t+1} = \mathrm{M}_B \dot{\mathbf{q}}_B^{t} + \Delta t \mathbf{f}_B^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_B \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

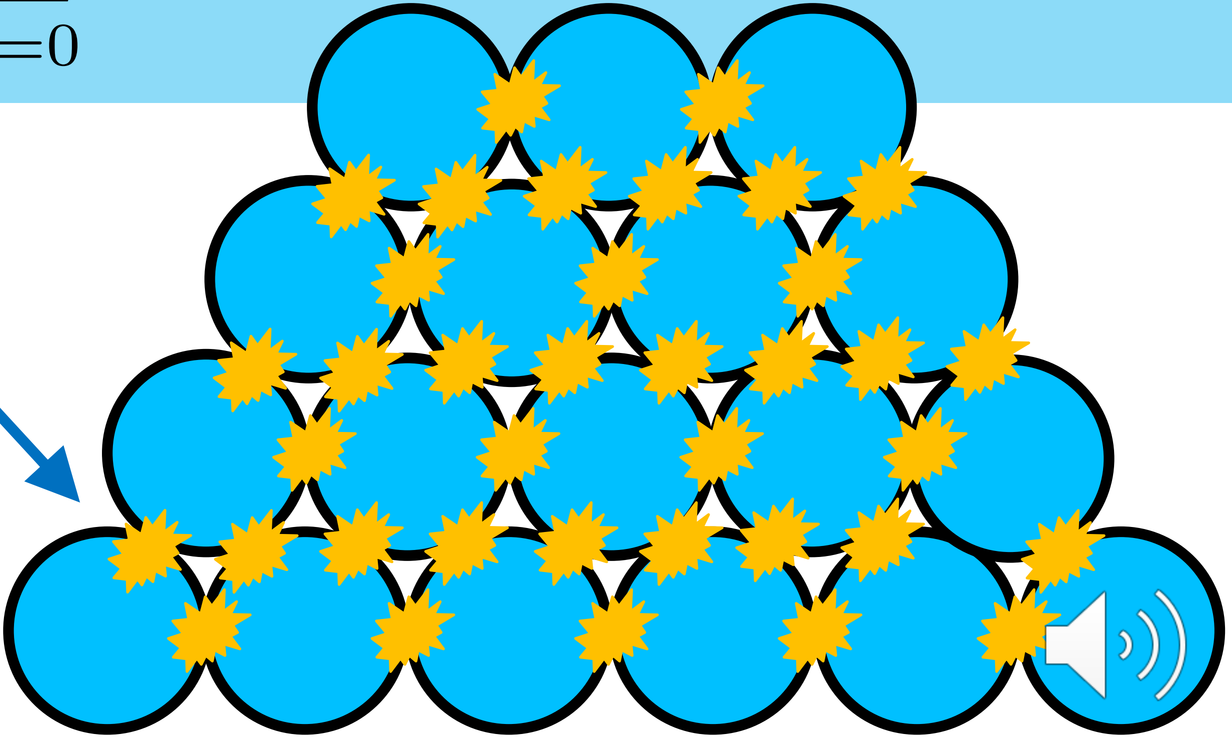
Constraints

$$\alpha_i \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \perp \alpha_i$$

Look Here !



Solving the Contact Problem Iteratively

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \underbrace{\mathrm{J}_A \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j}_{\mathbf{g}_j^A}$$

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \underbrace{\left(\sum_{j \neq i}^{n_c-1} \alpha_j \mathbf{g}_j^A\right)}_{\text{All other contacts}} + \underbrace{\alpha_i \Delta t \mathbf{g}_i^A}_{\text{Current contact}}$$

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \underbrace{\mathbf{f}_i^A}_{\text{Forces from all other contacts}} + \alpha_i \Delta t \mathbf{g}_i^A$$

Solving the Contact Problem Iteratively

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \mathbf{f}_i^A + \alpha_i \Delta t \mathbf{g}_i^A$$

Forces from all other contacts

$$\dot{\mathbf{q}}_A^{t+1} = \underbrace{\dot{\mathbf{q}}_A^{t} + \Delta t \mathrm{M}_A^{-1} \mathbf{f}_A^t}_{\text{Unconstrained velocity } \dot{\mathbf{q}}_A^*} + \underbrace{\Delta t \mathrm{M}_A^{-1} \mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1} \mathbf{g}_i^A}_{\text{Change due to contact}}$$

Solving the Contact Problem Iteratively

Dynamics Equations

$$\mathrm{M}_A \dot{\mathbf{q}}_A^{t+1} = \mathrm{M}_A \dot{\mathbf{q}}_A^{t} + \Delta t \mathbf{f}_A^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_A \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

$$\mathrm{M}_B \dot{\mathbf{q}}_B^{t+1} = \mathrm{M}_B \dot{\mathbf{q}}_B^{t} + \Delta t \mathbf{f}_B^t + \Delta t \sum_{j=0}^{n_c-1} \alpha_j \mathrm{J}_B \left(\mathbf{y}_j\right)^T \hat{\mathbf{n}}_j$$

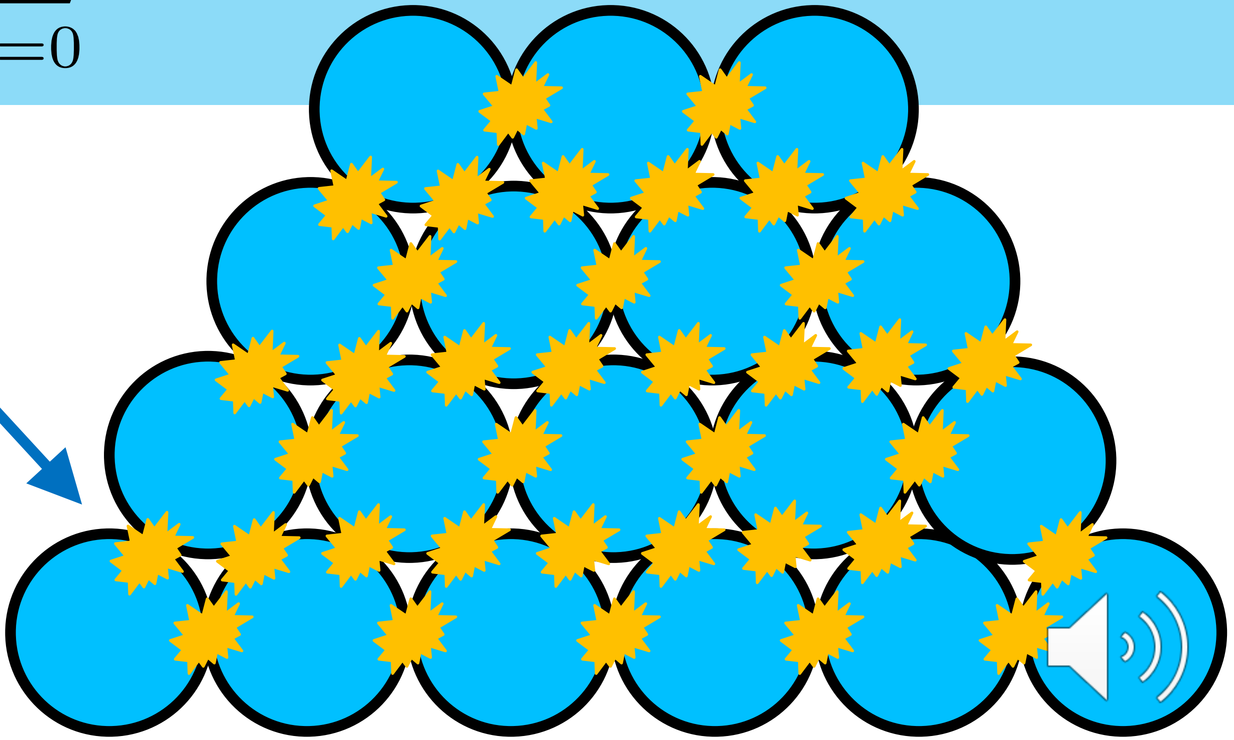
Constraints

$$\alpha_i \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \perp \alpha_i$$

Look Here !



Solving the Contact Problem Iteratively

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^* + \Delta t \mathrm{M}_A^{-1} \mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1} \mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^* + \Delta t \mathrm{M}_B^{-1} \mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1} \mathbf{g}_i^B$$

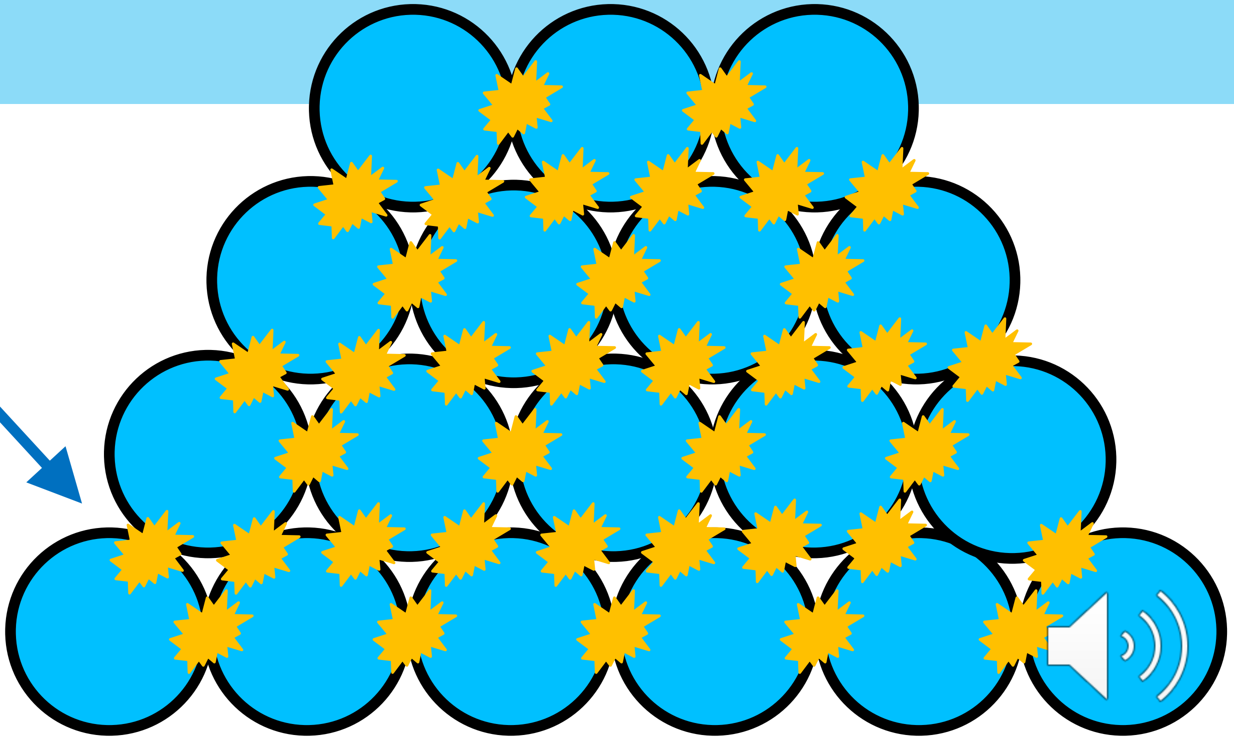
Constraints

$$\alpha_i \geq 0$$

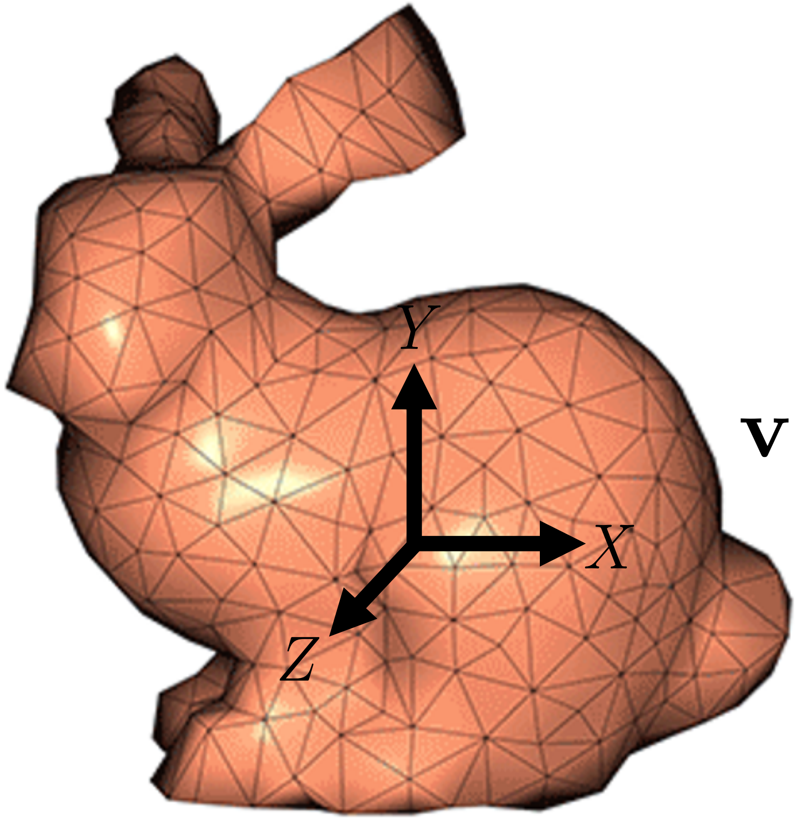
$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1} \right) \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1} \right) \perp \alpha_i$$

Look Here !



The Rigid Body Jacobian



$$\mathbf{v}\left(\bar{\mathbf{X}}, t\right) = \underbrace{\mathrm{R}\begin{pmatrix}[\bar{\mathbf{X}}]^T & \mathrm{I}\end{pmatrix}\begin{pmatrix}\mathrm{R}^T & 0 \\ 0 & \mathrm{R}^T\end{pmatrix}}_{\substack{\text{Jacobian} \\ \mathrm{J} \in \mathbb{R}^{3\times 6}}} \underbrace{\begin{pmatrix}\omega \\ \dot{\mathbf{p}}\end{pmatrix}}_{\dot{\mathbf{q}} \in \mathbb{R}^6}$$

Reference (Undeformed) Space

Solving the Contact Problem Iteratively

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^{*} + \Delta t \mathrm{M}_A^{-1}\mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1}\mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^{*} + \Delta t \mathrm{M}_B^{-1}\mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1}\mathbf{g}_i^B$$

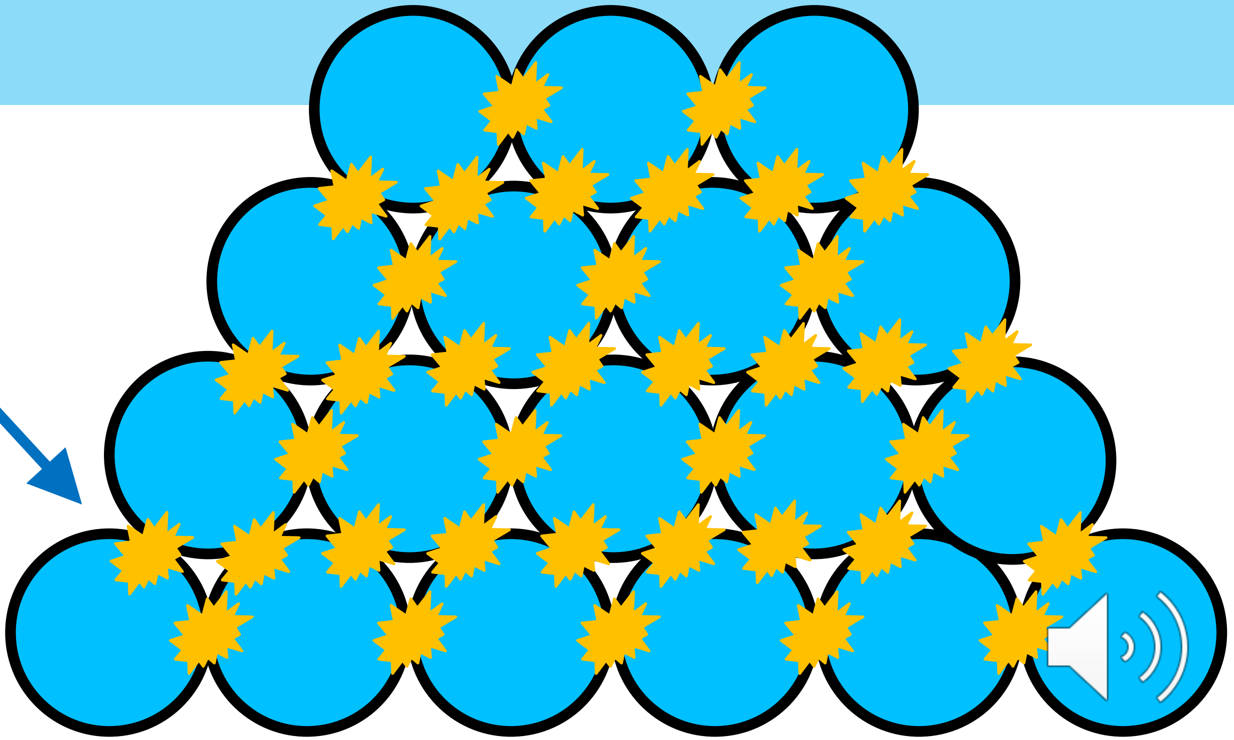
Constraints

$$\alpha_i \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \geq 0$$

$$\mathbf{n}_i^T \left(\dot{\mathbf{y}}_B^{t+1} - \dot{\mathbf{y}}_A^{t+1}\right) \perp \alpha_i$$

Look Here !



Solving the Contact Problem Iteratively

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^{*} + \Delta t \mathrm{M}_A^{-1}\mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1}\mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^{*} + \Delta t \mathrm{M}_B^{-1}\mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1}\mathbf{g}_i^B$$

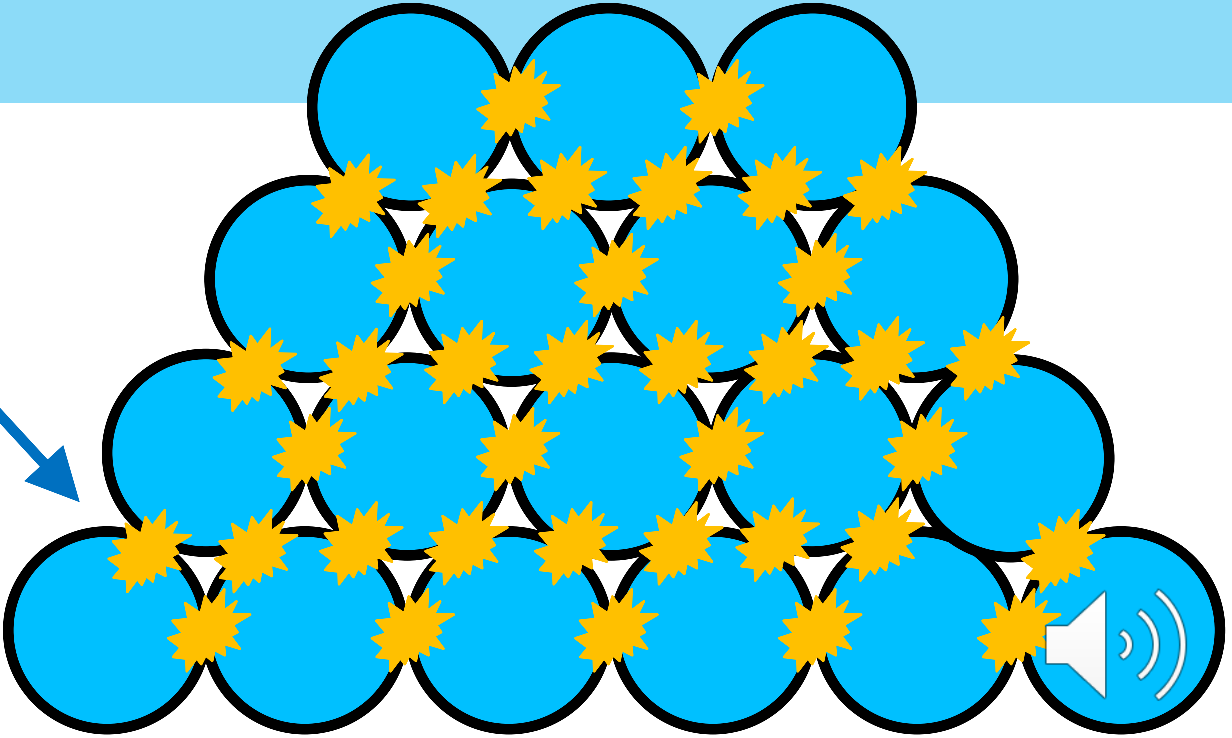
Constraints

$$\alpha_i \geq 0$$

$$\mathbf{n}_i^T \left(\mathrm{J}_B(\mathbf{y})\,\dot{\mathbf{q}}_B^{t+1} - \mathrm{J}_A(\mathbf{y})\,\dot{\mathbf{q}}_A^{t+1}\right) \geq 0$$

$$\mathbf{n}_i^T \left(\mathrm{J}_B(\mathbf{y})\,\dot{\mathbf{q}}_B^{t+1} - \mathrm{J}_A(\mathbf{y})\,\dot{\mathbf{q}}_A^{t+1}\right) \perp \alpha_i$$

Look Here !



Solving the Contact Problem Iteratively

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^* + \Delta t \mathrm{M}_A^{-1} \mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1} \mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^* + \Delta t \mathrm{M}_B^{-1} \mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1} \mathbf{g}_i^B$$

$$\mathbf{n}_i^T \left(\mathrm{J}_B \left(\mathbf{y} \right) \dot{\mathbf{q}}_B^{t+1} - \mathrm{J}_A \left(\mathbf{y} \right) \dot{\mathbf{q}}_A^{t+1} \right)$$

$$\left(\mathbf{g}_i^B \right)^T \dot{\mathbf{q}}_B^{t+1} + \left(\mathbf{g}_i^A \right)^T \dot{\mathbf{q}}_A^{t+1}$$

$$\underbrace{\Delta t \left[\left(\mathbf{g}_i^B \right)^T \mathrm{M}_B^{-1} \mathbf{g}_i^B + \left(\mathbf{g}_i^A \right)^T \mathrm{M}_A^{-1} \mathbf{g}_i^A \right]}_{\delta_i} \alpha_i + \underbrace{\left(\mathbf{g}_i^B \right)^T \left(\dot{\mathbf{q}}_B^* + \Delta t \mathrm{M}_B^{-1} \mathbf{f}_i^B \right) + \left(\mathbf{g}_i^A \right)^T \left(\dot{\mathbf{q}}_A^* + \Delta t \mathrm{M}_A^{-1} \mathbf{f}_i^A \right)}_{\gamma_i}$$

Solving the Contact Problem Iteratively

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^* + \Delta t \mathrm{M}_A^{-1} \mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1} \mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^* + \Delta t \mathrm{M}_B^{-1} \mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1} \mathbf{g}_i^B$$

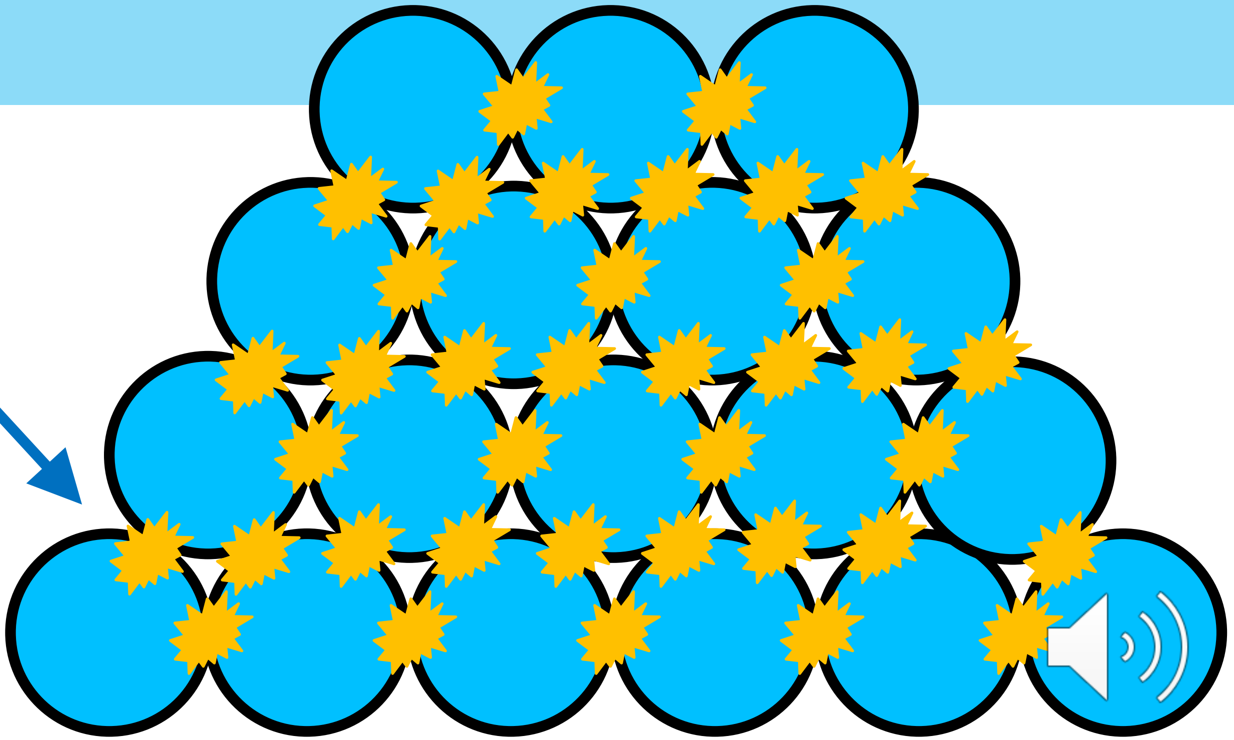
Constraints

$$\alpha_i \geq 0$$

$$\mathbf{n}_i^T \left(\mathrm{J}_B(\mathbf{y}) \dot{\mathbf{q}}_B^{t+1} - \mathrm{J}_A(\mathbf{y}) \dot{\mathbf{q}}_A^{t+1} \right) \geq 0$$

$$\mathbf{n}_i^T \left(\mathrm{J}_B(\mathbf{y}) \dot{\mathbf{q}}_B^{t+1} - \mathrm{J}_A(\mathbf{y}) \dot{\mathbf{q}}_A^{t+1} \right) \perp \alpha_i$$

Look Here !



Solving the Contact Problem Iteratively

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^{*} + \Delta t \mathrm{M}_A^{-1}\mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1}\mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^{*} + \Delta t \mathrm{M}_B^{-1}\mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1}\mathbf{g}_i^B$$

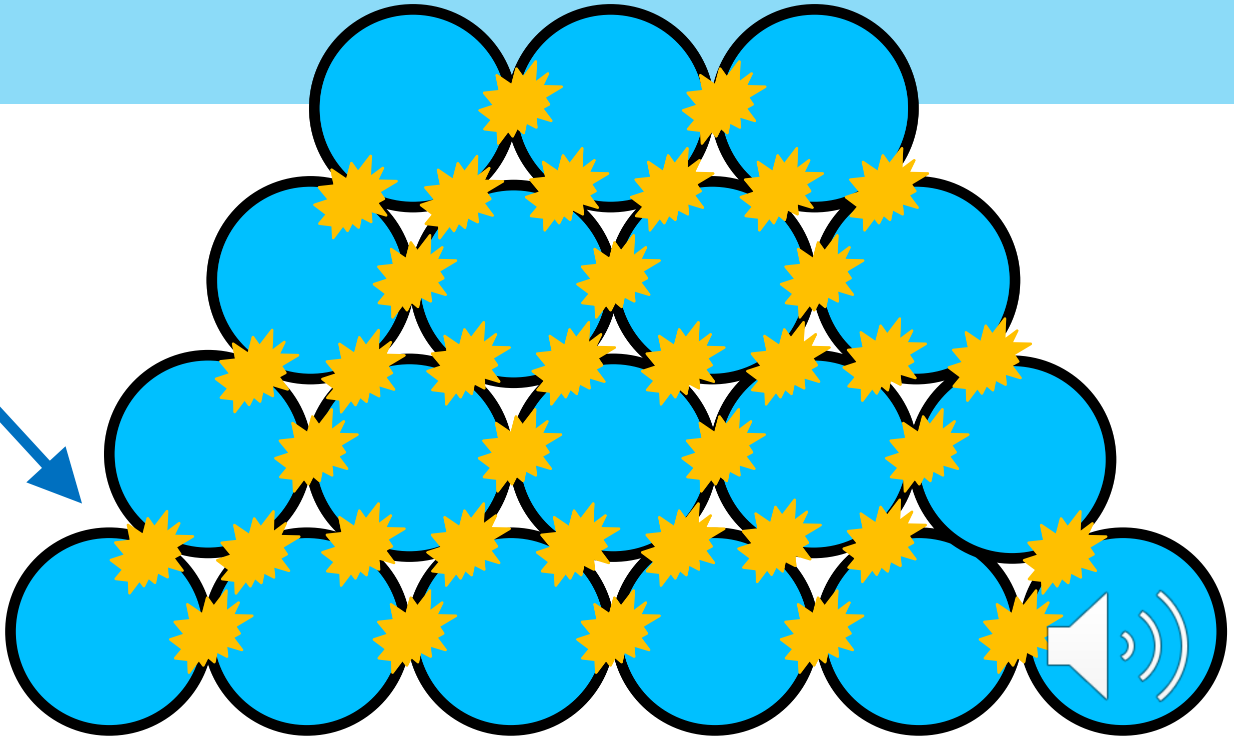
Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

$$\delta_i \alpha_i + \gamma \perp \alpha_i$$

Look Here !



Solving the Single Point Contact Problem

Dynamics Equations

$$\dot{\mathbf{q}}_A^{t+1} = \dot{\mathbf{q}}_A^* + \Delta t \mathrm{M}_A^{-1}\mathbf{f}_i^A + \alpha_i \Delta t \mathrm{M}_A^{-1}\mathbf{g}_i^A$$

$$\dot{\mathbf{q}}_B^{t+1} = \dot{\mathbf{q}}_B^* + \Delta t \mathrm{M}_B^{-1}\mathbf{f}_i^B + \alpha_i \Delta t \mathrm{M}_B^{-1}\mathbf{g}_i^B$$

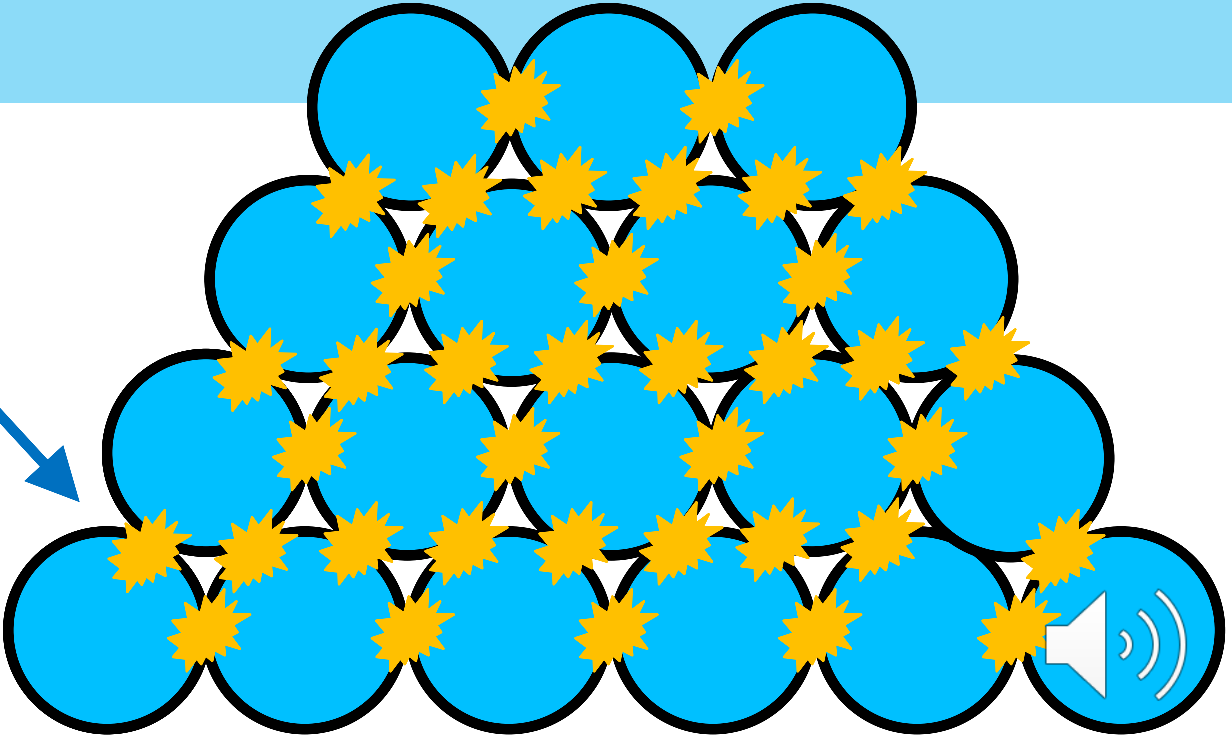
Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$

Look Here !



Solving the Single Point Contact Problem

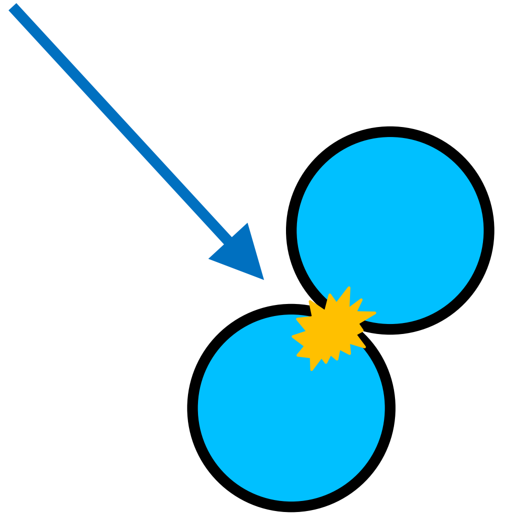
Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$

Look Here !



Assume $\delta_i \alpha_i + \gamma_i = 0 \rightarrow \alpha_i = \dfrac{-\gamma_i}{\delta_i} \rightarrow$ $\alpha_i \geq 0$

Yes → $\delta_i \alpha_i + \gamma_i = 0$, $\delta_i \alpha_i + \gamma_i \perp \alpha_i$

No, $\alpha_i = 0$ → $\delta_i \alpha_i + \gamma_i \geq 0$?, $\delta_i \alpha_i + \gamma_i \perp \alpha_i$

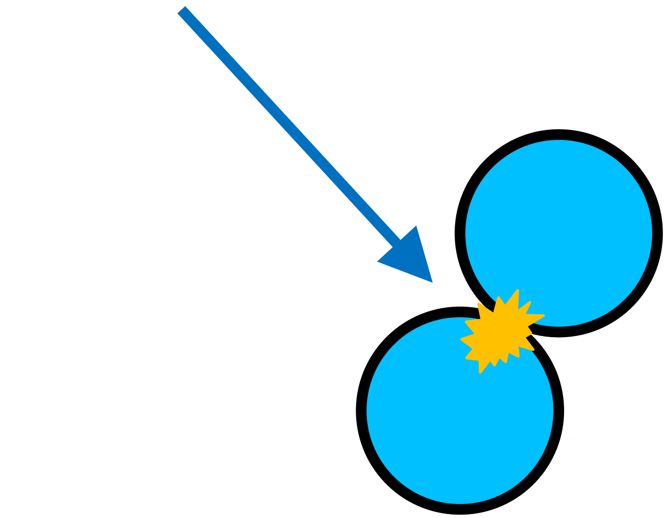
Solving the Single Point Contact Problem

Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$



If $\alpha_i = \dfrac{-\gamma_i}{\delta_i} < 0$ then $\gamma_i > 0$

So if we set $\alpha_i = 0$ then $\delta_i \alpha_i + \gamma_i = \gamma_i > 0$

Solving the Single Point Contact Problem

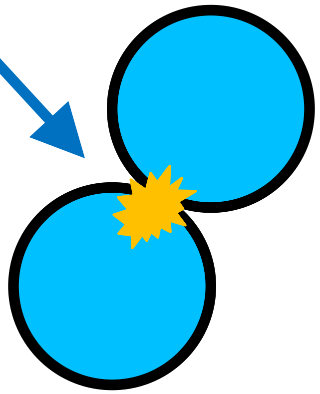
Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$

Look Here !



Assume $\delta_i \alpha_i + \gamma_i = 0 \rightarrow \alpha_i = \dfrac{-\gamma_i}{\delta_i} \rightarrow$ $\alpha_i \geq 0$

Yes $\rightarrow$

$$\delta_i \alpha_i + \gamma_i = 0$$
$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$

No, $\alpha_i = 0$ $\rightarrow$

$$\delta_i \alpha_i + \gamma_i \geq 0 \;?$$
$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$

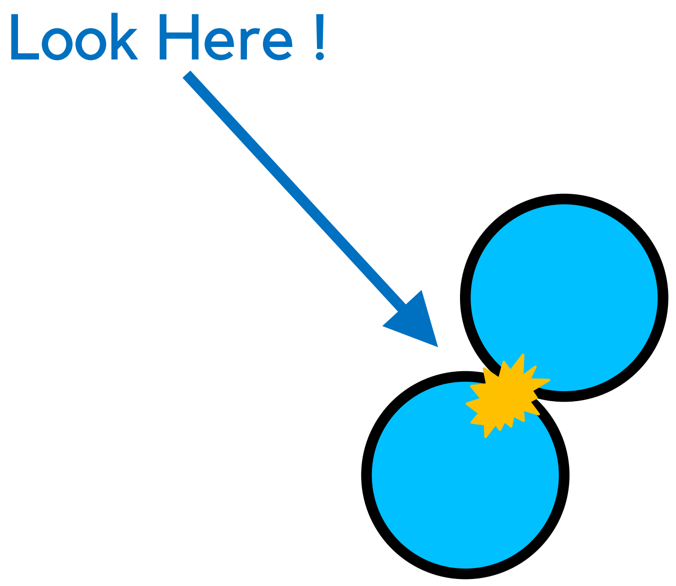
Solving the Single Point Contact Problem

Constraints

$$\alpha_i \geq 0$$

$$\delta_i \alpha_i + \gamma_i \geq 0$$

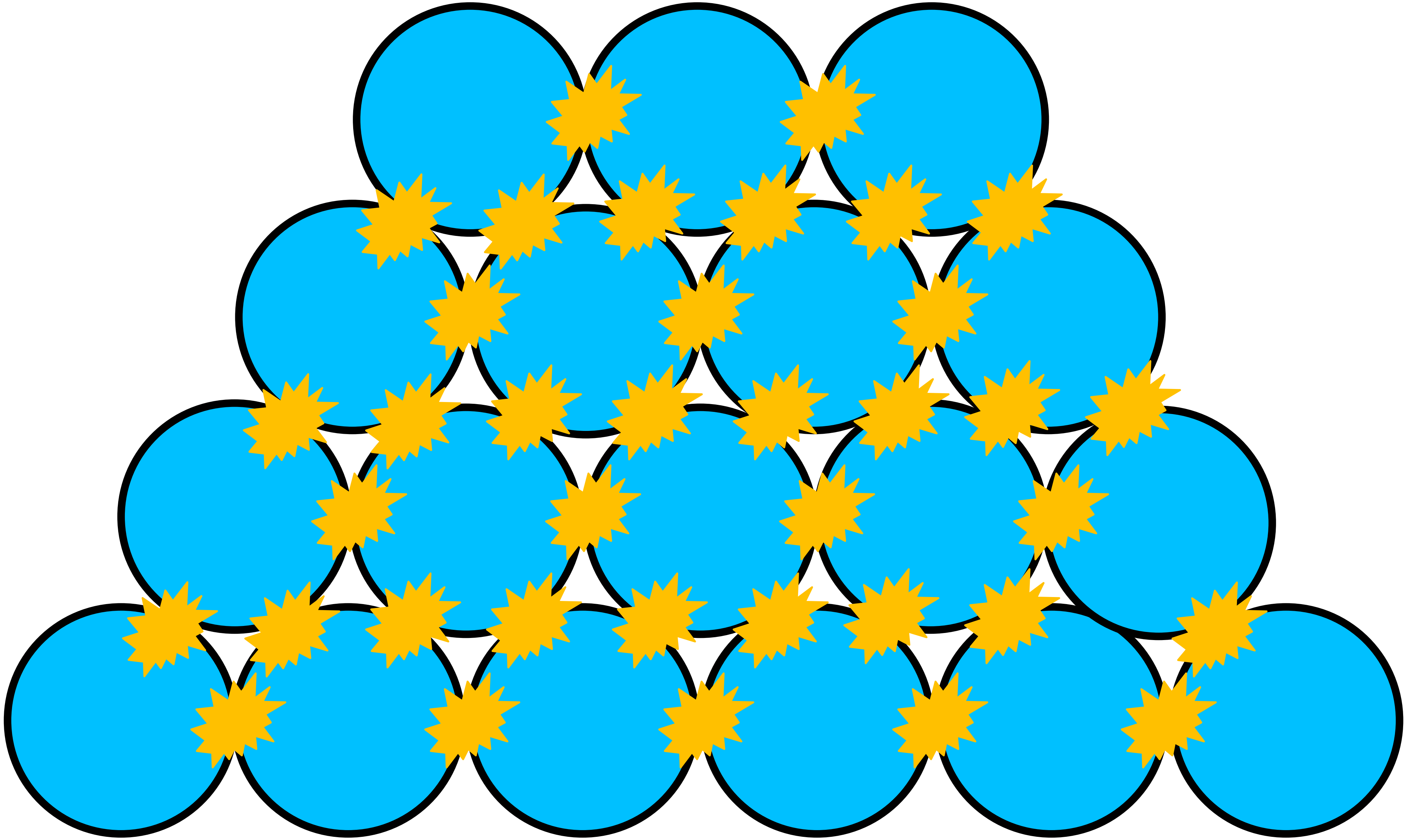
$$\delta_i \alpha_i + \gamma_i \perp \alpha_i$$



Solution

$$\alpha_i = \max\left(\frac{-\gamma_i}{\delta_i}, 0\right)$$

Solving the Contact Problem Iteratively



Solving the Contact Problem Iteratively

Compute $\dot{\mathbf{q}}^*$ for all rigid bodies in the scene

$$k = 0$$
$$\boldsymbol{\alpha} = \mathbf{0}$$

$$\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

While constraints still violated or maximum iterations not reached

$$k = k + 1$$

For $i = 1$ to number of contacts

Compute δ_i and γ_i using α

In-Place Update

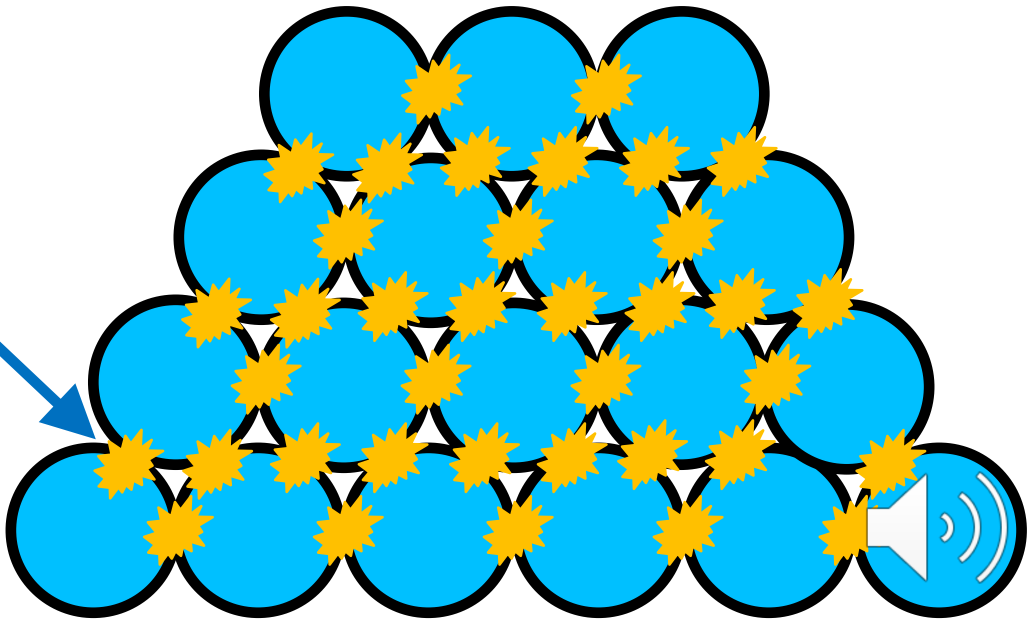
$$\alpha_i = \max\left(\frac{-\gamma_i}{\delta_i}, 0\right)$$

Done

Done

Compute $\dot{\mathbf{q}}^{t+1}$ for all rigid bodies in the scene

Update positions for all rigid bodies in the scene



Projected Gauss-Seidel

Compute $\dot{\mathbf{q}}^*$ for all rigid bodies in the scene

$$k = 0$$
$$\boldsymbol{\alpha} = \mathbf{0}$$

While constraints still violated or maximum iterations not reached

$$k = k + 1$$

For $i = 1$ to number of contacts

Compute δ_i and γ_i using $\boldsymbol{\alpha}$

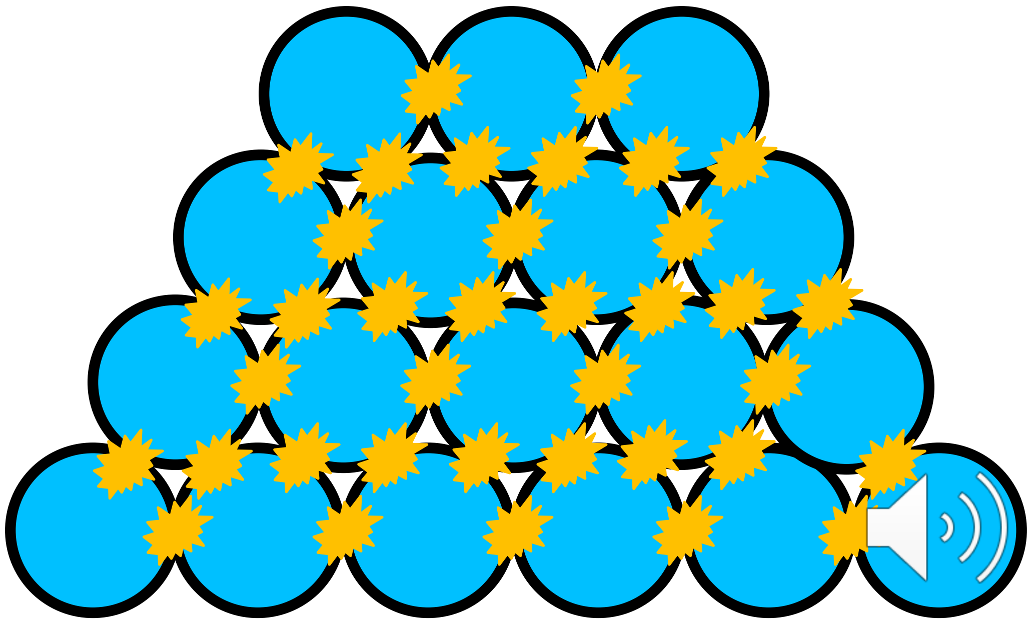
$$\boldsymbol{\alpha}_i = \max\left(\frac{-\gamma_i}{\delta_i}, 0\right)$$

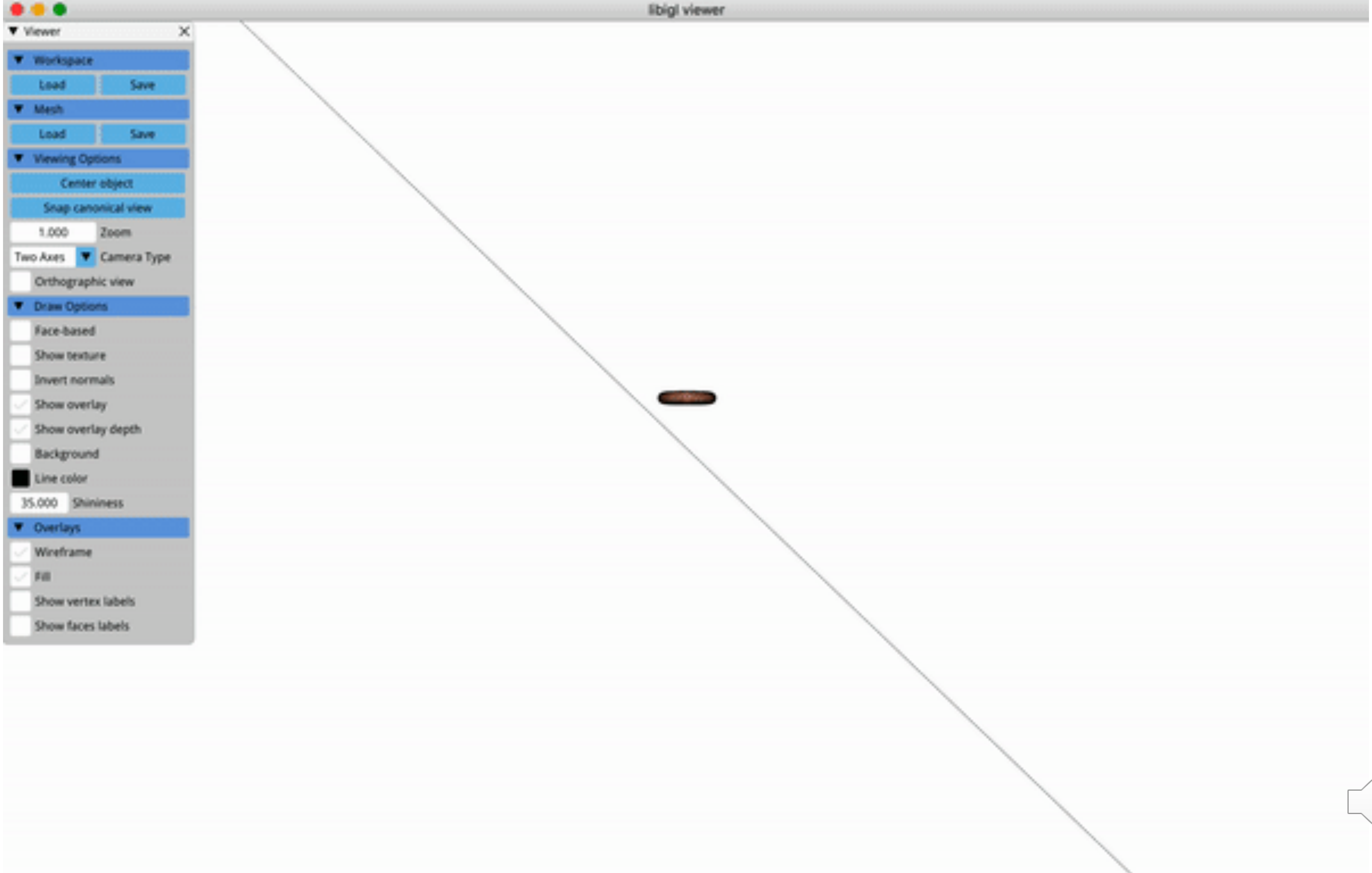
Done

Done

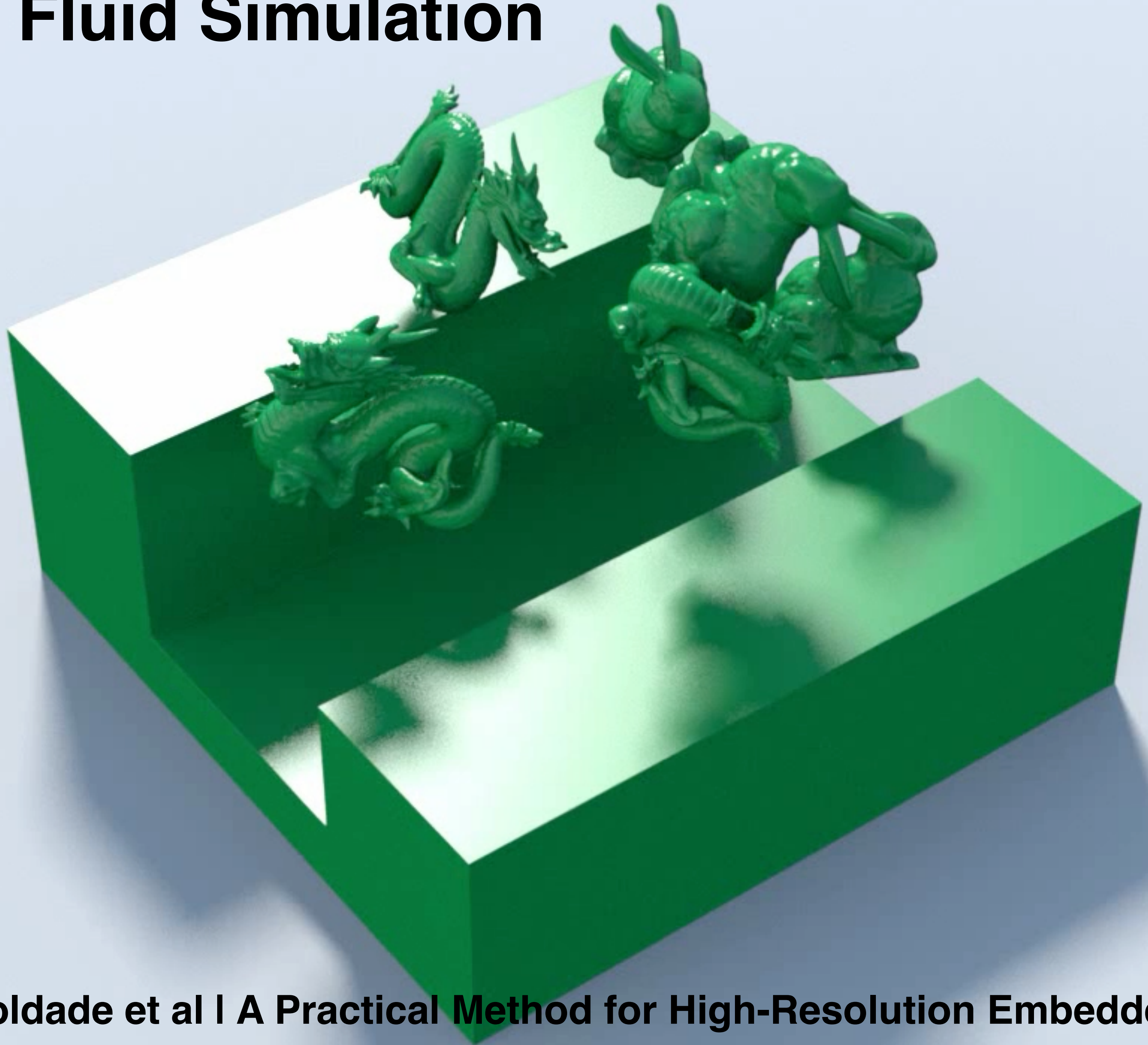
Compute $\dot{\mathbf{q}}^{t+1}$ for all rigid bodies in the scene

Update positions for all rigid bodies in the scene




libigl viewer
Viewer
Workspace
Load
Save
Mesh
Load
Save
Viewing Options
Center object
Snap canonical view
1.000
Zoom
Two Axes
Camera Type
Orthographic view
Draw Options
Face-based
Show texture
Invert normals
Show overlay
Show overlay depth
Background
Line color
35.000
Shininess
Overlays
Wireframe
Fill
Show vertex labels
Show faces labels

Next Video: Fluid Simulation



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