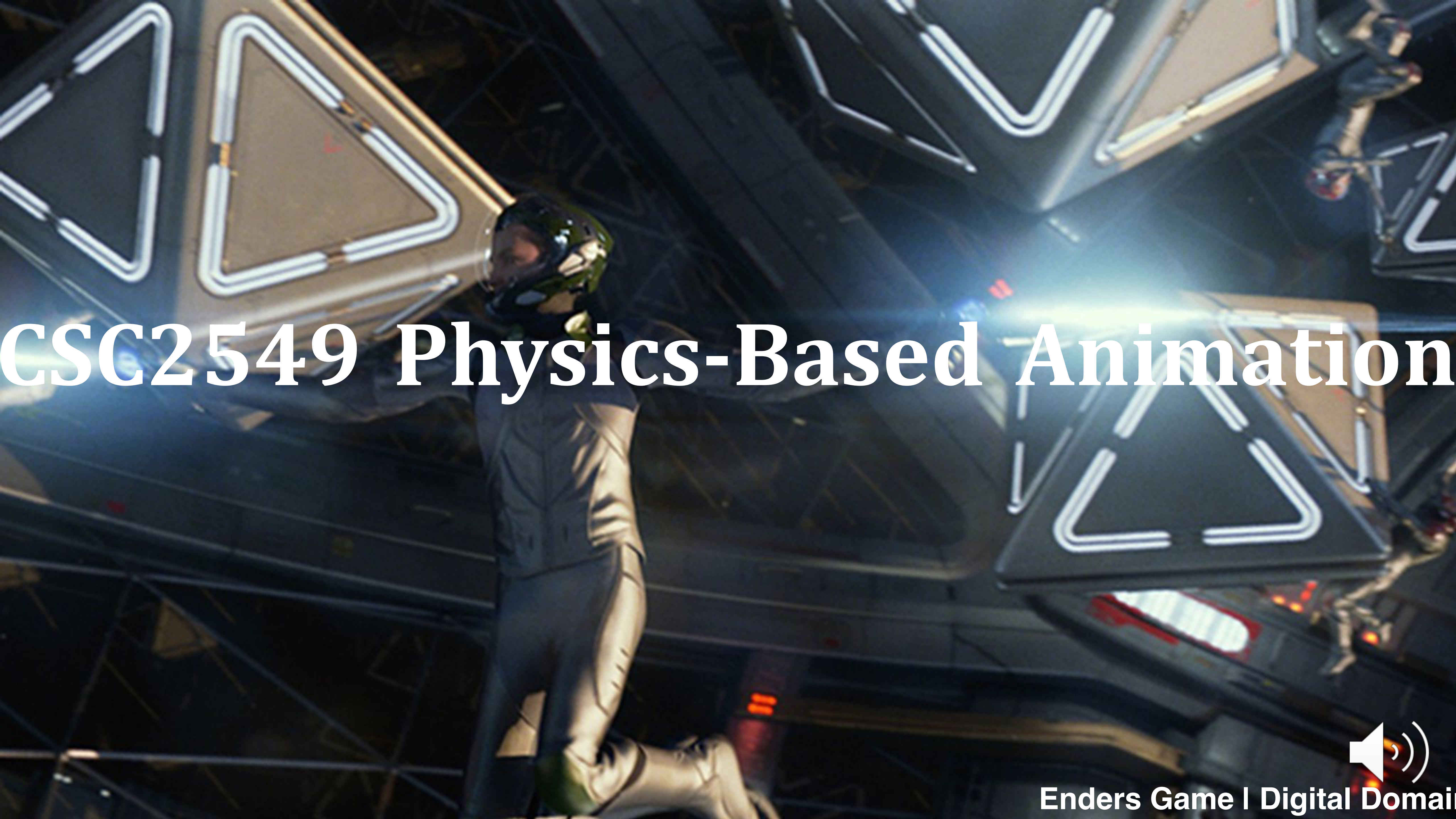
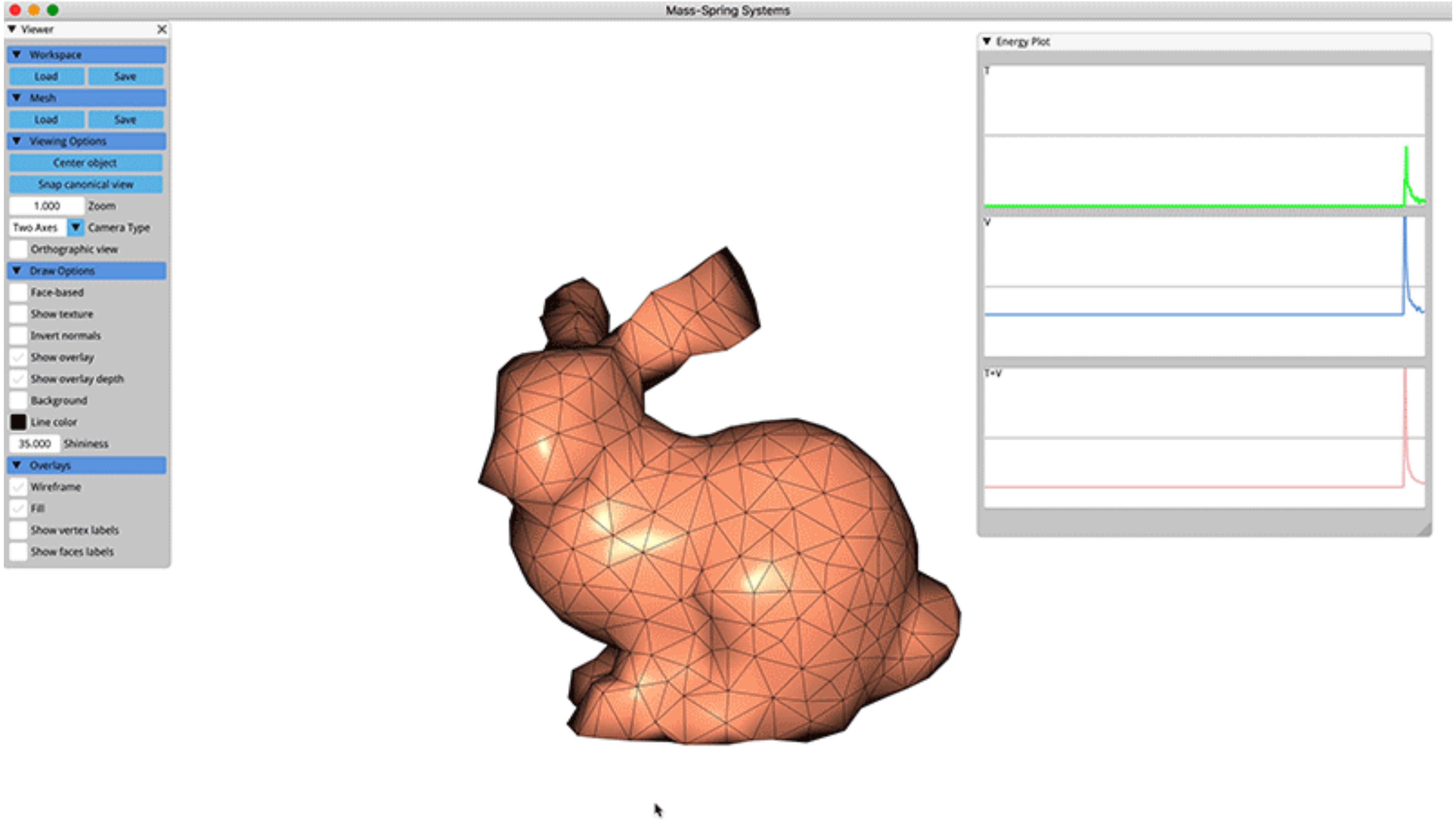
CSC2549 Physics-Based Animation



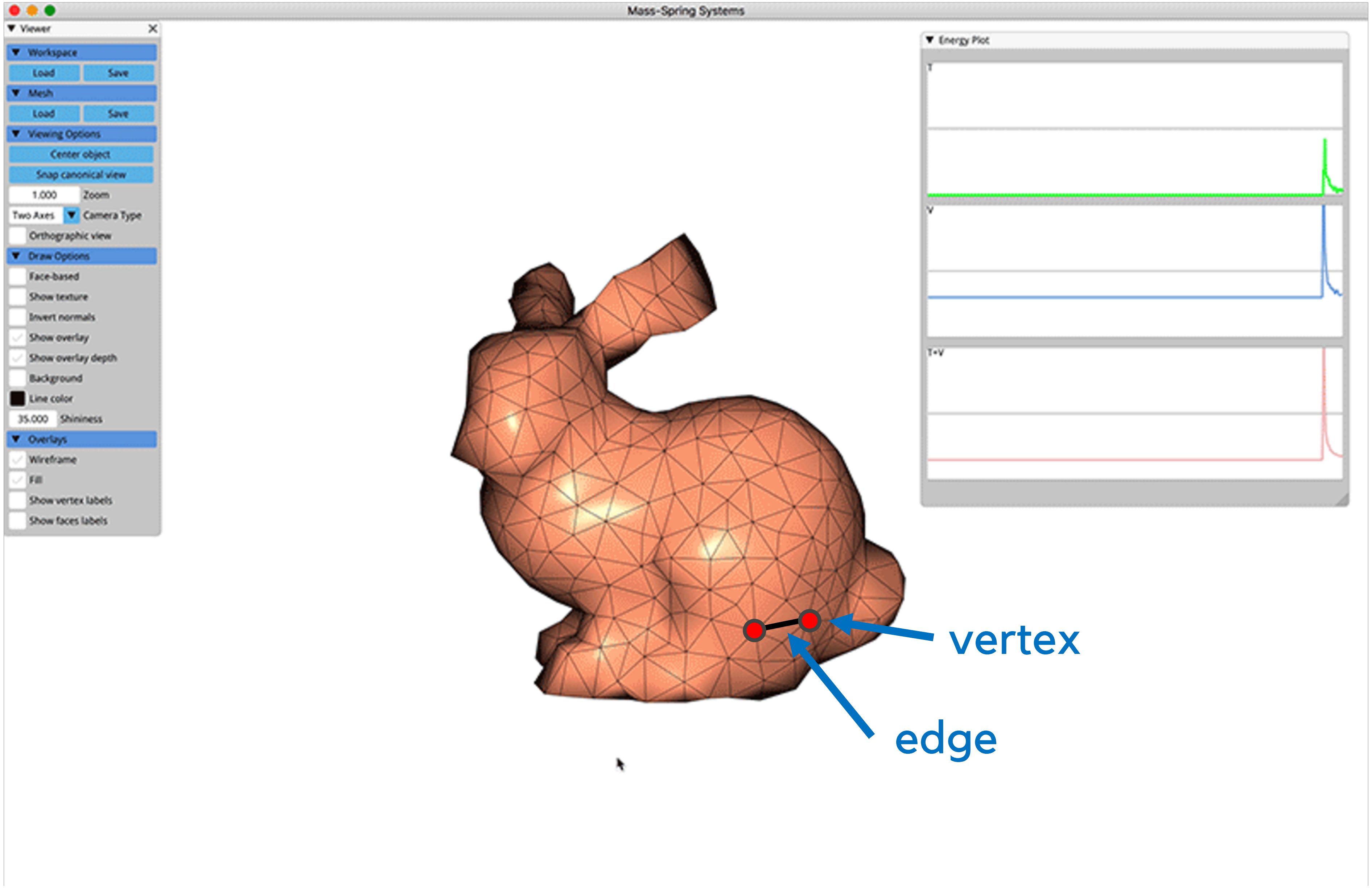
Enders Game | Digital Domai

Last Video: 3D Mass-Spring Systems

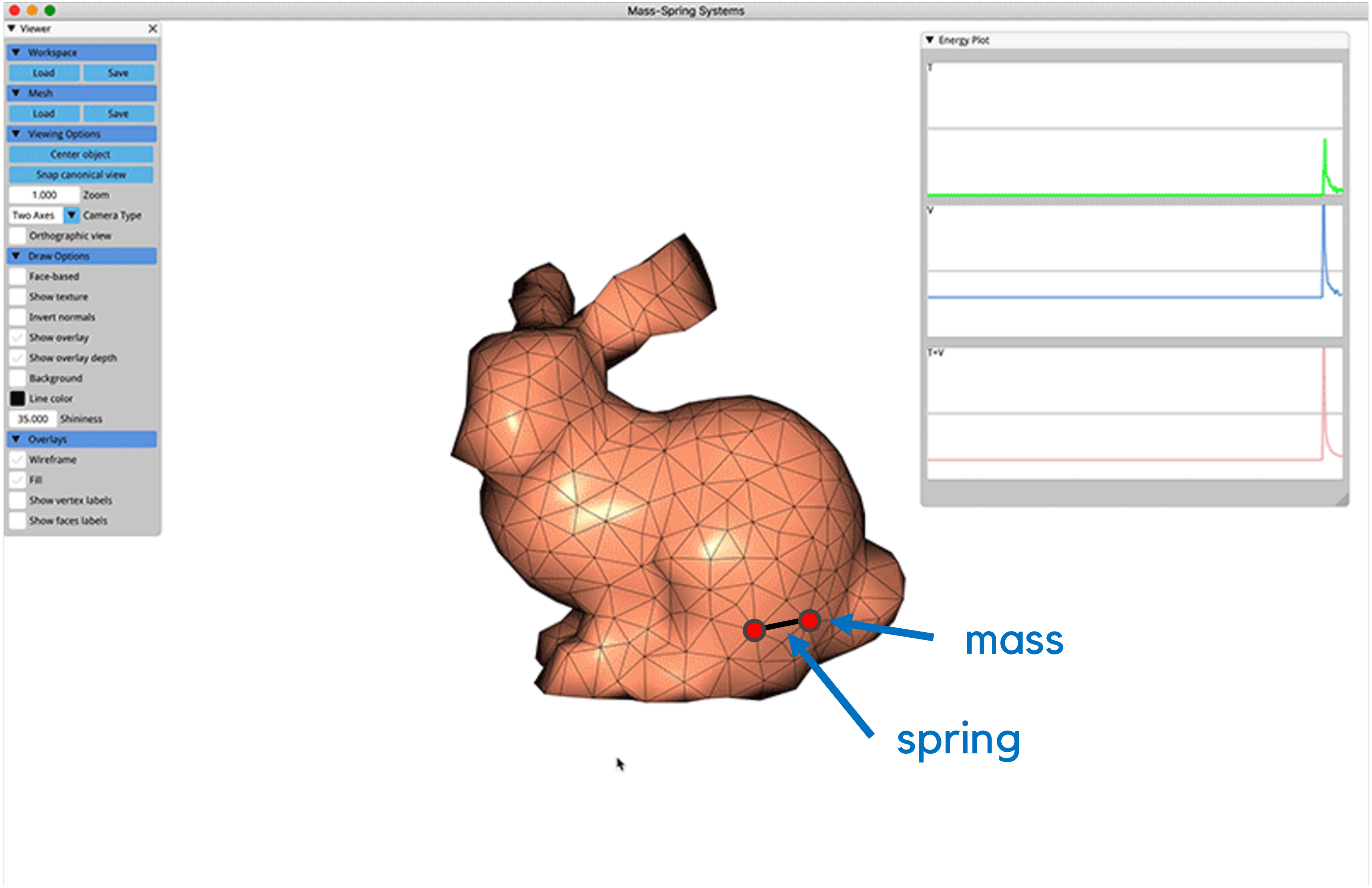


This Video: The Finite Element Method

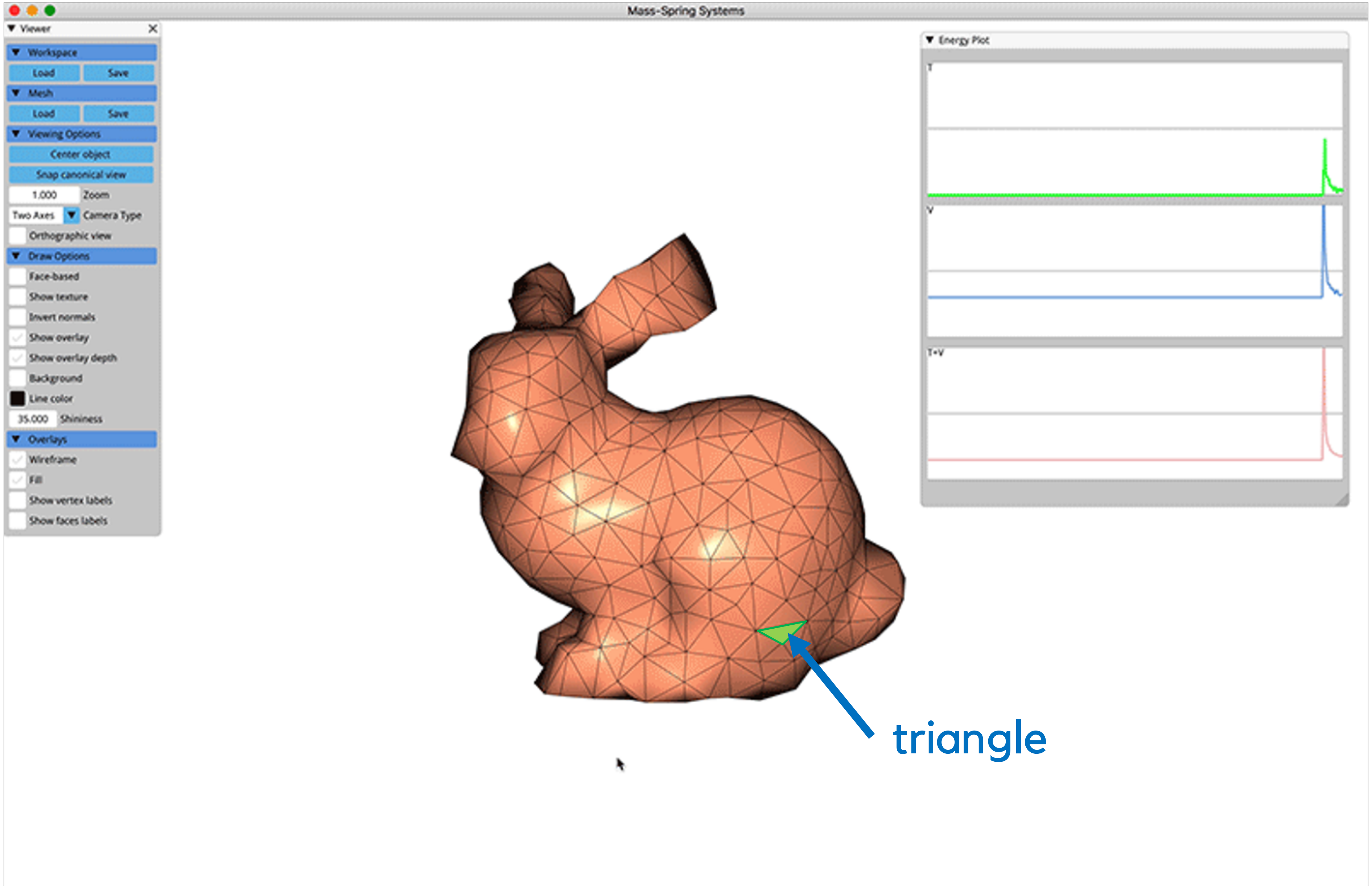
Mass-Spring Systems in 3D



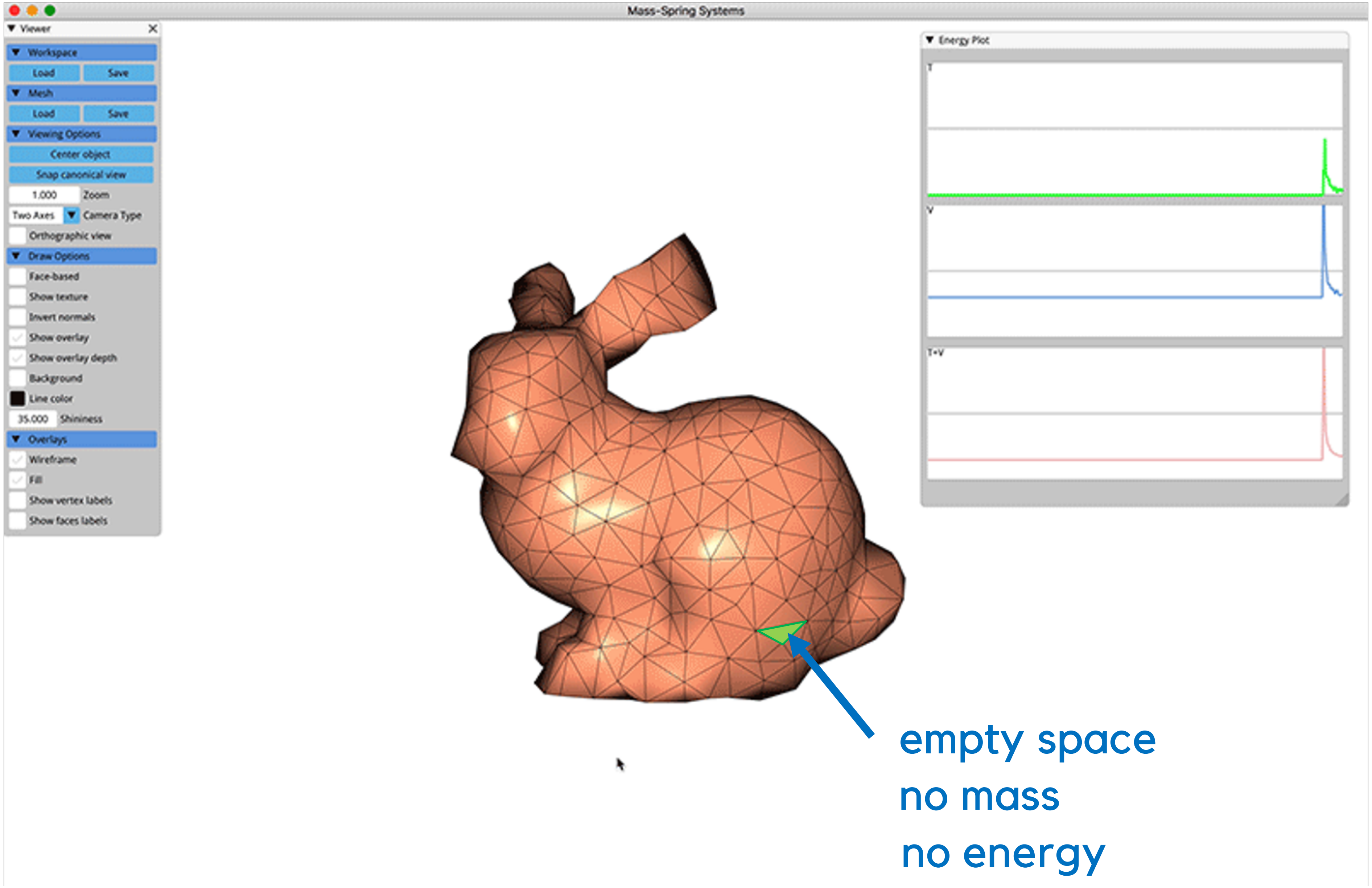
Mass-Spring Systems in 3D



Mass-Spring Systems in 3D

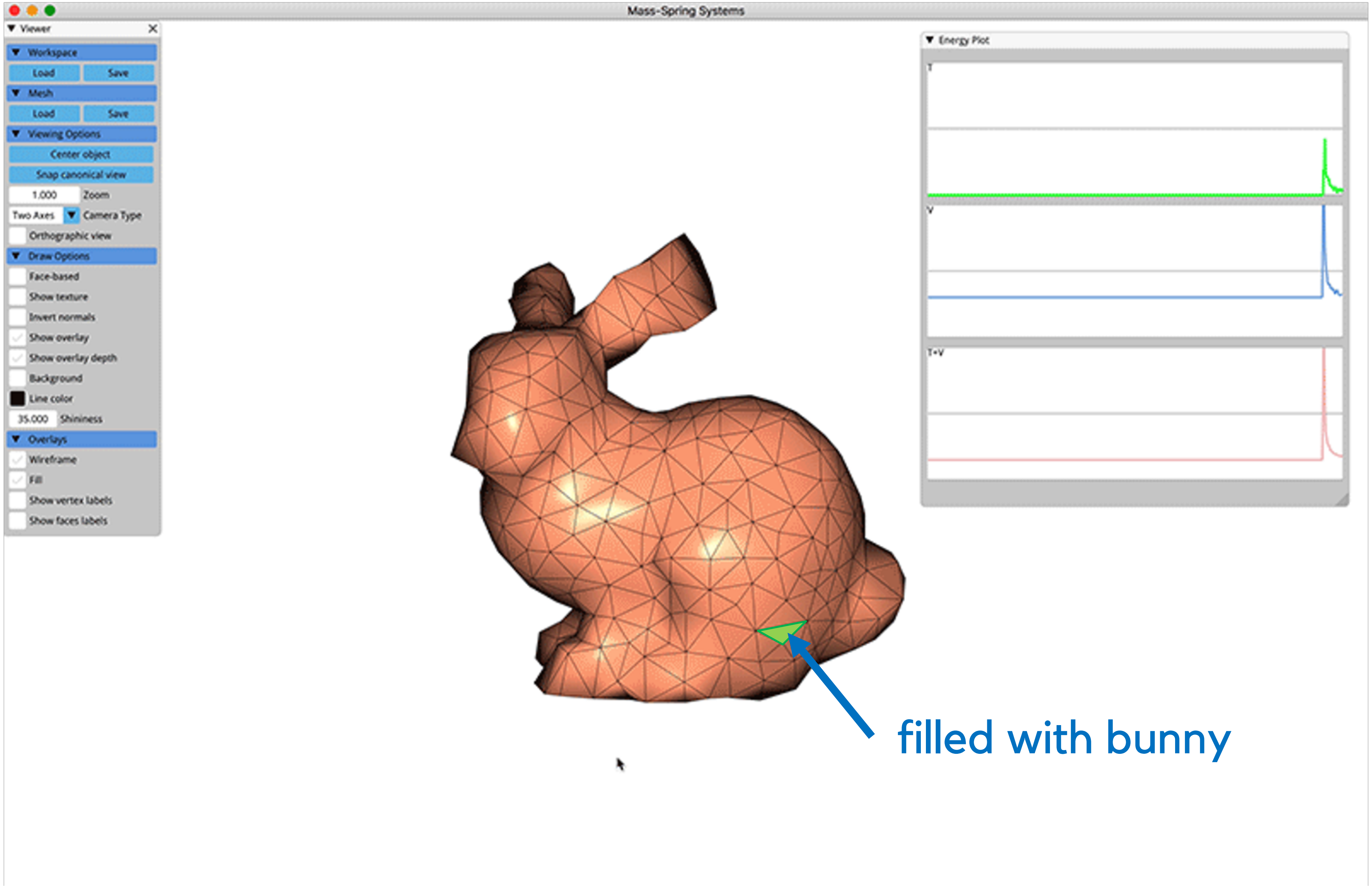


Mass-Spring Systems in 3D

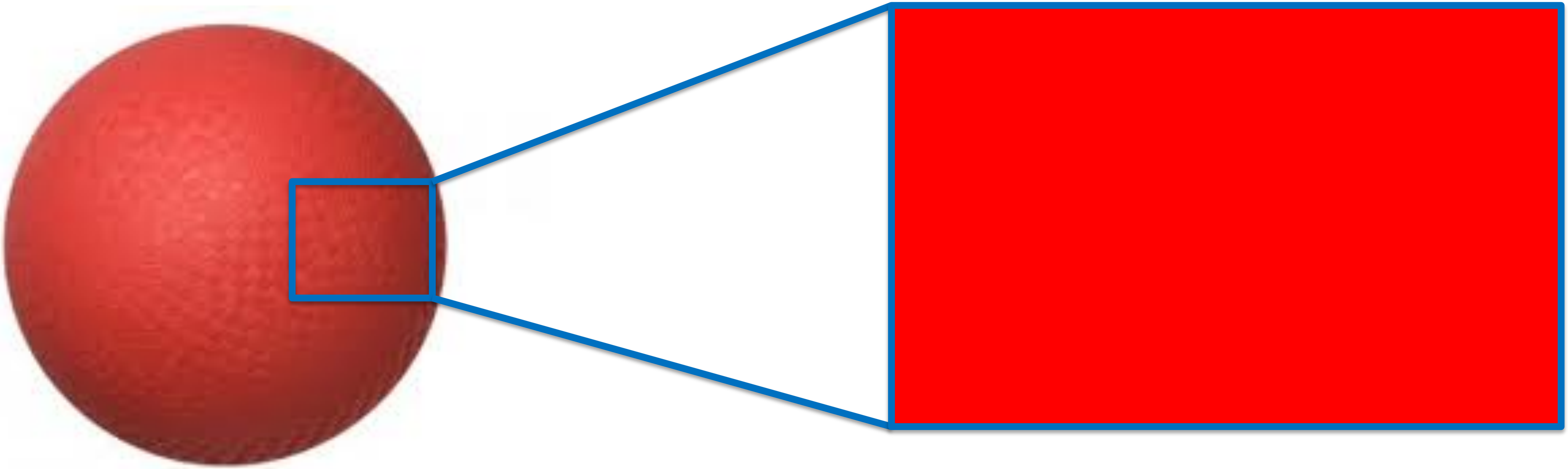


empty space
no mass
no energy

Finite Elements



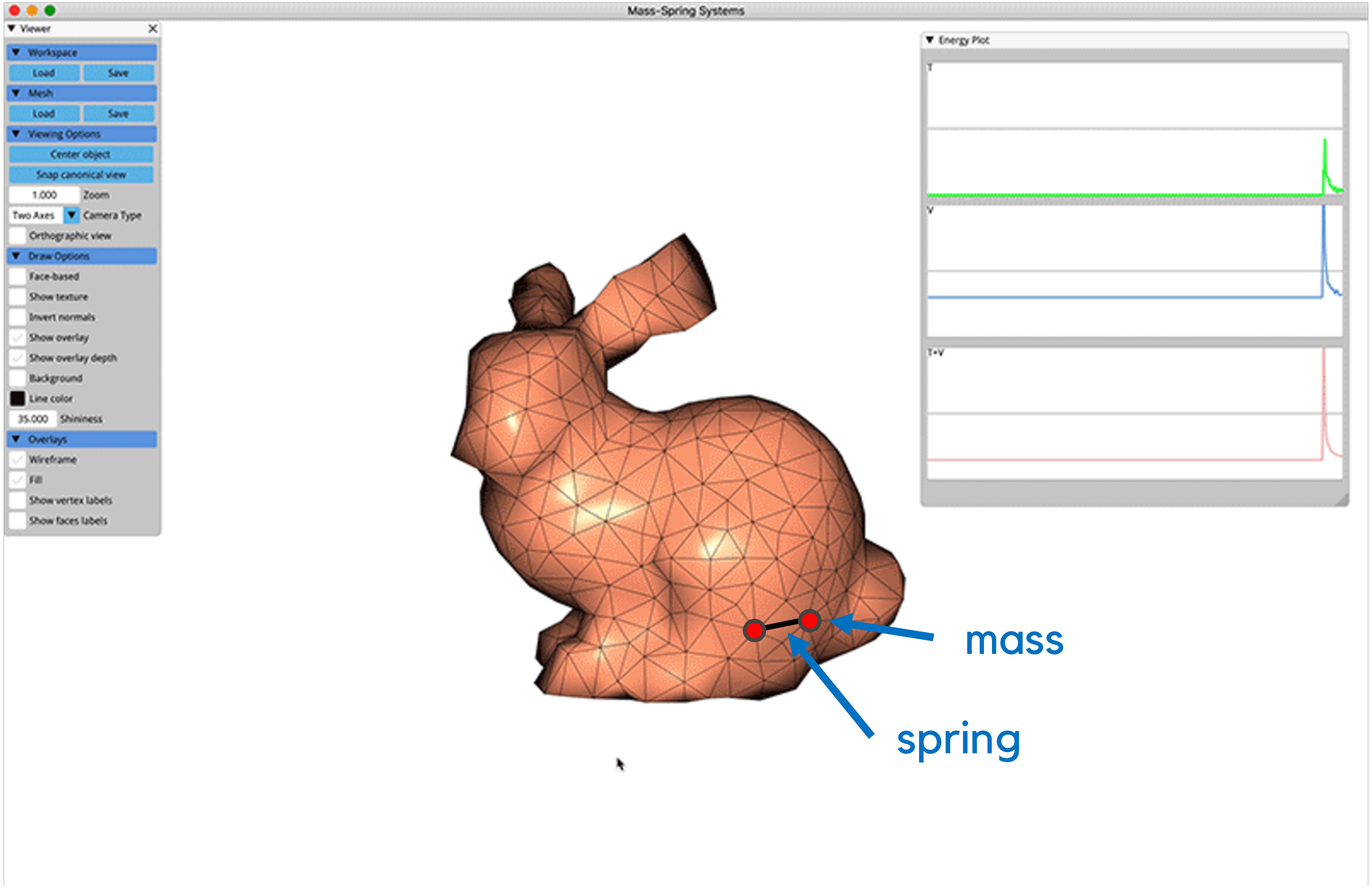
Continuum Hypothesis



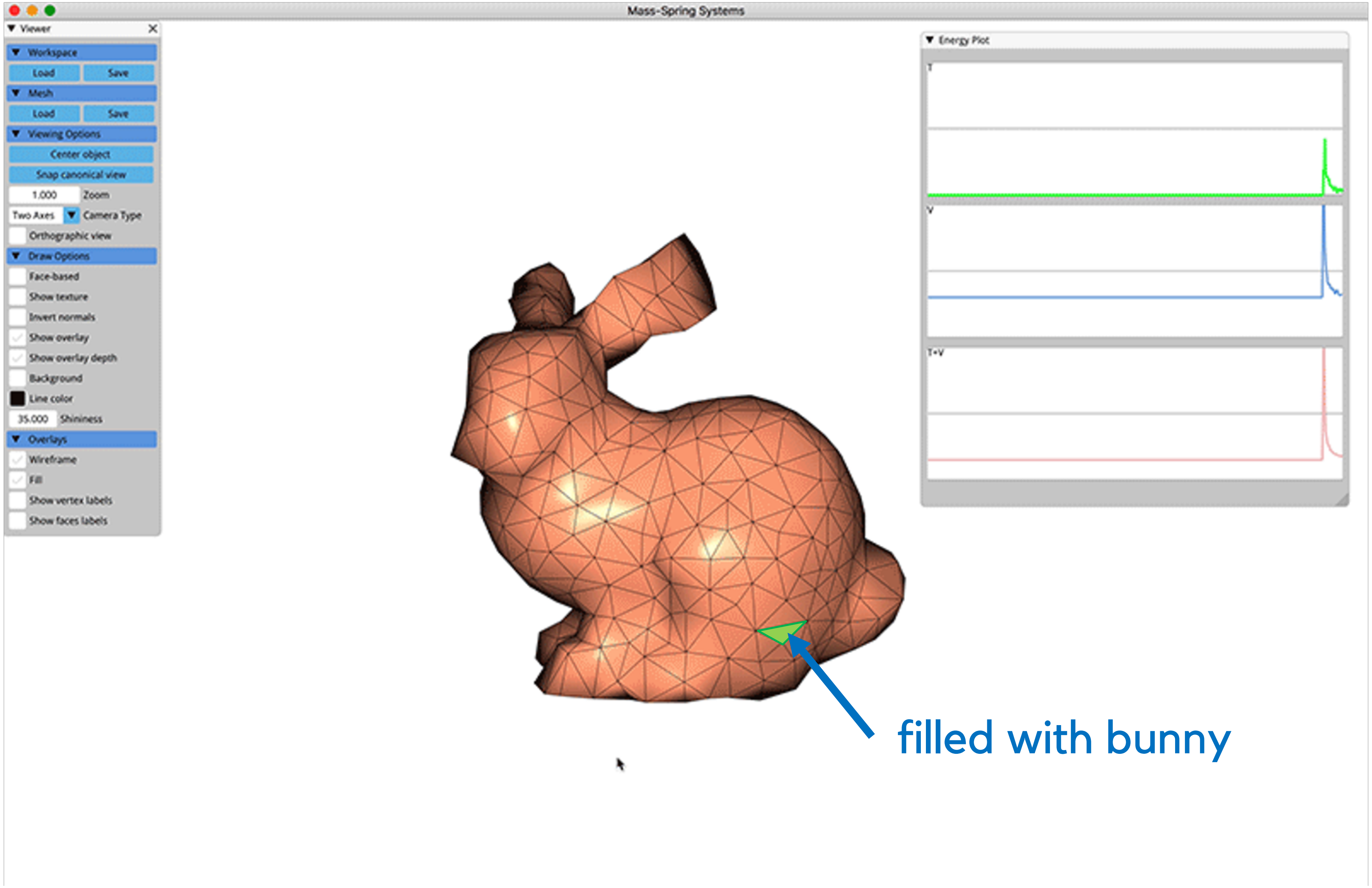
Continuum Mechanics

How did every point in this object change shape ?

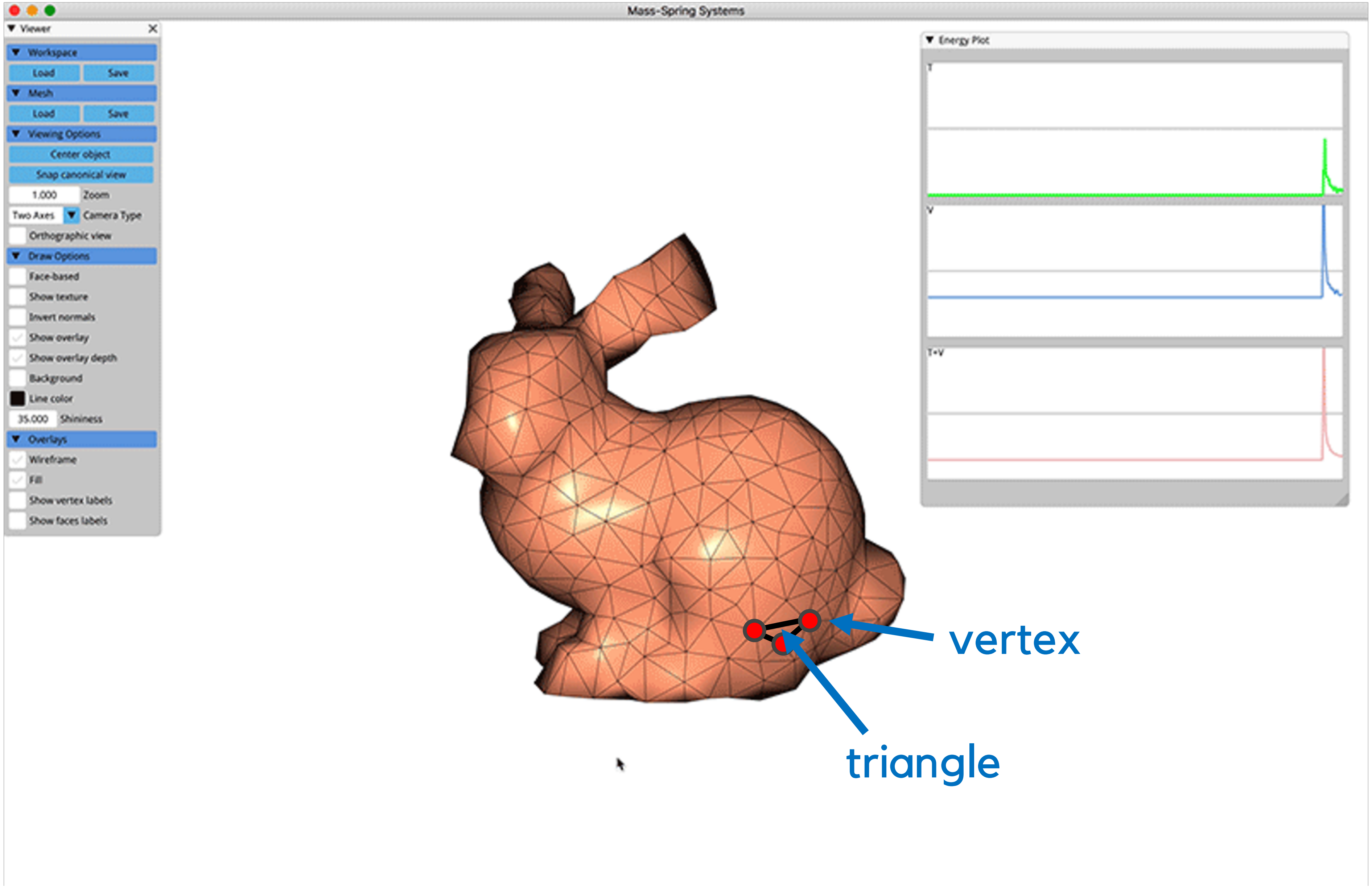
Mass-Spring Systems in 3D



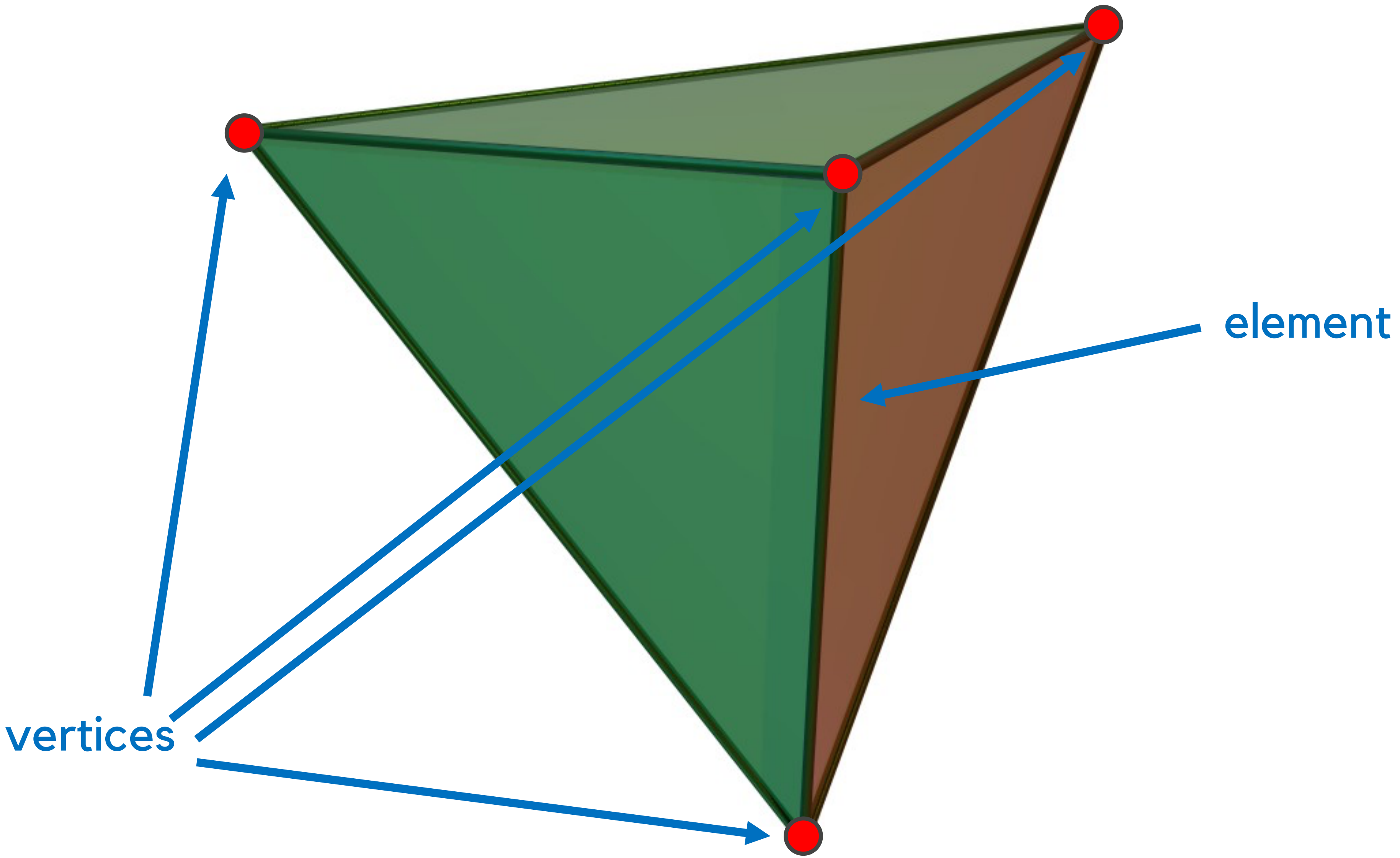
Finite Elements



Finite Elements

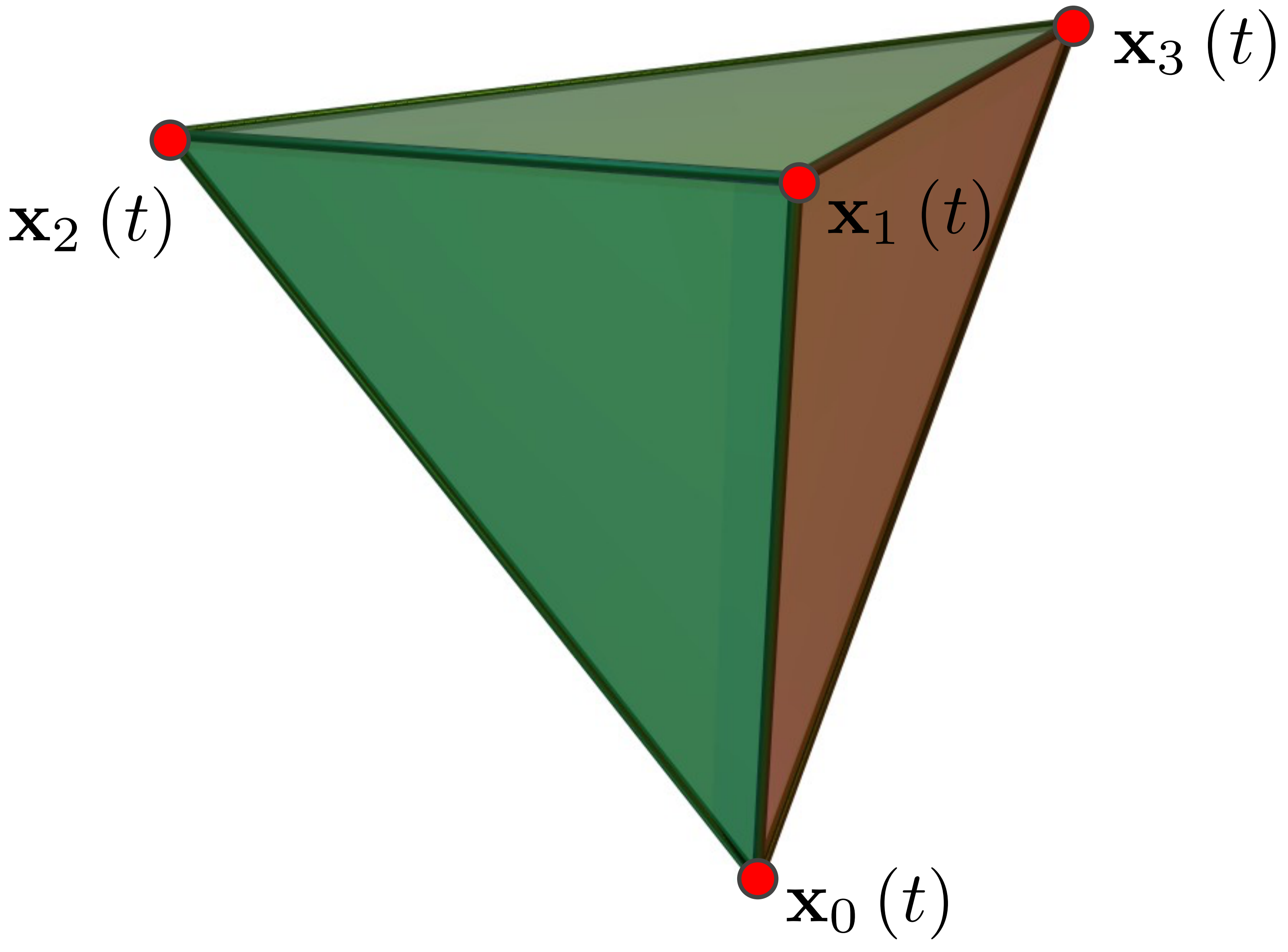


Tetrahedral Finite Elements

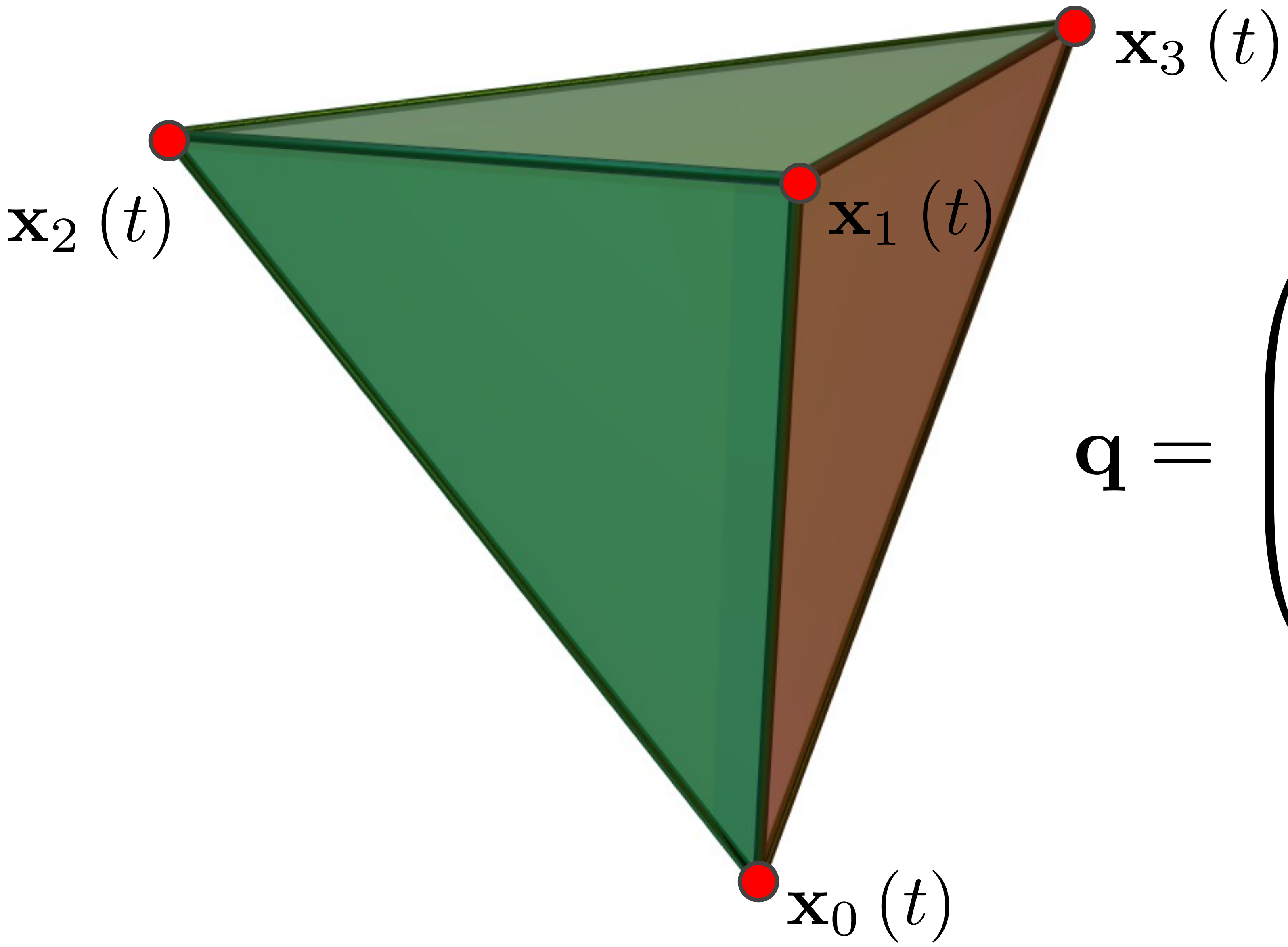


TetWild | https://github.com/Yixin-Hu/TetWild

Tetrahedral Finite Elements

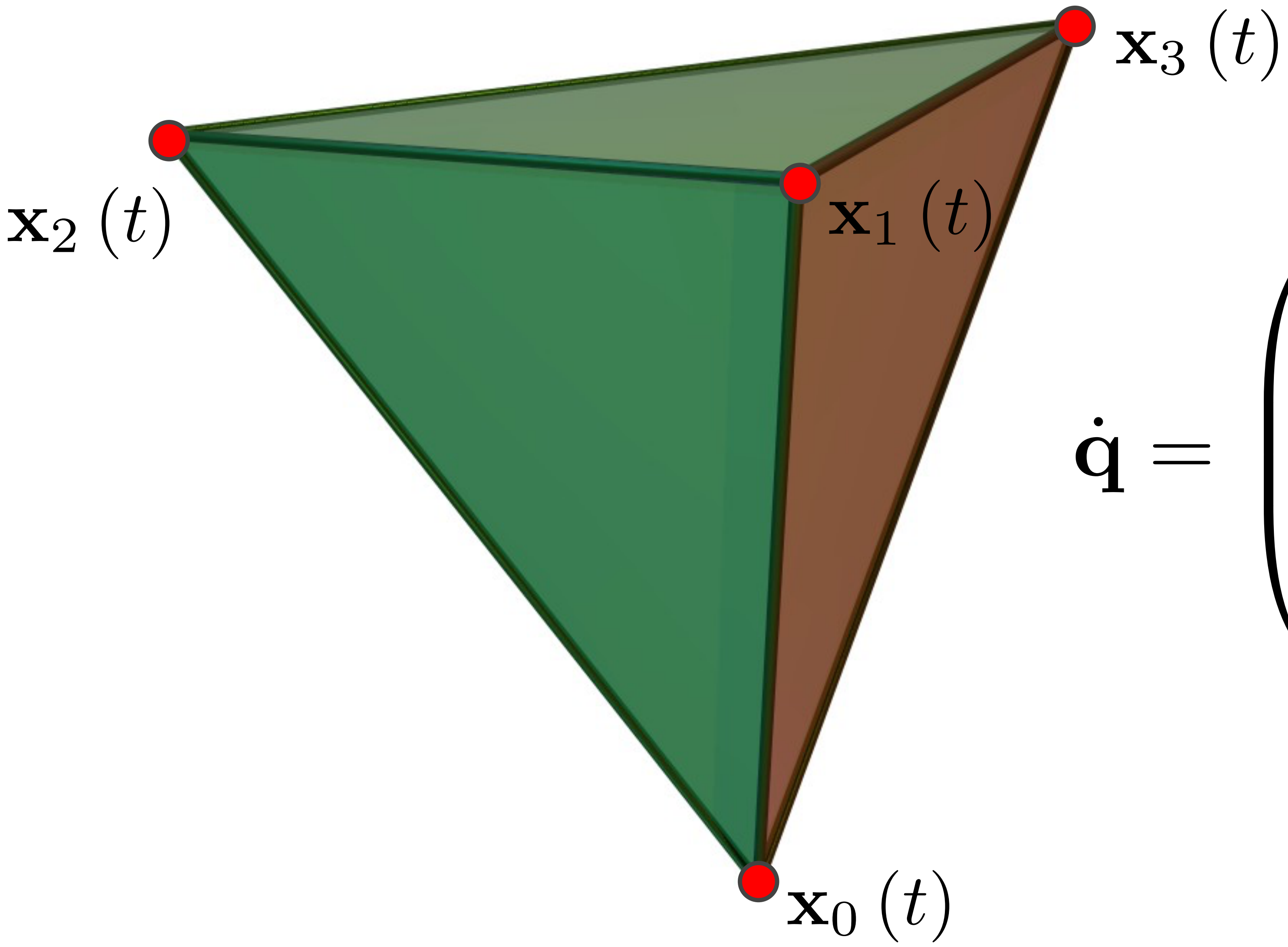


Tetrahedral Finite Elements



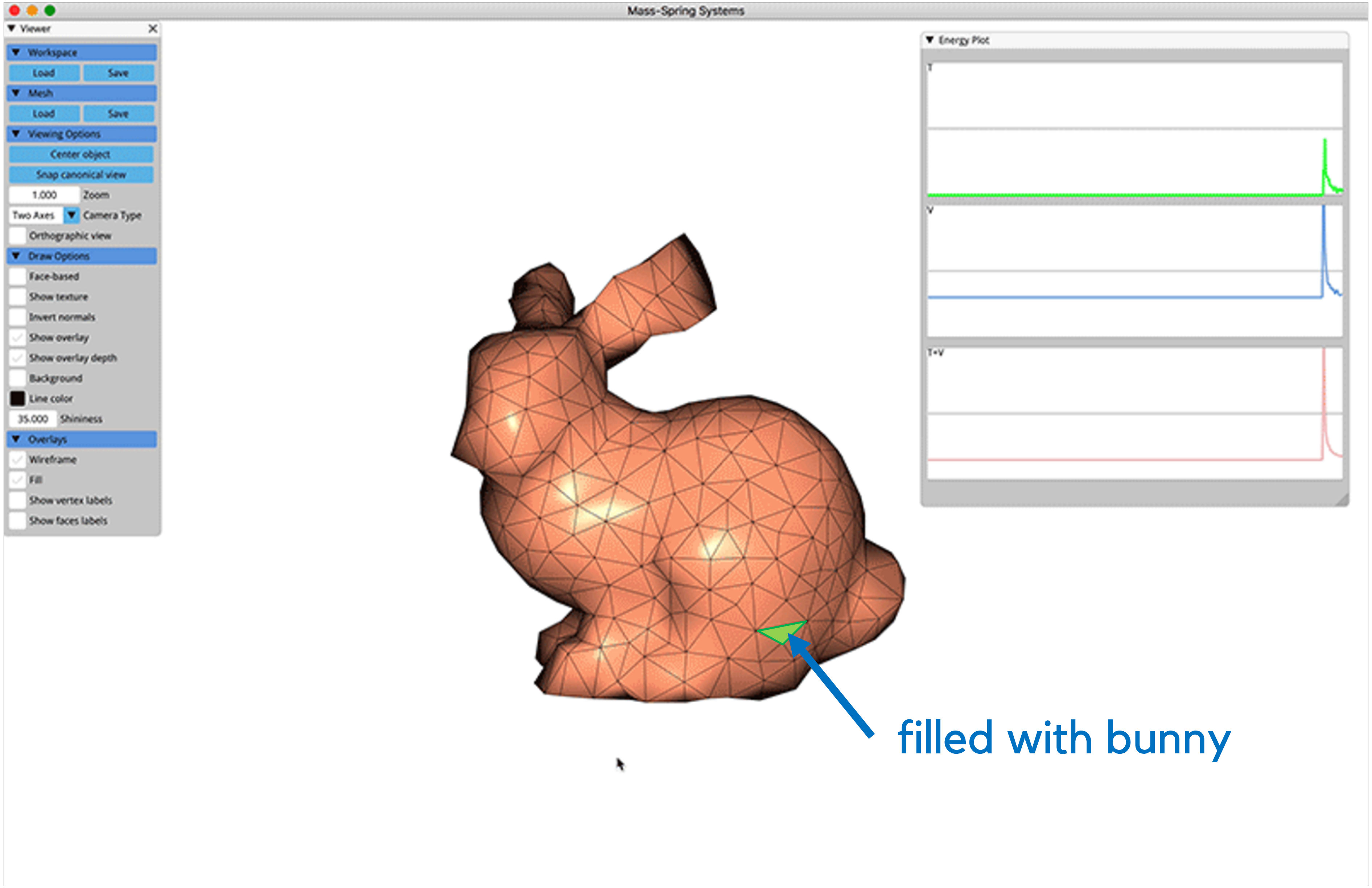
$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$$

Generalized Coordinates for Tetrahedral Element

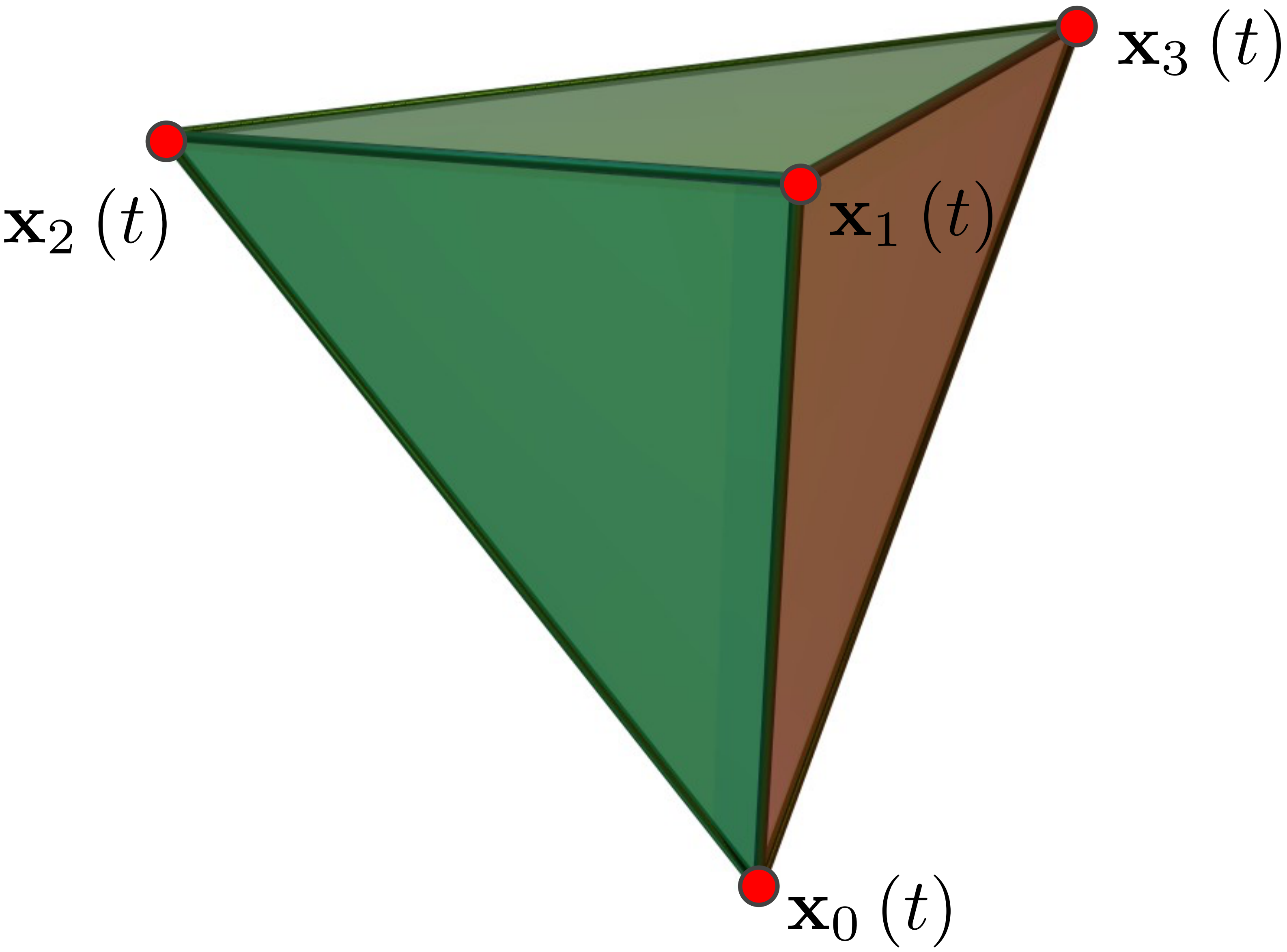


$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}$$

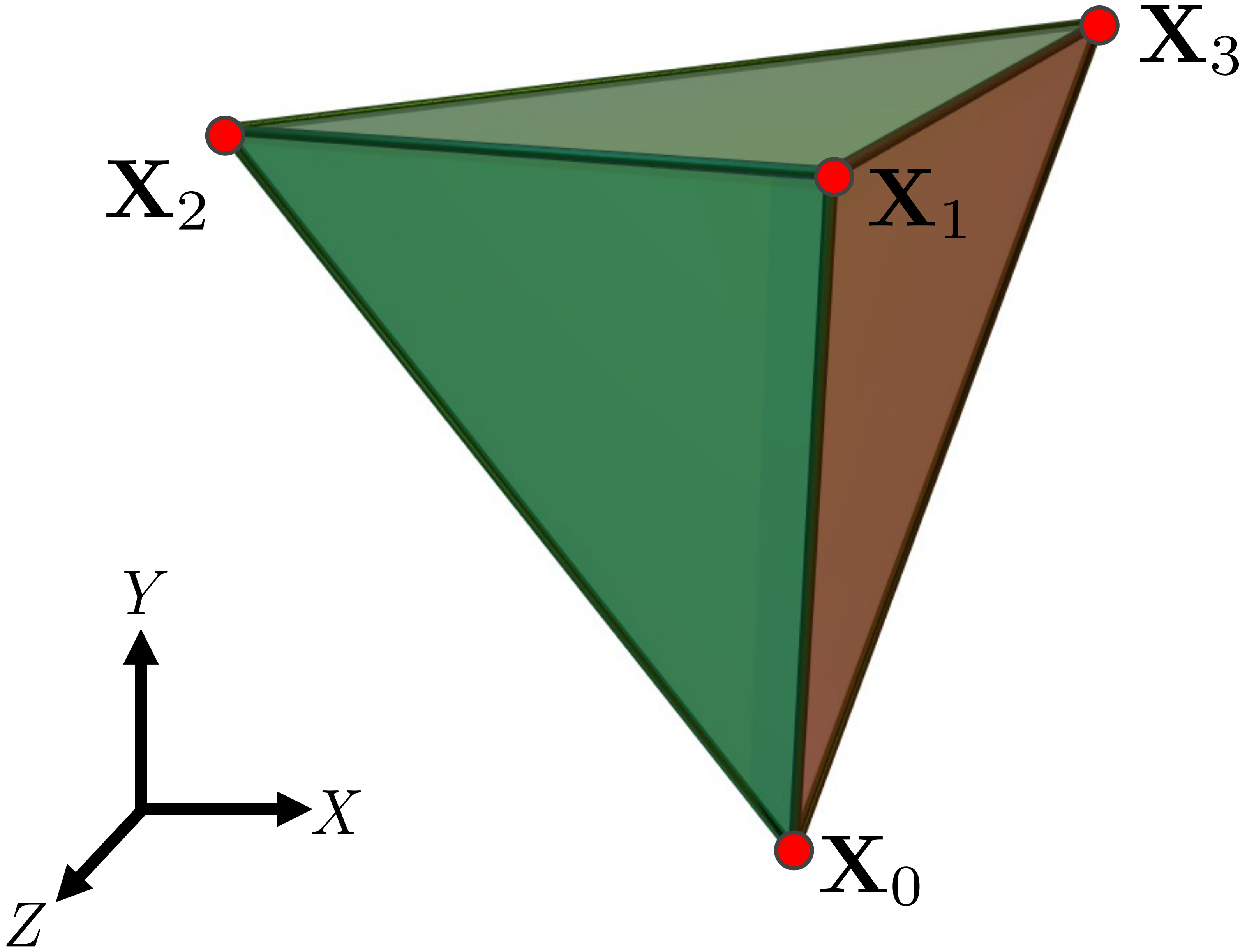
Finite Elements



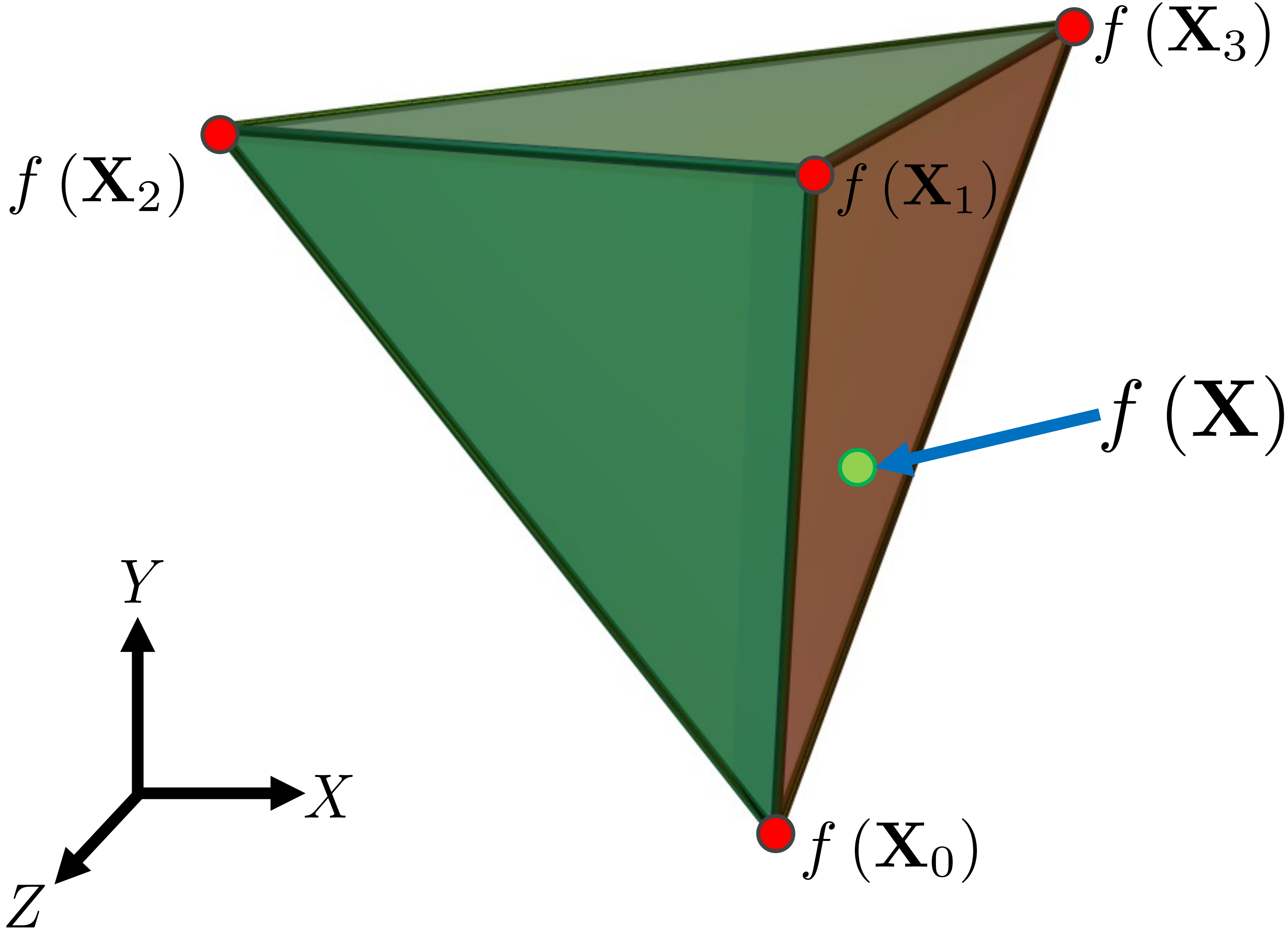
Finite Elements



Finite Elements

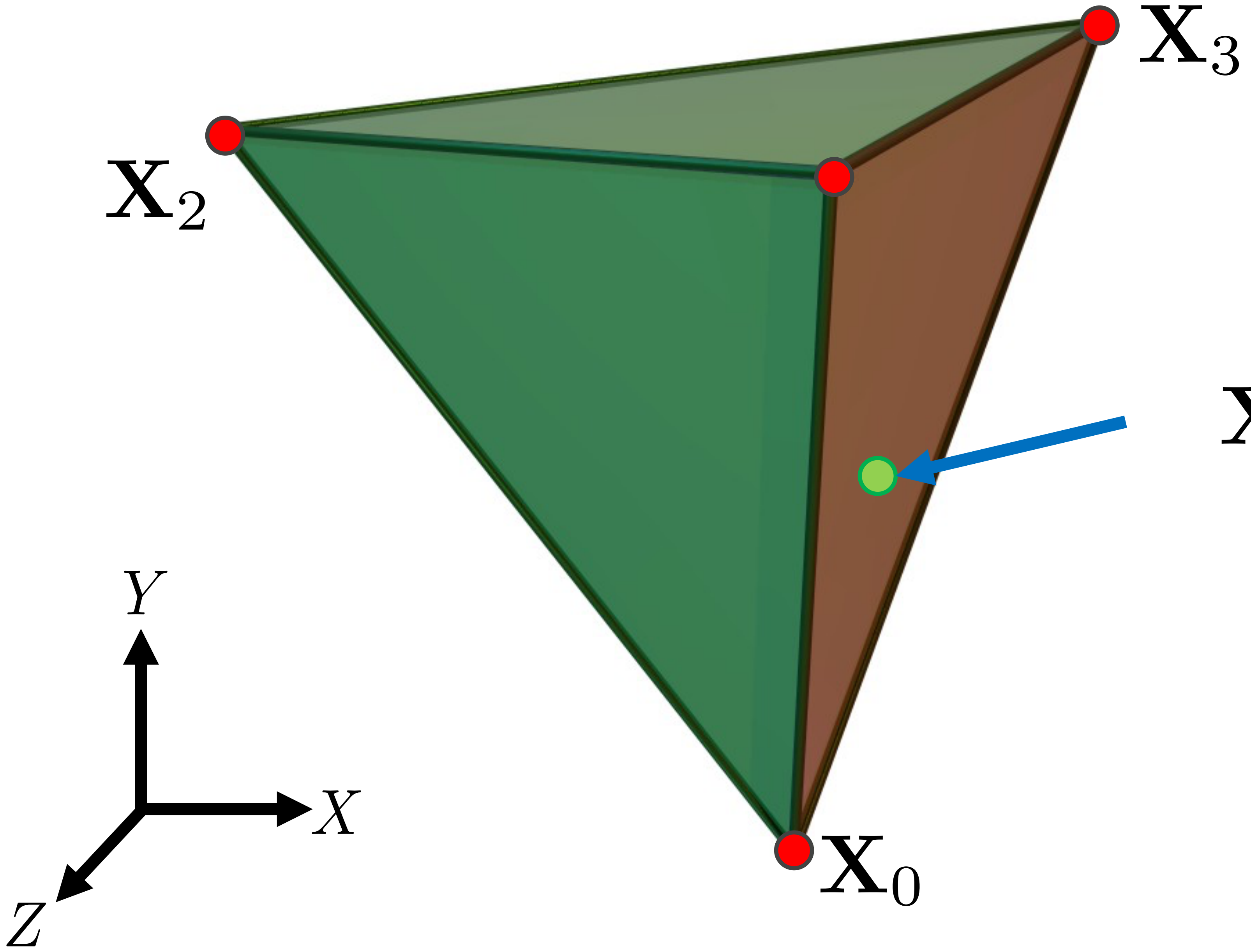


Finite Elements



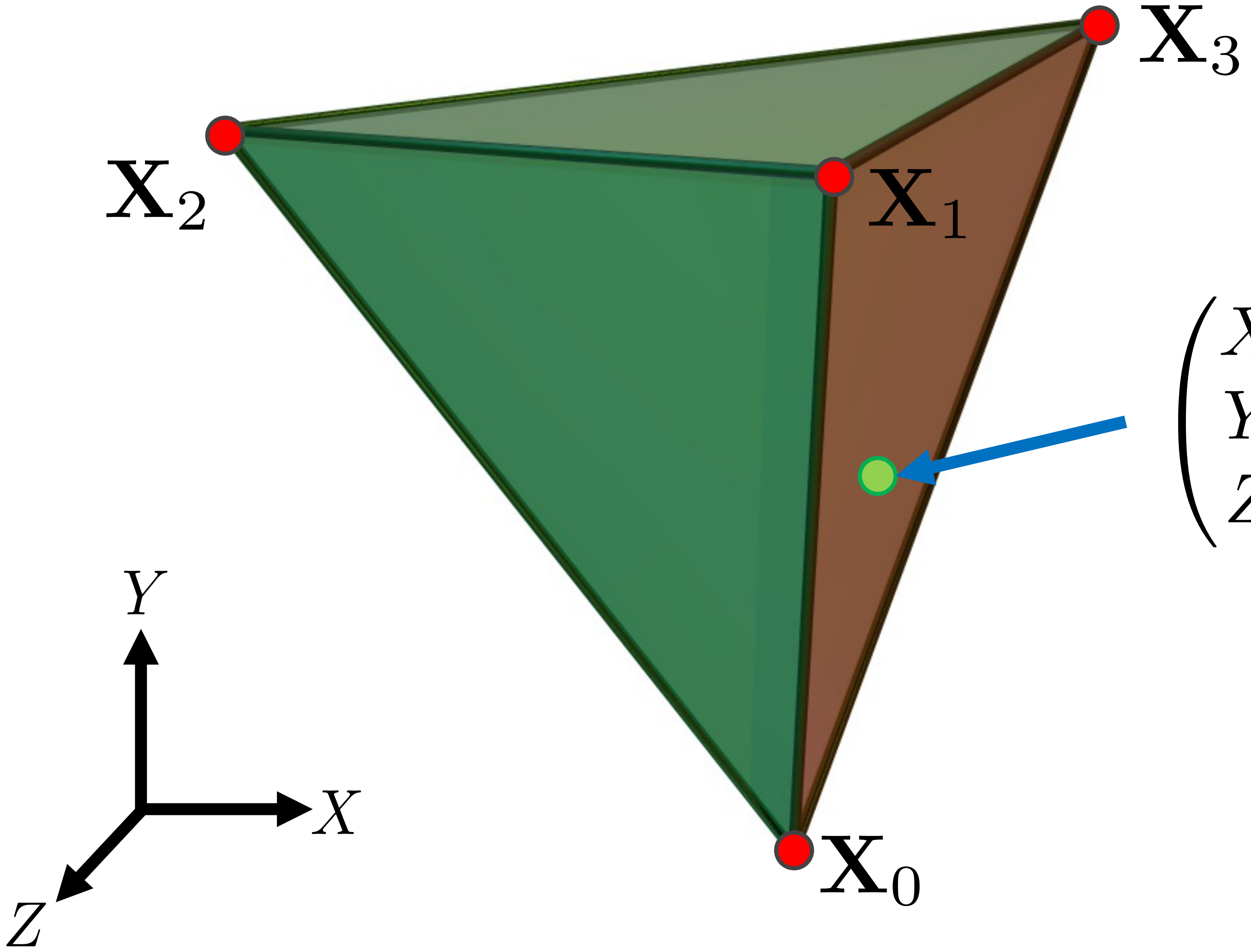
$$f(\mathbf{X}) = \sum_{i=0}^{i=3} \underbrace{f_i}_{f(\mathbf{X}_i)} \overbrace{\phi_i(\mathbf{X})}^{\text{shape function}}$$

Finite Elements



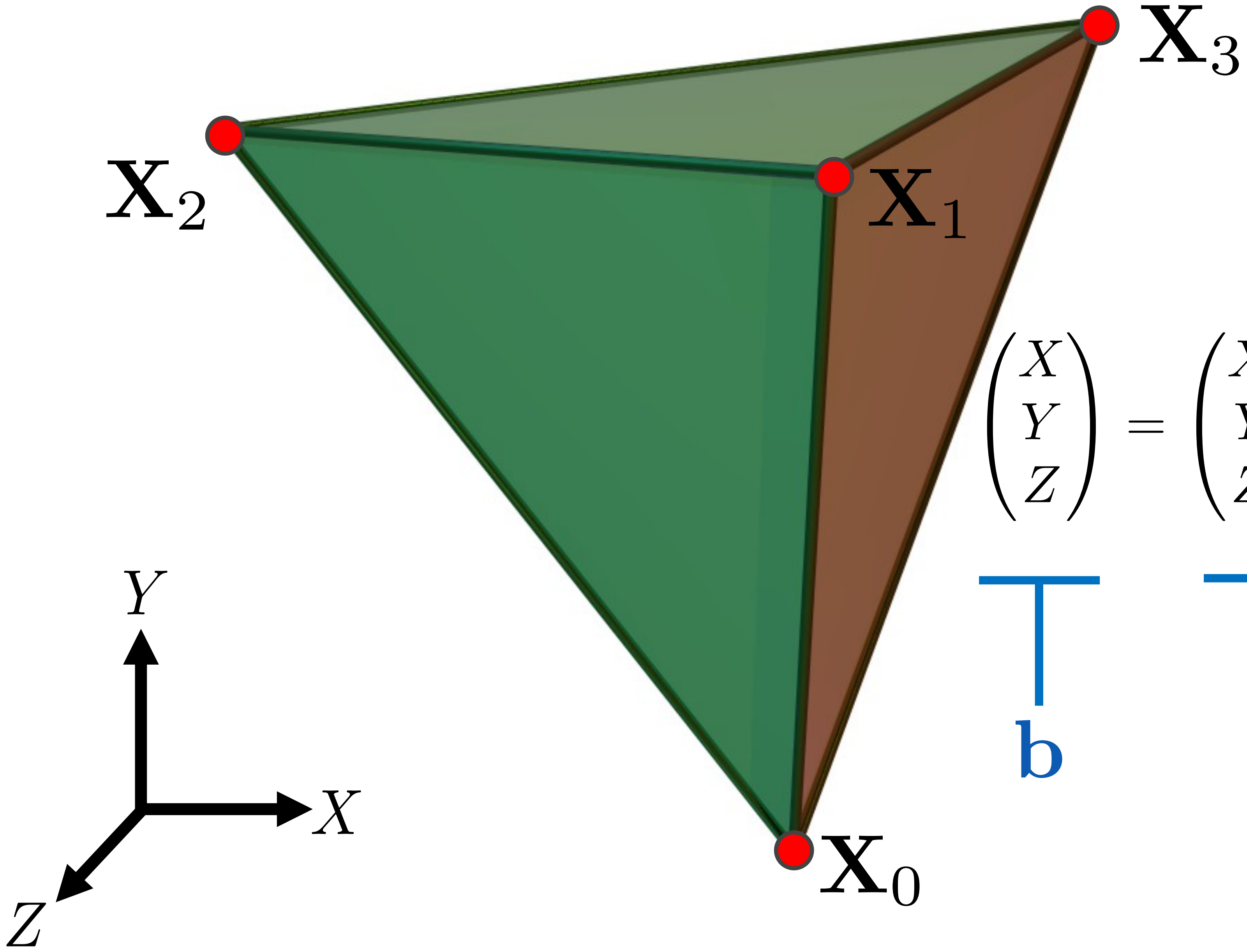
$$\mathbf{X} = \sum_{i=0}^{i=3} \underbrace{\mathbf{X}_i}_{\text{vertex coordinates}} \overbrace{\phi_i\left(\mathbf{X}\right)}^{\text{shape function}}$$

Finite Elements



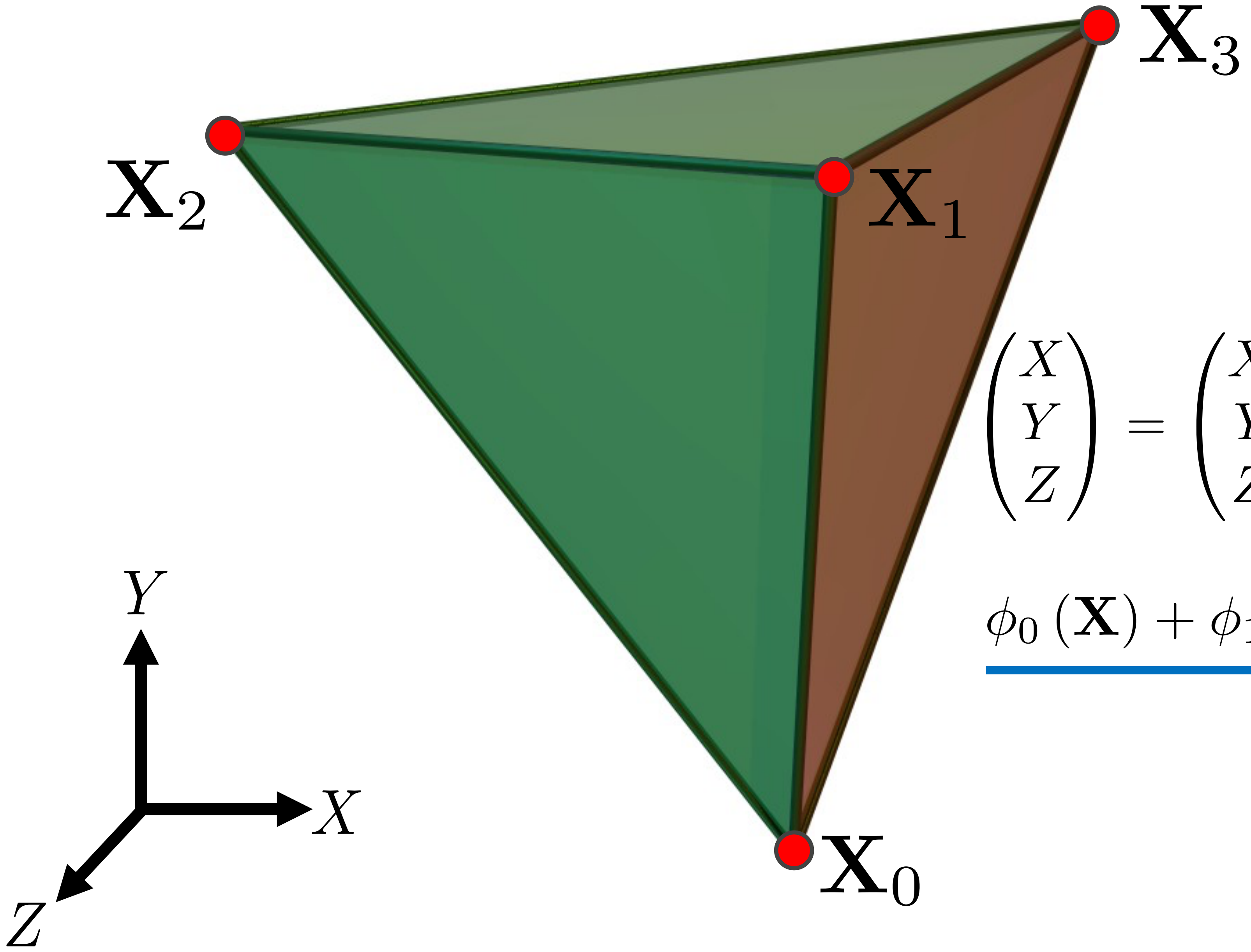
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \sum_{i=0}^{i=3} \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix} \phi_i(\mathbf{X})$$

Finite Elements



$$\underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}}_{\mathbf{b}} = \underbrace{\begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ Z_0 & Z_1 & Z_2 & Z_3 \end{pmatrix}}_{\mathsf{A}} \underbrace{\begin{pmatrix} \phi_0(\mathbf{X}) \\ \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix}}_{\mathbf{x}}$$

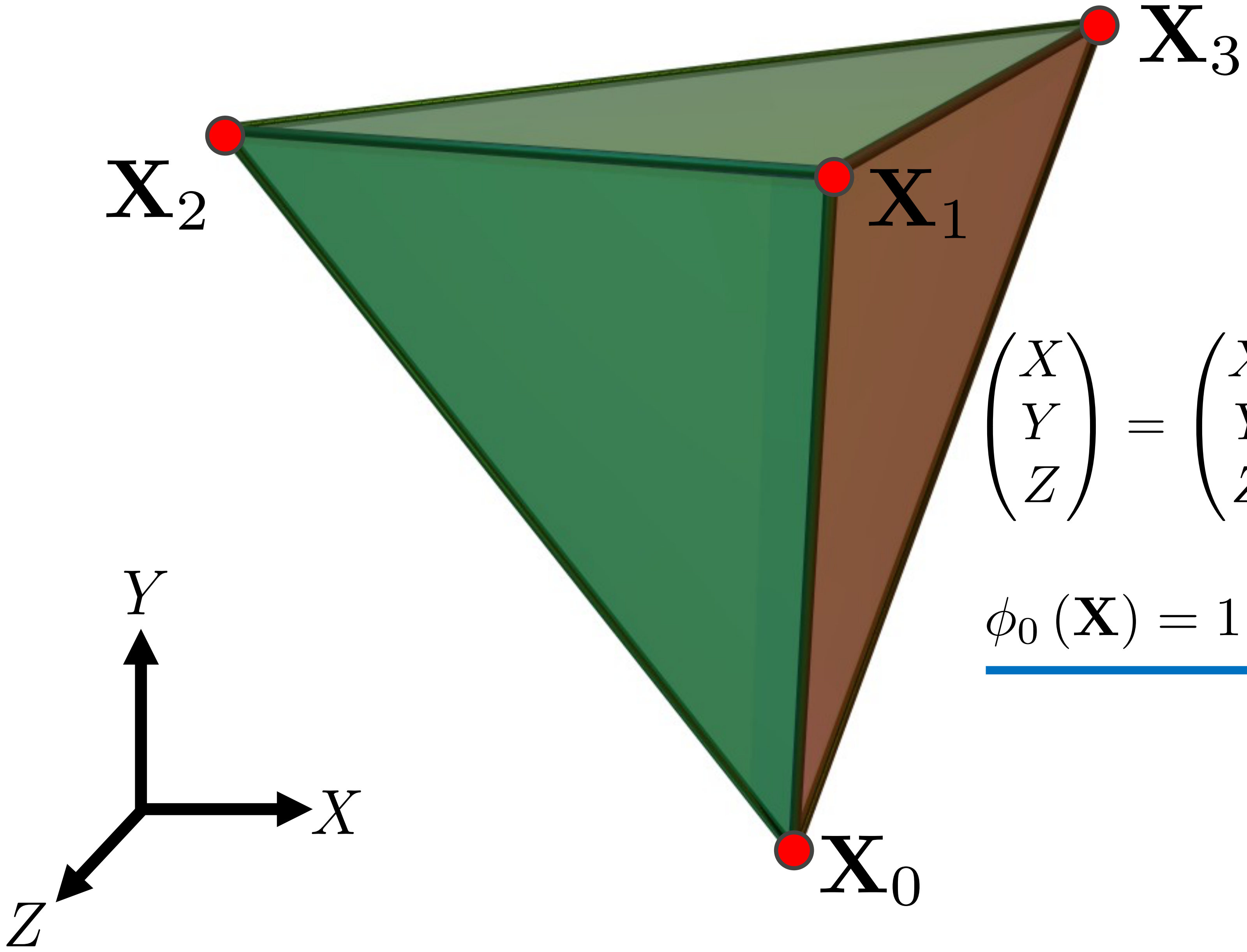
Finite Elements



$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ Z_0 & Z_1 & Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} \phi_0(\mathbf{X}) \\ \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix}$$

$$\phi_0(\mathbf{X}) + \phi_1(\mathbf{X}) + \phi_2(\mathbf{X}) + \phi_3(\mathbf{X}) = 1$$

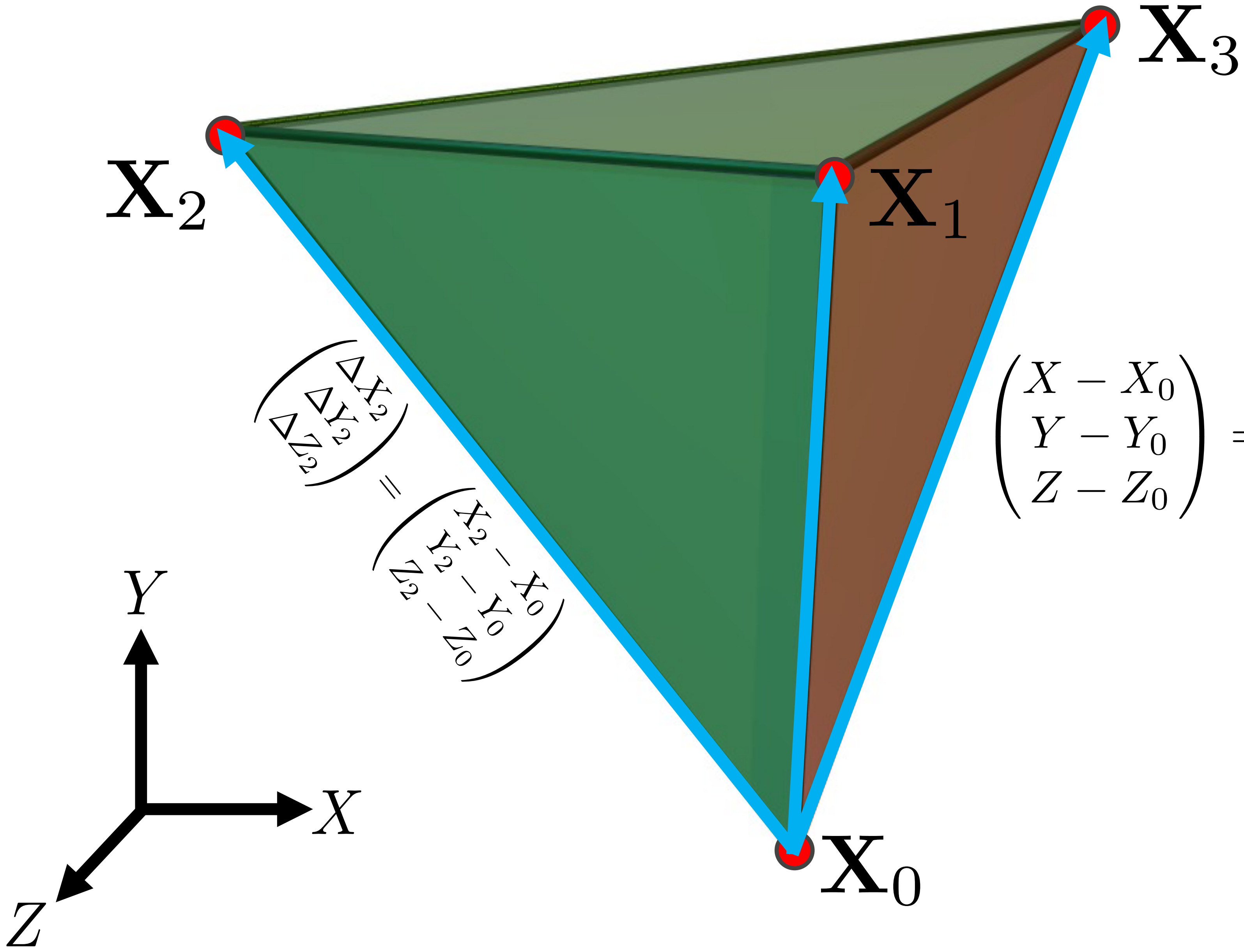
Finite Elements



$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \\ Z_0 & Z_1 & Z_2 & Z_3 \end{pmatrix} \begin{pmatrix} \phi_0(\mathbf{X}) \\ \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix}$$

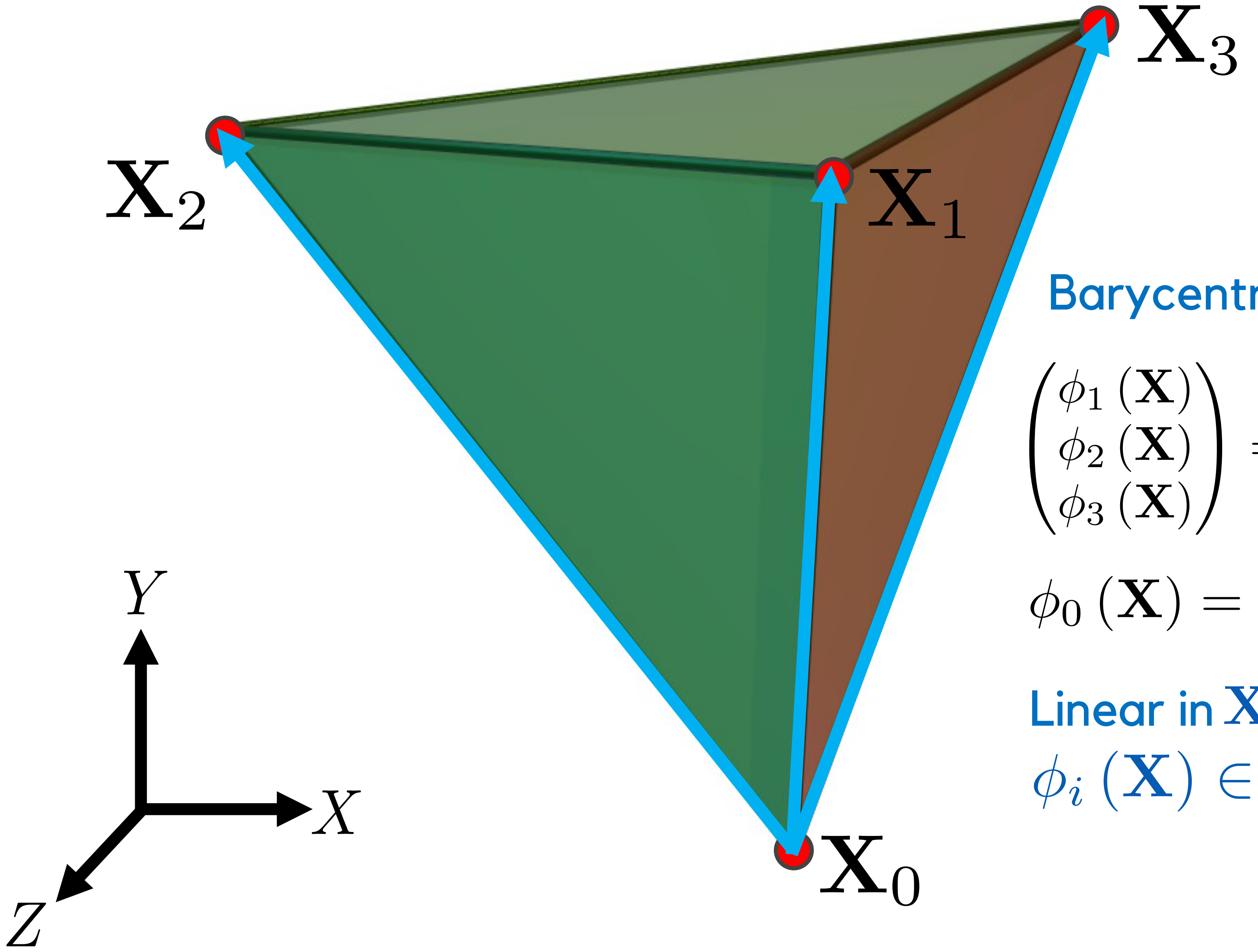
$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X}) - \phi_3(\mathbf{X})$$

Finite Elements



$$
\begin{pmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \Delta X_1 & \Delta X_2 & \Delta X_3 \\ \Delta Y_1 & \Delta Y_2 & \Delta X_3 \\ \Delta Z_1 & \Delta Z_2 & \Delta X_3 \end{pmatrix}}_{\mathrm{T}} \begin{pmatrix} \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix}
$$

Finite Elements



Barycentric Coordinates

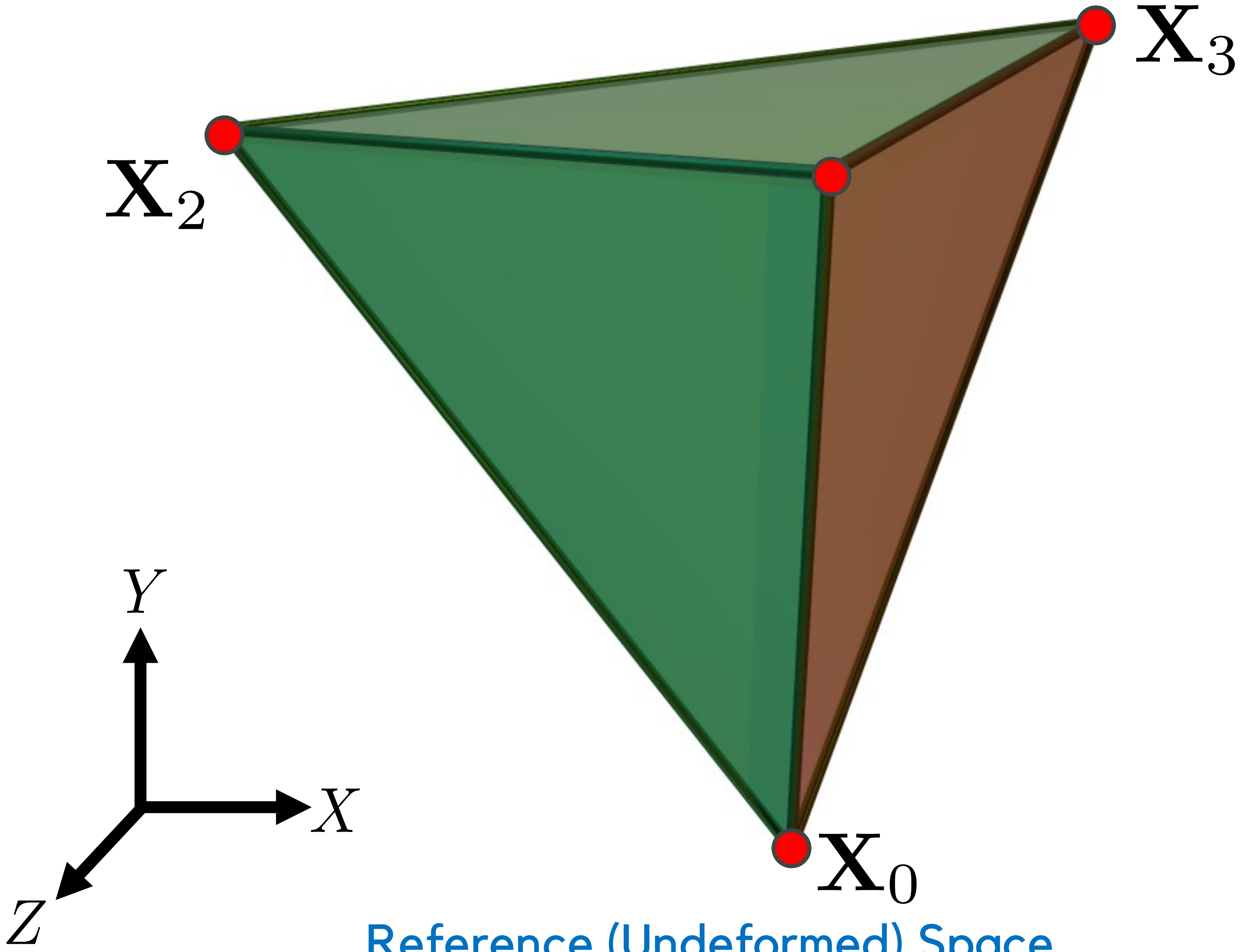
$$\begin{pmatrix} \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix} = \mathrm{T}^{-1}(\mathbf{X} - \mathbf{X}_0)$$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X}) - \phi_3(\mathbf{X})$$

Linear in $\mathbf{X}$

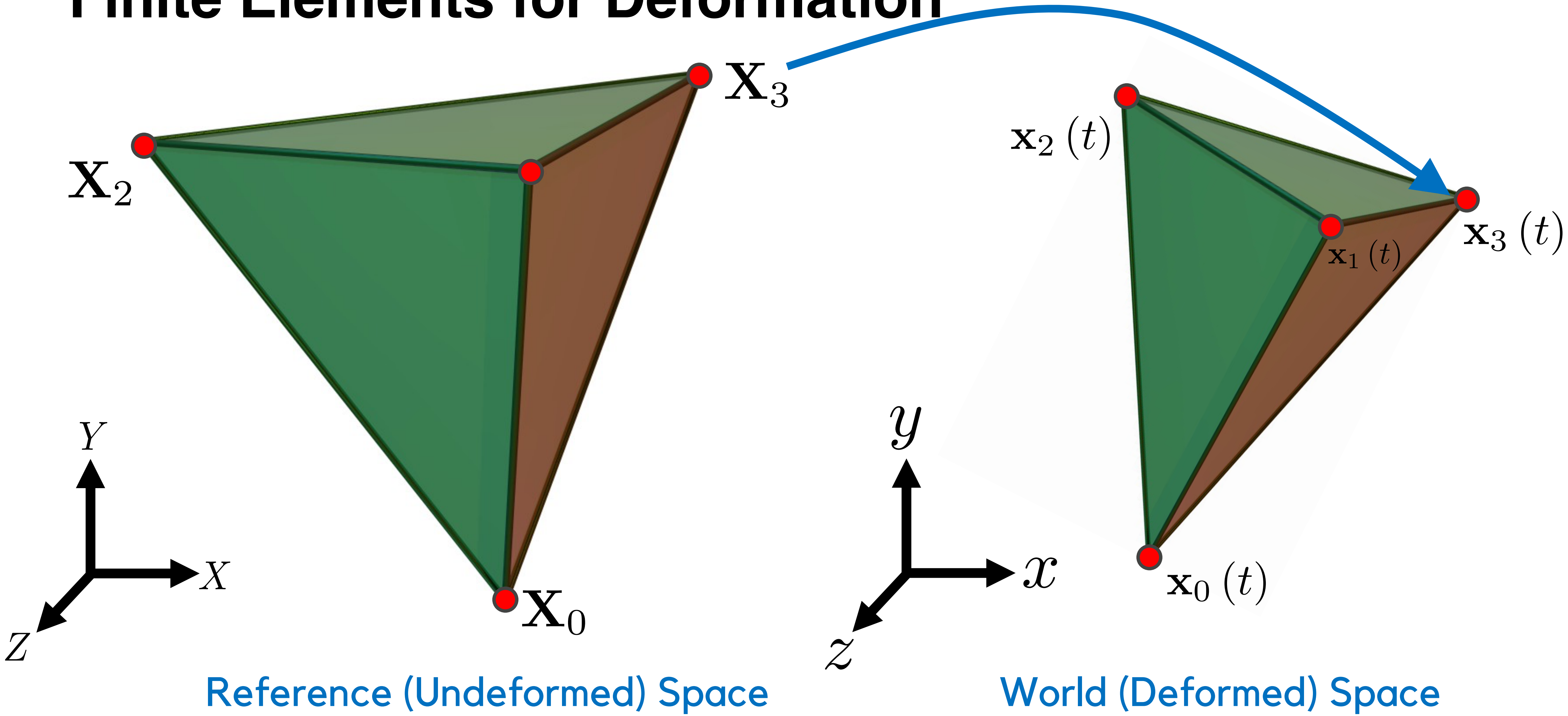
$\phi_i(\mathbf{X}) \in [0, 1]$ inside tetrahdron

Finite Elements for Deformation

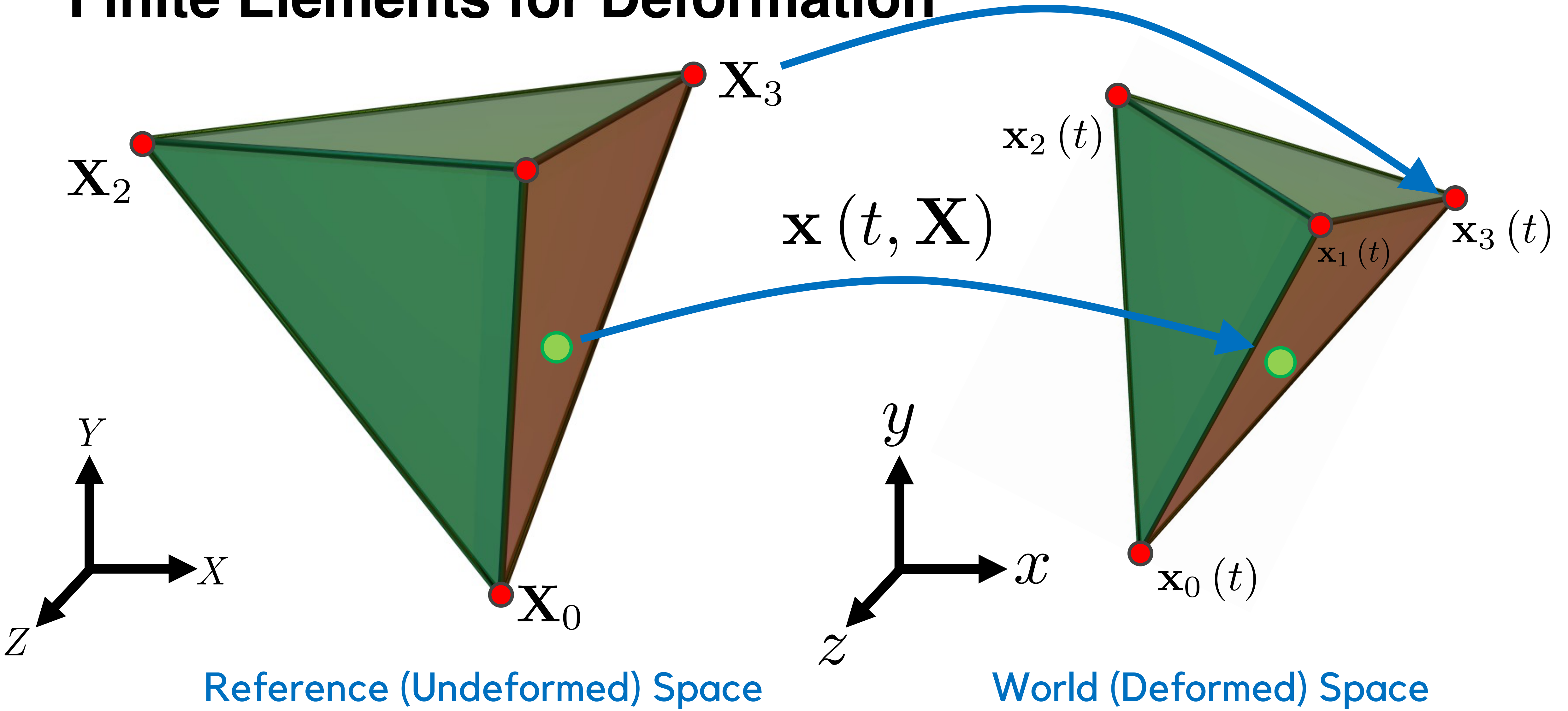


Reference (Undeformed) Space

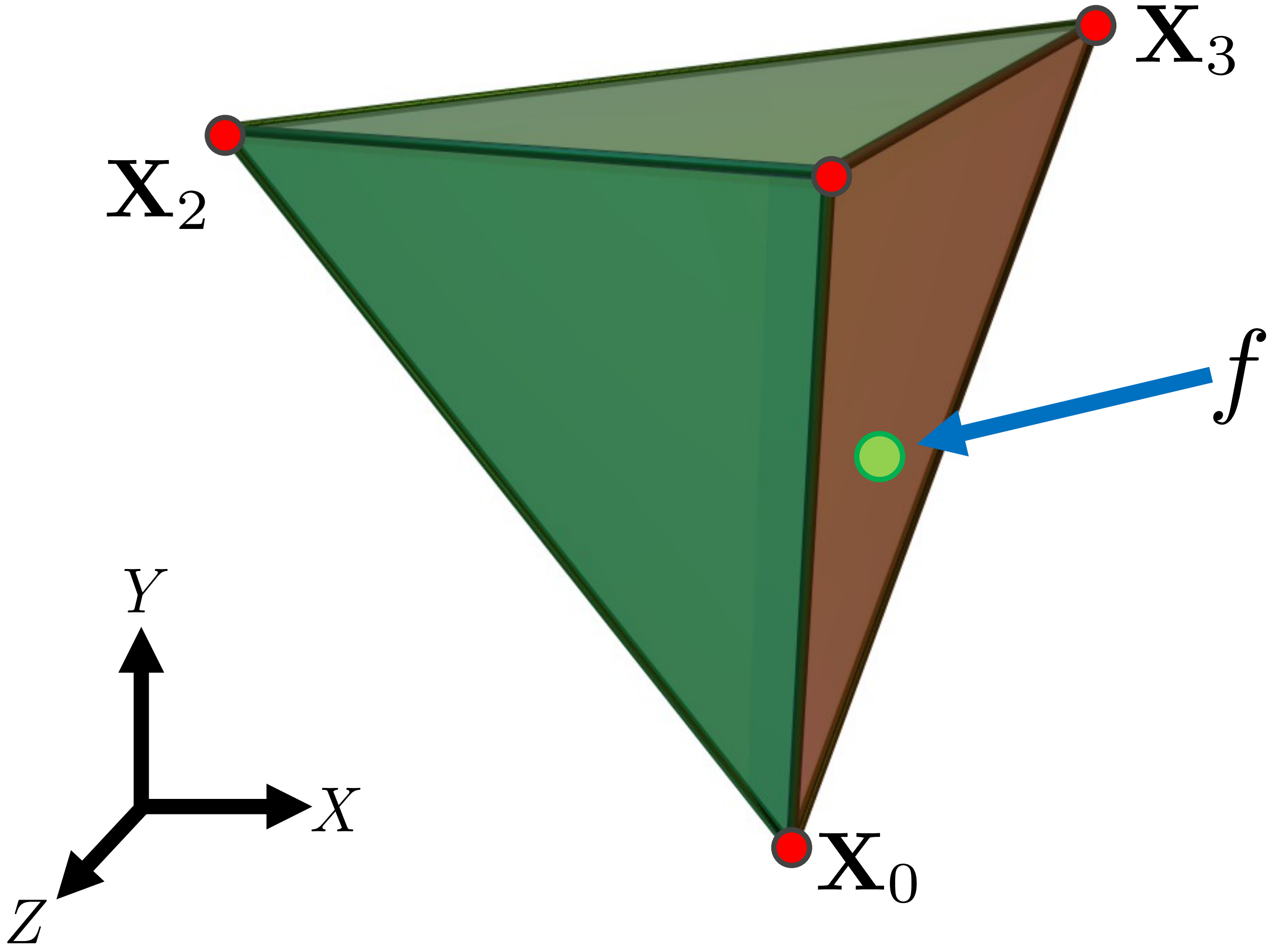
Finite Elements for Deformation



Finite Elements for Deformation



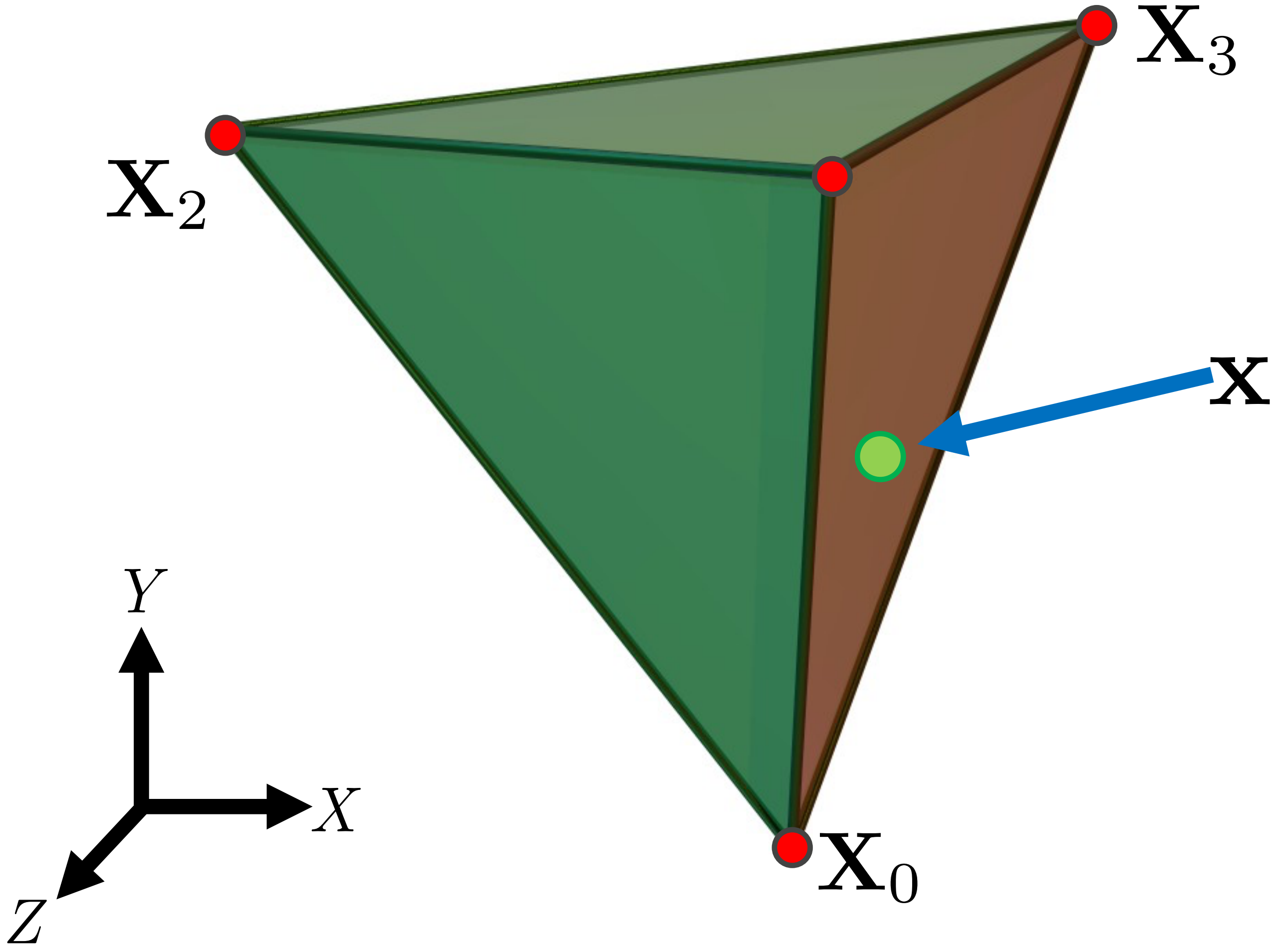
Finite Elements for Deformation



Reference (Undeformed) Space

$$f(\mathbf{X}) = \sum_{i=0}^{i=3} \underbrace{f_i}_{f(\mathbf{X}_i)} \overbrace{\phi_i(\mathbf{X})}^{\text{shape function}}$$

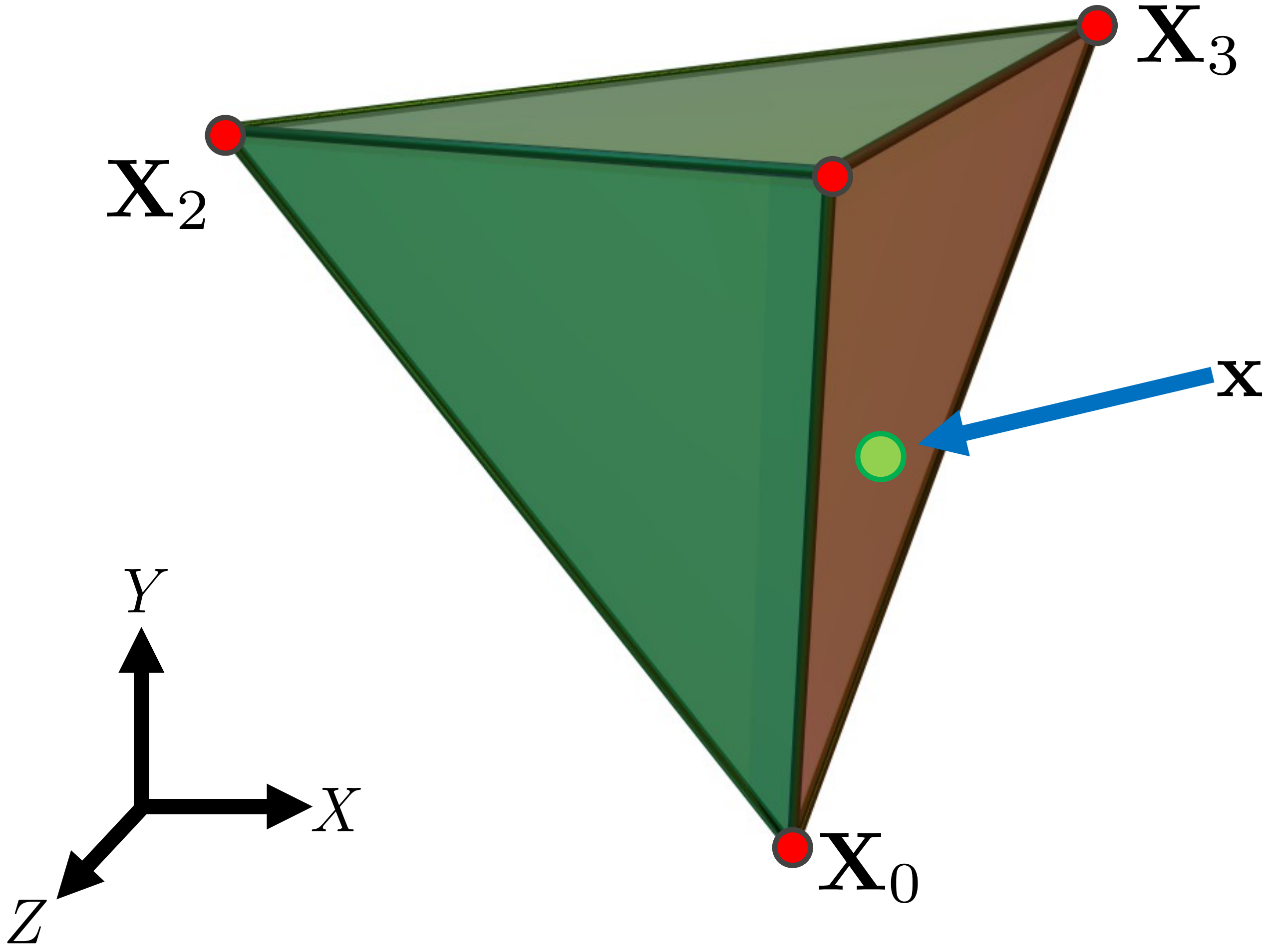
Finite Elements for Deformation



Reference (Undeformed) Space

$$\mathbf{x}\left(\mathbf{X}\right) = \sum_{i=0}^{i=3} \underbrace{\mathbf{x}_i\left(t\right)}_{\text{deformed vertex positions}} \overbrace{\phi_i\left(\mathbf{X}\right)}^{\text{shape function}}$$

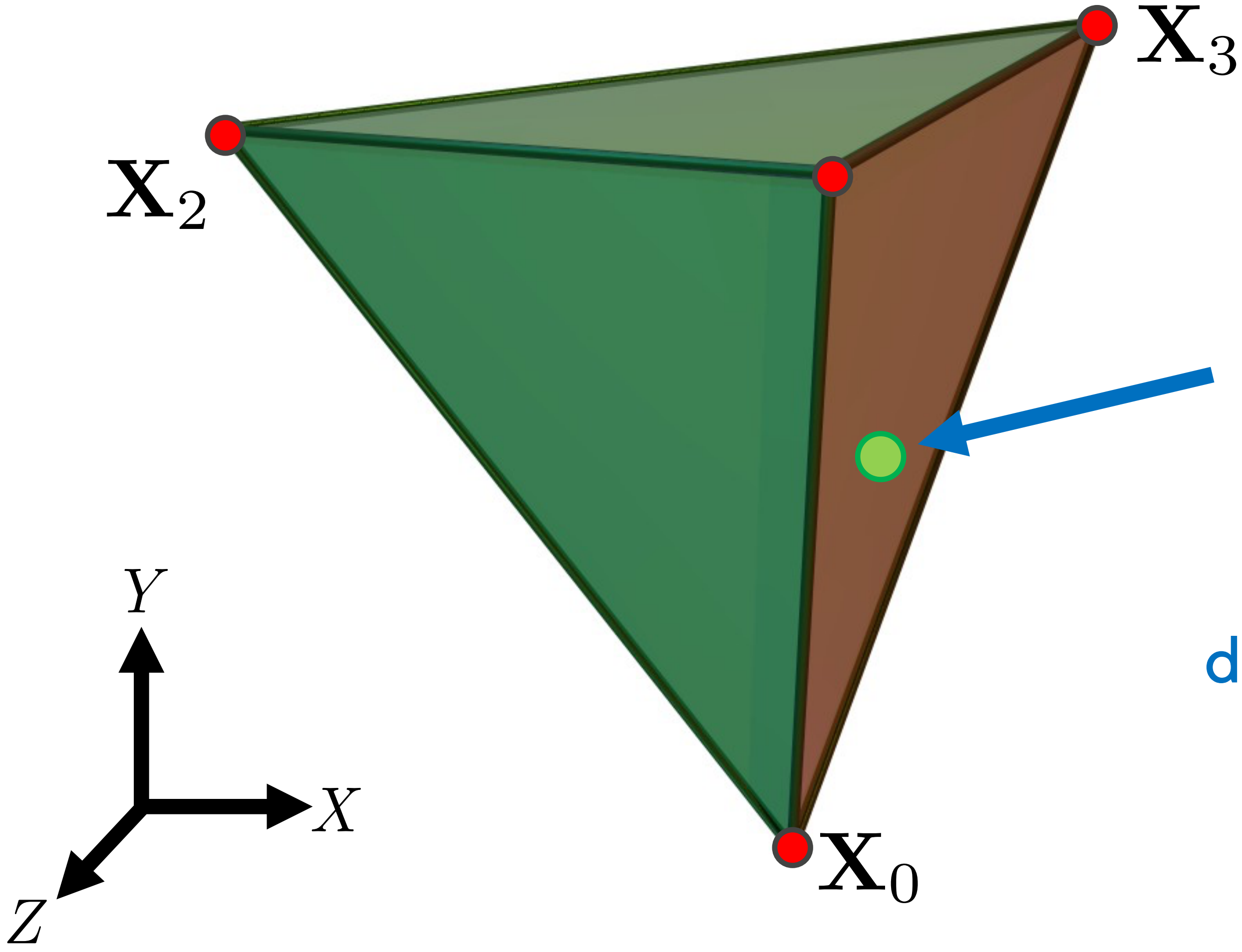
Finite Elements for Deformation



Reference (Undeformed) Space

$$\mathbf{x}\left(\mathbf{X}\right) = \underbrace{\begin{pmatrix}\phi_0 \mathrm{I} & \phi_1 \mathrm{I} & \phi_2 \mathrm{I} & \phi_3 \mathrm{I}\end{pmatrix}}_{\mathrm{N}\left(\mathbf{X}\right)} \underbrace{\begin{pmatrix}\mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3\end{pmatrix}}_{\mathbf{q}\left(t\right)}$$

Kinetic Energy of a Tetrahedron



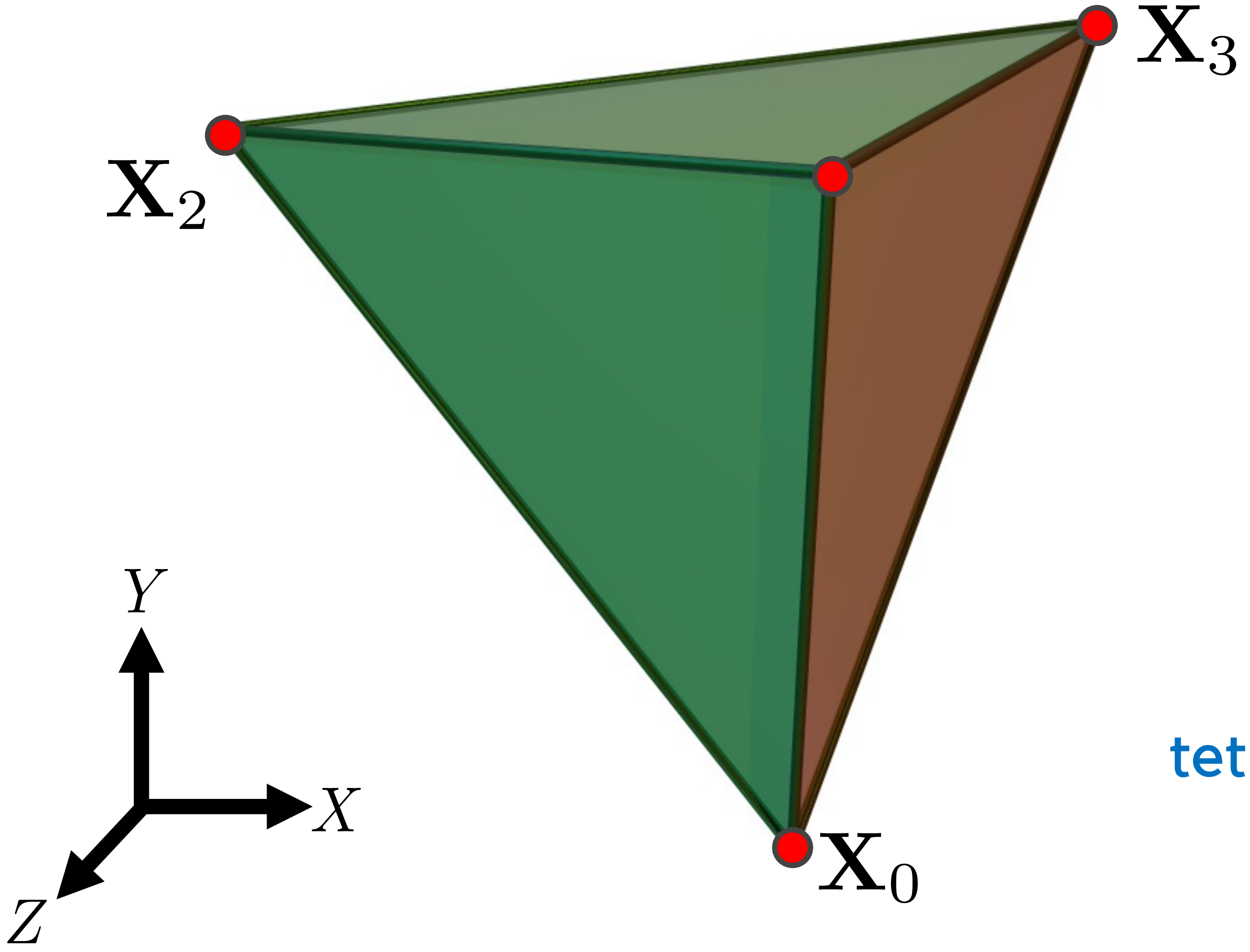
infinitesimal volume

$$\frac{1}{2}\rho\|\mathbf{v}\left(\mathbf{X}\right)\|_2^2 d\Omega$$

density ($\frac{kg}{m^3}$)

Reference (Undeformed) Space

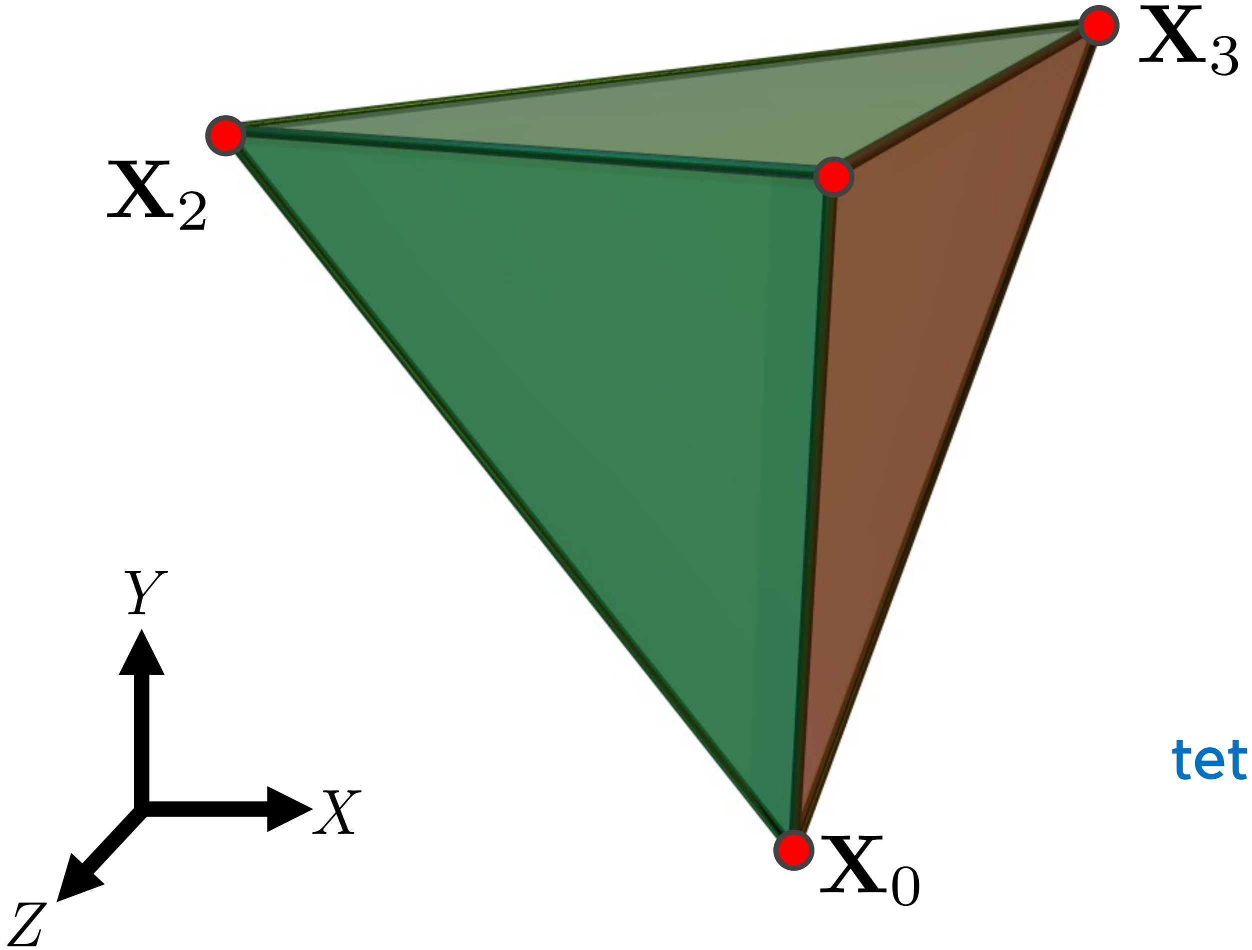
Kinetic Energy of a Tetrahedron



Reference (Undeformed) Space

$$\underbrace{\frac{1}{2}\int_{\Omega}}_{\text{tetrahedron domain}} \rho \| \mathbf{v}\left(\mathbf{X}\right) \|_2^2 d\Omega$$

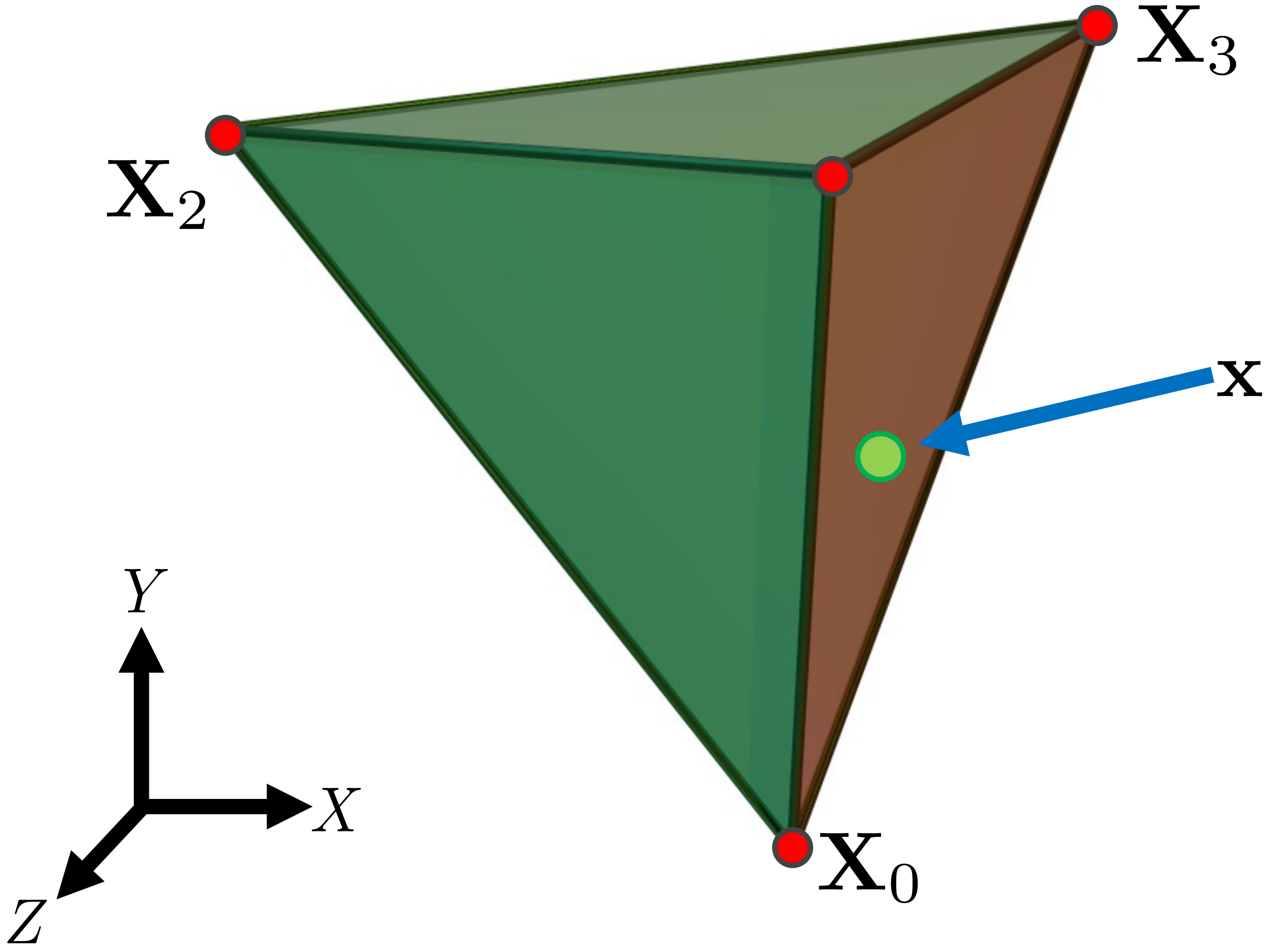
Kinetic Energy of a Tetrahedron



$$\frac{1}{2}\int_{\underbrace{\Omega}_{\text{tetrahedron domain}}} \rho \mathbf{v}^T \mathbf{v} \, d\Omega$$

Reference (Undeformed) Space

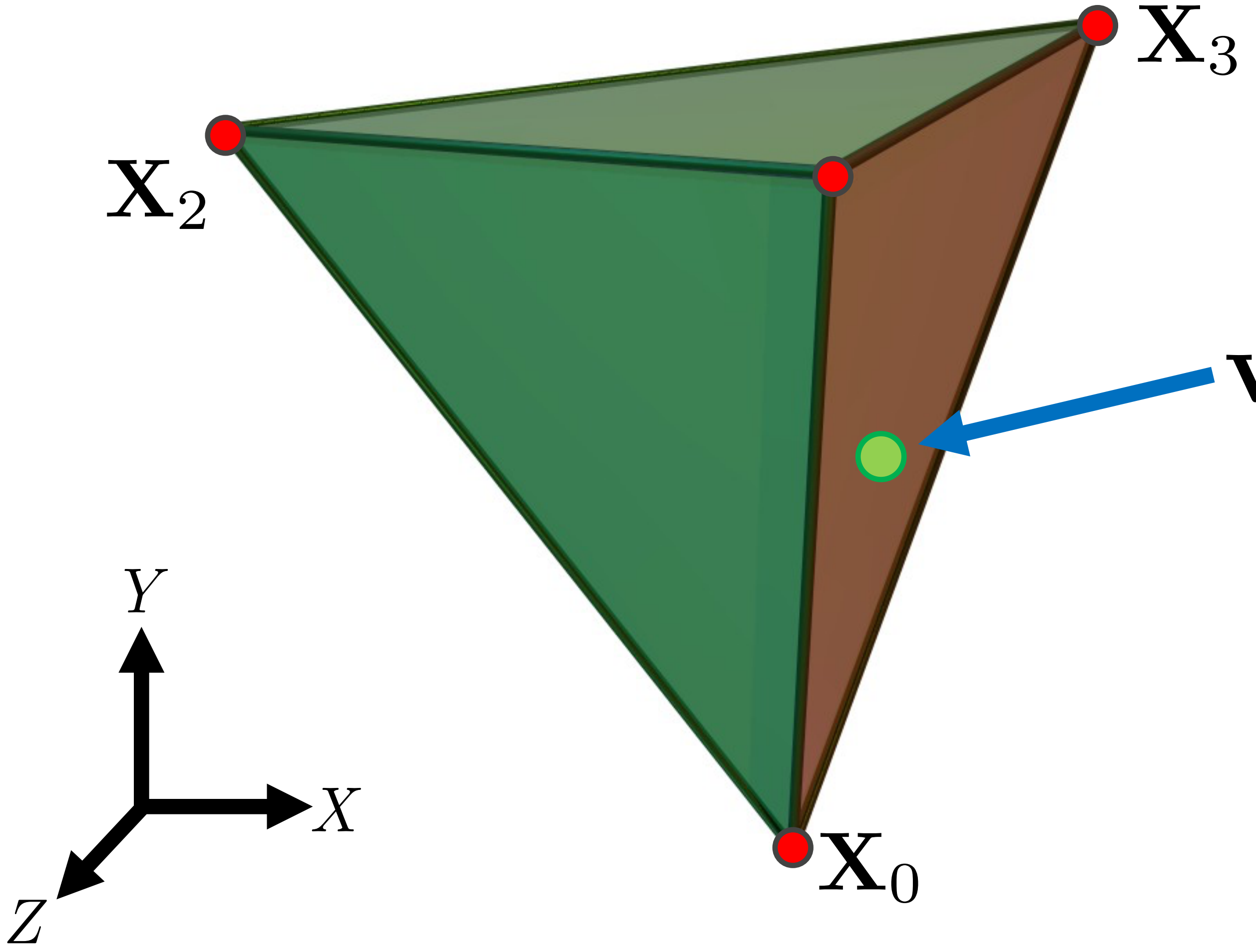
Finite Elements for Deformation



Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}) = \underbrace{\begin{pmatrix} \phi_0 \mathrm{I} & \phi_1 \mathrm{I} & \phi_2 \mathrm{I} & \phi_3 \mathrm{I} \end{pmatrix}}_{\mathrm{N}(\mathbf{X})} \underbrace{\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}}_{\mathbf{q}(t)}$$

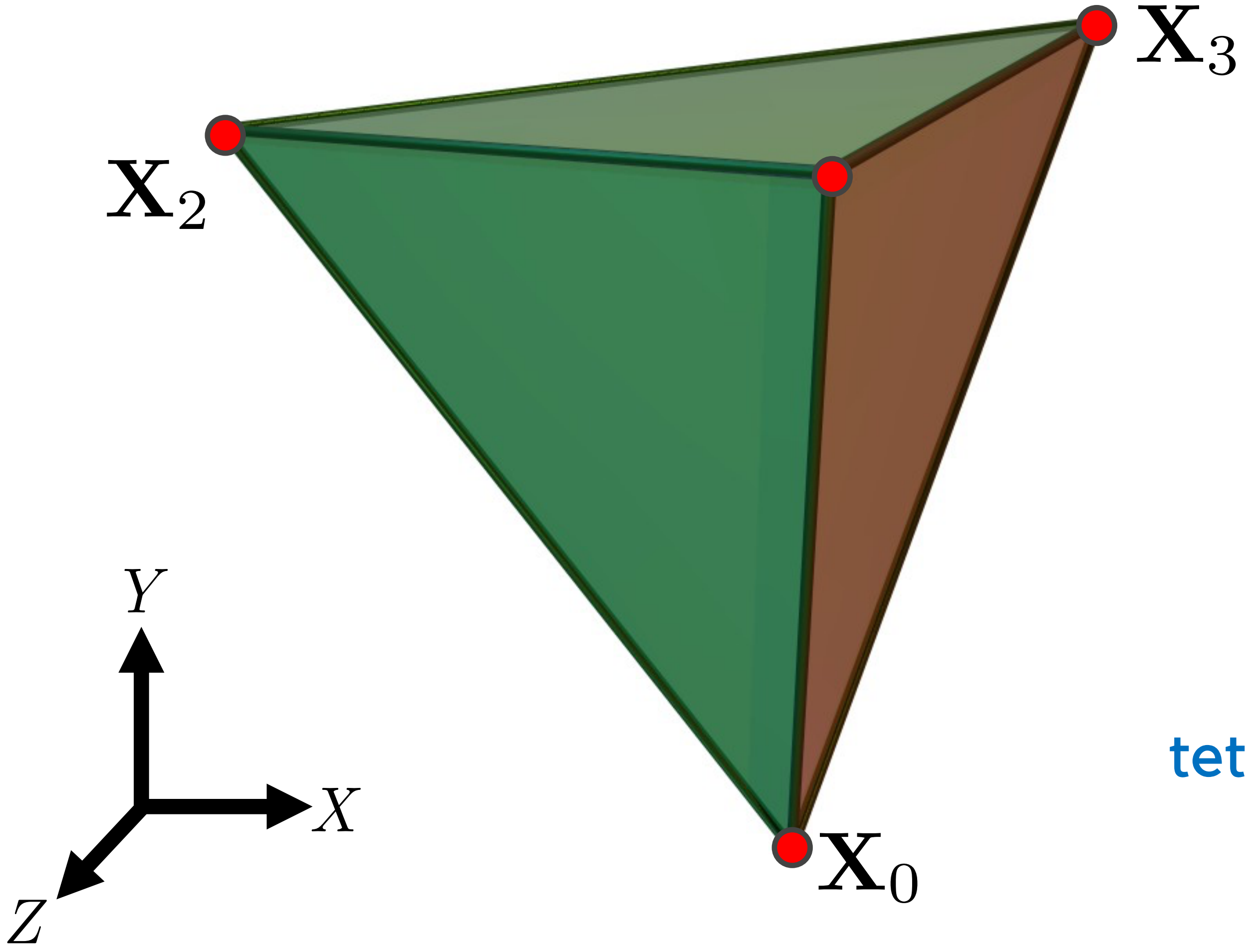
Finite Elements for Deformation



$$\mathbf{v}\left(\mathbf{X}\right) = \dot{\mathbf{x}}\left(\mathbf{X}\right) = \mathrm{N}\left(\mathbf{X}\right)\dot{\mathbf{q}}$$

Reference (Undeformed) Space

Kinetic Energy of a Tetrahedron

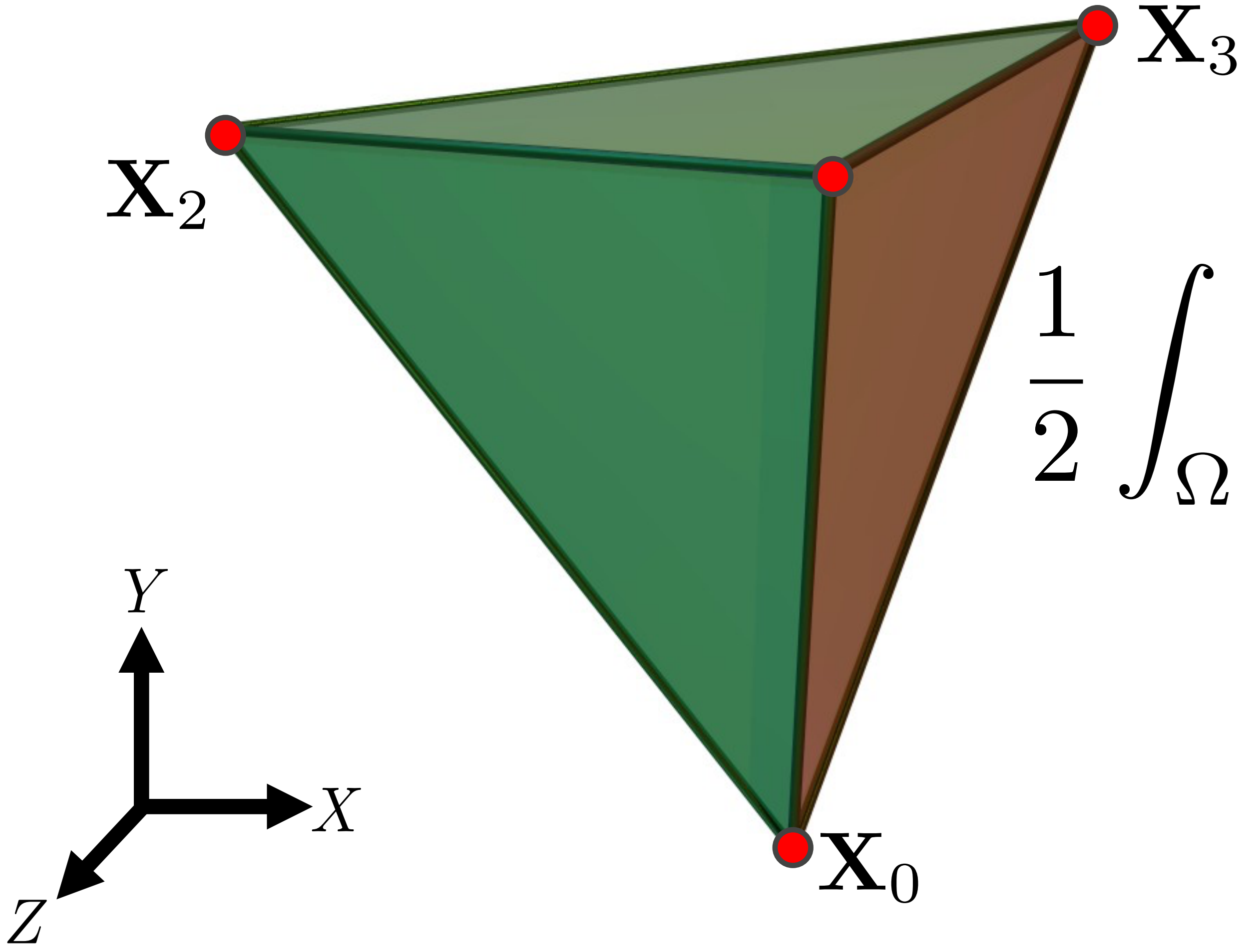


Reference (Undeformed) Space

$$\frac{1}{2}\int_{\Omega} \rho \mathbf{v}^T \mathbf{v} \, d\Omega$$

tetrahedron domain

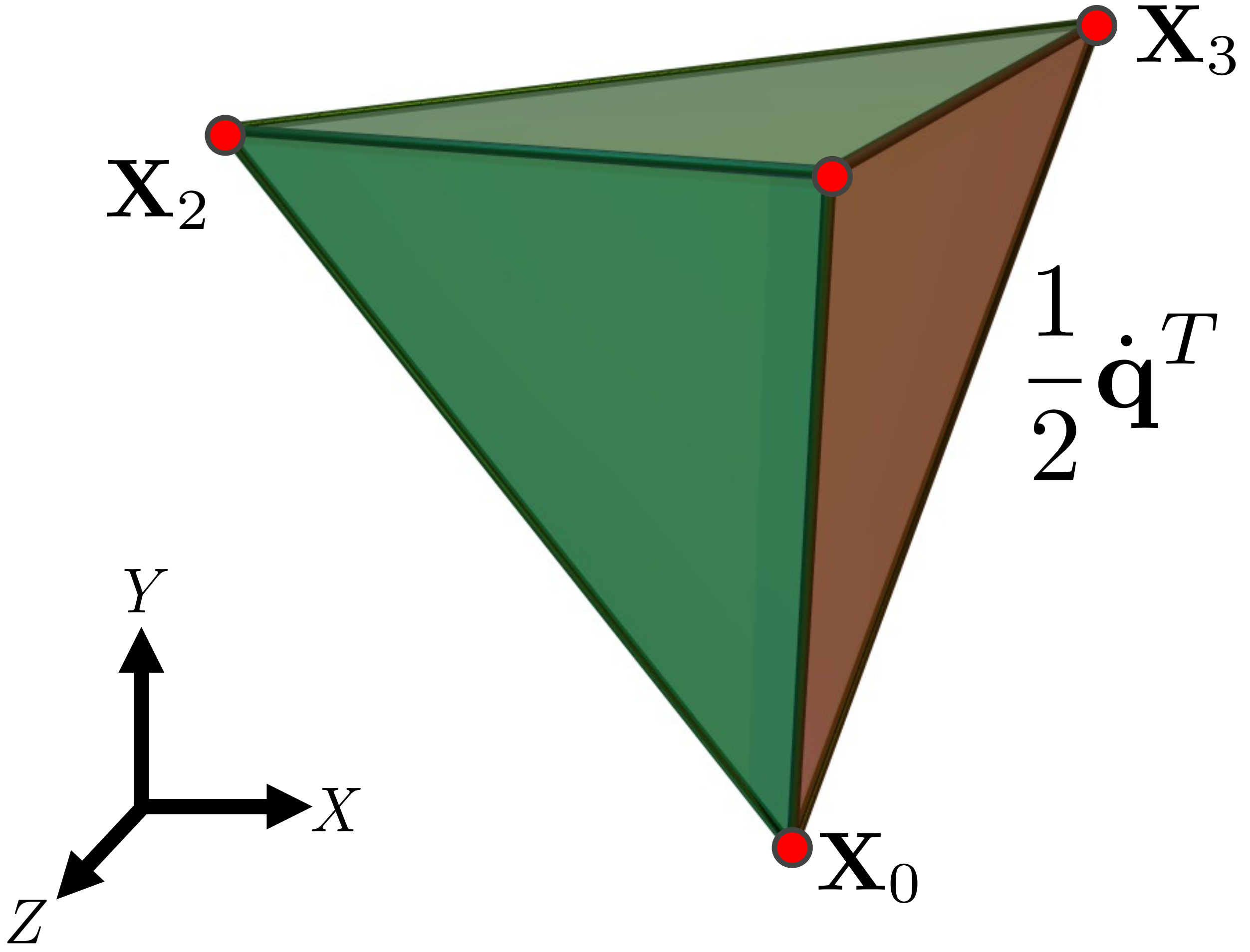
Kinetic Energy of a Tetrahedron



$$\frac{1}{2}\int_{\Omega} \rho \left(\dot{\mathbf{q}}^T \mathrm{N}\left(\mathbf{X}\right)^T \mathrm{N}\left(\mathbf{X}\right) \dot{\mathbf{q}} \right) d\Omega$$

Reference (Undeformed) Space

Kinetic Energy of a Tetrahedron



Reference (Undeformed) Space

$$\frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\int_\Omega \rho \mathrm{N}\left(\mathbf{X}\right)^T \mathrm{N}\left(\mathbf{X}\right) d\Omega\right)}_{\mathrm{M}_0} \dot{\mathbf{q}}$$

Integrating the Mass Matrix

$$\underbrace{\int_\Omega \rho \mathrm{N}\left(\mathbf{X}\right)^T \mathrm{N}\left(\mathbf{X}\right) d\Omega}_{\text{Integrate over tetrahedron}}$$

$$\int_\Omega \rho \begin{pmatrix} \phi_0\phi_0\mathrm{I} & \phi_0\phi_1\mathrm{I} & \phi_0\phi_2\mathrm{I} & \phi_0\phi_3\mathrm{I} \\ \phi_1\phi_0\mathrm{I} & \phi_1\phi_1\mathrm{I} & \phi_1\phi_2\mathrm{I} & \phi_1\phi_3\mathrm{I} \\ \phi_2\phi_0\mathrm{I} & \phi_2\phi_1\mathrm{I} & \phi_2\phi_2\mathrm{I} & \phi_2\phi_3\mathrm{I} \\ \phi_3\phi_0\mathrm{I} & \phi_3\phi_1\mathrm{I} & \phi_3\phi_2\mathrm{I} & \phi_3\phi_3\mathrm{I} \end{pmatrix} d\Omega$$

Integrating the Mass Matrix

$$\int_\Omega \rho \begin{pmatrix} \phi_0\phi_0\mathrm{I} & \phi_0\phi_1\mathrm{I} & \phi_0\phi_2\mathrm{I} & \phi_0\phi_3\mathrm{I} \\ \phi_1\phi_0\mathrm{I} & \phi_1\phi_1\mathrm{I} & \phi_1\phi_2\mathrm{I} & \phi_1\phi_3\mathrm{I} \\ \phi_2\phi_0\mathrm{I} & \phi_2\phi_1\mathrm{I} & \phi_2\phi_2\mathrm{I} & \phi_2\phi_3\mathrm{I} \\ \phi_3\phi_0\mathrm{I} & \phi_3\phi_1\mathrm{I} & \phi_3\phi_2\mathrm{I} & \phi_3\phi_3\mathrm{I} \end{pmatrix} d\Omega$$

evaluate each term separately

$$\rho \int_\Omega \phi_r\left(\mathbf{X}\right)\phi_s\left(\mathbf{X}\right) d\Omega\mathrm{I}$$

Integrating the Mass Matrix

evaluate each term separately

$$\rho \int_\Omega \phi_r\left(\mathbf{X}\right)\phi_s\left(\mathbf{X}\right) d\Omega \mathrm{I}$$

integration using barycentric coordinates

$$\overbrace{6\rho \cdot \underbrace{vol}_{\text{tetrahedron volume}}}^{\text{tetrahedron mass}} \cdot \int_0^1 \int_0^{1-\phi_1} \int_0^{1-\phi_1-\phi_2} \left(\phi_r \phi_s\right) d\phi_3 d\phi_2 d\phi_1$$

need this identity as well

$$\phi_0\left(\mathbf{X}\right) = 1 - \phi_1\left(\mathbf{X}\right) - \phi_2\left(\mathbf{X}\right) - \phi_3\left(\mathbf{X}\right)$$

Integrating the Mass Matrix – An Example

integration using barycentric coordinates

$$6\rho \cdot vol \cdot \int_0^1 \int_0^{1-\phi_1} \int_0^{1-\phi_1-\phi_2} (\phi_1 \phi_1)\, d\phi_3 d\phi_2 d\phi_1$$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X}) - \phi_3(\mathbf{X})$$

Integrating the Mass Matrix – An Example

integration using barycentric coordinates

$$6\rho \cdot vol \cdot \int_0^1 \int_0^{1-\phi_1} \int_0^{1-\phi_1-\phi_2} \left(\phi_1^2\right) d\phi_3 d\phi_2 d\phi_1$$

integrate from inside out

$$6\rho \cdot vol \cdot \int_0^1 \int_0^{1-\phi_1} {\phi_1}^2 \left(1 - \phi_1 - \phi_2\right) d\phi_2 d\phi_1$$

Integrating the Mass Matrix – An Example

integration using barycentric coordinates

$$6\rho \cdot vol \cdot \int_0^1 \int_0^{1-\phi_1} {\phi_1}^2 \; (1 - \phi_1 - \phi_2) \, d\phi_2 d\phi_1$$

integrate from inside out

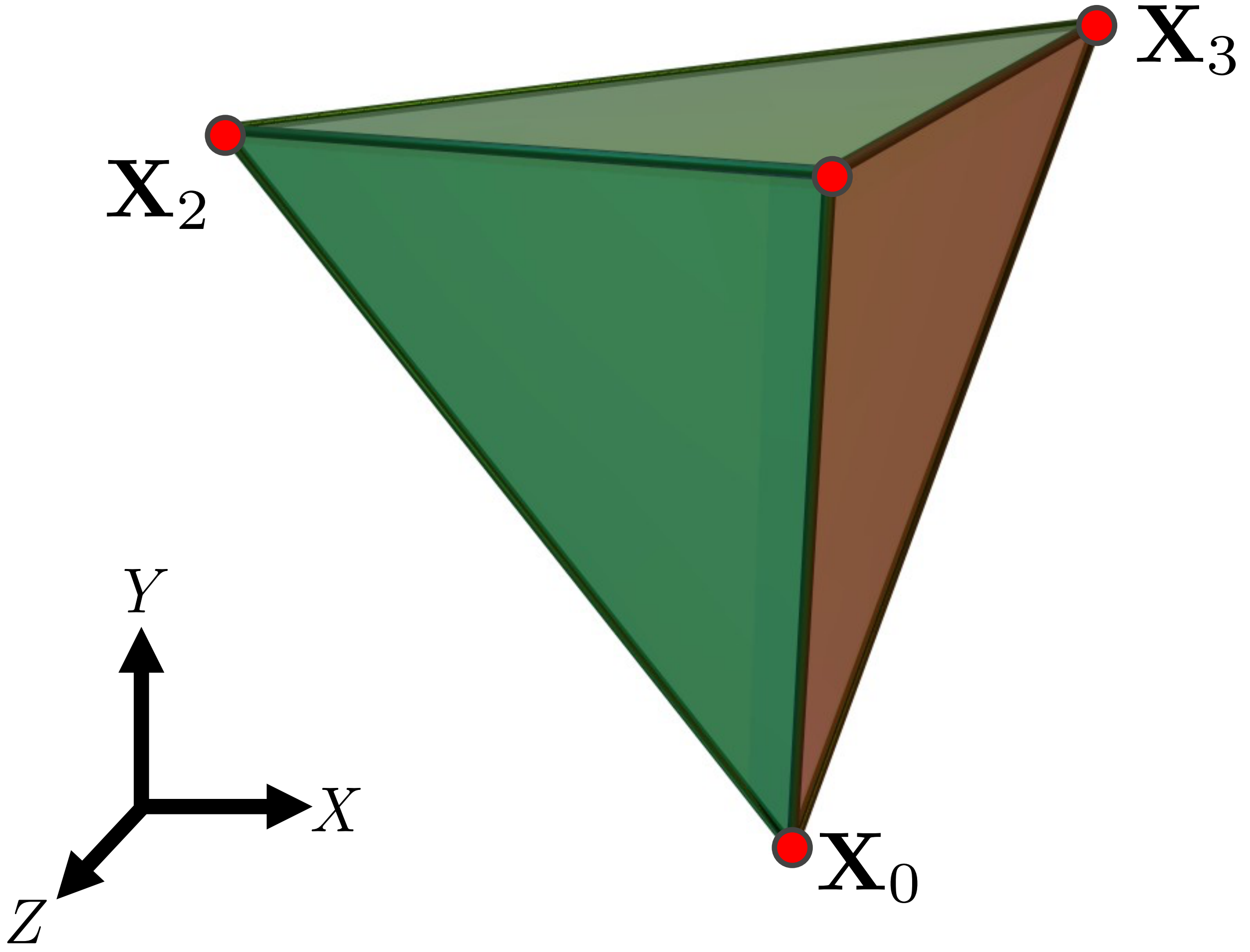
$$6\rho \cdot vol \cdot \int_0^1 \frac{{\phi_1}^2 \, (\phi_1 - 1)^2}{2} d\phi_1$$

$$6\rho \cdot vol \cdot \frac{1}{60} = \frac{\rho \cdot vol}{10}$$

Integrating the Mass Matrix

$$\underbrace{\int_\Omega \rho \begin{pmatrix} \phi_0\phi_0\mathrm{I} & \phi_0\phi_1\mathrm{I} & \phi_0\phi_2\mathrm{I} & \phi_0\phi_3\mathrm{I} \\ \phi_1\phi_0\mathrm{I} & \phi_1\phi_1\mathrm{I} & \phi_1\phi_2\mathrm{I} & \phi_1\phi_3\mathrm{I} \\ \phi_2\phi_0\mathrm{I} & \phi_2\phi_1\mathrm{I} & \phi_2\phi_2\mathrm{I} & \phi_2\phi_3\mathrm{I} \\ \phi_3\phi_0\mathrm{I} & \phi_3\phi_1\mathrm{I} & \phi_3\phi_2\mathrm{I} & \phi_3\phi_3\mathrm{I} \end{pmatrix} d\Omega}_{\mathrm{M}_0}$$

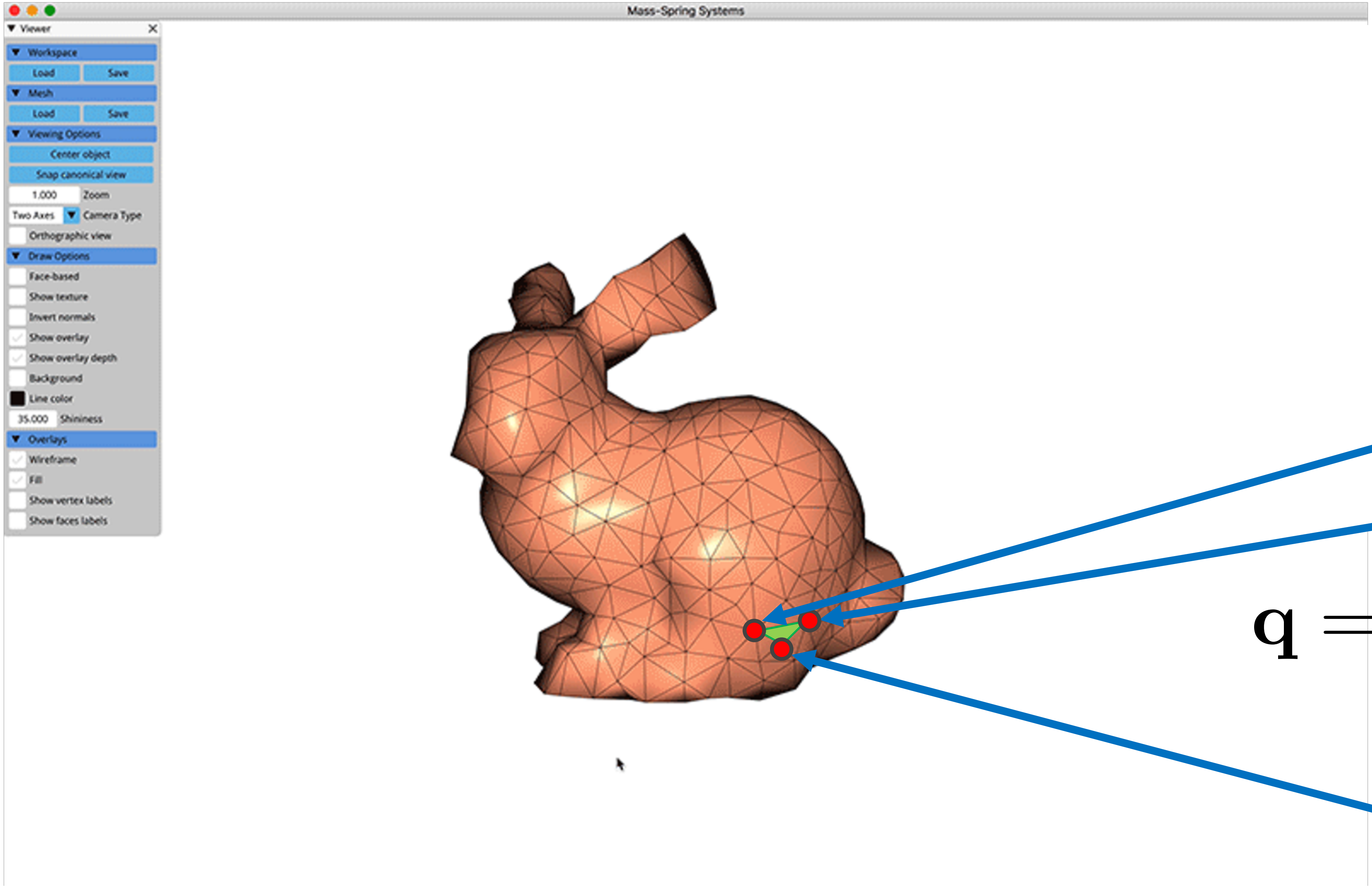
Kinetic Energy of a Tetrahedron



Reference (Undeformed) Space

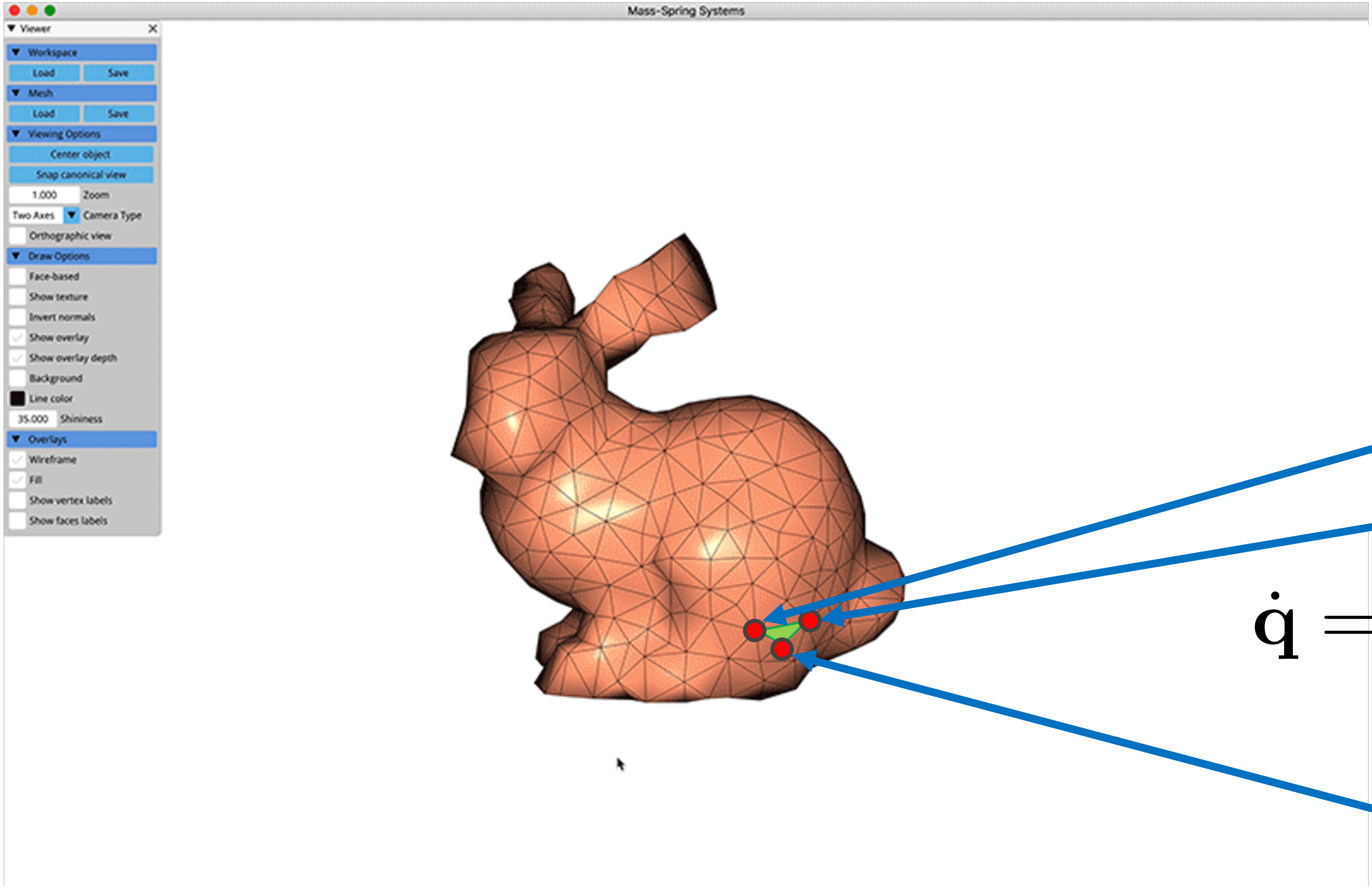
$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M}_0 \dot{\mathbf{q}}$$

Generalized Coordinates for Bunny FEM



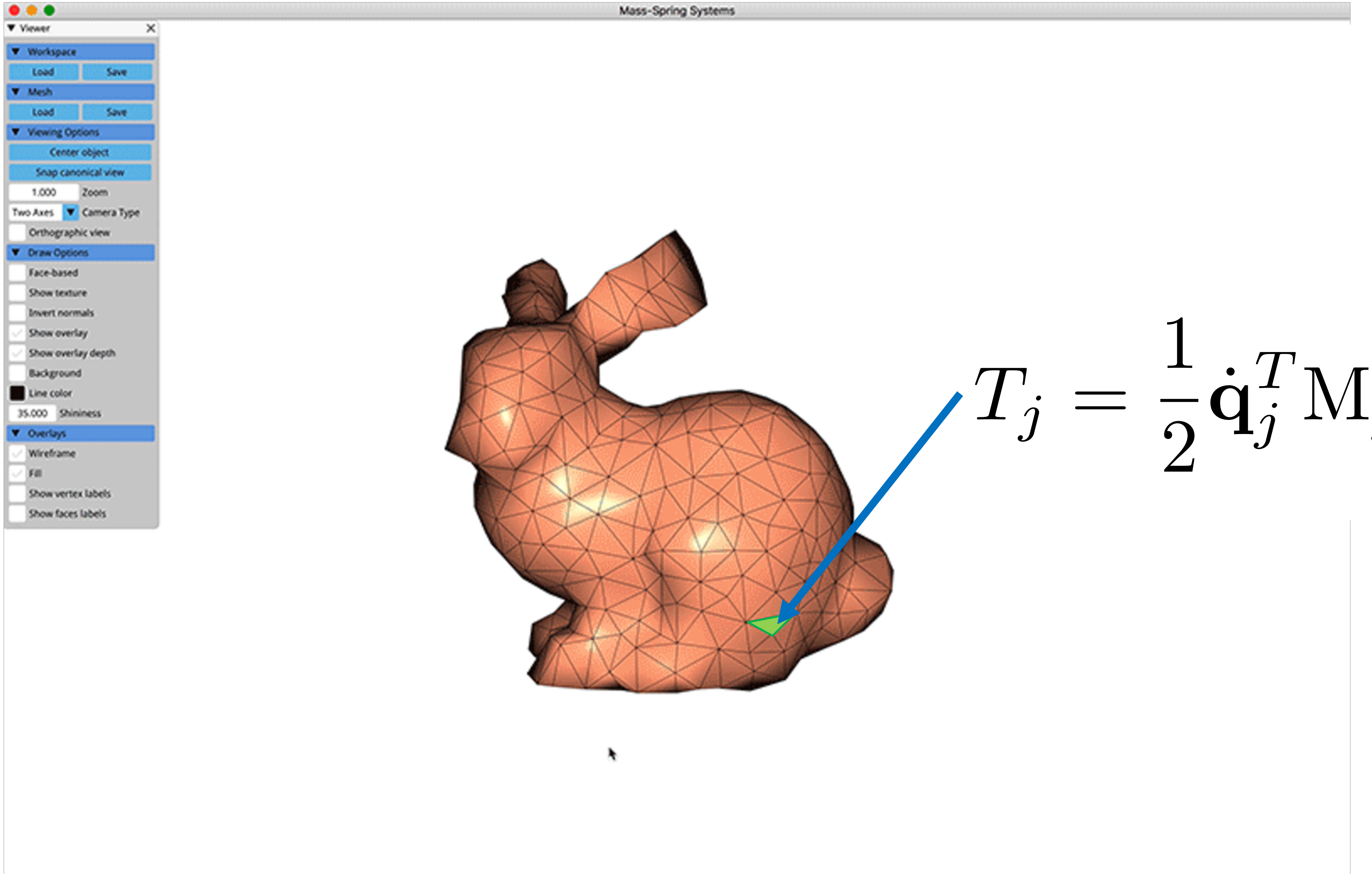
$$\mathbf{q} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$$

Generalized Coordinates for Bunny FEM



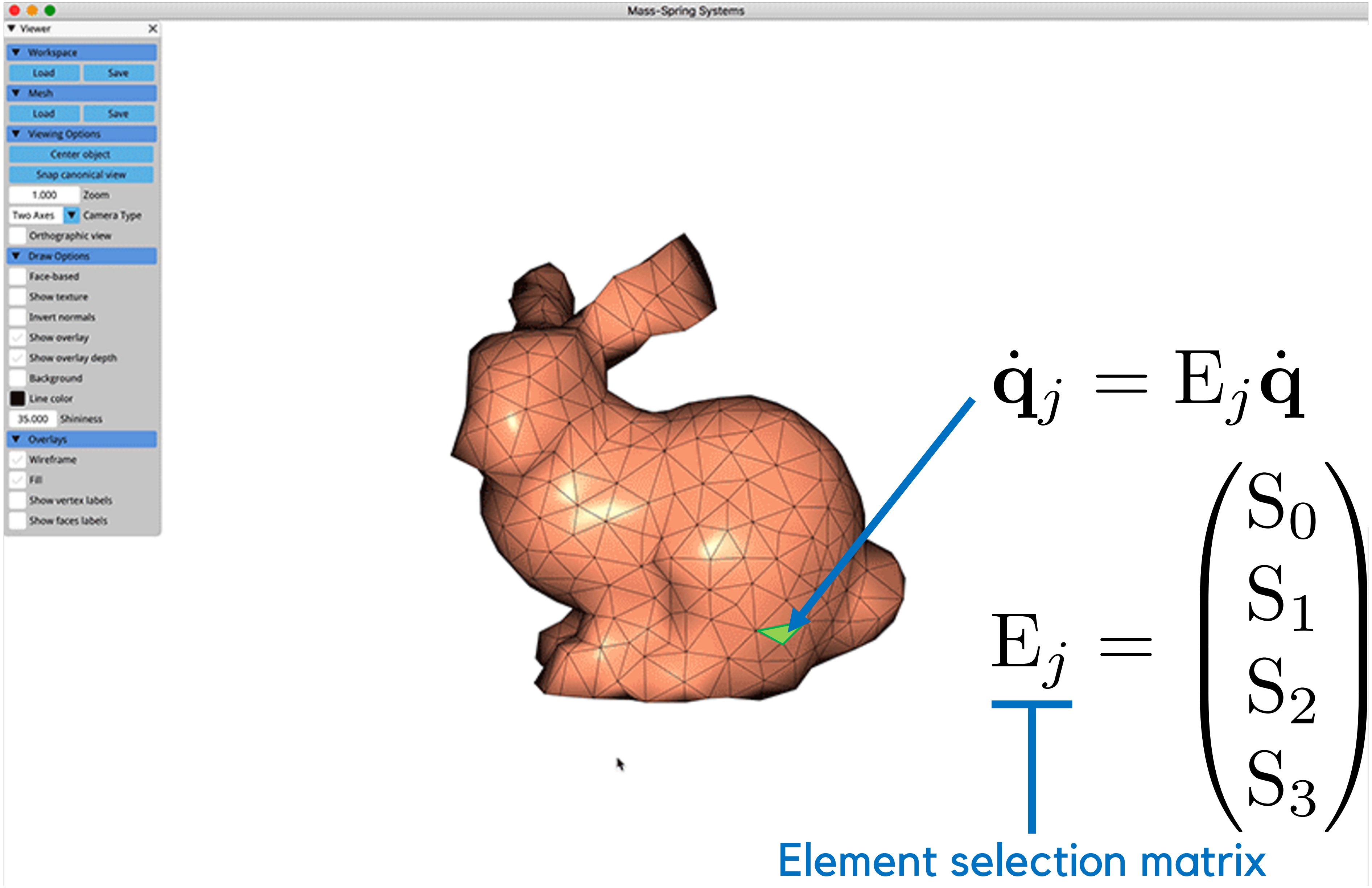
$$\dot{\mathbf{q}} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}$$

Kinetic Energy for a Bunny



$$T_j = \frac{1}{2}\dot{\mathbf{q}}_j^T \mathrm{M}_j \dot{\mathbf{q}}_j$$

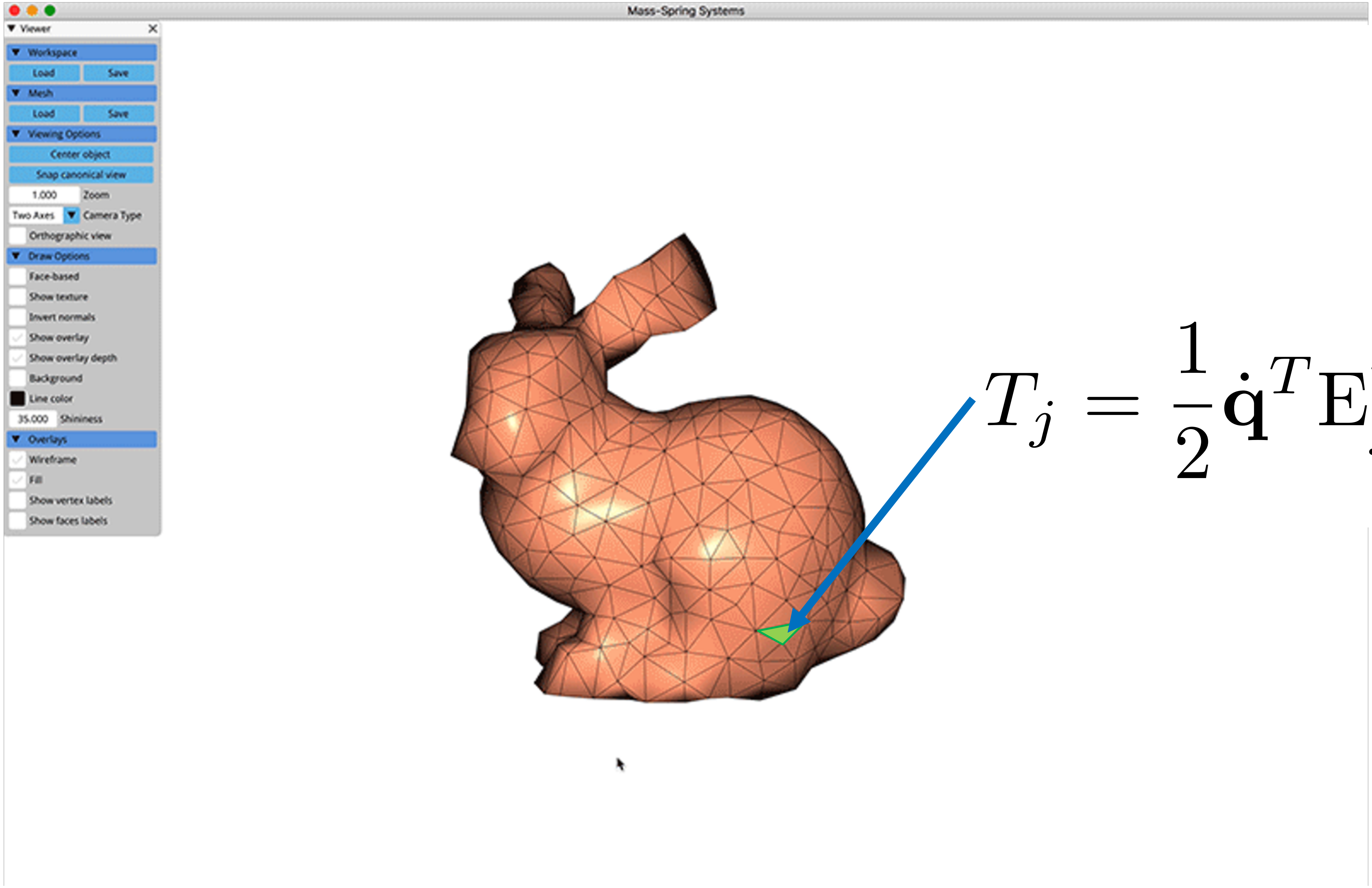
Kinetic Energy for a Bunny



$$\dot{\mathbf{q}}_j = \mathrm{E}_j \dot{\mathbf{q}}$$

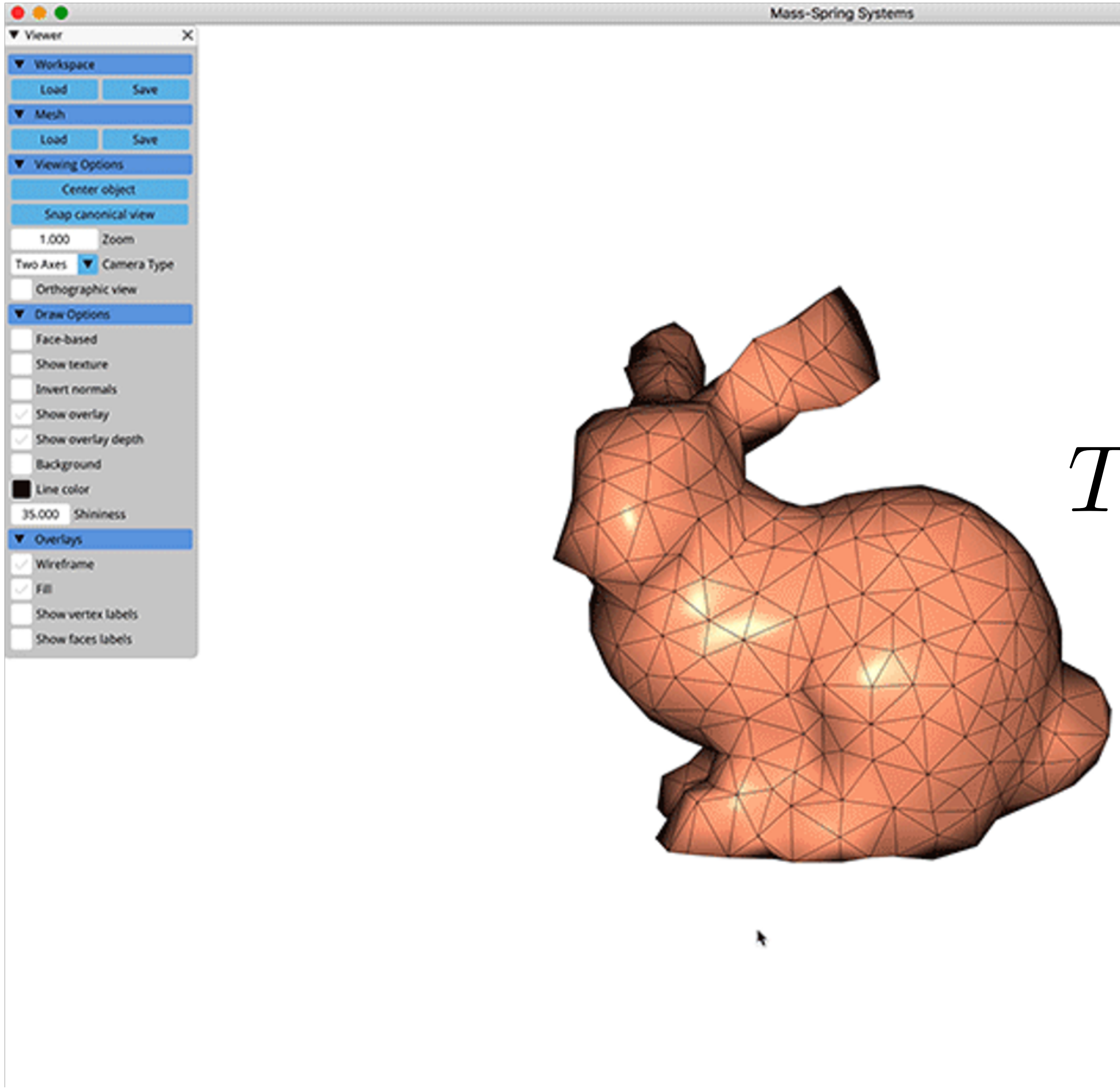
$$\underbrace{\mathrm{E}_j}_{\text{Element selection matrix}} = \begin{pmatrix} \mathrm{S}_0 \\ \mathrm{S}_1 \\ \mathrm{S}_2 \\ \mathrm{S}_3 \end{pmatrix}$$

Kinetic Energy for a Bunny



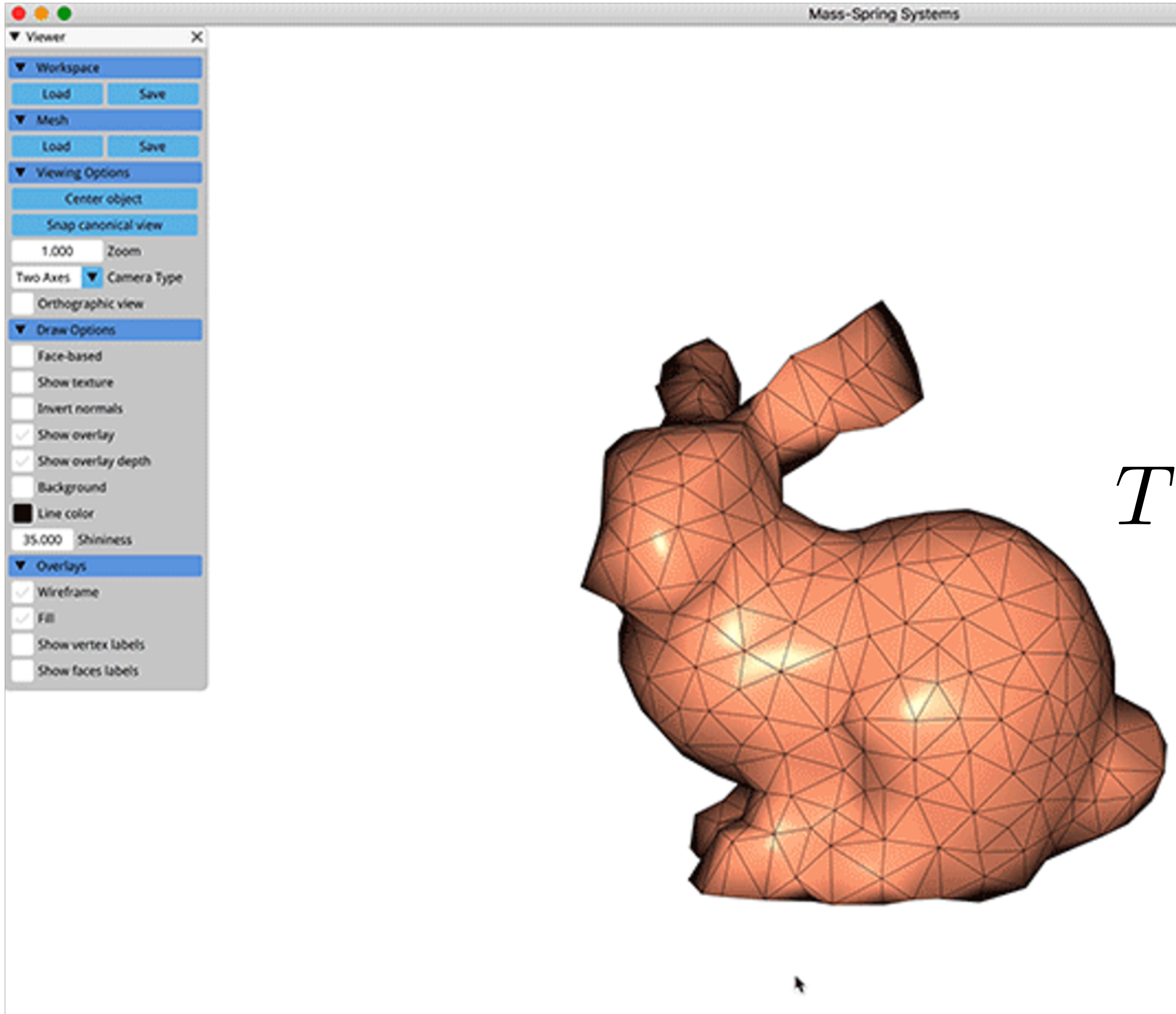
$$T_j = \frac{1}{2}\dot{\mathbf{q}}^T \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j \dot{\mathbf{q}}$$

Kinetic Energy for a Bunny



$$T = \sum_{j=0}^{j-1} \frac{1}{2} \dot{\mathbf{q}}^T \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j \dot{\mathbf{q}}$$

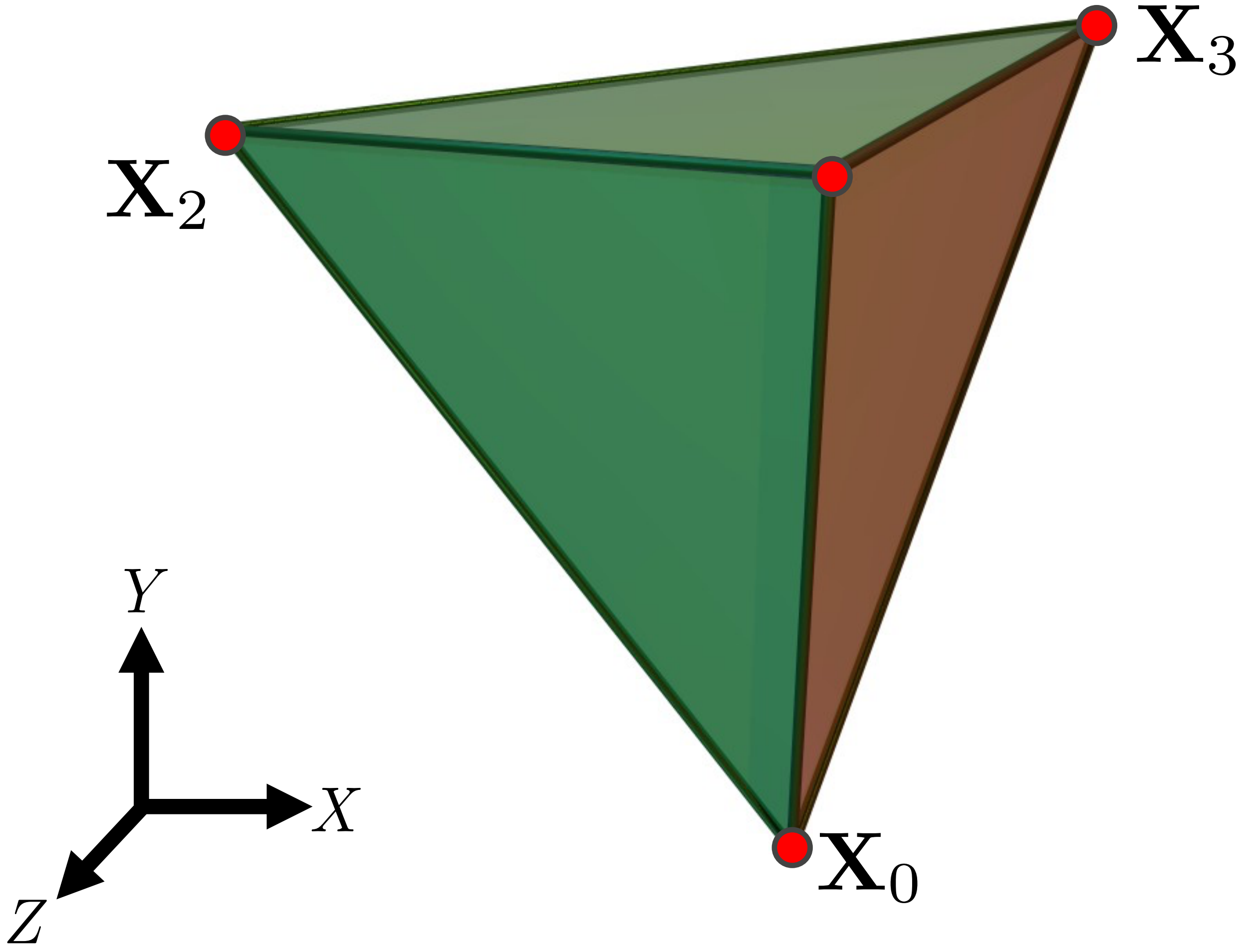
Kinetic Energy for a Bunny



$$T = \frac{1}{2}\dot{\mathbf{q}}^T \underbrace{\left(\sum_{j=0}^{j-1} \mathrm{E}_j^T \mathrm{M}_j \mathrm{E}_j\right)}_{\mathrm{M}} \dot{\mathbf{q}}$$

Assemble M by summing over all tetrahedra

Potential Energy for a Single Tetrahedron



Reference (Undeformed) Space

$$V = ?$$

Potential Energy for a 3D Spring

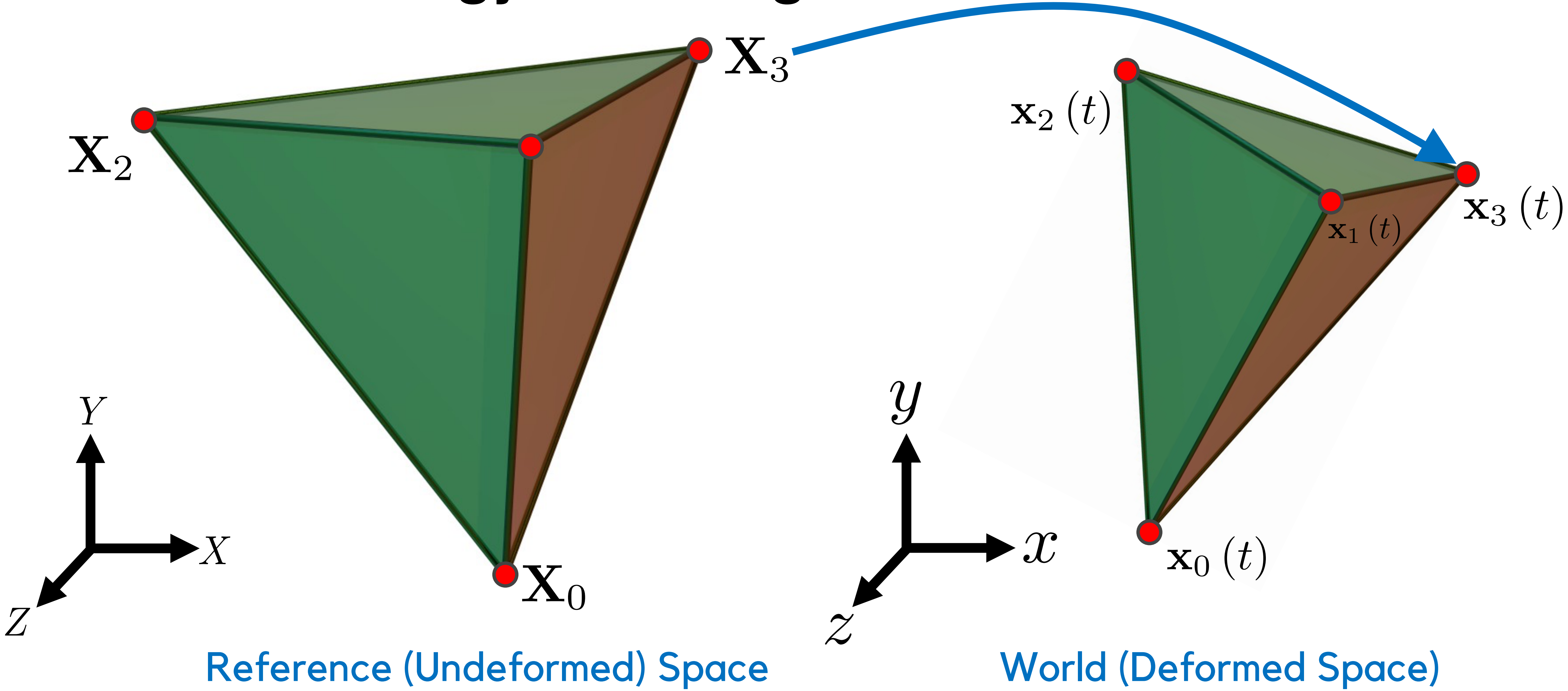
Strain $l - l_0$

deformed length

Potential Energy $\frac{1}{2}k\left(l - l_0\right)^2$

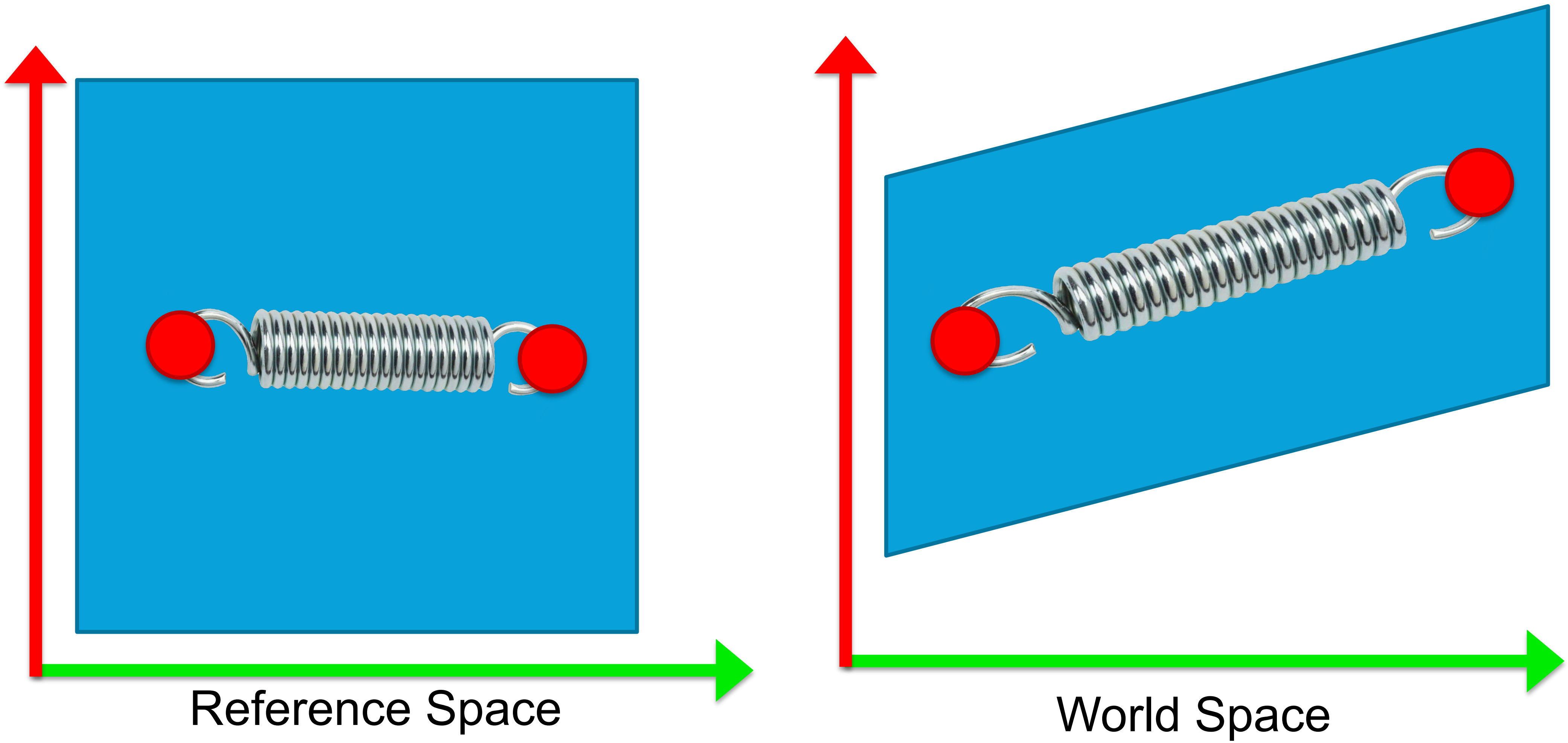
stiffness parameter

Potential Energy for a Single Tetrahedron

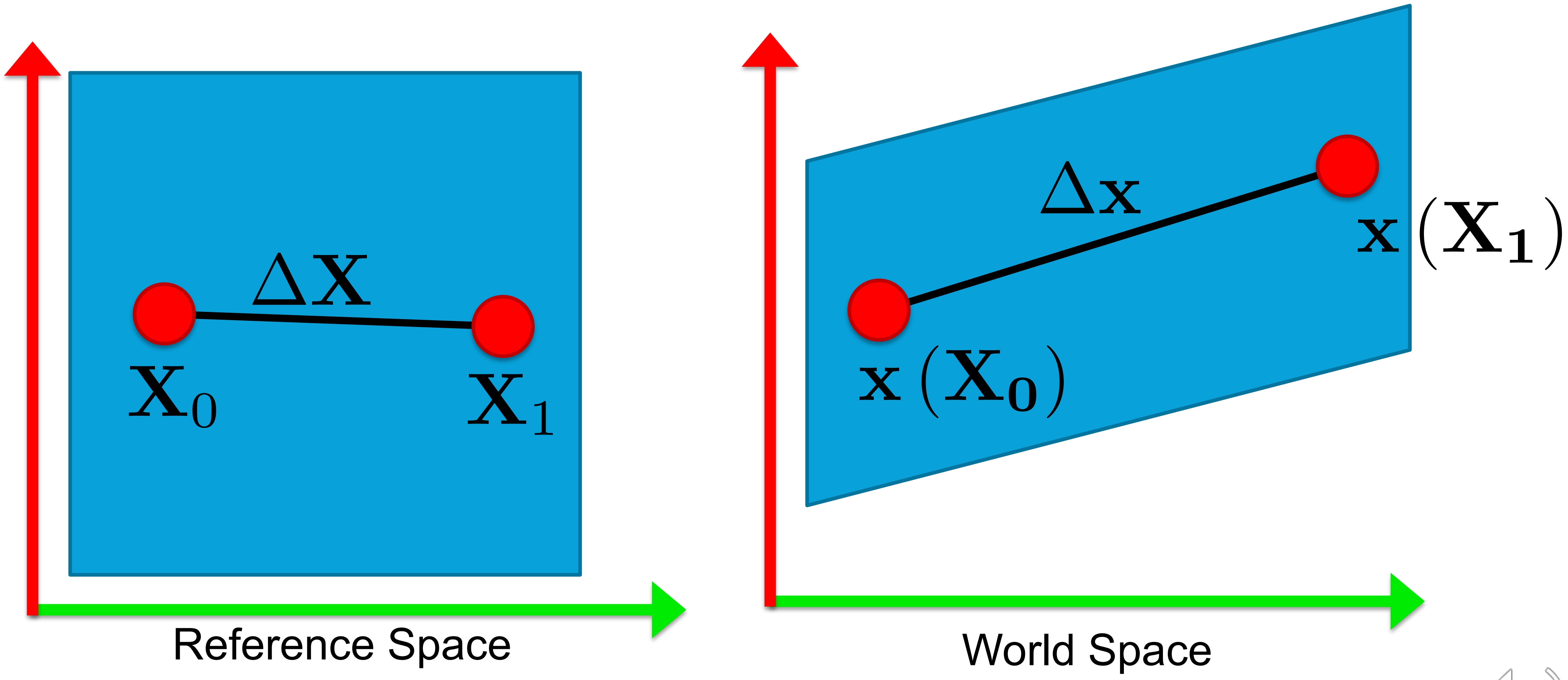


How do we measure strain ?

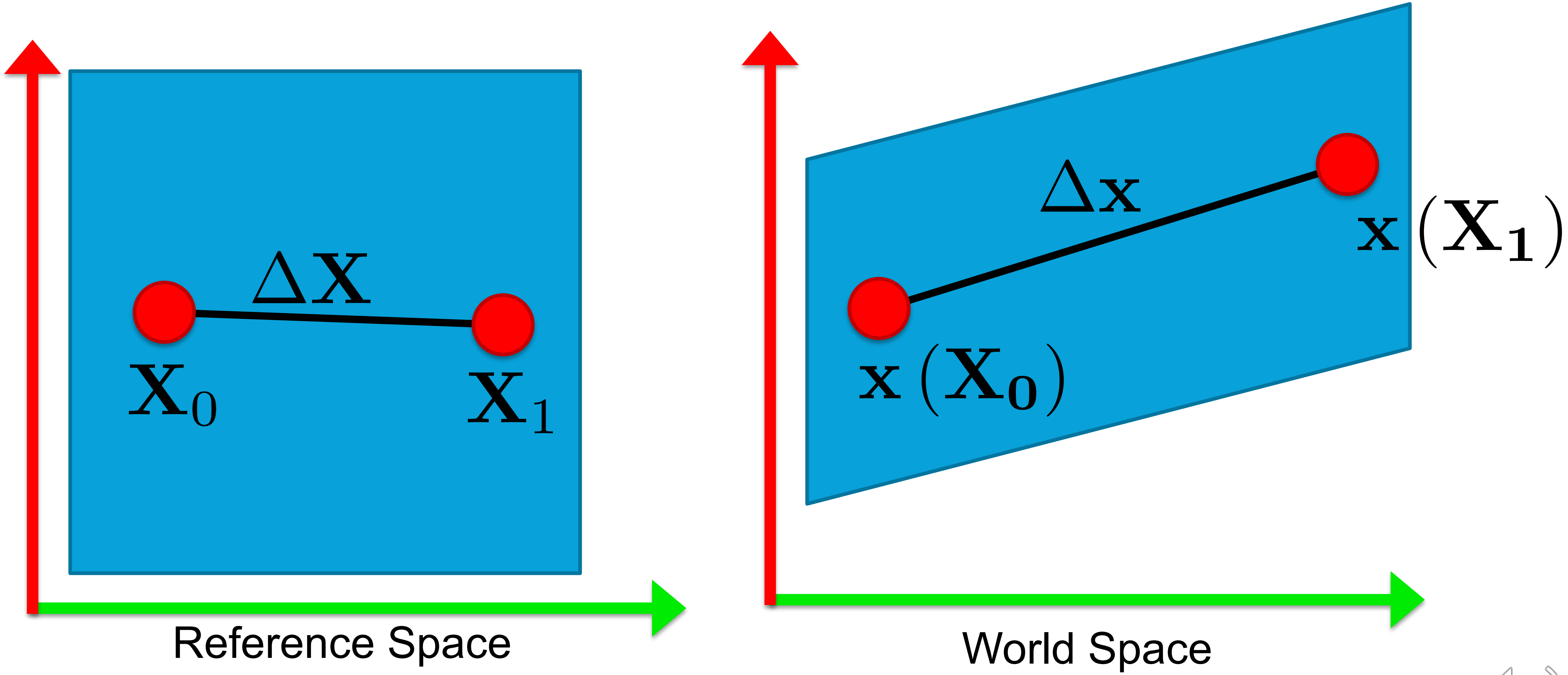
A Closer Look at Deformation



A Closer Look at Deformation

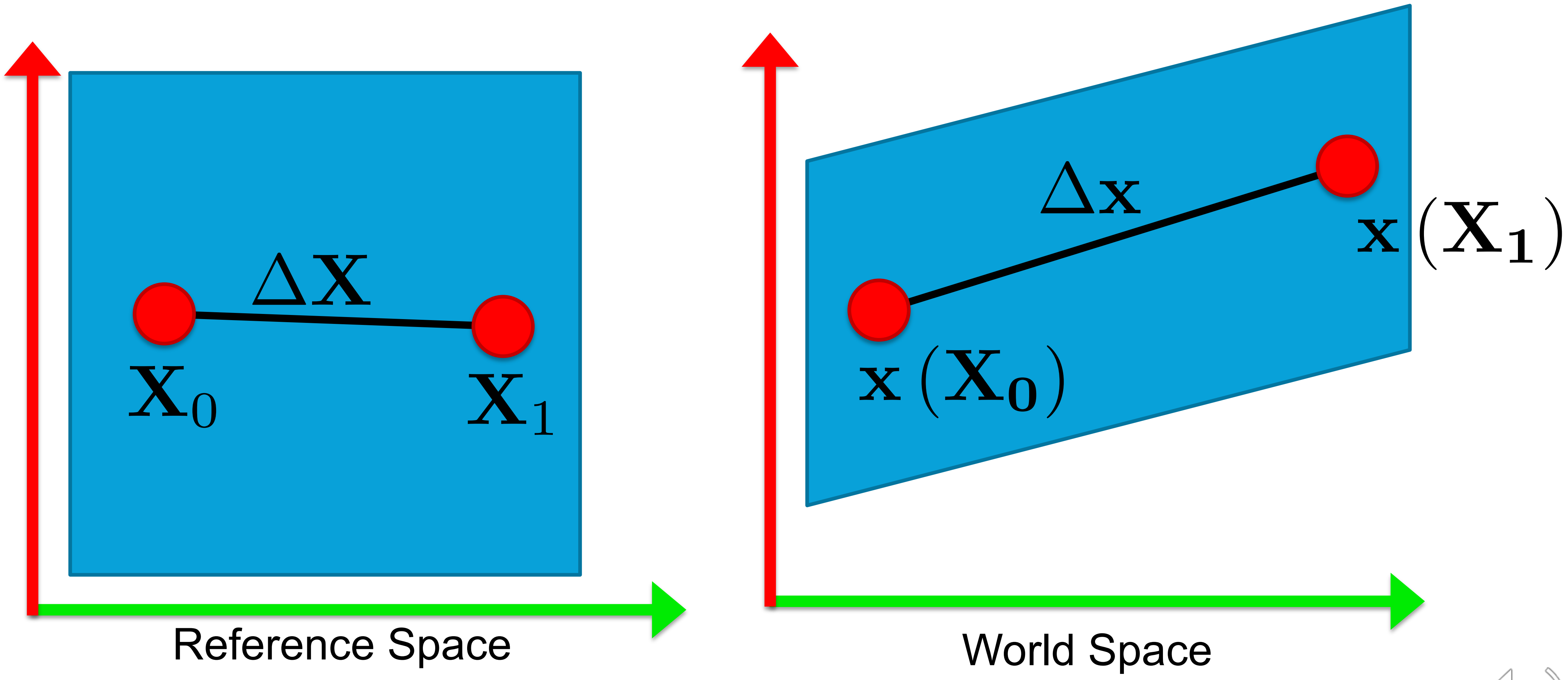


A Closer Look at Deformation



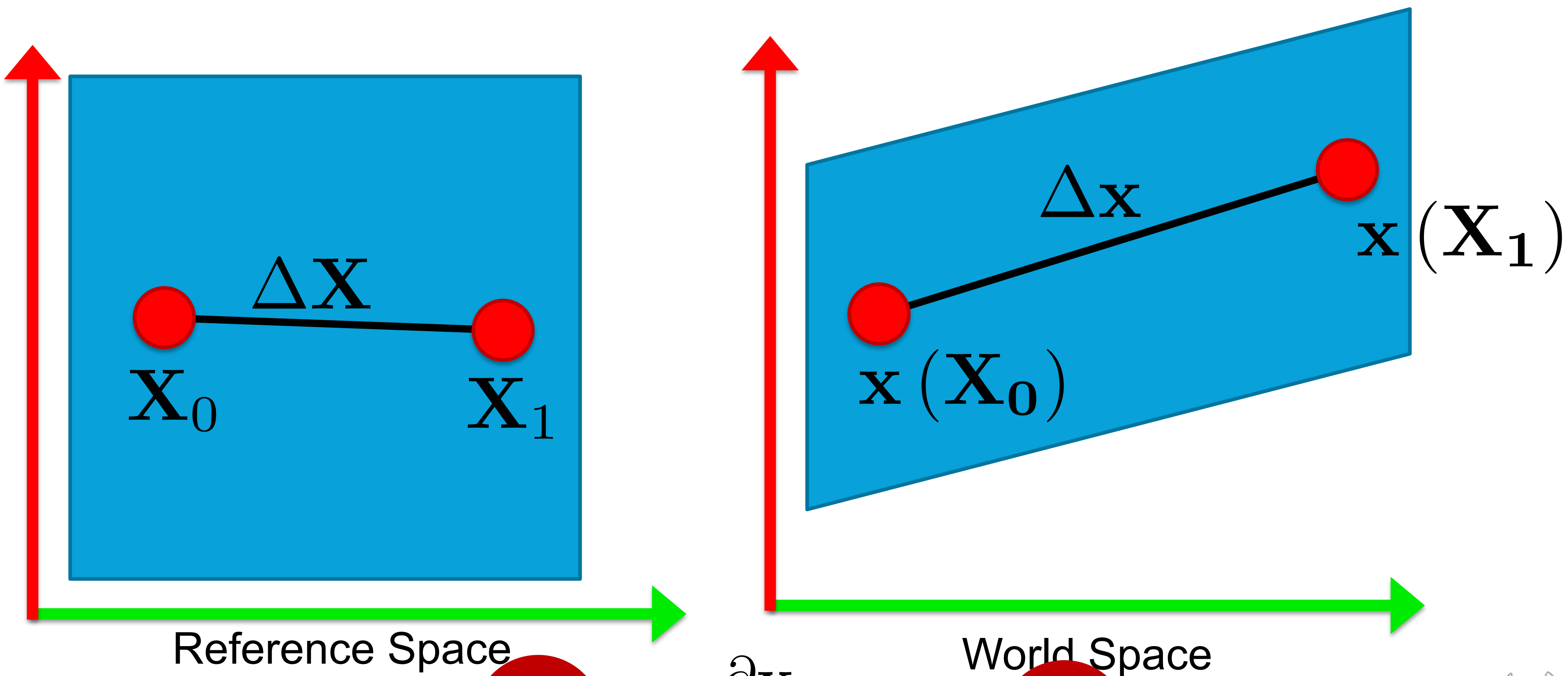
$$\Delta \mathbf{x} = \mathbf{x}(\mathbf{X}_1) - \mathbf{x}(\mathbf{X}_0)$$

A Closer Look at Deformation



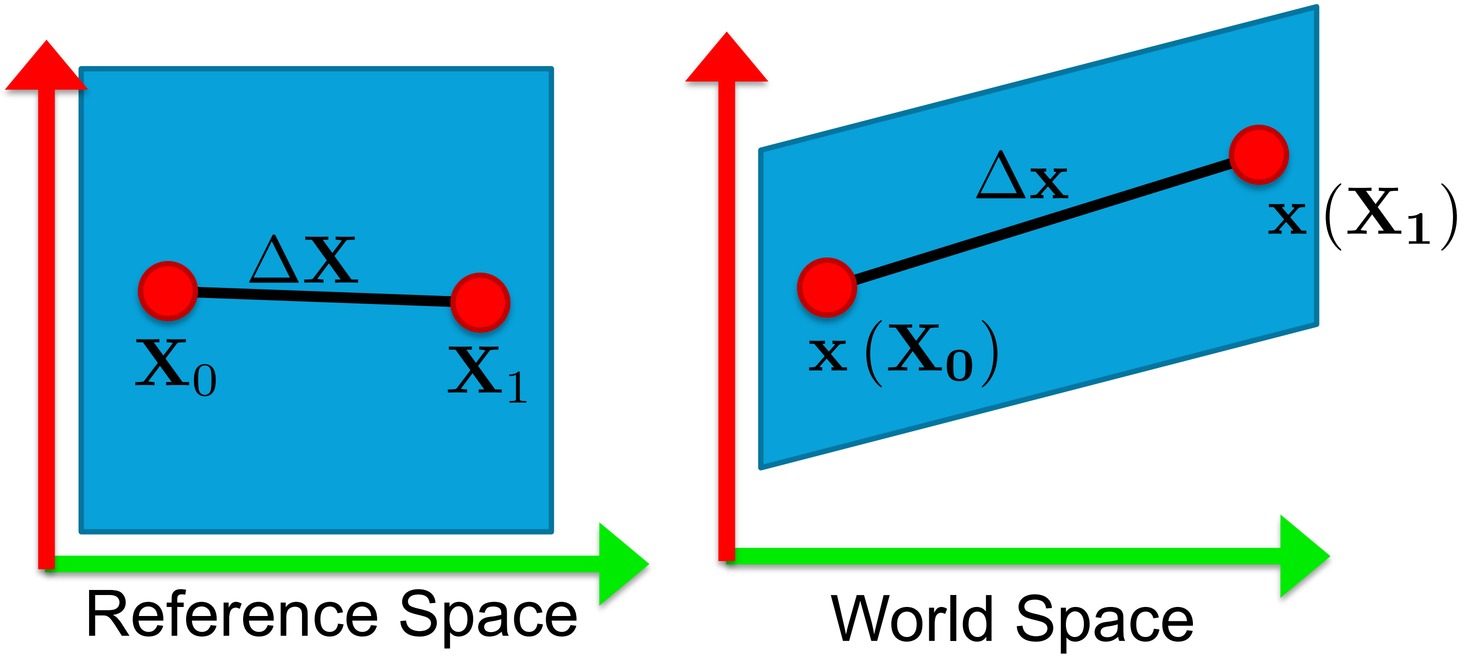
$$\Delta \mathbf{x} = \mathbf{x}\left(\mathbf{X}_0 + \Delta \mathbf{X}\right) - \mathbf{x}\left(\mathbf{X}_0\right)$$

A Closer Look at Deformation



$$\Delta \mathbf{x} \approx \cancel{\mathbf{x}(\mathbf{X}_0)} + \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \Delta \mathbf{X} - \cancel{\mathbf{x}(\mathbf{X}_0)}$$

A Closer Look at Deformation



$$\Delta \mathbf{x} \approx \cancel{\mathbf{x}(\mathbf{X}_1)} + \underbrace{\frac{\partial \mathbf{x}}{\partial \mathbf{X}}}_{\text{deformation gradient } \mathbf{F}} \Delta \mathbf{X} - \cancel{\mathbf{x}(\mathbf{X}_0)}$$

rest length squared

Strain $\underbrace{l^2}_{\text{deformed length squared}} - l_0^2$

$$l^2 = \Delta \mathbf{x}^T \Delta \mathbf{x}$$

$$l_0^2 = \Delta \mathbf{X}^T \Delta \mathbf{X}$$

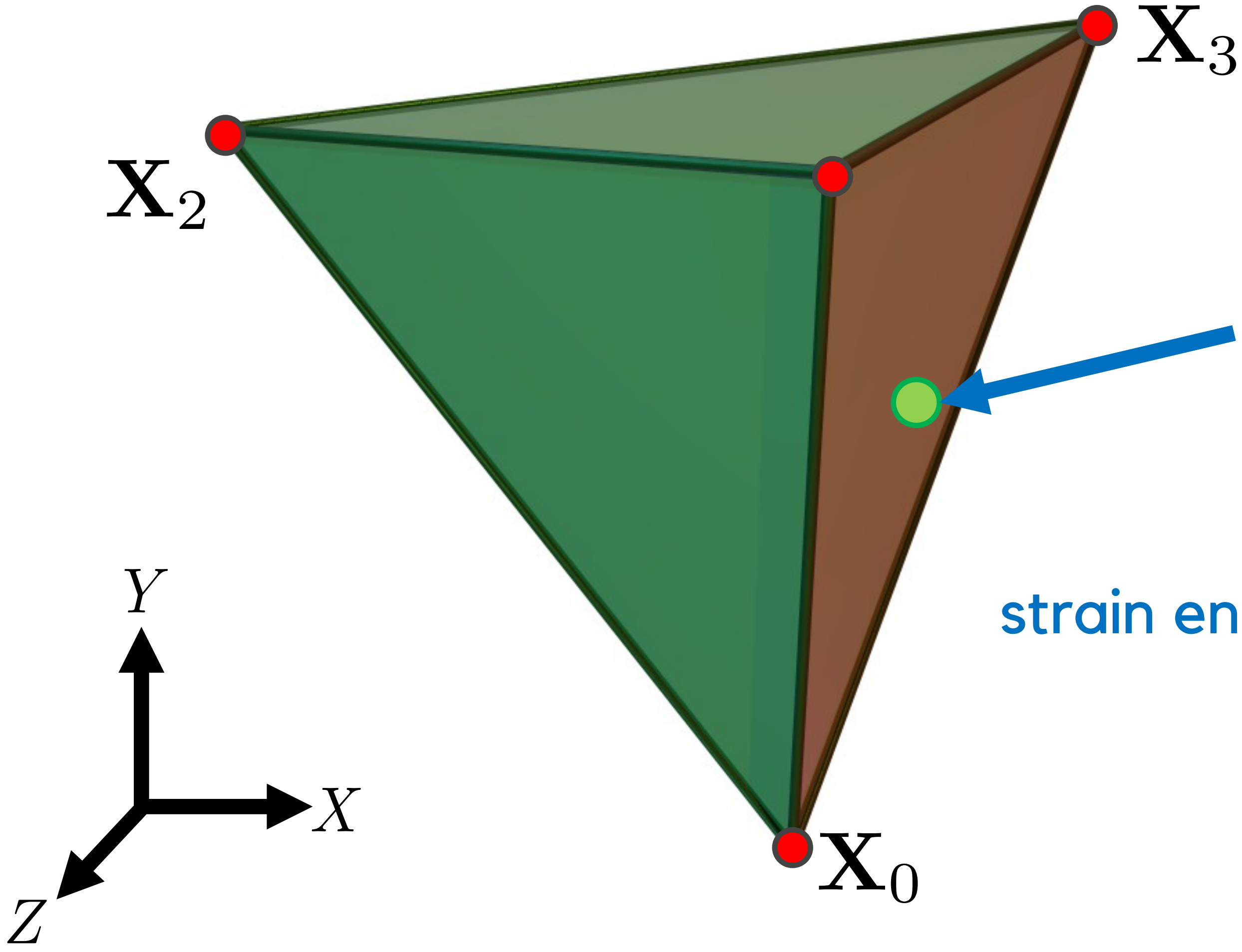
A Closer Look at Deformation

Strain $\Delta \mathbf{x}^T \Delta \mathbf{x} - \Delta \mathbf{X}^T \Delta \mathbf{X}$

$$\Delta \mathbf{X}^T \mathrm{F}^T \mathrm{F} \Delta \mathbf{X} - \Delta \mathbf{X}^T \Delta \mathbf{X}$$

$$\Delta \mathbf{X}^T \underbrace{\left(\overbrace{\mathrm{F}^T \mathrm{F}}^{\text{Right Cauchy Green Deformation}} - \mathrm{I} \right)}_{\text{Green Lagrange Strain}} \Delta \mathbf{X}$$

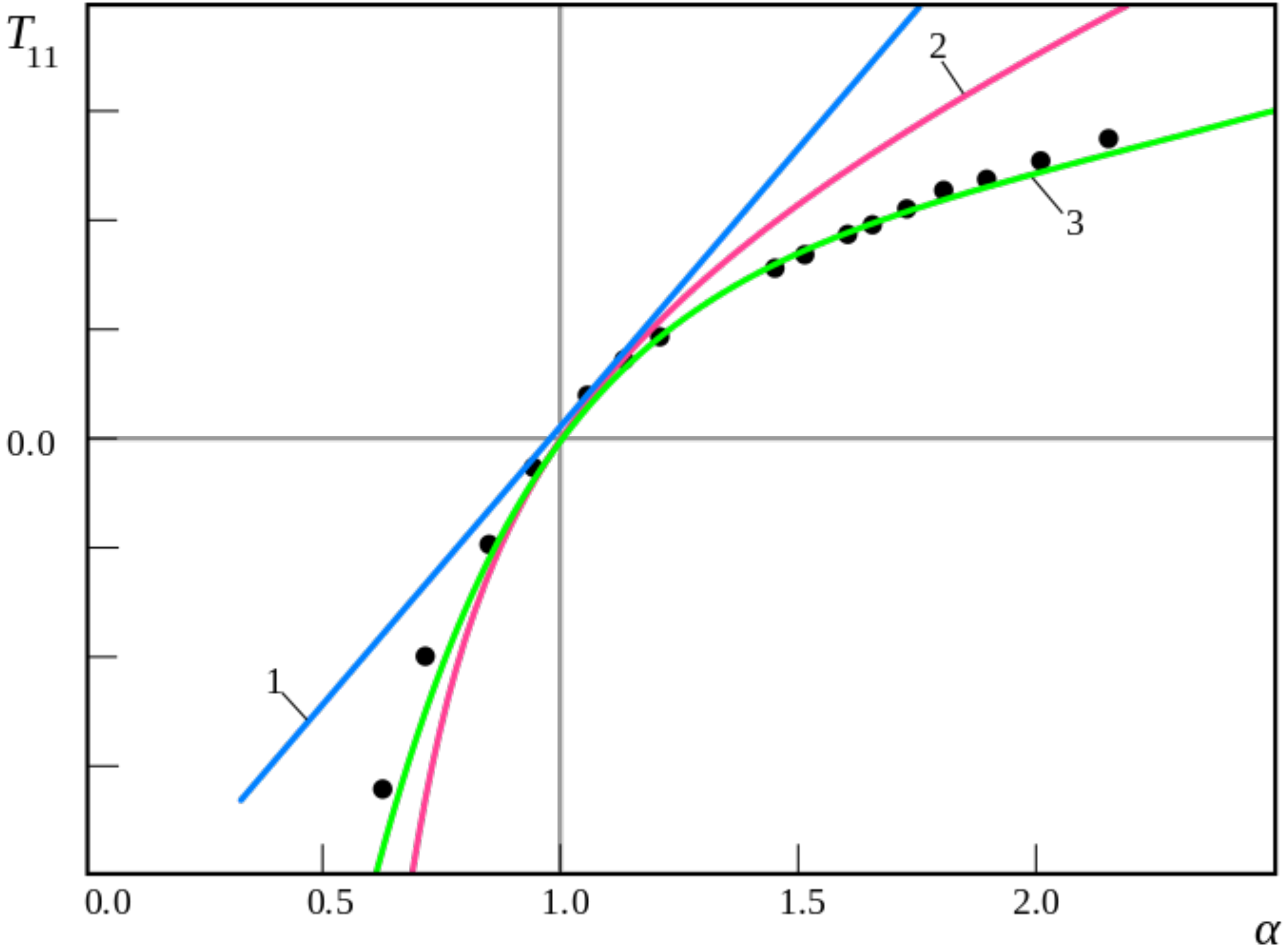
From Deformation to Potential Energy



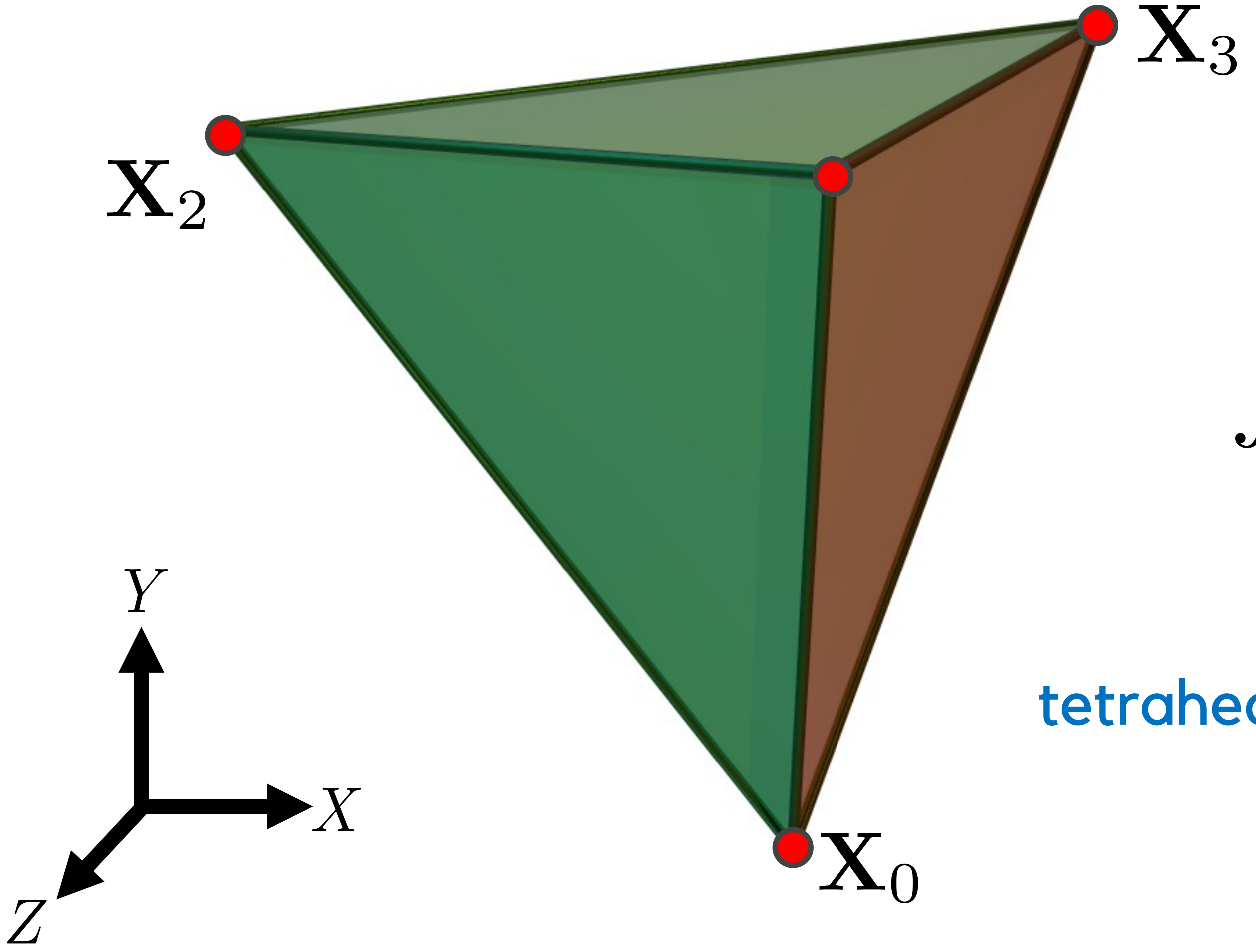
$$\underbrace{\psi}_{\text{strain energy density}}\left(\mathrm{F}\left(\mathbf{X}\right)\right)\overbrace{d\Omega}^{\text{infinitesimal volume}}$$

Reference (Undeformed) Space

Neohookean Strain Energy Density



From Deformation to Potential Energy

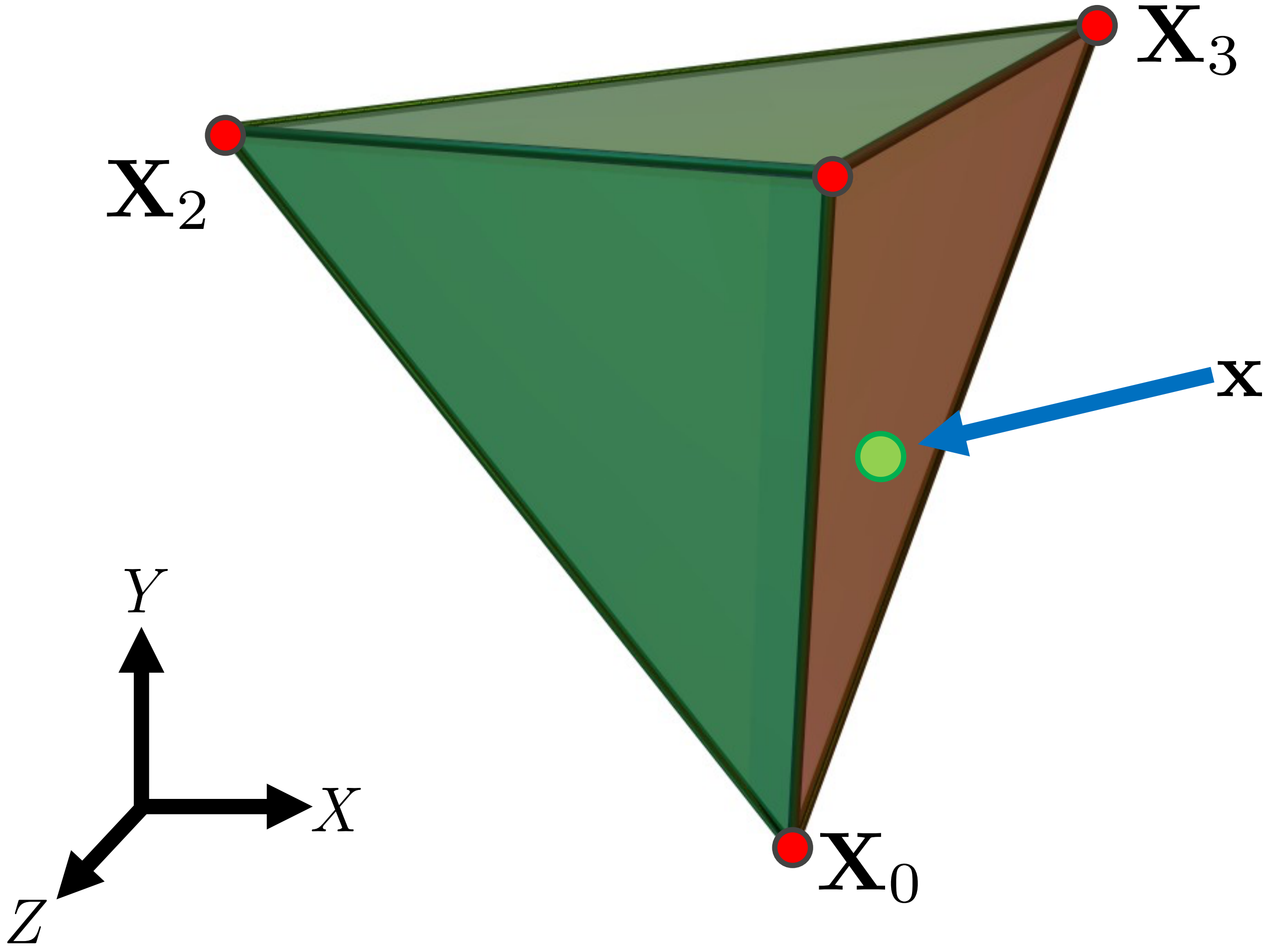


$$\int_{\Omega} \psi\left(\mathrm{F}\left(\mathbf{X}\right)\right) d\Omega$$

tetrahedron domain

Reference (Undeformed) Space

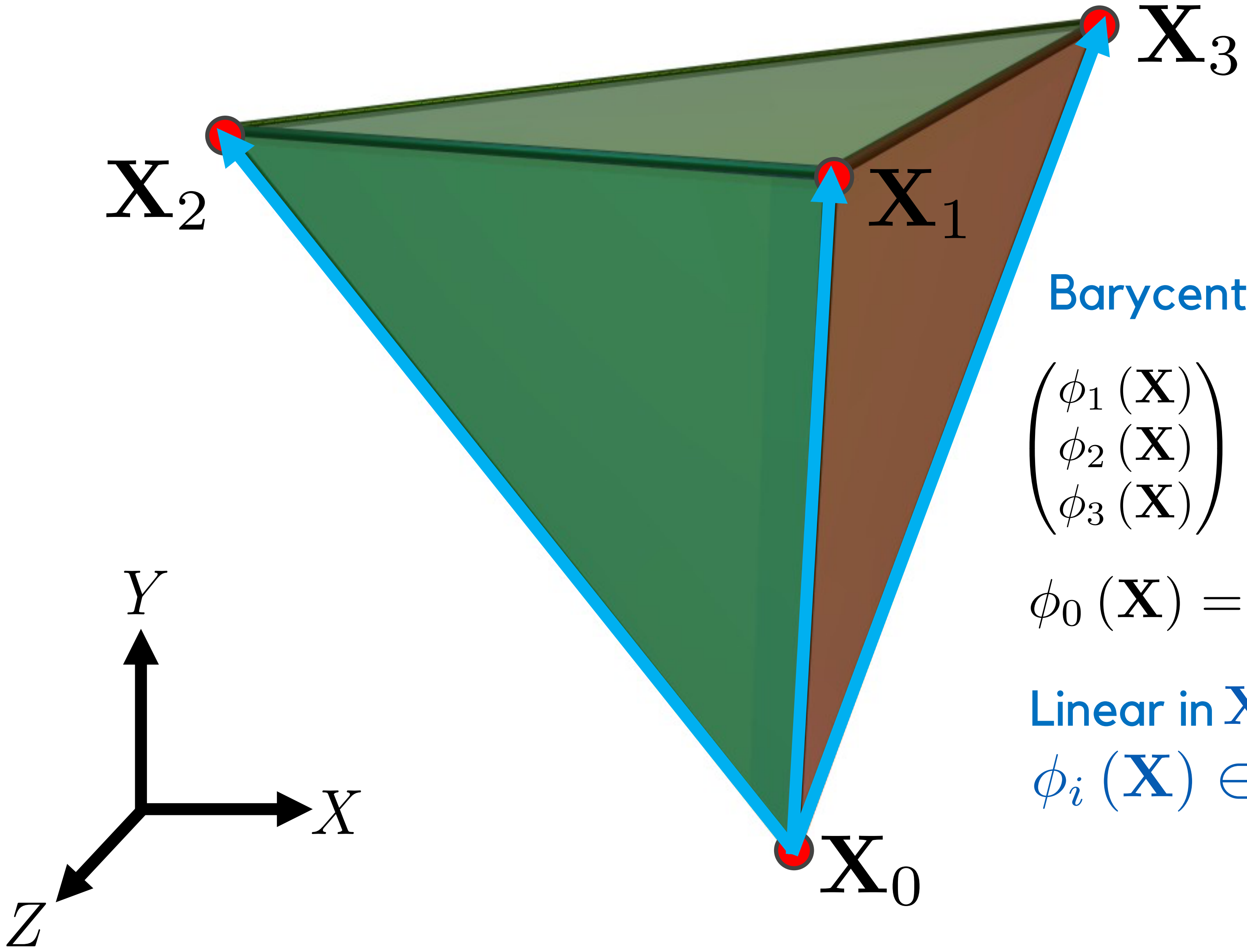
Finite Elements for Deformation



Reference (Undeformed) Space

$$\mathbf{x}(\mathbf{X}) = \underbrace{\begin{pmatrix}\phi_0 \mathrm{I} & \phi_1 \mathrm{I} & \phi_2 \mathrm{I} & \phi_3 \mathrm{I}\end{pmatrix}}_{\mathrm{N}(\mathbf{X})} \underbrace{\begin{pmatrix}\mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3\end{pmatrix}}_{\mathbf{q}(t)}$$

Finite Elements for Deformation



Barycentric Coordinates

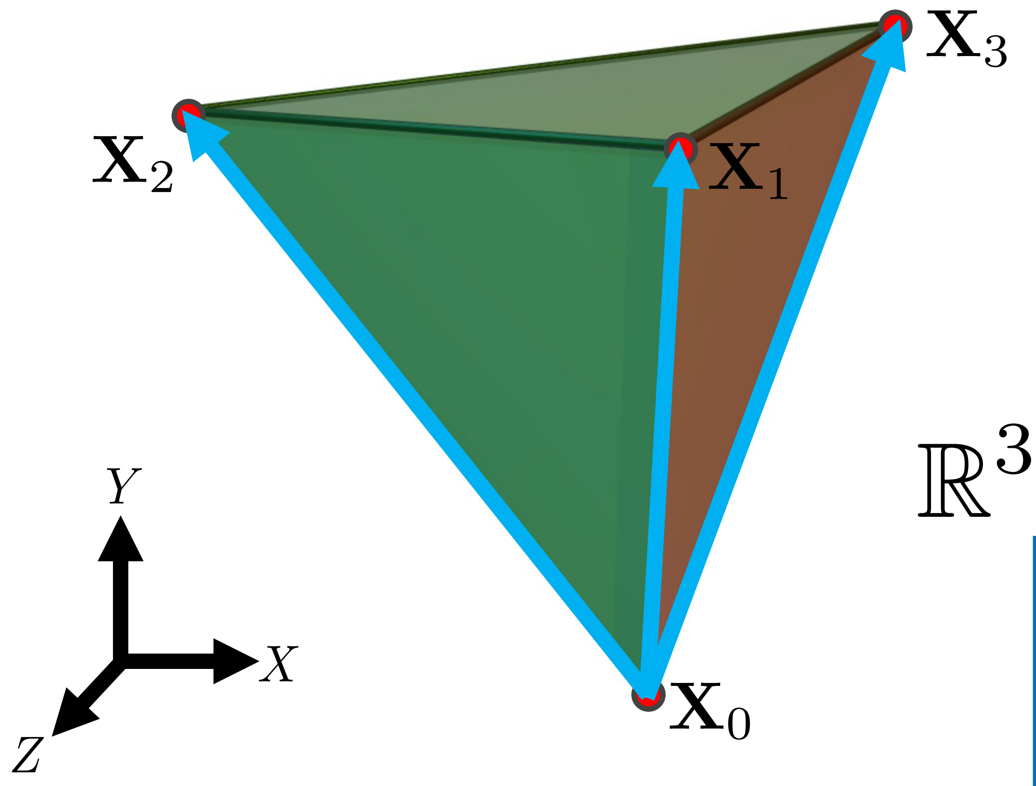
$$\begin{pmatrix} \phi_1(\mathbf{X}) \\ \phi_2(\mathbf{X}) \\ \phi_3(\mathbf{X}) \end{pmatrix} = \mathrm{T}^{-1}(\mathbf{X} - \mathbf{X}_0)$$

$$\phi_0(\mathbf{X}) = 1 - \phi_1(\mathbf{X}) - \phi_2(\mathbf{X}) - \phi_3(\mathbf{X})$$

Linear in $\mathbf{X}$

$\phi_i(\mathbf{X}) \in [0, 1]$ inside tetrahdron

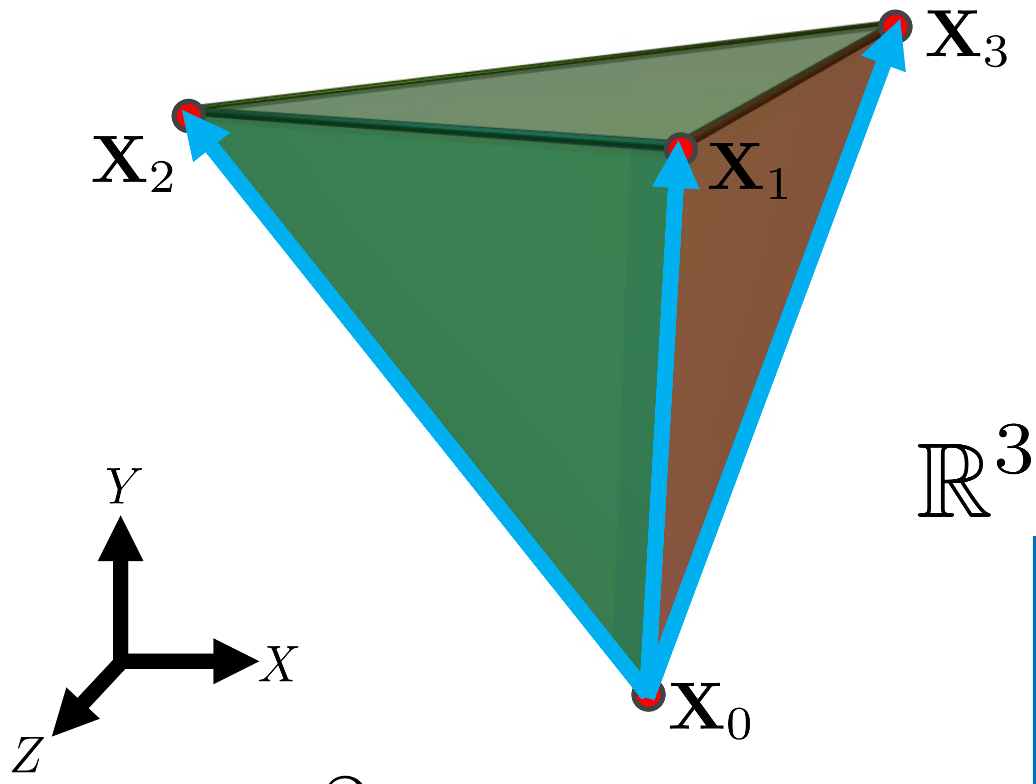
Finite Elements for Deformation



$$\mathbf{x}\left(\mathbf{X}\right) = \mathbf{x}_0 + \overbrace{\begin{pmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix}}^{\mathbb{R}^{3\times 4}} \overbrace{\begin{pmatrix} -\mathbf{1}^T \mathrm{T}^{-1} \\ \mathrm{T}^{-1} \end{pmatrix}}^{\mathbb{R}^{4\times 3}} \left(\mathbf{X} - \mathbf{X}_0\right)$$

$$\mathbf{1}^T = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

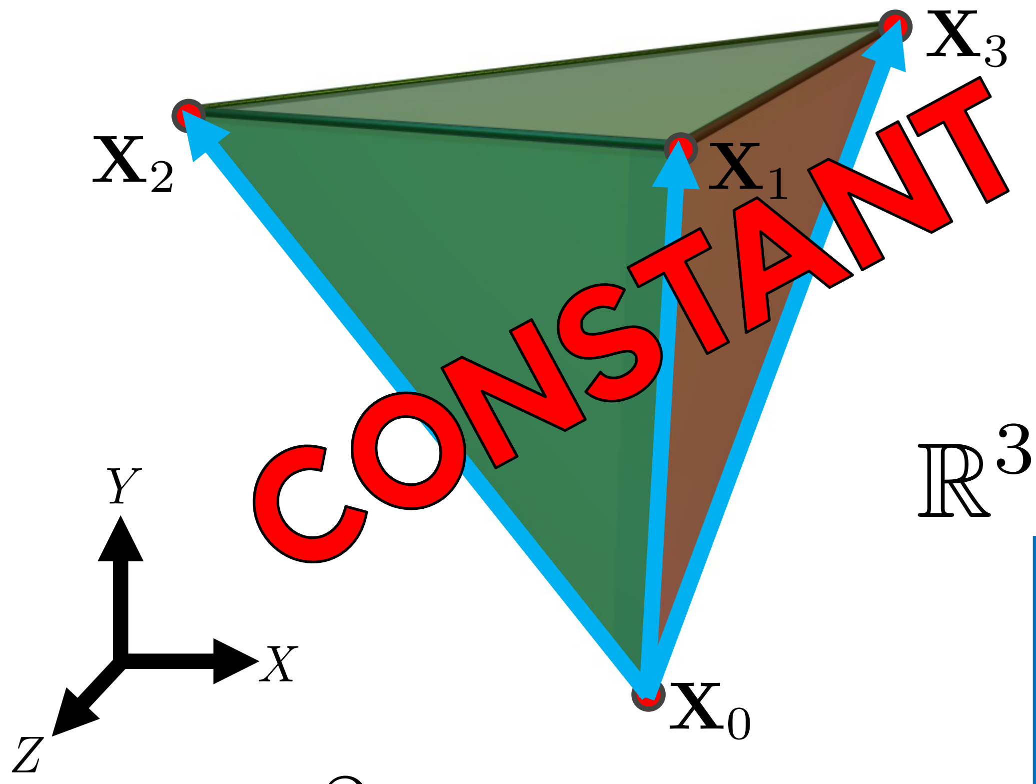
Finite Elements for Deformation



$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \overbrace{\begin{pmatrix}\mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{pmatrix}}^{\mathbb{R}^{3\times 4}} \overbrace{\begin{pmatrix} -\mathbf{1}^T \mathrm{T}^{-1} \\ \mathrm{T}^{-1}\end{pmatrix}}^{\mathbb{R}^{4\times 3}}$$

$$\mathbf{1}^T = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$$

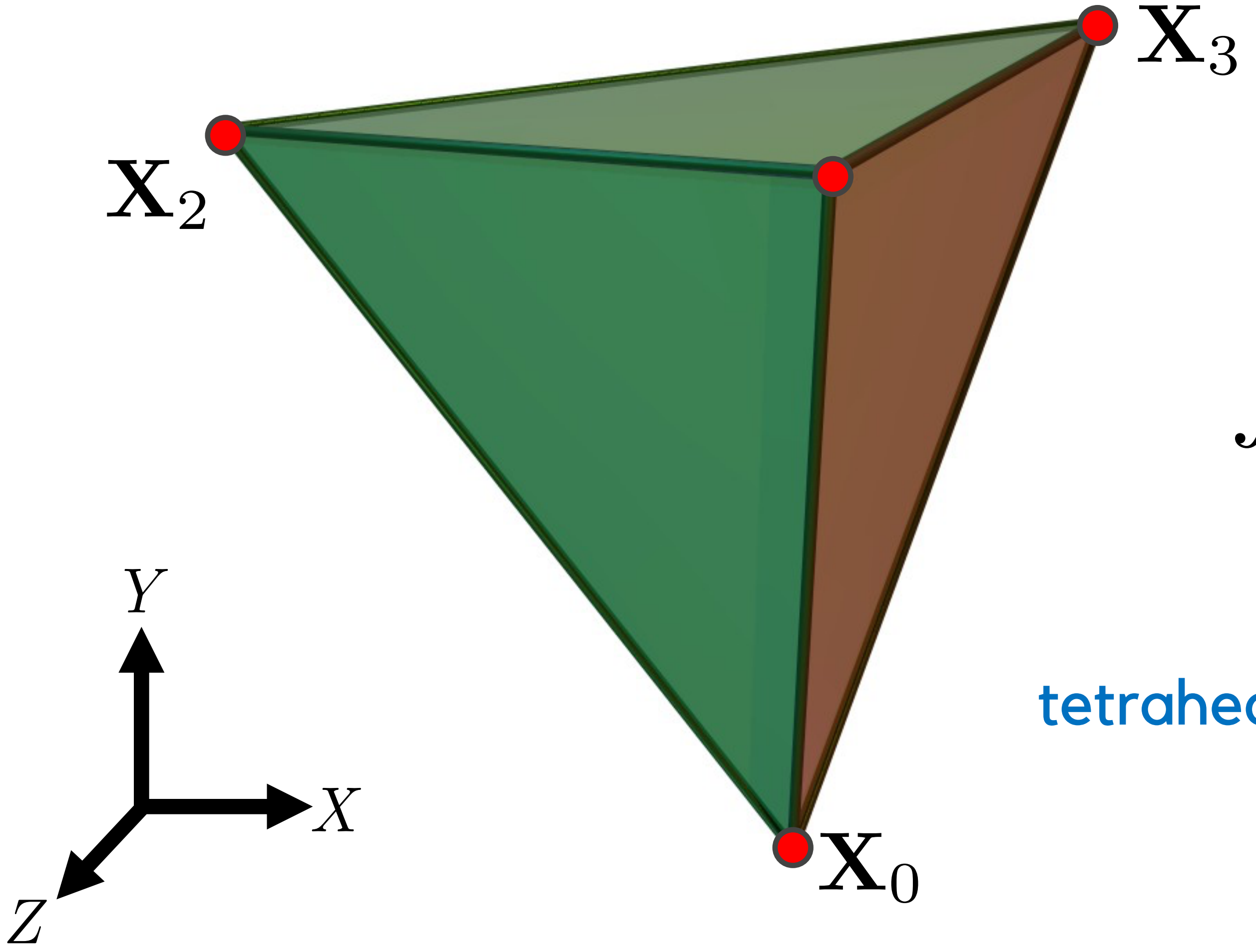
Finite Elements for Deformation



$$\mathrm{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \overbrace{\begin{pmatrix}\mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{pmatrix}}^{\mathbb{R}^{3\times 4}} \overbrace{\begin{pmatrix} -\mathbf{1}^T \mathrm{T}^{-1} \\ \mathrm{T}^{-1}\end{pmatrix}}^{\frac{\partial \phi}{\partial \mathbf{X}}}$$

$$\mathbf{1}^T = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$$

From Deformation to Potential Energy

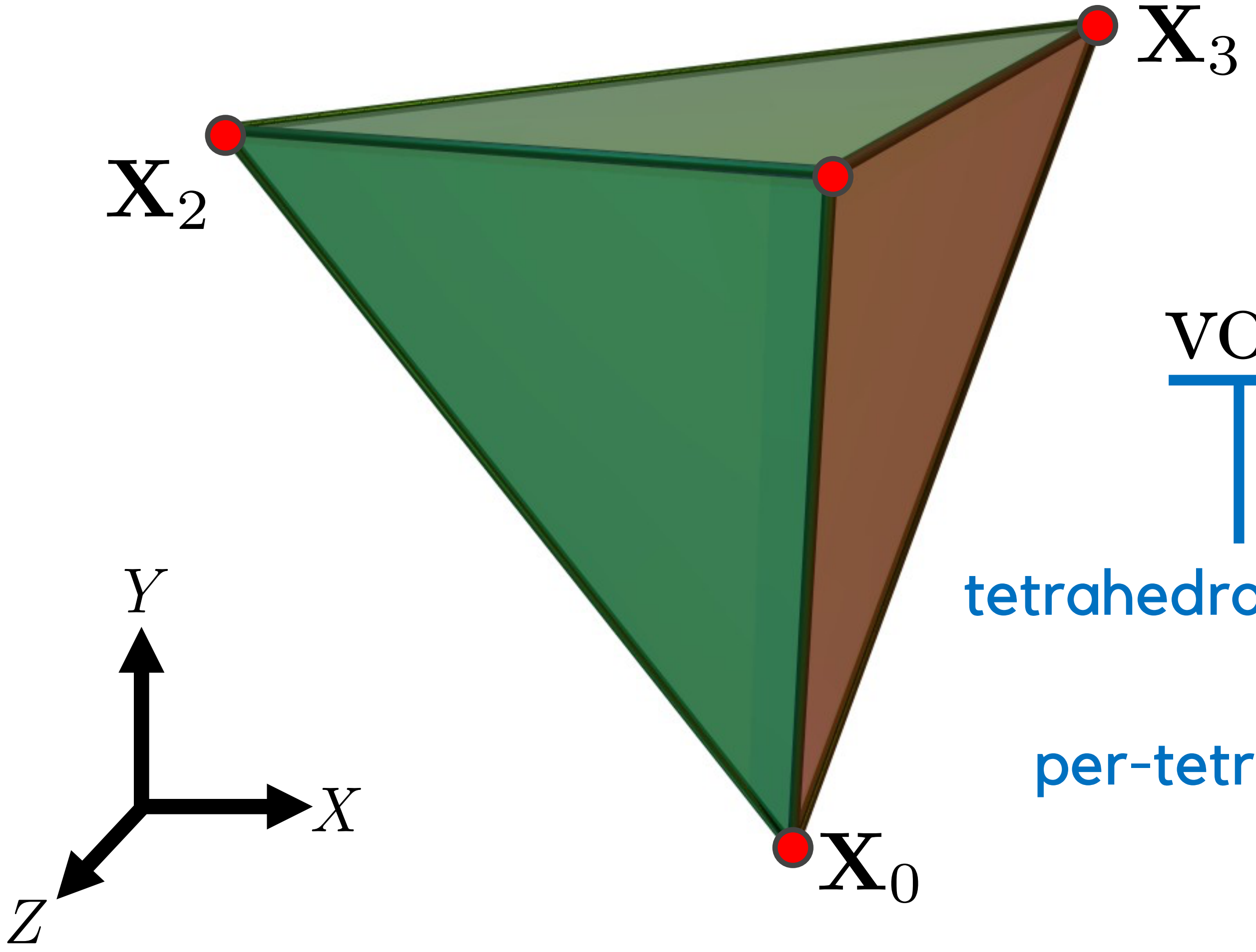


$$\int_{\Omega} \psi\left(\mathrm{F}\left(\mathbf{X}\right)\right) d\Omega$$

tetrahedron domain

Reference (Undeformed) Space

From Deformation to Potential Energy

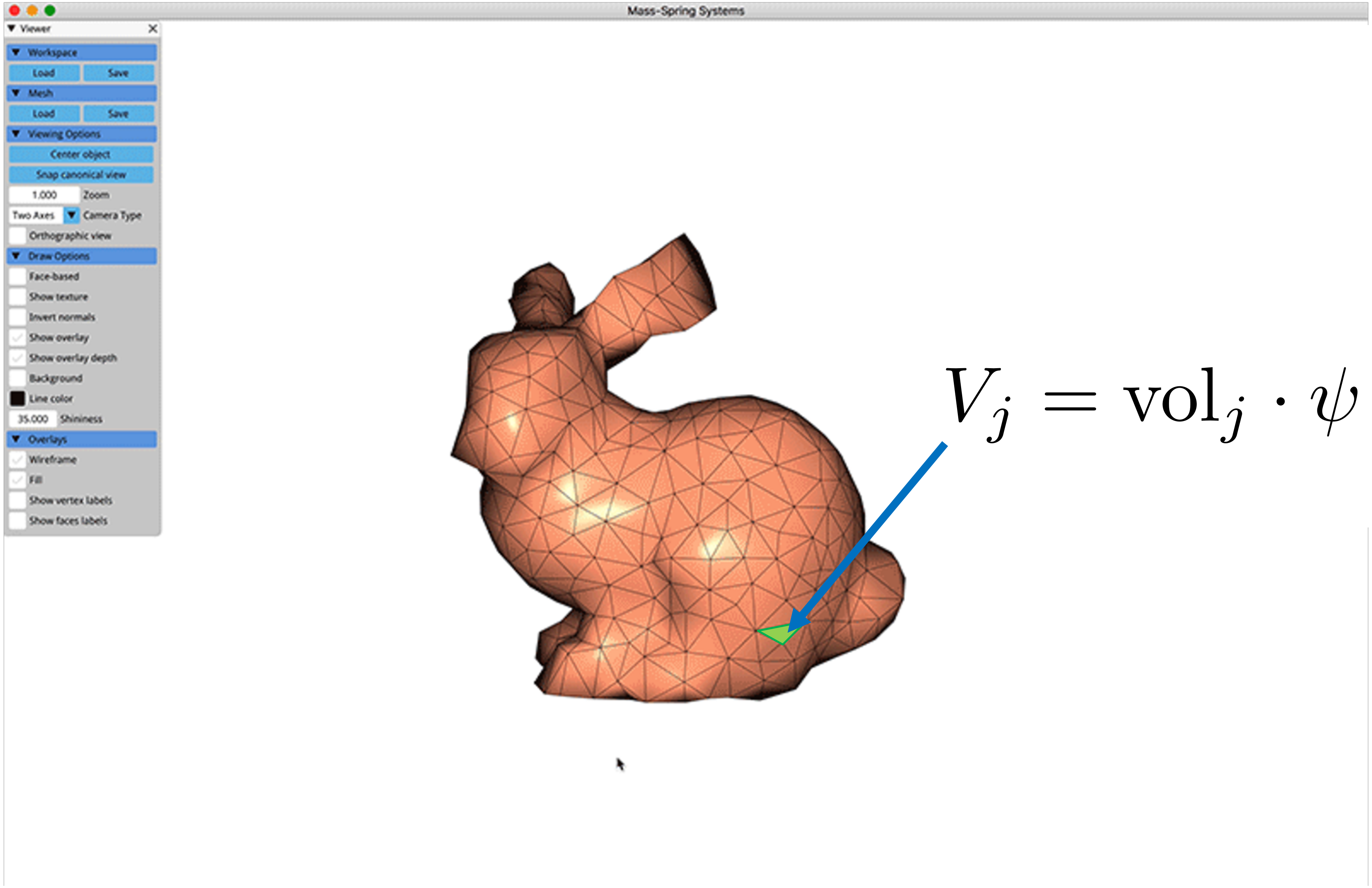


Reference (Undeformed) Space

$$\underbrace{\mathrm{vol}}_{\text{tetrahedron volume}} \cdot \psi\left(\underbrace{\mathrm{F}_0}_{\text{per-tetrahedron deformation gradient}}\right)$$

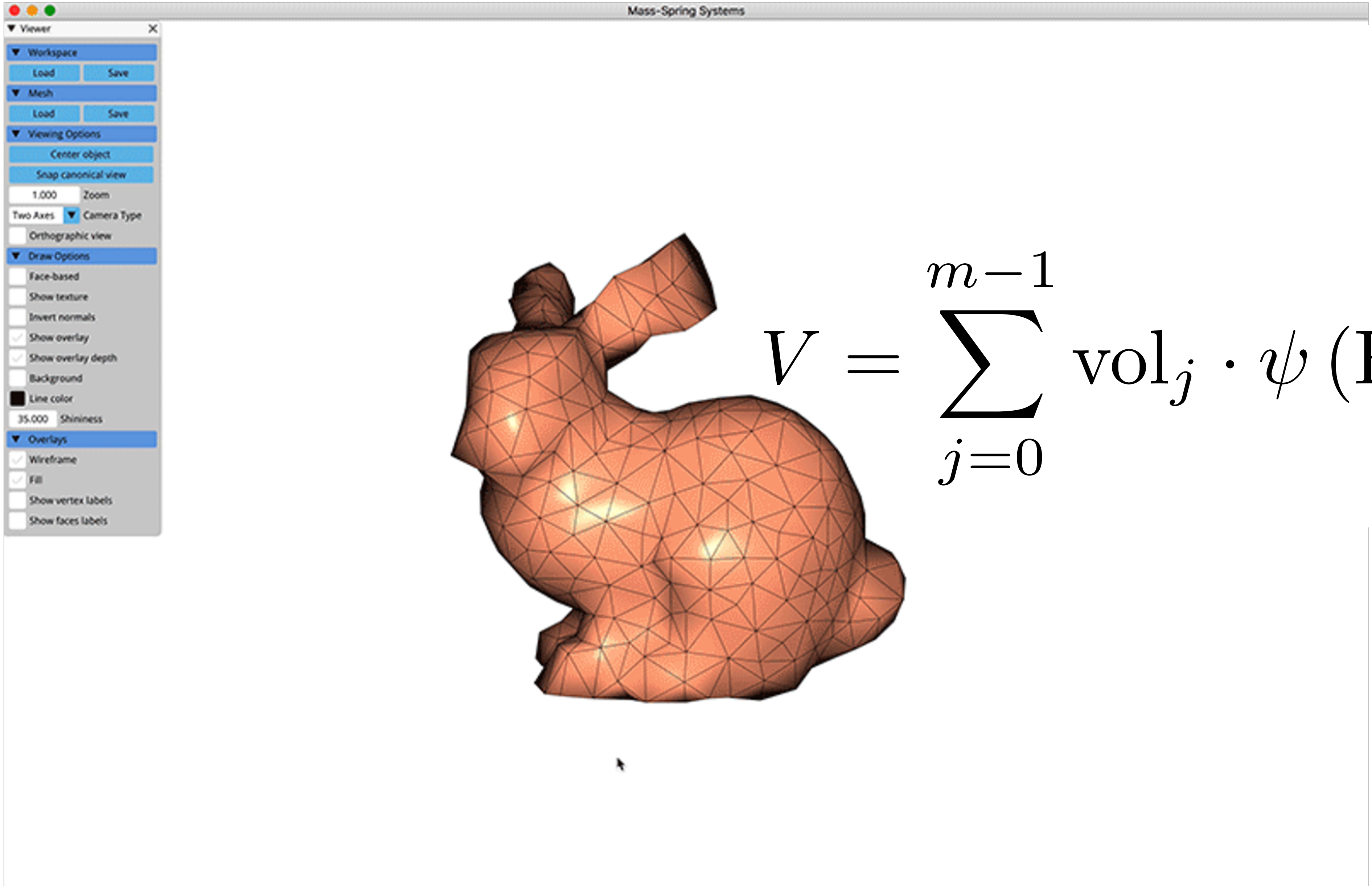
Single-Point Numerical Quadrature

Potential Energy for a Bunny



$$V_j = \mathrm{vol}_j \cdot \psi\left(\mathrm{F}_j\left(\mathbf{q_j}\right)\right)$$

Potential Energy for a Bunny

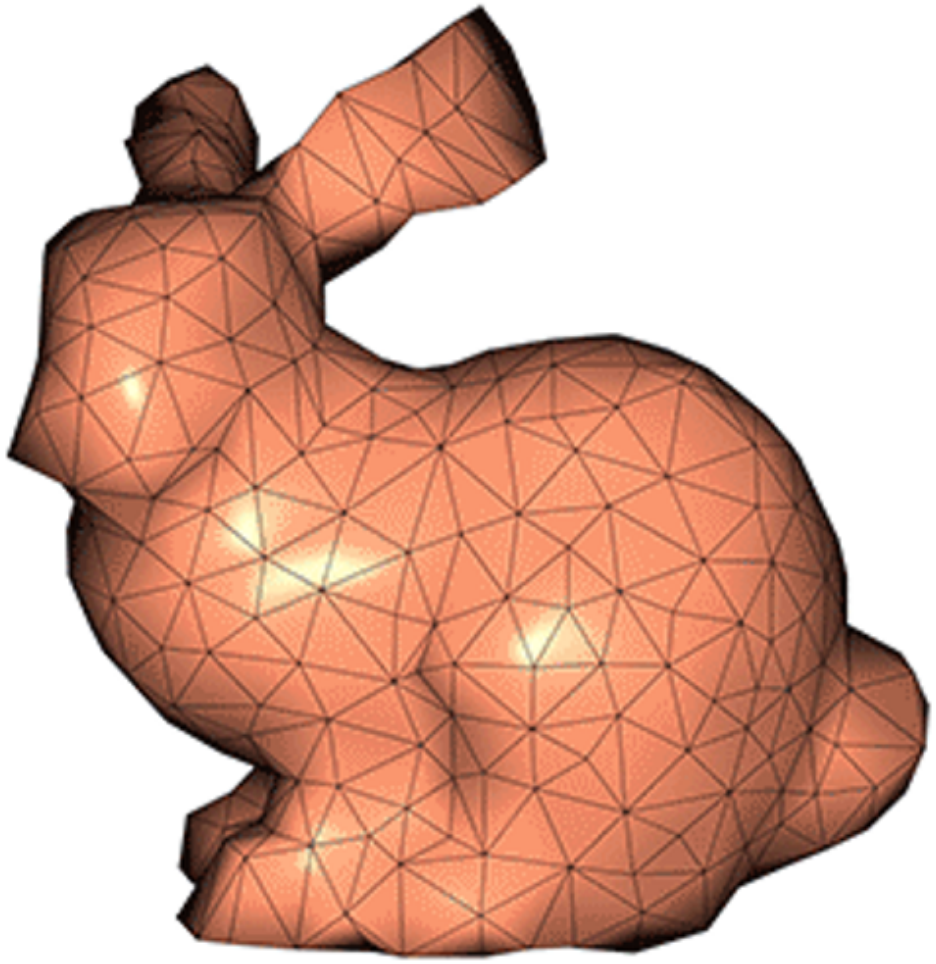


$$V = \sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \psi\left(\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)\right)$$

The Lagrangian

$$V = \sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \psi\left(\mathrm{F}_j\left(\mathbf{q_j}\right)\right)$$

$$L = \underbrace{T}_{\frac{1}{2}\dot{\mathbf{q}}^T \mathrm{M}\dot{\mathbf{q}}} - V$$



Euler-Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \underbrace{-\frac{\partial V}{\partial \mathbf{q}}}_{\text{Generalized Forces } \mathbf{f}}$$

Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \frac{\partial}{\partial \mathbf{q}} \psi\left(\underbrace{\mathrm{F}_j\left(\mathrm{E}_j \mathbf{q}\right)}\right)$$

Because F is a matrix, this is tricky

We can CONVERT F to a vector

Vectorized Deformation Gradient

$$\mathrm{F} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial X} \\ \frac{\partial x}{\partial Y} \\ \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} \\ \frac{\partial y}{\partial Y} \\ \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} \\ \frac{\partial z}{\partial Y} \\ \frac{\partial z}{\partial Z} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -\mathbf{1}^T \mathrm{T}^{-1} \\ \mathrm{T}^{-1} \end{pmatrix}}_{\mathrm{D} \in \mathbb{R}^{4 \times 3}}$$

Vectorized Deformation Gradient

$$
\begin{pmatrix}
\frac{\partial x}{\partial X} \\
\frac{\partial x}{\partial Y} \\
\frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial X} \\
\frac{\partial y}{\partial Y} \\
\frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial X} \\
\frac{\partial z}{\partial Y} \\
\frac{\partial z}{\partial Z}
\end{pmatrix}
=
\underbrace{\begin{pmatrix}
D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 & D_{30} & 0 & 0 \\
D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 & D_{31} & 0 & 0 \\
D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 & D_{32} & 0 & 0 \\
0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 & D_{30} & 0 \\
0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 & D_{31} & 0 \\
0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 & D_{32} & 0 \\
0 & 0 & D_{00} & 0 & 0 & D_{10} & 0 & 0 & D_{20} & 0 & 0 & D_{30} \\
0 & 0 & D_{01} & 0 & 0 & D_{11} & 0 & 0 & D_{21} & 0 & 0 & D_{31} \\
0 & 0 & D_{02} & 0 & 0 & D_{12} & 0 & 0 & D_{22} & 0 & 0 & D_{32}
\end{pmatrix}}_{\mathrm{B}_j}
\underbrace{\begin{pmatrix}
x_0 \\ y_0 \\ z_0 \\ x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ x_3 \\ y_3 \\ z_3
\end{pmatrix}}_{\mathbf{q}_j}
$$

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \frac{\partial}{\partial \mathbf{q}} \psi\left(\underbrace{\mathbf{F}_j\left(\mathbf{E}_j \mathbf{q}\right)}_{\text{Because F is a matrix, this is tricky}}\right)$$

We can CONVERT F to a vector

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \frac{\partial}{\partial \mathbf{q}} \psi \underbrace{\left(\mathrm{B}_j \mathrm{E}_j \mathbf{q}\right)}_{\text{vectorized}}$$

Now we can compute the derivatives

Generalized Forces

$$-\frac{\partial V}{\partial \mathbf{q}} = -\sum_{j=0}^{m-1} \mathrm{vol}_j \cdot \mathrm{E}_j^T \mathrm{B}_j^T \frac{\partial \psi\left(\mathrm{F}_j\right)}{\partial \mathrm{F}}$$

$$\underbrace{\mathbf{f} = \sum_{j=0}^{m-1} \mathrm{E}_j^T \mathbf{f}_j}_{\text{assemble per-tetrahedron forces}} \qquad \underbrace{\mathbf{f}_j = -\mathrm{vol}_j \mathrm{B}_j^T \frac{\partial \psi\left(\mathrm{F}_j\right)}{\partial \mathrm{F}}}_{\text{per-tetrahedron generalized force}}$$

Equations of Motion

$$M\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}$$

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Next Video: More Finite Elements!