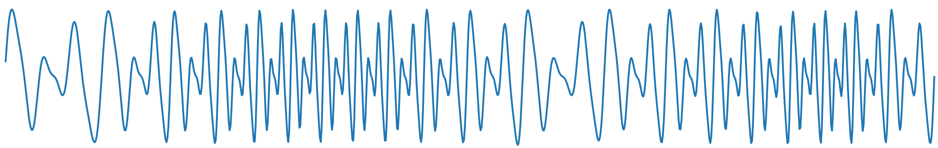
Topological techniques for inference on periodic functions with phase variation

Wojciech Reise

Under the supervision of Frédéric Chazal and Bertrand Michel

Dec 6, 2023, Orsay



TopAI chair
ANR-19-CHIA-0001

Data with phase variation

Signals with phase variation

A sample $S_1, \dots, S_N : [0,1] \to \mathbb{X}$ has **phase variation** if

$$S_n = f(\gamma_n) + W_n, \qquad \text{for each } n \in \{1, \dots, N\}, \tag{1}$$

where $\gamma_1, \dots, \gamma_N : [0,1] \to [0,1]$ are increasing homeomorphisms, $f : [0,1] \to \mathbb{X}$ is continuous and $W_n : [0, T] \to \mathbb{R}$ is a noise process.

Litterature

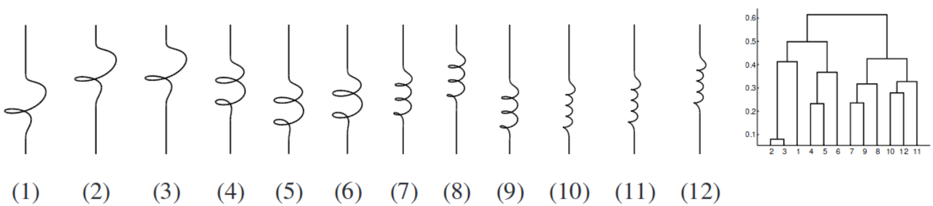
- Curve registration: estimating $\gamma_n \circ \gamma_{n'}^{-1}$ (Tang and Muller 2008, Zhao and Itti 2018)
- Computing a representative of f (Su et al. 2014)
- Clustering of $S_1, \dots, S_n$ (Srivastava et al. 2011)

See Marron et al. 2015 for a review.

Fixed endpoints assumption

For all $1 \leq n \leq N$,

$$\begin{aligned} \gamma_1(0) &= \gamma_n(0), \\ \gamma_1(1) &= \gamma_n(1). \end{aligned} \tag{2}$$



Source: Srivastava et al. 2011.

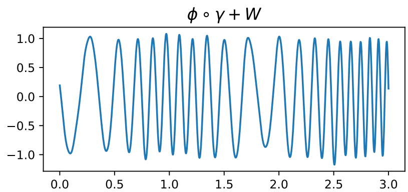
Periodic data with phase variation

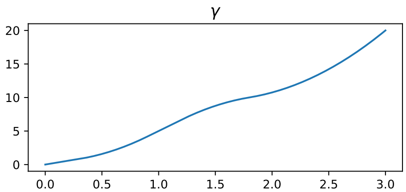
Definition

We call $S : [0,1] \to \mathbb{X}$ a **periodic function with phase variation** if

$$S(t) = \phi(\gamma(t)) + W(t) \tag{3}$$

where $\phi : \mathbb{R} \to \mathbb{X}$ is 1-periodic, $\gamma : [0,1] \to [0,R]$ is an increasing homeomorphism and $W : [0,1] \to \mathbb{X}$ is a noise process.





Example (Instantaneous phase estimation,Boashash, O'Shea, and Arnold 1990)

Decompose $s(t) = a(t)\cos(\gamma_0(t))$ into an amplitude $a(t)$, and a phase-variation component $\gamma_0(t) = \arctan(H(s(t))/s(t))$.

Topological data analysis for periodic time series

We study S using persistent homology, a technique from Topological Data Analysis (TDA).

Contributions

We describe the structure of a topological descriptor of $\phi \circ \gamma$. (Chapter 3)

Let S be a periodic function with phase variation.

1. We propose an estimator of γ from S, (Chapter 5)
2. We construct a descriptor of ϕ from S. (Chapter 4)

Topological data analysis for periodic time series

We study S using persistent homology, a technique from Topological Data Analysis (TDA).

Contributions

We describe the structure of a topological descriptor of $\phi \circ \gamma$. (Chapter 3)

Let S be a periodic function with phase variation.

1. We propose an estimator of γ from S, (Chapter 5)
2. We construct a descriptor of ϕ from S. (Chapter 4)

TDA for time series

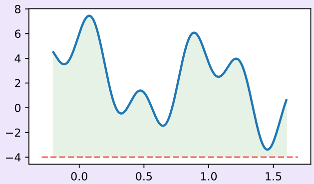
- Detecting periodicity in a time series (Perea 2019),
- Detecting financial crashes (Gidea and Katz 2018)
- Robust zero-crossings (Khasawneh and Munch 2018; Tanweer, Khasawneh, and Munch 2023),
- Analysis of gate signals for the study of multiple sclerosis (Bois et al. 2022).

Outline

Persistence diagram of sub level sets

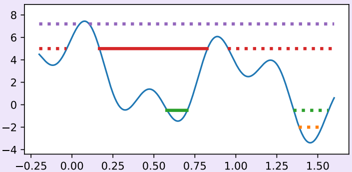
Intuition

The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



Definition (Chazal et al. 2016)

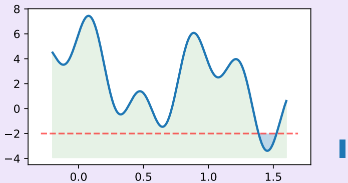
For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.



Persistence diagram of sub level sets

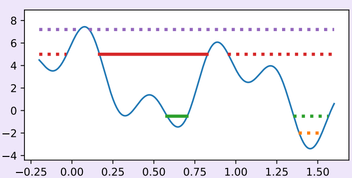
Intuition

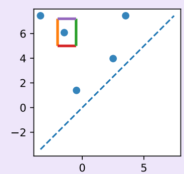
The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



Definition (Chazal et al. 2016)

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.

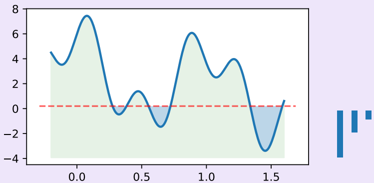




Persistence diagram of sub level sets

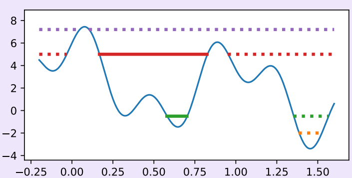
Intuition

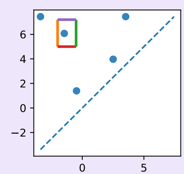
The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



Definition (Chazal et al. 2016)

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.

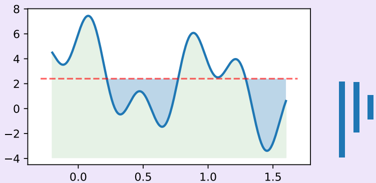




Persistence diagram of sub level sets

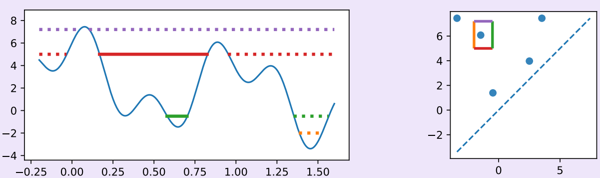
Intuition

The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



Definition (Chazal et al. 2016)

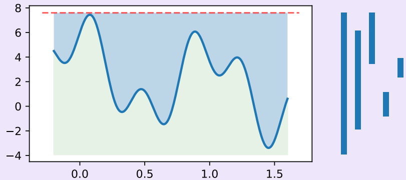
For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.



Persistence diagram of sub level sets

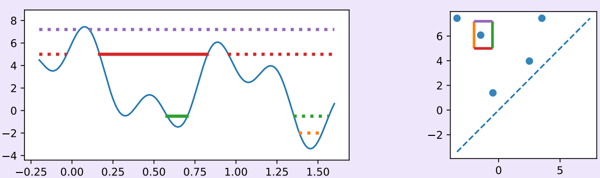
Intuition

The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



Definition (Chazal et al. 2016)

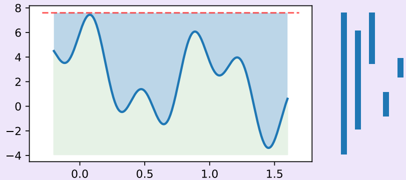
For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.

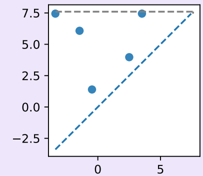


Persistence diagram of sub level sets

Intuition

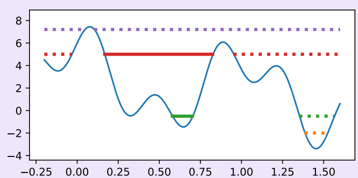
The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.

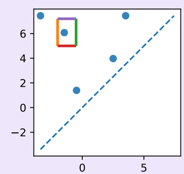




Definition (Chazal et al. 2016)

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$, and the ranks of $(\iota_s^t)_{s\leq t}$ define $D(f)$.





Stability: bottleneck distance

Definition (Bottleneck distance)

We call a ϵ-matching between two persistence diagrams D and D' a bijection $\Gamma : A \to A'$ between some subsets of $A \subset D$ and $A' \subset D'$, considered with multiplicity, if

$$\begin{aligned} d_\infty(a, \Gamma(a)) &\leq \epsilon, && \text{for any } a \in A, \\ d_\infty(a, \Delta) &\leq \epsilon, && \text{for any } a \in (D \setminus A) \cup (D' \setminus A'). \end{aligned}$$

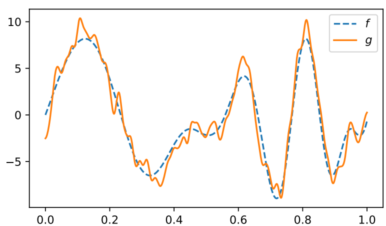
where $\Delta = \{(x, x) \in \mathbb{R}^2\}$ denotes the diagonal.

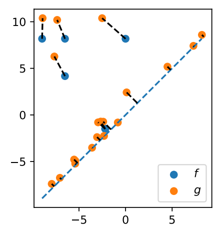
$$d_B(D, D') := \inf\{\epsilon > 0 \mid \Gamma \text{ is an } \epsilon\text{-matching between } D' \text{ and } D'\}.$$

Theorem (Bottleneck stability, Edelsbrunner and Harer 2010)

Let $f, g : \mathbb{X} \to \mathbb{R}$ be two continuous functions on a compact space $\mathbb{X}$. Then,

$$d_B(D(f), D(g)) \leq \|f - g\|_\infty.$$





Total p-persistence

Definition (Total persistence)

The **persistence** of $(b, d) \in D$ is $d - b$. The **total p-persistence** of a diagram D is

$$\mathrm{pers}_p(D) := \left(\sum_{(b,d) \in D} (d-b)^p \right)^{1/p} .$$

Proposition (Plonka and Zheng 2016, Perez 2022)

For $p = 1$,

$$\mathrm{pers}_1(D(f)) + \mathrm{pers}_1(D(-f)) = TV(f).$$

If f is α-Hölder for $p > 1 + 1/\alpha$, then, $\mathrm{pers}_p(D(f)) < \infty$.

Persistence diagrams of periodic functions

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function and denote by $\phi|_{[a,b]}$ the restriction of ϕ to an interval $[a, b]$.

Proposition (Invariance to reparametrisation)

Let $\gamma : [0,1] \to [0,1]$ be an increasing homeomorphism. Then, $D(\phi \circ \gamma) = D(\phi|_{[0,1]})$.

Theorem (Additivity of persistence diagrams for periodic functions)

For $R \in \mathbb{N}^*$, there exists $c \in [0,1]$ such that

$$D(\phi|_{[0,R]}) = RD(\phi|_{[c,c+1]}). \tag{4}$$

For any $R > 1$,

$$D(\phi|_{[0,R]}) = \lfloor R-1 \rfloor D(\phi|_{[c,c+1]}) + D', \quad \textit{with} \quad \mathrm{pers}_p(D') \leq 2\mathrm{pers}_p(D(\phi|_{[c,c+1]})). \tag{5}$$

Persistence diagrams of periodic functions

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function and denote by $\phi|_{[a,b]}$ the restriction of ϕ to an interval $[a, b]$.

Proposition (Invariance to reparametrisation)

Let $\gamma : [0, 1] \to [0, 1]$ be an increasing homeomorphism. Then, $D(\phi \circ \gamma) = D(\phi|_{[0,1]})$.

Theorem (Additivity of persistence diagrams for periodic functions)

For $R \in \mathbb{N}^*$, there exists $c \in [0, 1]$ such that

$$D(\phi|_{[0,R]}) = RD(\phi|_{[c,c+1]}). \tag{4}$$

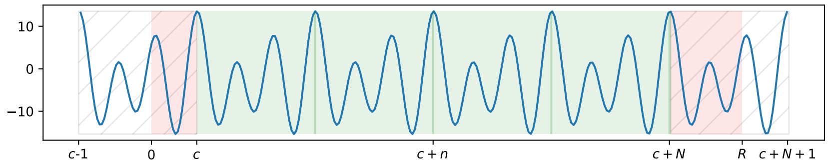
For any $R > 1$,

$$D(\phi|_{[0,R]}) = \lfloor R - 1 \rfloor D(\phi|_{[c,c+1]}) + D', \quad \textit{with} \;\; \mathrm{pers}_p(D') \leq 2\mathrm{pers}_p(D(\phi|_{[c,c+1]})). \tag{5}$$

Conclusion

The persistence diagram $D(\phi \circ \gamma)$ contains information about
- extrema of ϕ,
- number of periods $(\gamma(1) - \gamma(0))$.

Proof of (5)



Proof.

Let $c := \inf\{x \in [0,1[\mid \phi(x) = \max\phi\}$, $N = \max\{n \in \mathbb{N} \mid c + n \leq R\}$ and denote by $X_t := \phi^{-1}([-\infty, t[)$.

Step 1: For any $t < M$, $X_t \cap [0,c] \cap [c, c+1] = \emptyset$, so

$$H_0(X_t \cap [0,R]) \simeq H_0(X_t \cap [0,c]) \oplus H_0(X_t \cap [c, c+N]) \oplus H_0(X_t \cap [c+N, R])), \tag{6}$$

Step 2: similarly,

$$H_0(X_t \cap [c, c+N]) \simeq \bigoplus_{n=1}^{N} H_0(X_t \cap [c + (n-1), c+n]) \tag{7}$$

$$(x \mapsto x + n) \quad \simeq \bigoplus_{n=1}^{N} H_0(X_t \cap [c, c+1]) \tag{8}$$

Step 3: The inclusion $[0,c] \subset [c-1, c]$ induces an injective morphism

$$H_0(X_t \cap [0,c]) \hookrightarrow H_0(X_t \cap [c-1, c]).$$

□

Outline

Additivity of persistence diagrams of periodic functions

Segmentation of periodic signals and phase estimation

Signatures of periodic signals with phase variation

Phase estimation

Setting

Consider S a periodic function with phase variation

$$\begin{array}{rccl} S: & [0,T] & \to & \mathbb{R} \\ & t & \mapsto & \phi(\gamma(t)) + W(t), \end{array}$$

where

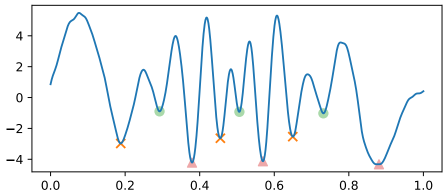
1. $\phi : [0,1] \to \mathbb{R}$ is 1-periodic and unknown,
2. $\gamma : [0,T] \to [0,N]$ with $N \in \mathbb{N}$ unknown,
3. $W : [0,T] \to \mathbb{R}$ is a continuous noise process.

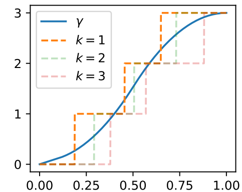
Goal: Given S, estimate γ.

Proposed solution: segmenting the curve into periods

1. Estimate N using $D(S)$.
2. Find $t_1 < \ldots < t_N$ such that $\gamma(t_n) - \gamma(t_{n-1}) = 1$ for all $n = 2, \ldots, N$.

Let $\hat{\gamma} : [0,T] \to \mathbb{R}^*$ be such that $\hat{\gamma}(t_n) = n$ and interpolate.



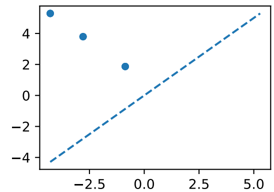


Developed and studied in (Bonis et al. 2022).

Estimation of N: noiseless setting

We will denote by $D(S)(A)$ the number of points from $D(S)$ that are in $A \subset \{(x, y) \in \mathbb{R}^2 \mid y - x > 0\}$.

$$\hat{N}(S) := \mathrm{gcd}\{D(S)(x) \mid x \text{ in } \mathrm{supp}(D(S))\}.$$



Proposition

Assume $W = 0$, so $S = \phi \circ \gamma$ with $\gamma : [0, T] \to [0, N]$. For any $A \subset \mathbb{R}^2$,

$$D(\phi \circ \gamma)(A) = ND(\phi|_{[0,1]})(A). \tag{9}$$

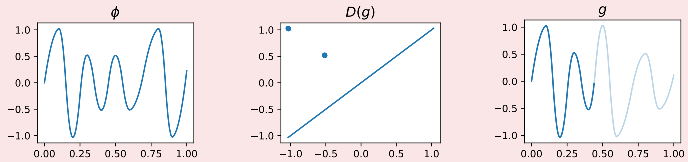
In particular,

$$\hat{N}(\phi \circ \gamma) = N\hat{N}(\phi|_{[0,1]}).$$

Estimation of N: correctness in the noiseless setting

Identifiability

There exists a 1-periodic function g such that $D(g|_{[0,1]}) = D(\phi|_{[0,1]})/\hat{N}(\phi|_{[0,1]})$!



Non-degeneracy

We say that $\phi|_{[0,1]}$ is **non-degenerate** if $\hat{N}(\phi|_{[0,1]}) = 1$.

Example

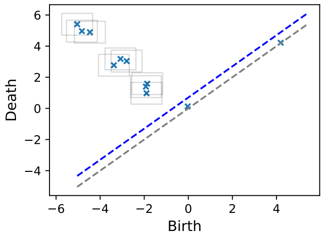
If ϕ has at least one unique critical value, it is non-degenerate.

Corollary

When ϕ is non-degenerate, $\hat{N}(\phi \circ \gamma) = N$.

Estimation of N: noisy signal

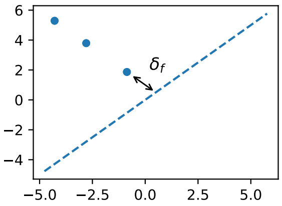
For the noisy signal $S = \phi \circ \gamma + W$, the points in $D(S)$ have multiplicity 1.



Estimator

For $\tau > 0$, we define

$$\hat{N}_\tau(S) := \gcd\{|D(S)(B(x,\tau))| \mid x \in D(S), \mathrm{pers}(x) > \tau\}. \quad (10)$$



Separation constant

The separation constant is the smallest distance between points in $D(\phi)$,

$$\delta_\phi := \min(d(x_1,x_2), d(x_1,\Delta) \mid x_1,x_2 \in D(\phi)).$$

Proposition (Stability)

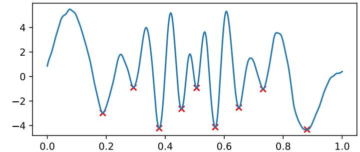
If ϕ is non-degenerate, then for any $\tau > 0$ such that $2\|W\|_\infty < \tau < \delta/3$, we have

$$\hat{N}_\tau(S) = N.$$

Estimation of γ

Persistent minima

Let $\tau > 0$ and $\hat{\mathcal{C}}_\tau = \{t_1, \dots, t_M\} \subset [0, T]$ be the set of local minima of S, corresponding to points in the diagram with persistence more than τ.



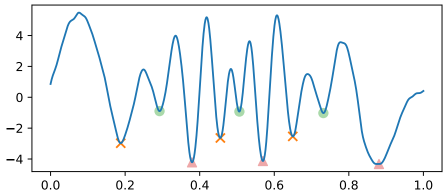
Proposition

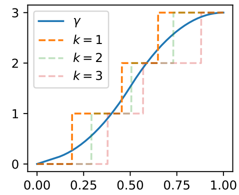
If $\tau \in]2\|W\|_\infty, \delta/3[$, then, for some $K \in \mathbb{N}$,

$$|\hat{\mathcal{C}}_\tau| = NK.$$

For each $k \in \{1, \dots, K\}$, we can define an estimator of γ

$$\begin{array}{rccc} \hat{\gamma} : & [0, T] & \to & \mathbb{R} \\ & t & \mapsto & \sum_{n=1}^{N} 1_{t_{(n-1)K+k} \le t}. \end{array} \tag{11}$$



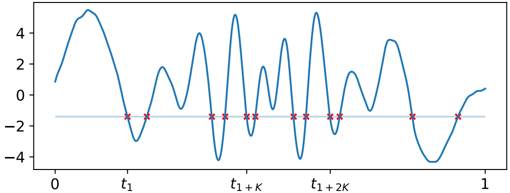


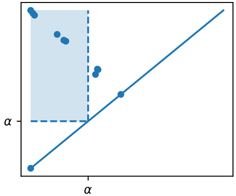
Zero-crossings[1]

Let $K := |\phi^{-1}(\alpha) \cap [0,1[|$ and assume that $0 < K < \infty$, for some $\alpha \in \mathbb{R}$.

Estimation of N

If K is known, $N_\alpha(S) := \frac{|S^{-1}(\alpha)|}{K}$ is an estimator of N.





Segmentation of the signal

If $S^{-1}(\alpha) = \{t_1, \ldots, t_{NK}\}$, then $\gamma(t_{n+k}) - \gamma(t_n) = 1$ for $1 \le k \le K$ and $n \le NK - k$.

Issues

- K is not known (and not necessarily finite),
- N_α is not stable. (Tanweer, Khasawneh, and Munch 2023)

[1] Boualem Boashash, Peter O'Shea, and Morgan Arnold (1990). "Algorithms for Instantaneous Frequency Estimation: A Comparative Study". In: *Advanced Signal Processing Algorithms, Architectures, and Implementations*. Vol. 1348. SPIE, pp. 126–148. DOI: 10.1117/12.23471.

Application: estimating the speed of a moving vehicle

Context

The magnetic signal measured in a car is $\mathbf{B}(\theta, \theta_h) = Q(\theta_h)\mathbf{B}_E + \mathbf{B}_u(\theta) \in \mathbb{R}^3$, where θ_h is the orientation of the vehicle and θ the angular position of a wheel[2].
As the car moves, we observe $\mathbf{S}(t) = \mathbf{B}(\gamma(t), \gamma_h(t))$.

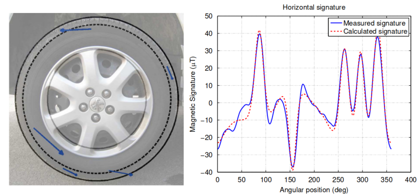
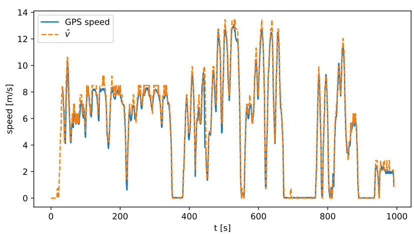
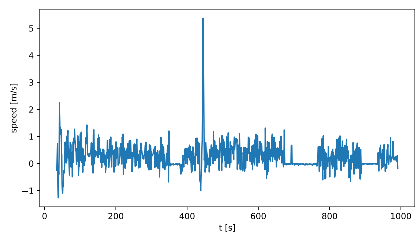


Fig. 2. Results of the optimization. Left: magnetic dipoles found after optimization. Right: measured and calculated signatures.

Proposed solution

Choose a vector $v \in \mathbb{S}^2$ and construct $\hat{\gamma}$ for on $S := \langle \mathbf{S}, v \rangle$. Estimate the speed by $(\hat{\gamma}(t) - \hat{\gamma}(t - t_0))/t_0$, for some small delay t_0.

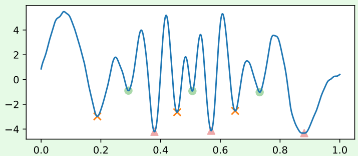
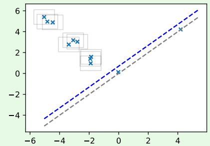
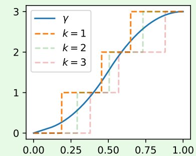




[2] Pierre-Jean Bristeau (2012). "Techniques d'estimation du déplacement d'un véhicule sans GPS et autres exemples de conception de systèmes de navigation MEMS". PhD thesis. Ecole Nationale Superieure des Mines de Paris

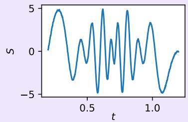
Conclusion and future work

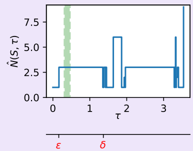
Conclusion

Limitations and future work

1. Identifiability
   - Use the order of local minima to lift the identifiability issue
2. More robust estimators
   - Extend the guarantees to $\hat{N}_c$ and $\hat{N}_T$
   - Choose the sets to count multiplicity differently





3. The method is applicable only to $N \in \mathbb{N}$.
   - In practice, it is not a problem as ϕ is often simple.
   - Use the approximate greatest common divisor.

Outline

Additivity of persistence diagrams of periodic functions

Segmentation of periodic signals and phase estimation

Signatures of periodic signals with phase variation

Problem statement

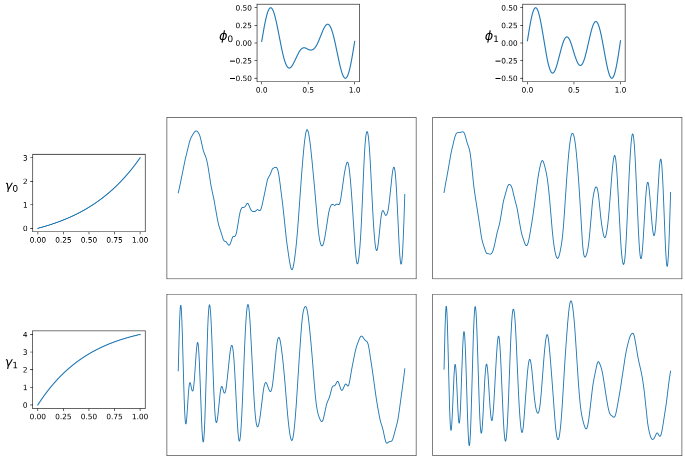
Data

Consider $S = \phi \circ \gamma + W$, where

- $\phi : \mathbb{R} \to \mathbb{R}$ is 1–periodic,
- $\gamma : [0, T] \to \mathbb{R}$ an increasing bijection,
- $W : [0, T] \to \mathbb{R}$ is a cont. stoch. proc.

Aim

Given S, construct a signature of ϕ.



Studied in Reise, Michel, and Chazal 2023.

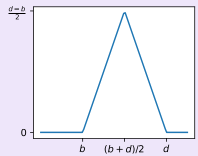
Functional representations of persistence diagrams

Defining a mean of a collection of persistence diagrams is not necessarily easy, so it is common to compute statistics of diagrams in a functional space[3].

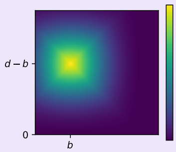
Functional representation

Let $\mathcal{H}$ be a functional Banach space.

$$\begin{array}{rcccccc} \kappa : & \mathbb{R}^2 & \to & \mathcal{H} & & & \\ & (b,d) & \mapsto & \kappa_{(b,d)} : & \mathbb{T} & \to & \mathbb{R} \\ & & & & x & \mapsto & \kappa_{(b,d)}(x). \end{array}$$



Persistence silhouette [4]



Persistence image[5]

Definition

For $p \geq 1$ and $\epsilon > 0$, **the ϵ-truncated p-persistence** of (b,d) is $w(d-b) = \max(d-b-\epsilon, 0)^p$. We define the **normalized functional** of D as persistence-weighted average of κ,

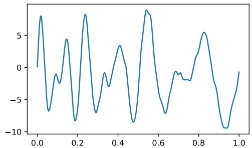
$$\begin{array}{rccc} \bar{\rho}(D) : & \mathbb{T} & \to & \mathbb{R} \\ & t & \mapsto & \frac{\sum_{(b,d)\in D} w(d-b)\kappa_{(b,d)}}{\sum_{(b,d)\in D} w(d-b)}. \end{array} \tag{12}$$

[3] Frédéric Chazal and Bertrand Michel (2021). "An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists". In: *Frontiers in Artificial Intelligence* 4. ISSN: 2624-8212.

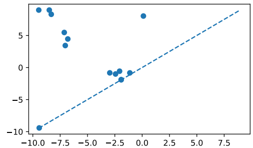
[4] Peter Bubenik (2015). "Statistical Topological Data Analysis Using Persistence Landscapes". In: *Journal of Machine Learning Research* 16.1, pp. 77–102

[5] Henry Adams et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252

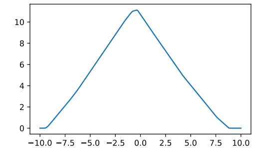
Proposed approach: normalized functionals of persistence



$\xrightarrow{D}$



$\xrightarrow{\bar{\rho}}$



$$F(S) := \mathbb{E}[\bar{\rho}(D(S))]. \tag{13}$$

Properties of signatures

1. Consistency: thanks to the additivity of persistence,

$$\bar{\rho}(D(\phi|_{[0,R]})) \xrightarrow{\|\cdot\|_{\mathcal{H}}} \bar{\rho}(D(\phi|_{[c,c+1]})), \qquad \text{as } R \to \infty. \tag{14}$$

2. Stability: when γ and W are random and independent, how does $F(S)$ depend on the law of γ?
3. Estimation: how to estimate the signature from a sampled time series?

Stability of the signature

A model for S

Let μ be a probability measure on $(\Gamma_{0,R,v_{\min}}, \mathcal{B}(\|\cdot\|_\infty))$ for some $v_{\min} > 0$, where

$$\Gamma_{0,R,v_{\min}} = \{\gamma \in C([0,T],\mathbb{R}) \mid \gamma(0) = 0,\ \gamma(T) = R,\ \gamma(s) - \gamma(t) \geq v_{\min}(s-t),\ \text{for all } s \geq t\}, \tag{15}$$

Let ν be a probability measure on $(C([0,T],\mathbb{R}), \mathcal{B}(\|\cdot\|_\infty))$, such that

$$\|W\|_\infty \leq (\max\phi - \min\phi)/2 - \epsilon \text{ almost-surely,} \tag{16}$$

$$t \mapsto W(t) \text{ has an } \alpha\text{-Hölder version.} \tag{17}$$

Let $S := \phi \circ \gamma + W$, where $\gamma \sim \mu$ and $W \sim \nu$ are independent.

Theorem

If μ_1, μ_2 are two probability measures on $\Gamma_{0,R,v_{\min}}$ and $S_k = \phi \circ \gamma_k + W$, then

$$\|F(S_1) - F(S_2)\|_{\mathcal{H}} \leq \frac{C}{v_{\min}^\alpha} \mathcal{W}_1(\mu_1,\mu_2)^\alpha, \tag{18}$$

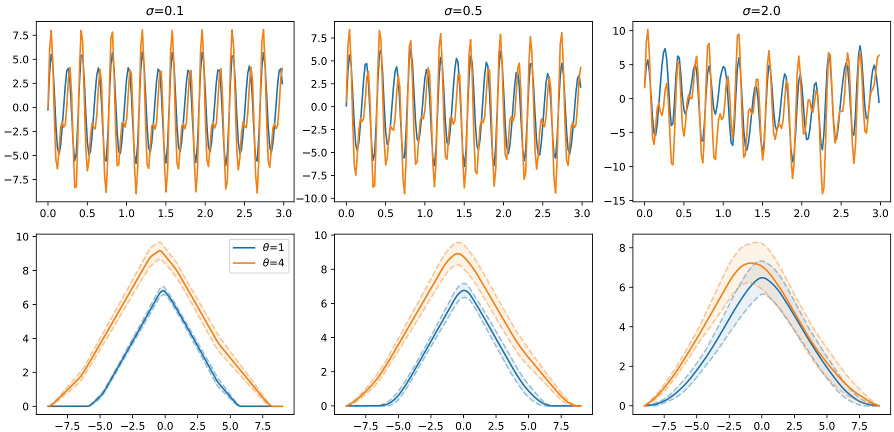
where $\mathcal{W}_1$ is the Wasserstein distance, and C depends on the regularity of W, $\|W\|_\infty$, ϵ, p and κ.

Comment

- ✓ As $c \to 0$, $\|F(\phi\circ\gamma_1) - F(\phi\circ\gamma_2 + cW)\|_{\mathcal{H}} \to 0$ (continuity of $\bar{\rho}$).
- ✗ How to remove the fixed-endpoints assumption in (15)?

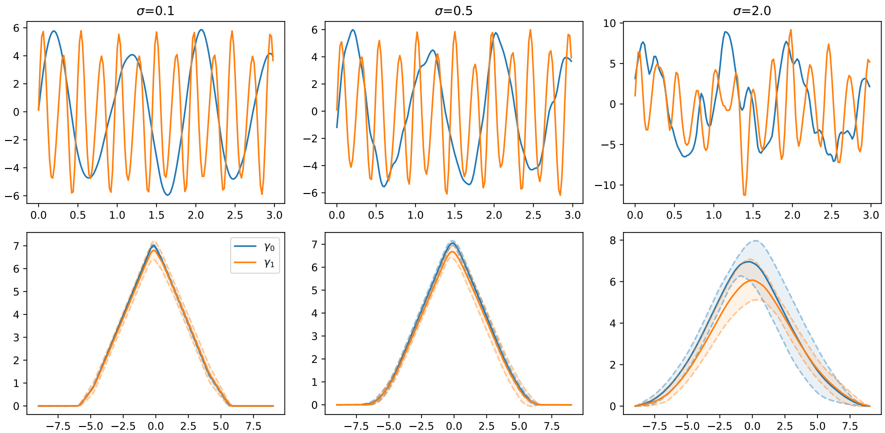
Numerical examples: stability

$$\phi_1 \neq \phi_2, \qquad \mu_1 = \mu_2,$$



Numerical examples: stability

$$\phi_1 = \phi_2, \qquad \mu_1 \neq \mu_2,$$



Estimation of signatures: introduction

Assume that only a single time series $(S_n)_{n=1}^N \subset \mathbb{R}$ is given,

$$S_n = \phi(\gamma(t_n)) + W(t_n).$$

Can we estimate the signature?

Proposition (Chazal et al. 2014[6], Berry et al. 2018[7])

Let be $D_1, \ldots, D_N$ i.i.d.persistence diagrams. When the (bracketing) entropy of $(\bar{\rho}_x)_{x \in \mathbb{T}}$ is finite,

$$\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}\bar{\rho}(D_n) - \bar{\rho}^*\right) \xrightarrow{d} \mathbb{G}, \tag{19}$$

for a zero-mean stochastic process $\mathbb{G}$.

Procedure

We fix $M \in \mathbb{N}$ and we generate $\mathbf{S}_1, \ldots \mathbf{S}_{N-M+1}$, where

$$\mathbf{S}_n = (S_n, \ldots, S_{n+M-1}).$$

Challenge

$\mathbf{S}_1, \ldots, \mathbf{S}_{N-M+1}$ are not independent! Under what assumptions on $(\gamma(t_n))_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ does an analogue of (19) hold?

[6] Frédéric Chazal et al. (2014). "Stochastic Convergence of Persistence Landscapes and Silhouettes". In: *Annual Symposium on Computational Geometry - SOCG'14*. Kyoto, Japan: ACM Press, pp. 474–483. ISBN: 978-1-4503-2594-3. DOI: 10.1145/2582112.2582128

[7] Eric Berry et al. (2018). *Functional Summaries of Persistence Diagrams*. arXiv: 1804.01618

Quantifying dependence

Definition (β-mixing coefficients, Dedecker et al. 2007)

Let $(X_n)_{n \in \mathcal{Z}}$ be a stationary sequence of random variables on a common measurable space. Then,

$$\beta(k) := \sup_{\mathcal{A},\mathcal{B}} \sum_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|,$$

where $\mathcal{A} \subset \sigma^X_{-\infty,0}$, $\mathcal{B} \subset \sigma^X_{k,\infty}$ are finite partitions of the sample space and $\sigma^X_{a,b} := \sigma((X_n)_{a \leq n \leq b})$.

Proposition (Kosorok 2008)

If $(\bar{\rho}_x)_{x \in \mathbb{T}}$ has finite bracketing entropy and $(\mathbf{S}_n)_{n \in \mathbb{N}}$ is stationary with $\beta_{\mathbf{S}}(k) = O(k^{-3})$, then

$$\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} \bar{\rho}(D_n) - \bar{\rho}^*\right) \xrightarrow{d} \mathbb{G}_{dep}, \tag{20}$$

where $\mathbb{G}_{dep}$ is a zero-mean stochastic process.

Proposition

For $k \geq M + 1$,

$$\beta_{\mathbf{S}}(k) \leq \beta_S(k - M + 1) \leq \beta_{\phi(\gamma)}(k - M + 1) + \beta_W(k - M + 1), \tag{21}$$

and

$$\beta_{\phi(\gamma)}(k) \leq \beta_{\mathrm{frac}(\gamma)}(k),$$

where $\mathrm{frac}(x) := x - \lfloor x \rfloor$.

Model for $(\gamma_n)_{n\in\mathbb{N}}$

Random walk model

For some $h > 0$, we set

$$\gamma_{n+1} = \gamma_n + hV_n,$$

for $(V_n)_{n\in\mathbb{N}} \sim \mathbf{P}$ i.i.d. We assume that

- $\mathrm{supp}(\mathbf{P}) \subseteq [v_{\min}, v_{\max}] \subset]0, \infty[$,
- for some $c > 0$ and a non-trivial interval $I \subset [v_{\min}, v_{\max}]$,

$$\mathbf{P}(A) \geq c\lambda(A), \qquad \text{for all } A \in \mathcal{B}(I). \tag{22}$$

Proposition

If $\gamma_0 \sim \mathcal{U}([0,1])$, then $(\mathrm{frac}(\gamma_n))_{n\in\mathbb{N}}$ is stationary and $\beta_{\mathrm{frac}(\gamma)}(k) = O(e^{-ak})$ for some $a > 0$.

Idea of the proof

1. By Thm 1 in Section 2.4 of Doukhan 1995, it suffices to show the **Doeblin condition**:

   There is μ_0 and $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ uniformly in x_0,

   $$P(\mathrm{frac}(\gamma_n) \in A \mid \gamma_0 = x_0) \geq \mu_0(A).$$

2. $\sum_{k=1}^{n} V_k \sim \mathbf{P}^{\star n}$, with $\mathbf{P}^{\star n}$ lower-bounded by a uniform measure with growing support.
3. For $n_0 \in \mathbb{N}$ big enough, the support is of length at least 1, and we obtain a lower–bound for the distribution of $\mathrm{frac}(\sum_{k=1}^{n_0} V_k)$ on $]0, 1[$.

Conclusion and future work

Conclusion

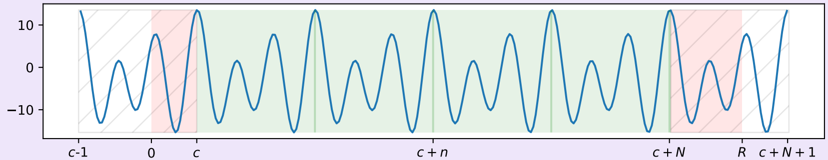
$F(S)$ is a **stable** signature of ϕ and can be **estimated with standard techniques**.

Limitations and future work

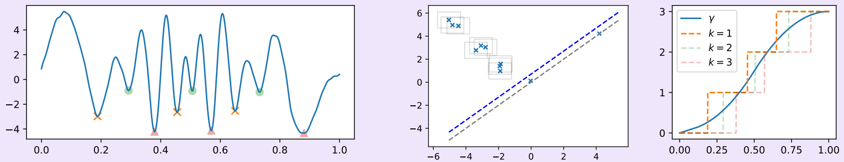
- Remove the assumption of fixed endpoints from the stability
  - Technical difficulties in defining the probability measures
  - Understand the distance between $D(f|_{[0,T]})$ and $D(f|_{[0,t]}) \cup D(f|_{[t,T]})$
- Numerical experiments to understand the discriminative power
  - Compare with registration-based methods.
  - Understand how the choice of the kernel

Summary

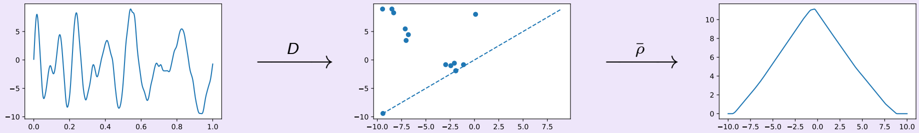
Additivity of diagrams (chapter 3)



Phase estimation (chapter 5)



Signatures of periodic functions (chapter 4)



Refernces I

Adams, Henry et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252.

Berry, Eric et al. (2018). *Functional Summaries of Persistence Diagrams*. arXiv: 1804.01618.

Boashash, Boualem, Peter O'Shea, and Morgan Arnold (1990). "Algorithms for Instantaneous Frequency Estimation: A Comparative Study". In: *Advanced Signal Processing Algorithms, Architectures, and Implementations*. Vol. 1348. SPIE, pp. 126–148. DOI: 10.1117/12.23471.

Bois, Alexandre et al. (2022). "A Topological Data Analysis-Based Method for Gait Signals with an Application to the Study of Multiple Sclerosis". In: *PLOS ONE* 17.5. Ed. by Chan Hwang See, e0268475. ISSN: 1932-6203. DOI: 10.1371/journal.pone.0268475.

Bonis, Thomas et al. (2022). *Topological Phase Estimation Method for Reparameterized Periodic Functions*. arXiv: 2205.14390.

Bristeau, Pierre-Jean (2012). "Techniques d'estimation du déplacement d'un véhicule sans GPS et autres exemples de conception de systèmes de navigation MEMS". PhD thesis. Ecole Nationale Superieure des Mines de Paris.

Bubenik, Peter (2015). "Statistical Topological Data Analysis Using Persistence Landscapes". In: *Journal of Machine Learning Research* 16.1, pp. 77–102.

Chazal, Frédéric and Bertrand Michel (2021). "An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists". In: *Frontiers in Artificial Intelligence* 4. ISSN: 2624-8212.

Chazal, Frédéric et al. (2014). "Stochastic Convergence of Persistence Landscapes and Silhouettes". In: *Annual Symposium on Computational Geometry - SOCG'14*. Kyoto, Japan: ACM Press, pp. 474–483. ISBN: 978-1-4503-2594-3. DOI: 10.1145/2582112.2582128.

Refernces II

Chazal, Frédéric et al. (2016). *The Structure and Stability of Persistence Modules*. SpringerBriefs in Mathematics 2191-8198. Springer, Cham. ISBN: 978-3-319-42543-6.

Dedecker, Jérôme et al. (2007). *Weak Dependence: With Examples and Applications*. 1st ed. Lecture Notes in Statistics 190. New York: Springer. ISBN: 978-0-387-69951-6.

Doukhan, Paul (1995). *Mixing*. Vol. 85. Lecture Notes in Statistics. Springer New York, NY. ISBN: 978-1-4612-2642-0.

Edelsbrunner, Herbert and John Harer (2010). *Computational Topology: An Introduction*. American Mathematical Society. ISBN: 978-0-8218-4925-5.

Gidea, Marian and Yuri Katz (2018). "Topological Data Analysis of Financial Time Series: Landscapes of Crashes". In: *Physica A: Statistical Mechanics and its Applications* 491, pp. 820–834. ISSN: 03784371. DOI: 10.1016/j.physa.2017.09.028.

Khasawneh, Firas A. and Elizabeth Munch (2018). "Topological Data Analysis for True Step Detection in Periodic Piecewise Constant Signals". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 474.2218, p. 20180027. DOI: 10.1098/rspa.2018.0027. (Visited on 11/30/2020).

Kosorok, Michael R. (2008). *Introduction to Empirical Processes and Semiparametric Inference*. Springer Series in Statistics. New York, NY: Springer New York. ISBN: 978-0-387-74977-8 978-0-387-74978-5. DOI: 10.1007/978-0-387-74978-5.

Marron, J. S. et al. (2015). "Functional Data Analysis of Amplitude and Phase Variation". In: *Statistical Science* 30.4, pp. 468–484. ISSN: 0883-4237. DOI: 10.1214/15-STS524.

Perea, Jose A. (2019). "Topological Time Series Analysis". In: *Notices of the American Mathematical Society* 66.05, p. 1. ISSN: 0002-9920, 1088-9477. DOI: 10.1090/noti1869.

Perez, Daniel (2022). *On C0-persistent Homology and Trees*. arXiv: 2012.02634v3.

Refernces III

Plonka, Gerlind and Yi Zheng (2016). "Relation between Total Variation and Persistence Distance and Its Application in Signal Processing". In: *Advances in Computational Mathematics* 42.3, pp. 651–674. ISSN: 1572-9044. DOI: 10.1007/s10444-015-9438-8.

Reise, Wojciech, Bertrand Michel, and Frédéric Chazal (2023). *Topological Signatures of Periodic-like Signals*. arXiv: 2306.13453.

Srivastava, A et al. (2011). "Shape Analysis of Elastic Curves in Euclidean Spaces". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 33.7, pp. 1415–1428. ISSN: 0162-8828. DOI: 10.1109/TPAMI.2010.184.

Su, Jingyong et al. (2014). "Statistical Analysis of Trajectories on Riemannian Manifolds: Bird Migration, Hurricane Tracking and Video Surveillance". In: *The Annals of Applied Statistics* 8.1, pp. 530–552. ISSN: 1932-6157. DOI: 10.1214/13-AOAS701.

Tang, R. and H.-G. Muller (2008). "Pairwise Curve Synchronization for Functional Data". In: *Biometrika* 95.4, pp. 875–889. ISSN: 0006-3444, 1464-3510. DOI: 10.1093/biomet/asn047.

Tanweer, Sunia, Firas A. Khasawneh, and Elizabeth Munch (2023). *Robust Zero-crossings Detection in Noisy Signals Using Topological Signal Processing*. arXiv: 2301.07703 [cs, eess].

Zhao, Jiaping and Laurent Itti (2018). "shapeDTW: Shape Dynamic Time Warping". In: *Pattern Recognition* 74, pp. 171–184. ISSN: 00313203. DOI: 10.1016/j.patcog.2017.09.020.

Thank you!

Persistence diagram of sub level sets: Definition

1. Persistence module

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$.

2. Rectangle measure

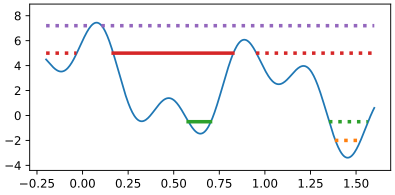
A measure m on rectangles of $\mathbb{R}^2$.

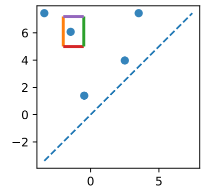
$$m(]a, b] \times [c, d[) = \dim \left(\frac{im(\iota_b^c) \cap \ker(\iota_c^d)}{im(\iota_a^c) \cap \ker(\iota_c^d)} \right),$$

3. Persistence diagram

The persistence diagram $D(f)$ is a multi-set in $\mathbb{R}^2$, where $(s, t) \in \mathbb{R}^2$ has multiplicity

$$m(s, t) = \lim_{\delta \to 0^+} m(]s - \delta, s + \delta] \times [t - \delta, t + \delta[).$$





Proof of additivity of sub level sets: details

Proof.

Let $c := \inf\{x \in [0,1[| \ \phi(x) = \max \phi\}$, $N = \max\{n \in \mathbb{N} \mid c + n \leq R\}$ and denote by $\mathbb{X}_t := \phi^{-1}([-\infty, t[)$.

Step 1: For any $t < M$, $\mathbb{X}_t \cap [0, c] \cap [c, c+1] = \emptyset$, so

$$H_0(\mathbb{X}_t \cap [0, R]) \simeq H_0(\mathbb{X}_t \cap [0, c]) \oplus H_0(\mathbb{X}_t \cap [c, c+N]) \oplus H_0(\mathbb{X}_t \cap [c+N, R])), \tag{23}$$

Step 2: similarly,

$$H_0(\mathbb{X}_t \cap [c, c+N]) \simeq \bigsqcup_{n=1}^{N} H_0(\mathbb{X}_t \cap [c + (n-1), c+n]) \tag{24}$$

$$(x \mapsto x+n) \quad \simeq \bigsqcup_{n=1}^{N} H_0(\mathbb{X}_t \cap [c, c+1]) \tag{25}$$

Step 3: The inclusion $[0, c] \subset [c-1, c]$ induces an injective morphism

$$H_0(\mathbb{X}_t \cap [0, c]) \hookrightarrow H_0(\mathbb{X}_t \cap [c-1, c]).$$

□

Stability: bottleneck distance (detailed)

Definition (Edelsbrunner and Harer 2010, p. VIII.2)

We call a ϵ-matching between two persistence diagrams D and D' a bijection $\Gamma : A \to A'$ between some subsets of $A \subset D$ and $A' \subset D'$, considered with multiplicity, if

$$\begin{aligned} d_\infty(a, \Gamma(a)) &\leq \epsilon, && \text{for any } a \in A, \\ d_\infty(a, \Delta) &\leq \epsilon, && \text{for any } a \in (D \setminus A) \cup (D' \setminus A'). \end{aligned}$$

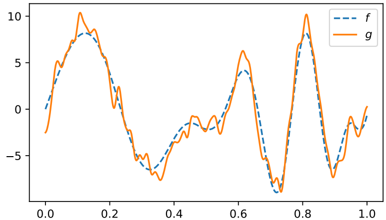
where $\Delta = \{(x, x) \in \mathbb{R}^2\}$ denotes the diagonal.

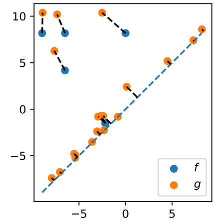
$$d_B(D, D') := \inf\{\epsilon > 0 \mid \Gamma \text{ is an } \epsilon\text{-matching between } D' \text{ and } D'\}.$$

Theorem (Bottleneck stability of diagrams)

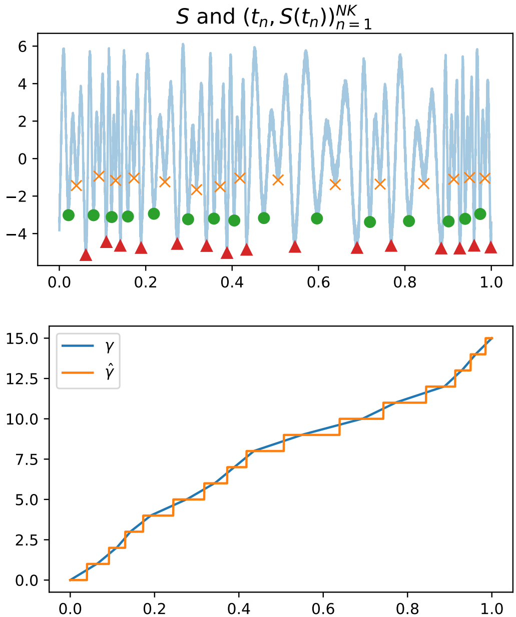
Let $f, g : \mathbb{X} \to \mathbb{R}$ be two continuous functions on a compact space $\mathbb{X}$. Then,

$$d_B(D(f), D(g)) \leq \|f - g\|_\infty.$$



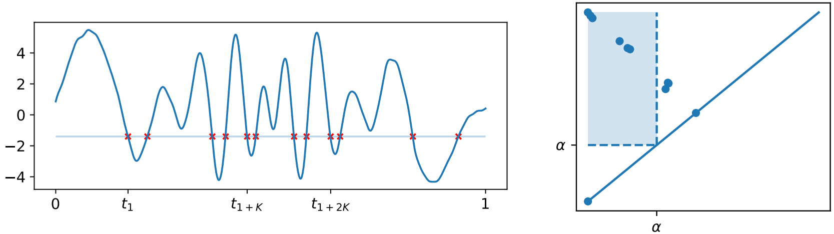


Landmarks for multiple periods



Zero-crossings from the persistence diagram

$$|S^{-1}(\alpha)| = 2 \lim_{\delta \to 0^+} |D(S) \cap (]-\infty, \alpha - \delta] \times [\alpha + \delta, \infty[)|.$$



Counting measure

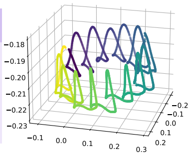
The persistence diagram D is also a counting measure on rectangles $A \subset \Delta_+ = \{(b,d) \in \mathbb{R}^2 \mid x < y\}$. By (4),

$$|D(\phi \circ \gamma) \cap A| = N \; |D(\phi|_{[0,1]}) \cap A|$$

Application: magnetic odometry and speed estimation

Problem

Using the magnetic signal **B**, recorded in a moving car, estimate the cars' trajectory. The angular position $t \mapsto \gamma(t)$ of a wheel in time is visible through $\mathbf{S}(t) = \mathbf{B}(\gamma(t), \gamma_h(t))$



Proposed solution

1. $S := \langle \mathbf{S}, v \rangle$, project **S** along a suitable direction $v \in \mathbb{S}^2$
2. $\hat{N}_{c,\tau}(S)$, for an appropriate scale τ,
3. Derive an odometric sequence $t_1, \dots, t_{\hat{N}_\tau(S)}$ from $\mathcal{C}_\tau$.
4. Construct $\hat{\gamma} : [0, T] \to \mathbb{R}$.

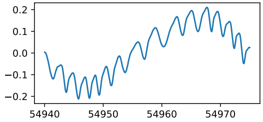
Results

	E_O		E_I	
Method	$\mathbf{S}_{v_1}$	$(\nabla \mathbf{S})_{v_1}$	$\mathbf{S}_{v_1}$	$(\nabla \mathbf{S})_{v_1}$
$\hat{N}_{c,\tau}$	15.75	16.66	3.02	3.01
$\hat{N}_{0,\tau}$	15.75	16.66	3.02	3.01
ZC	9.91	5.62	6.35	16.51

Application: magnetic odometry and speed estimation

Problem

Using the magnetic signal $\mathbf{B}$, recorded in a moving car, estimate the cars' trajectory. The angular position $t \mapsto \gamma(t)$ of a wheel in time is visible through $\mathbf{S}(t) = \mathbf{B}(\gamma(t), \gamma_h(t))$



Proposed solution

1. $S := \langle \mathbf{S}, v \rangle$, project $\mathbf{S}$ along a suitable direction $v \in \mathbb{S}^2$
2. $\hat{N}_{c,\tau}(S)$, for an appropriate scale τ,
3. Derive an odometric sequence $t_1, \ldots, t_{\hat{N}_\tau(S)}$ from $\mathcal{C}_\tau$.
4. Construct $\hat{\gamma} : [0, T] \to \mathbb{R}$.

Results

	E_O		E_I	
Method	$\mathbf{S}_{v_1}$	$(\nabla \mathbf{S})_{v_1}$	$\mathbf{S}_{v_1}$	$(\nabla \mathbf{S})_{v_1}$
$\hat{N}_{c,\tau}$	15.75	16.66	3.02	3.01
$\hat{N}_{0,\tau}$	15.75	16.66	3.02	3.01
ZC	9.91	5.62	6.35	16.51

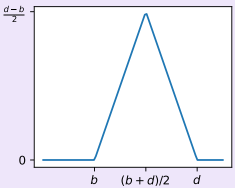
Normalized functionals of persistence

Functional representation

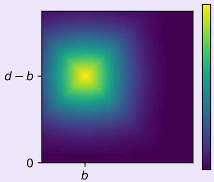
Let $\mathcal{H}$ be a functional Banach space

$$\begin{array}{rccccccc} \kappa: & \mathbb{R}^2 & \to & \mathcal{H} & & & & \\ & (b,d) & \mapsto & \kappa_{(b,d)}: & \mathbb{T} & \to & \mathbb{R} & \\ & & & & x & \mapsto & \kappa_{(b,d)}(x). & \end{array}$$

1. $\mathrm{supp}(\kappa_{(b,d)}) \subset K$, K bounded,
2. $x \mapsto \kappa_{(b,d)}(x)$ (uniformly) Lipschitz,
3. $\|\kappa_{(b,d)} - \kappa_{(b',d')}\|_{\mathcal{H}} \leq L_\kappa \|(b,d) - (b',d')\|$,
4. $\|\kappa_{(b,b)}\|_{\mathcal{H}} \leq C$.



Persistence silhouette[8]



Persistence image[9]

Normalized functionals of persistence diagrams

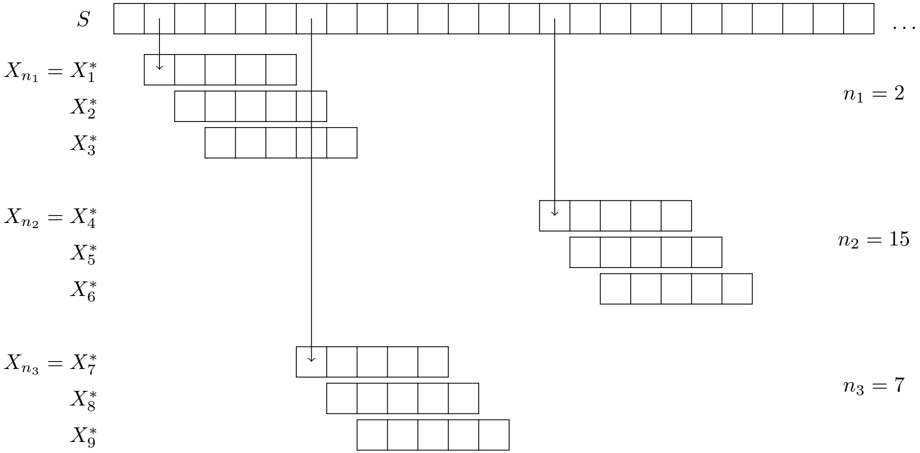
For some $p \geq 1$ and $\epsilon > 0$,

$$\bar{\rho}(D) := \frac{\sum_{(b,d)\in D} w(d-b)\kappa_{(b,d)}}{\sum_{(b,d)\in D} w(d-b)}, \qquad \text{where } w(d-b) = \max(d-b-\epsilon, 0)^p. \tag{26}$$

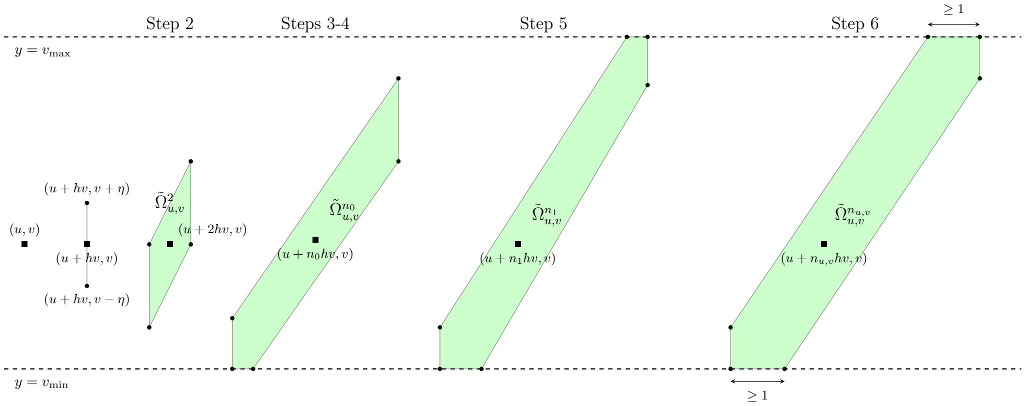
[8] Peter Bubenik (2015). "Statistical Topological Data Analysis Using Persistence Landscapes". In: *Journal of Machine Learning Research* 16.1, pp. 77–102

[9] Henry Adams et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252

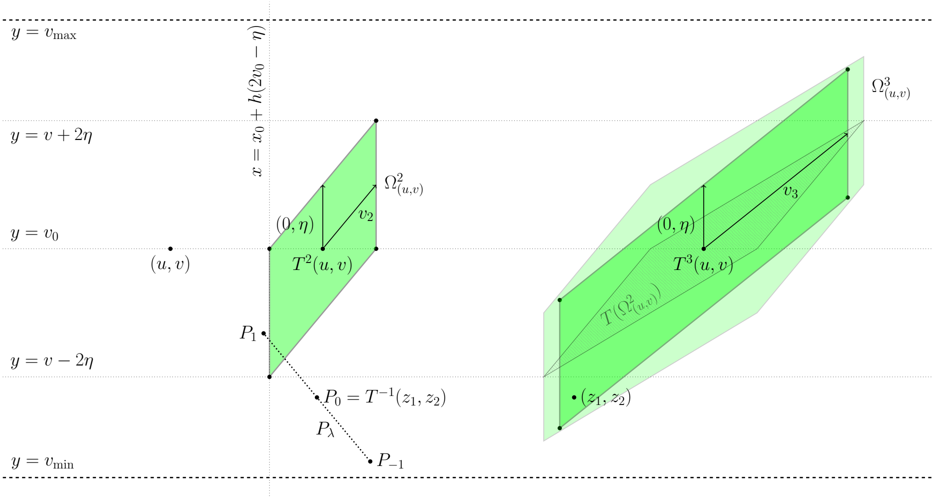
Bootstrap



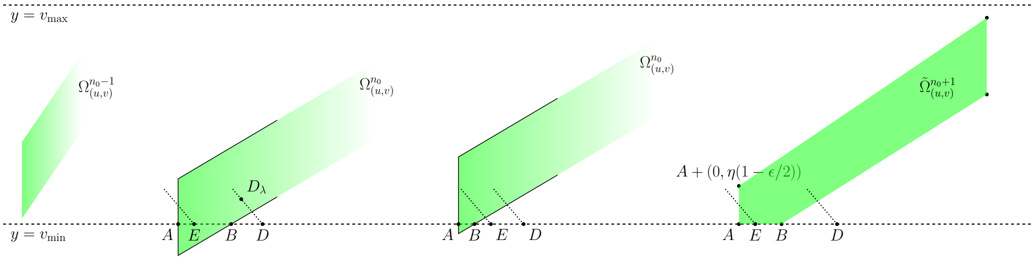
Mixing: overview



Mixing: initial



Mixing: boundary



Measures of dependence

Types of dependence

There are different ways to measure dependence in a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$:

- m-dependence,
- strong-mixing,
- weak-dependence,

Strong mixing

The β-mixing coefficient of a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ is

$$\beta_X(k) = \sup_{\mathcal{A},\mathcal{B}} \sum_{A\in\mathcal{A},B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|,$$

where $\mathcal{A} \subset \sigma^X_{-\infty,0}$, $\mathcal{B} \subset \sigma^X_{k,\infty}$ are finite partitions of the sample space and $\sigma^X_{a,b} := \sigma((X_n)_{a\leq n\leq b})$.

Example

1. If $(X_n)_n$ is m-dependent, then $\beta_X(k) = 0$ for $k \geq m$.
2. Markov chains: irreducible and aperiodic.

Proposition

For any measurable function $f : \mathbb{X} \to \mathbb{Y}$, $\beta_X(k) \leq \beta_{f(X)}(k)$.

Measures of dependence

Types of dependence

There are different ways to measure dependence in a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$:

- m-dependence,
- strong-mixing, $\longrightarrow$ **preserved by measurable functions!**
- weak-dependence,

Strong mixing

The β-mixing coefficient of a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ is

$$\beta_X(k) = \sup_{\mathcal{A},\mathcal{B}} \sum_{A\in\mathcal{A}, B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|,$$

where $\mathcal{A} \subset \sigma^X_{-\infty,0}$, $\mathcal{B} \subset \sigma^X_{k,\infty}$ are finite partitions of the sample space and $\sigma^X_{a,b} := \sigma((X_n)_{a\leq n\leq b})$.

Example

1. If $(X_n)_n$ is m-dependent, then $\beta_X(k) = 0$ for $k \geq m$.
2. Markov chains: irreducible and aperiodic.

Proposition

For any measurable function $f : \mathbb{X} \to \mathbb{Y}$, $\beta_X(k) \leq \beta_{f(X)}(k)$.