Topological techniques for inference on periodic functions with phase variation

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Add a picture of a periodic function with phase variation

TopAI chair
ANR-19-CHIA-0001

Data with phase variation

Signals with phase variation

A sample $S_1, \ldots, S_N$ has phase variation if

$$S_n = f(\gamma_n) + W_n, \qquad \text{for each } n \in \{1, \ldots, N\}, \tag{1}$$

where $f : [0,1] \to \mathbb{X}$ and $\gamma_1, \ldots, \gamma_N : [0,1] \to [0,1]$ are increasing homeomorphisms.

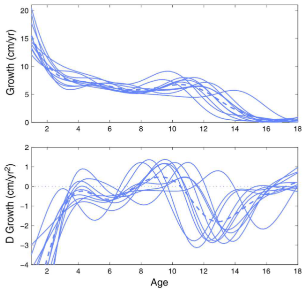

FIG. 3. *The top panel plots the growth, understood as the first derivative of height, of ten girls, and the bottom panel contains the corresponding height-acceleration or growth-derivative curves. The dashed curve in both plots is the cross-sectional mean. Both these plots indicate both phase and amplitude variability.*

Source: ???

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Problems

- Computing a representative of f (Su et al. 2014)

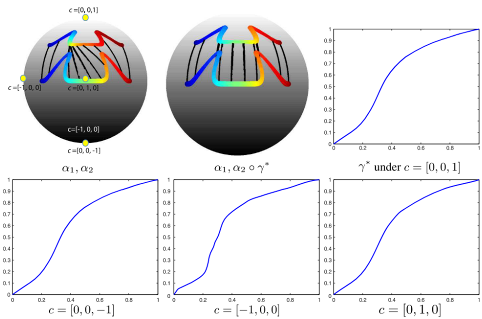

Fig. 2. *Registration of trajectories on $\mathbb{S}^2$.*

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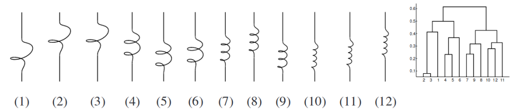

Fig. 4. A set of helices with different numbers and placements of spirals and their clustering using the elastic distance function.

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- Computing a representative of f (Su et al. 2014)
- Clustering of $S_1, \ldots, S_n$ (Srivastava et al. 2011)
- Synchronising curves: estimating $\gamma_n \circ \gamma_{n'}^{-1}$ (Tang and Muller 2008, Zhao and Itti 2018)

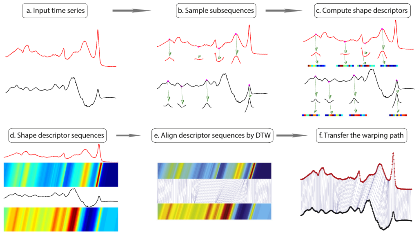

Fig. 2. Pipeline of shapeDTW. shapeDTW consists of two major steps: encode local structures by shape descriptors and align descriptor sequences by DTW. Concretely, we sample a subsequence from each temporal point, and further encode it by some shape descriptor. As a result, the original time series is converted into a descriptor sequence of the same length. Then we align two descriptor sequences by DTW and transfer the found warping path to the original time series.

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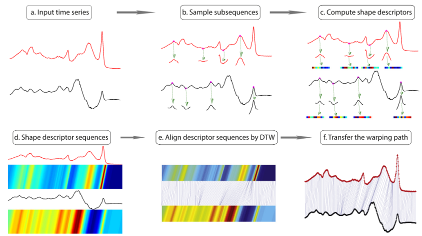

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Same endpoints!

For all $1 \leq n \leq N$, $\gamma_1(0) = \gamma_n(0)$ and $\gamma_1(1) = \gamma_n(1)$.

Periodic data with phase variation

Periodic function with variation

A periodic signal with phase variation is $S : [0,1] \to \mathbb{X}$

$$S = \phi(\gamma) + W_n. \tag{2}$$

where $\phi : \mathbb{R} \to \mathbb{X}$ is 1-periodic and $\gamma : [0,1] \to [0,R]$ is an increasing homeomorphism.

Phase estimation

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- Instantaneous phase estimation (Boashash 1992)

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Phase estimation

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- Zero-crossings (Khasawneh and Munch 2018, Tanweer, Khasawneh, and Munch 2023)

Topological data analysis for time series data

Persistent homology from Topological data analysis (TDA) quantifies the *shape of data*.

Time series

- Detecting periodicity (Perea, Munch, and Khasawneh 2019)
- Robust zero-crossings (Khasawneh and Munch 2018)
- The number of periods in the signal is reflected by the number of prominent features (Bois et al. 2022.)

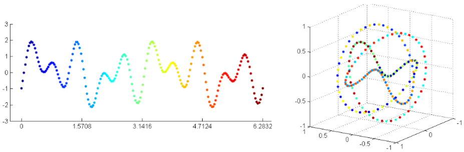


FIGURE 1. From a periodic function to its sliding window point cloud. **Left:** A periodic function f. **Right:** Multidimensional scaling into $\mathbb{R}^3$ for $SW_{20,\tau}f$. For each t, we use the same color for $f(t)$ and $SW_{20,\tau}f(t)$. Please refer to an electronic version for colors.

Contributions

Apply persistent homology to study periodic data with phase variation, with guarantees.

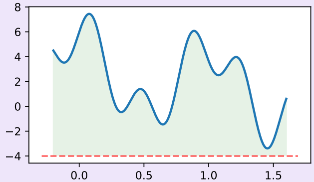
1. Describe the structure of a topological descriptor of S
2. Estimate $\gamma(T) - \gamma(0)$ and construct $\hat{\gamma}$ an estimator of γ.
3. Construct a descriptor of ϕ.

Outline

Persistence diagram of sub level sets

Intuition

The persistence diagram $D(f)$ of a continuous function $f : [0, T] \to \mathbb{R}$ is a multi-set of points in $\mathbb{R}^2$, which reflect when connected components appear and merge in $(f^{-1}(]-\infty, t]))_{t\in\mathbb{R}}$ as t increases.



[1] Frédéric Chazal et al. (2016). *The Structure and Stability of Persistence Modules*. SpringerBriefs in Mathematics 2191-8198. Springer, Cham. ISBN: 978-3-319-42543-6.

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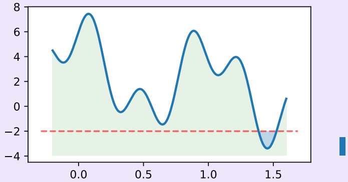

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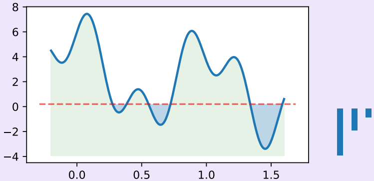

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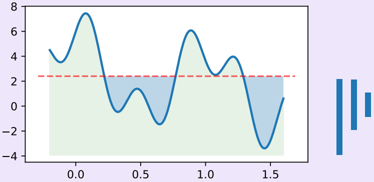

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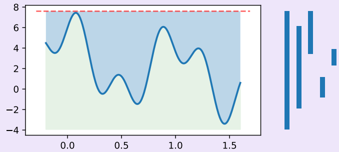

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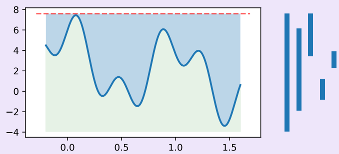

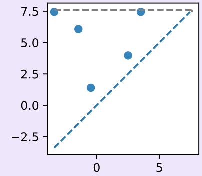

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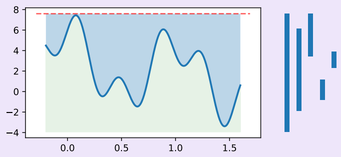
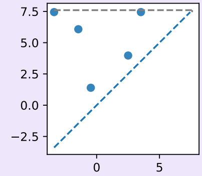
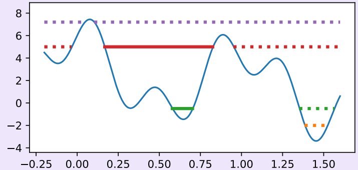
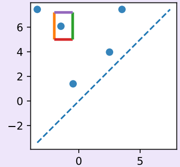




Definition

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$.





If $f : \mathbb{X} \to \mathbb{R}$ is continuous and $\mathbb{X}$ compact, then $D(f)$ is well-defined[1].

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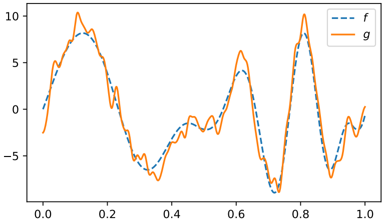
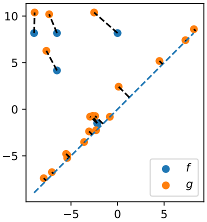
Stability: bottleneck distance

Theorem (Bottleneck stability of diagrams)

Let $f, g : \mathbb{X} \to \mathbb{R}$ be two continuous functions on a compact space $\mathbb{X}$. Then,

$$d_B(D(f), D(g)) \leq \|f - g\|_\infty.$$





Total p-persistence

Definition

The total p-persistence of a diagram D is

$$\mathrm{pers}_p(D) := \left(\sum_{(b,d)\in D} (d-b)^p \right)^{1/p} .$$

Proposition (Plonka and Zheng 2016, Perez 2022)

For $p = 1$, $\mathrm{pers}_1(D(f)) + \mathrm{pers}_1(D(-f)) = TV(f)$.
If f is α-Hölder for $p > 1 + 1/\alpha$, then, $\mathrm{pers}_p(D(f)) < \infty$.

[2] Olympio Hacquard et al. (2021). "Topologically Penalized Regression on Manifolds". In: *arXiv:2110.13749 [cs, math, stat]*. arXiv: `2110.13749 [cs, math, stat]`. (Visited on 01/04/2022), Section 3.2.

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Persistence diagrams of periodic functions

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function and denote by $\phi|_{[a,b]}$ the restriction of ϕ to an interval $[a, b]$.

Proposition (Invariance to reparametrisation)

Let $\gamma : [0, 1] \to [0, 1]$ be an increasing homeomorphism. Then, $D(\phi \circ \gamma) = D(\phi|_{[0,1]})$.

Theorem (Additivity of persistence diagrams for periodic functions)

For $R > 1$, there exists $c \in [0, 1]$ such that

$$D(\phi|_{[0,R]}) = \lfloor R - 1 \rfloor D(\phi|_{[c,c+1]}) + D', \quad \textit{with} \ \ \mathrm{pers}_p(D') \leq 2\mathrm{pers}_p(D(\phi|_{[c,c+1]})). \tag{3}$$

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If $R \in \mathbb{N}^*$, then

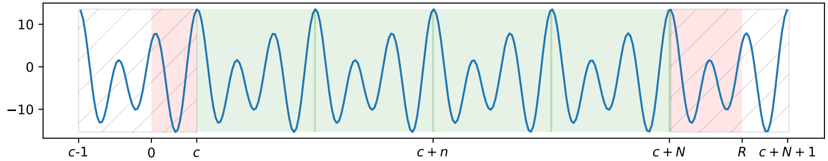
$$D(\phi|_{[0,R]}) = RD(\phi|_{[c,c+1]}). \tag{4}$$

Conclusion

The persistence diagram $D(\phi \circ \gamma)$ contains information about

- extrema of ϕ,
- number of periods $(\gamma(1) - \gamma(0))$.

Proof



Proof.

Let $c := \inf\{x \in [0,1[\mid \phi(x) = \max \phi\}$, $N = \max\{n \in \mathbb{N} \mid c + n \leq R\}$ and denote by $X_t := \phi^{-1}([-\infty, t[)$.

Step 1: For any $t < M$, $X_t \cap [0, c] \cap [c, c+1] = \emptyset$, so

$$H_0(X_t \cap [0, R]) \simeq H_0(X_t \cap [0, c]) \oplus \color{green}{H_0(X_t \cap [c, c+N])} \oplus H_0(X_t \cap [c+N, R])), \tag{5}$$

□

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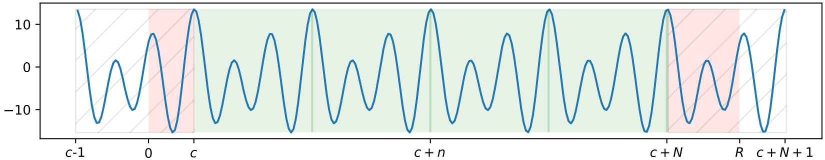

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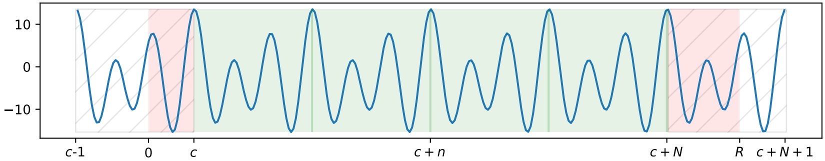
Step 2: similarly,

$$H_0(X_t \cap [c, c+N]) \simeq \bigsqcup_{n=1}^{N} H_0(X_t \cap [c + (n-1), c+n]) \tag{6}$$

$$(x \mapsto x + n) \quad \simeq \bigsqcup_{n=1}^{N} H_0(X_t \cap [c, c+1]) \tag{7}$$

$\square$

Proof



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Step 3: The inclusion $[0, c] \subset [c-1, c]$ induces an injective morphism

$$H_0(X_t \cap [0, c]) \hookrightarrow H_0(X_t \cap [c-1, c]).$$

$\square$

Plan

Additivity of persistence diagrams of periodic functions

Segmentation of periodic signals

Signatures of periodic signals with phase variation

Segmenting a periodic curve with phase variation

Problem: phase estimation

Given S, estimate γ

$$\begin{array}{rccl} S: & [0,T] & \to & \mathbb{R} \\ & t & \mapsto & \phi(\gamma(t)) + W(t). \end{array}$$

Setting:

1. ϕ is unknown.
2. The number of periods N is an integer: $\gamma : [0, T] \to [0, N]$ with $N \in \mathbb{N}$.

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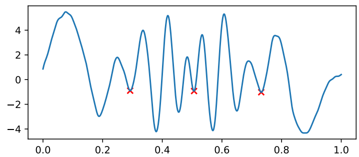
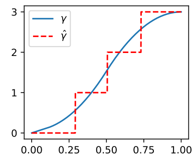
Proposed solution: odometry

1. Estimate N.
2. Find $t_1 < \ldots < t_N$ such that $\gamma(t_n) - \gamma(t_{n-1}) = 1$ for all $n = 2, \ldots, N$.

Let $\hat{\gamma} : [0,T] \to \mathbb{R}^*$ be such that $\hat{\gamma}(t_n) = n$ and interpolate.





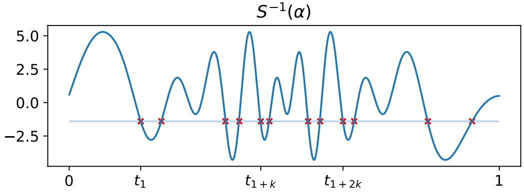
Developed and studied in (Bonis et al. 2022).

Zero-crossings

Let $K := |\phi^{-1}(\alpha) \cap [0, 1[|$ and assume that $0 < K < \infty$, for some $\alpha \in \mathbb{R}$.

Estimation of N

If K is known, $N_\alpha(S) := \frac{|S^{-1}(\alpha)|}{K}$ is an estimator of N.



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If $S^{-1}(\alpha) = \{t_1, \ldots, t_{NK}\}$, then $\gamma(t_{n+k}) - \gamma(t_n) = 1$ for $1 \le k \le K$ and $n \le NK - k$.

[3] Sunia Tanweer, Firas A. Khasawneh, and Elizabeth Munch (2023). *Robust Zero-crossings Detection in Noisy Signals Using Topological Signal Processing.* arXiv: 2301.07703 [cs, eess]. (Visited on 10/17/2023).

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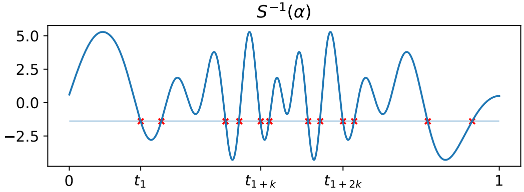

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Issues

- K is not known (and not necessarily finite)
- N_α is not stable[3]

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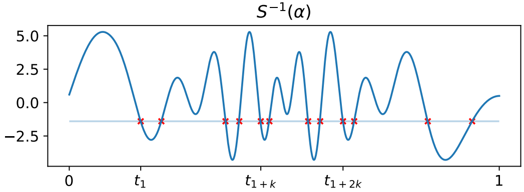

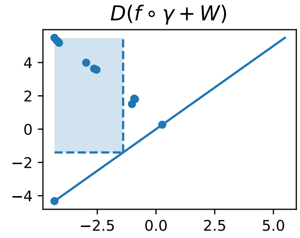

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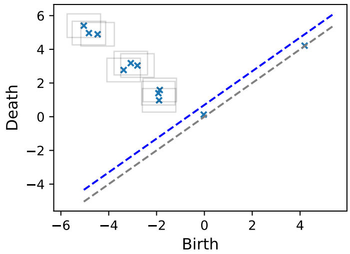
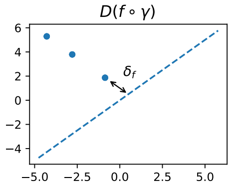
Estimation of N: stability



Estimator

For $\tau > 0$, we define

$$\hat{N}_\tau(S) := \gcd\{|D(S) \cap B(x, \tau)| \mid x \in D(S), \mathrm{pers}(x) > \tau\}. \tag{8}$$



Separation constant

$$\delta_\phi := \min(d(x_1, x_2), d(x_1, \Delta) \mid x_1, x_2 \in D(\phi))$$

Proposition (Stability)

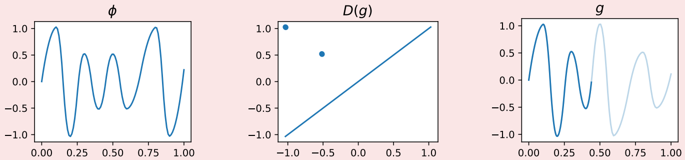
For $\tau > 0$ satisfying $2\|W\|_\infty < \tau < \delta/3$, we have that

$$\hat{N}_\tau(S) = \hat{N}_\tau(\phi \circ \gamma).$$

Correctness

Identifiability with the diagram

There exists a 1-periodic function g such that $D(g|_{[0,1]}) = D(\phi|_{[0,1]})/\hat{N}(\phi|_{[0,1]})$!



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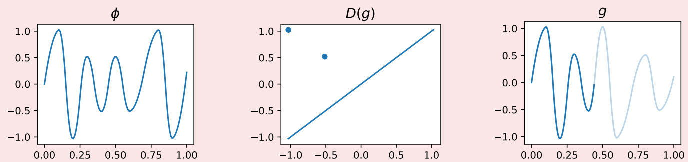


Non-degeneracy

We say that $\phi|_{[0,1]}$ is **non-degenerate** if $\hat{N}(\phi|_{[0,1]}) = 1$,

$$\hat{N}(\phi|_{[0,1]}) := \gcd \left\{ \lim_{\tau \to 0^+} |D(\phi|_{[0,1]}) \cap B(x,\tau)| \mid x \in D(\phi) \right\}. \tag{9}$$

Example

If ϕ has at least one unique critical value, it is non-degenerate.

Corollary (Stability)

If ϕ is non-degenerate, then for any $\tau > 0$ such that $2\|W\|_\infty < \tau < \delta/3$, we have

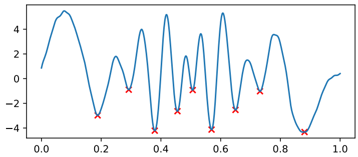
$$\hat{N}_\tau(S) = N.$$

Odometric sequence

Proposition

Let $\tau > 0$ and $\hat{\mathcal{C}}_\tau$ be the set of local minima of S, corresponding to points in the diagram with persistence more than τ. If $\tau \in]2\epsilon, \delta/3[$, then

$$|\hat{\mathcal{C}}_\tau| = NK.$$



Put the figure with $K = 3$ sequences!

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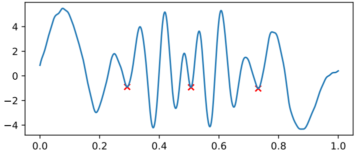

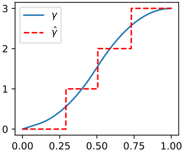

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Conclusion and future work

Example

We can infer the number of periods of ϕ in S by counting points in the persistence diagram. We can also construct a segmentation of $[0, T]$ into periods.

Limitations and future work

1. Identifiability
2. More robust estimators
3. The method is applicable only to $N \in \mathbb{N}$: boundary effects when $\gamma(1) - \gamma(0) \notin \mathbb{N}$.

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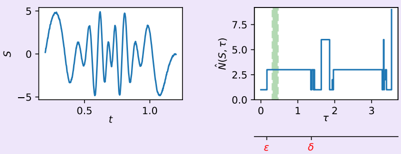
We can infer the number of periods of ϕ in S by counting points in the persistence diagram. We can also construct a segmentation of $[0, T]$ into periods.

Limitations and future work

1. Identifiability
   - Use merge trees to verify that the segmentation is correct
2. More robust estimators
   - Extend the guarantees to $\hat{N}_c$ and $\hat{N}_T$
   - Choose the sets to count multiplicity differently



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Outline

Problem statement

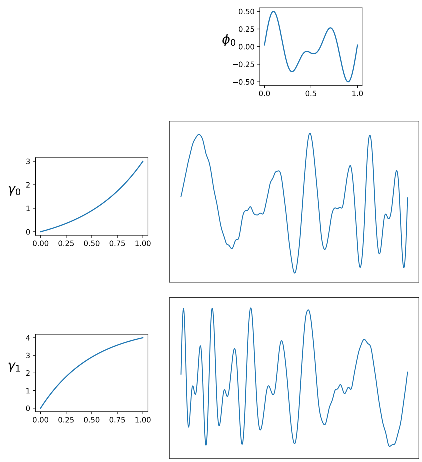
Data

Consider $S = \phi \circ \gamma + W$, where

- $\phi : \mathbb{R} \to \mathbb{R}$ is 1–periodic,
- $\gamma : [0, T] \to \mathbb{R}$ an increasing bijection,
- $W : [0, T] \to \mathbb{R}$ is a cont. stoch. proc.

Aim

Given S, construct a signature of ϕ.



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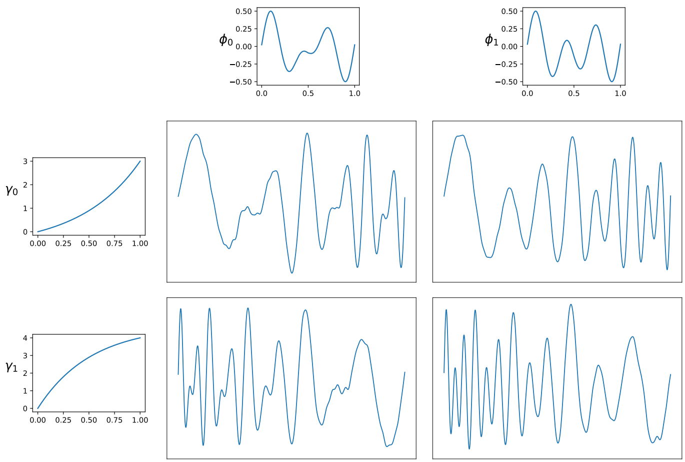


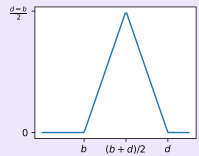
Functional representations of persistence diagrams

Persistence diagrams can be seen as multi-sets or discrete measures: even when a mean persistence diagram exists, it is not necessarily unique![4,5] It is common to represent a persistence diagram D in a functional space[6,7].

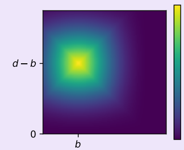
Functional representation

Let $\mathcal{H}$ be a functional Banach space

$$\begin{array}{rccccccc} \kappa: & \mathbb{R}^2 & \to & \mathcal{H} & & & & \\ & (b,d) & \mapsto & \kappa_{(b,d)}: & \mathbb{T} & \to & \mathbb{R} & \\ & & & & x & \mapsto & \kappa_{(b,d)}(x). & \end{array}$$



Persistence silhouette[8]



Persistence image[9]

Normalized functionals of persistence diagrams

For some $p \geq 1$ and $\epsilon > 0$,

$$\bar{\rho}(D) := \frac{\sum_{(b,d)\in D} w(d-b)\kappa_{(b,d)}}{\sum_{(b,d)\in D} w(d-b)}, \qquad \text{where } w(d-b) = \max(d-b-\epsilon, 0)^p. \tag{10}$$

[4] Yuriy Mileyko, Sayan Mukherjee, and John Harer (2011). "Probability Measures on the Space of Persistence Diagrams". In: *Inverse Problems* 27.12, p. 124007. ISSN: 0266-5611, 1361-6420. DOI: 10.1088/0266-5611/27/12/124007. (Visited on 08/18/2021).

[5] Vincent Divol and Théo Lacombe (2021). "Estimation and Quantization of Expected Persistence Diagrams". In: *arXiv:2105.04852 [math, stat]*. arXiv: 2105.04852 [math, stat]. (Visited on 05/31/2021).

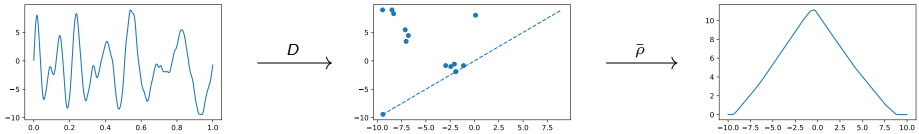
[6] Frédéric Chazal and Bertrand Michel (2021). "An Introduction to Topological Data Analysis: Fundamental and Practical Aspects for Data Scientists". In: *Frontiers in Artificial Intelligence* 4. ISSN: 2624-8212. (Visited on 12/15/2022).

[7] Eric Berry et al. (2020). "Functional Summaries of Persistence Diagrams". In: *Journal of Applied and Computational Topology* 4.2, pp. 211–262. ISSN: 2367-1734. DOI: 10.1007/s41468-020-00048-w. (Visited on 08/24/2023).

[8] Peter Bubenik (2015). "Statistical Topological Data Analysis Using Persistence Landscapes". In: *Journal of Machine Learning Research* 16.1, pp. 77–102

[9] Henry Adams et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252

Proposed approach: normalized functionals of persistence



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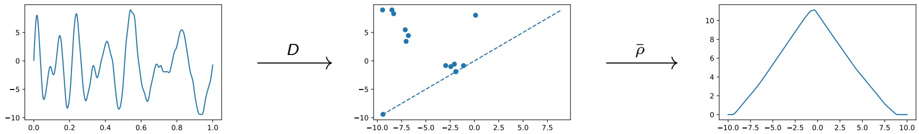

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Properties

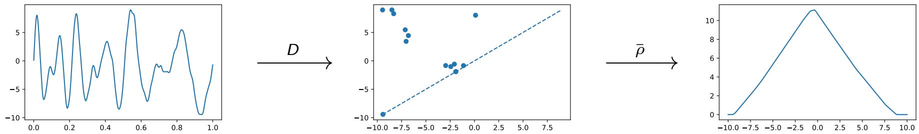
1. Consistency

$$\bar{\rho}(D(\phi|_{[0,R]})) \xrightarrow{\|\cdot\|_{\mathcal{H}}} \bar{\rho}(D(\phi|_{[c,c+1]})), \qquad \text{as } R \to \infty. \tag{12}$$

2. Stability

3. Estimation of a signature from a sampled time series

Proposed approach: normalized functionals of persistence



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Let μ_1 and μ_2 be probability measures on a space of reparametrisations with fixed endpoints $\gamma(0), \gamma(T)$.
$$\|\mathbb{E}_{\gamma_1 \sim \mu_1, W}[\bar{\rho}(\phi \circ \gamma_1 + W)] - \mathbb{E}_{\gamma_2 \sim \mu_2, W}[\bar{\rho}(\phi \circ \gamma_2 + W)]\|_{\mathcal{H}} \leq C W_1(\mu_1, \mu_2)^{\alpha}. \tag{13}$$
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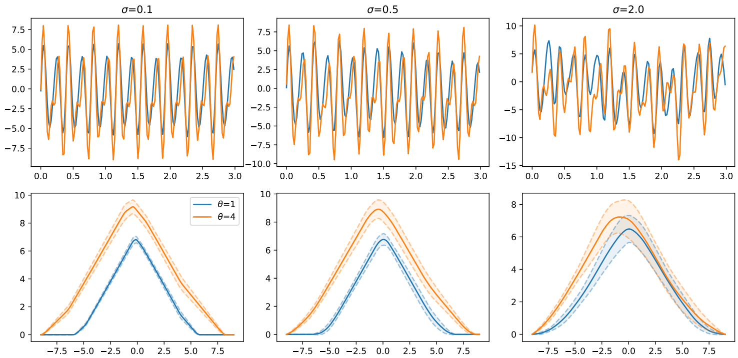
Numerical examples

$$\phi_1 \neq \phi_2, \qquad \mu_1 = \mu_2,$$



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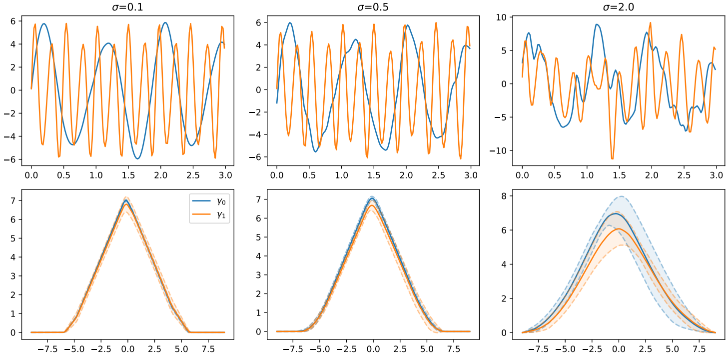

Estimation of signatures: introduction

Assume that only a single time series $\mathbf{S} = (S_n)_{n=1}^N \subset \mathbb{R}$ is given,

$$S_n = \phi(\gamma(t_n)) + W(t_n).$$

Can we estimate the signature of $\mathbf{S}$?

Proposition (Chazal et al. 2014[10], Berry et al. 2018[11])

Let be $D_1, \ldots, D_N$ i.i.d.persistence diagrams. When the (bracketing) entropy of $(\bar{\rho}_x)_{x \in \mathbb{T}}$ is finite,

$$\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}\bar{\rho}(D_n) - \bar{\rho}^*\right) \xrightarrow{d} \mathbb{G},$$

for a zero-mean stochastic process $\mathbb{G}$.

Challenges

- $\bar{\rho}$ is calculated on a window, not on a single element S_m
- $S_1, \ldots, S_N$ are not independent

[10] Frédéric Chazal et al. (2014). "Stochastic Convergence of Persistence Landscapes and Silhouettes". In: *Annual Symposium on Computational Geometry - SOCG'14*. Kyoto, Japan: ACM Press, pp. 474–483. ISBN: 978-1-4503-2594-3. DOI: 10.1145/2582112.2582128. (Visited on 03/05/2021)

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- $S_1, \ldots, S_N$ are not independent
  - Dependence in the window $\mathbf{S}$.
  - Dependence in γ_n and W_n.

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Key insight

The following map is measurable

$$(\gamma_n, W_n) \mapsto (\phi(\gamma_n), W_n) \mapsto \phi(\gamma_n) + W_n, \qquad \text{and } \phi(\gamma_n) = \phi(\mathrm{frac}(\gamma_n)).$$

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Model for γ

For some $h > 0$, we set

$$\gamma_{n+1} = \gamma_n + hV_n,$$

for $(V_n)_{n\in\mathbb{N}}$ a stationary Markov chain on $[v_{\min}, v_{\max}] \subset]0, \infty[$. Then, $\mathrm{frac}(\gamma_n)$ is stationary and the strong-mixing coefficients of $(\gamma_n)_{n\in\mathbb{N}}$ decrease exponentially.

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Idea of the proof

Show that for n big enough, some $c > 0$ and μ the uniform measure on $[0, 1]$,

$$P(\mathrm{frac}(\gamma_n) \in [a, b] \mid \gamma_0 = x_0) \geq c^n \mu([a, b]).$$

Estimation of signatures: procedure

Statistical model

A time series $(S_n)_{n=1}^N \subset \mathbb{R}$,

$$S_n = \phi(\gamma_n) + W_n, \tag{14}$$

- for $(V_n)_{n\in\mathbb{N}}$ a stationary Markov chain on $[v_{\min}, v_{\max}] \subset]0,\infty[$,

$$\gamma_{n+1} = \gamma_n + hV_n,$$

- $(W_n)_{n\in\mathbb{N}}$ a stationary, real-valued noise process, with strong-mixing coefficients $\beta_W(k) = O(k^{-3})$.

Procedure

1. Fix $M \in \mathbb{N}$,
2. Generate a sample $(\mathbf{S}_n)_{n=1}^{N-M-1}$, where $\mathbf{S}_n = (S_n, \dots, S_{n+M-1})$,
3. Calculate $\hat{F} := \frac{1}{N-M-1}\sum_{n=1}^{N-M-1} \bar{\rho}(\mathbf{S}_n)$.

Theorem

If the bracketing entropy of $(\bar{\rho}_x)_{x\in\mathbb{T}}$ is finite, then

$$\sqrt{N-M+1}(\hat{F} - \mathbb{E}[\bar{\rho}(\mathbf{S}_1)]) \to \mathbb{G} \tag{15}$$

where $\mathbb{G}$ is a zero–mean Gaussian process.

Numerical illustration

Hypothesis testing

Conclusion and future work

Conclusion

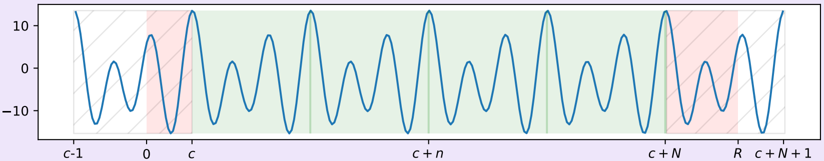
Normalized functionals of persistence of S yield **stable** signatures of ϕ and can be **estimated with standard techniques**.

Limitations and future work

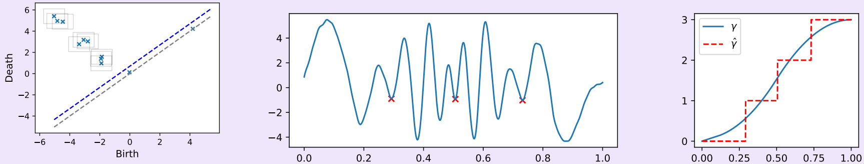
- Removing the assumption of fixed marginals from the stability
  - Understand the distance between $D(f|_{[0,T]})$ and $D(f|_{[0,t]}) \cup D(f|_{[t,T]})$
- Extensive numerical experiments and comparison with other methods.

Summary

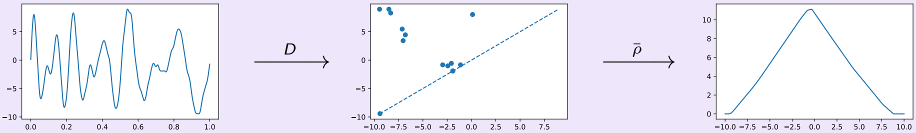
Additivity of diagrams



Odometry and phase estimation



Signatures



Refernces I

Adams, Henry et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252.

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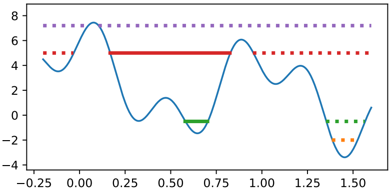
Thank you!

Persistence diagram of sub level sets: Definition

1. Persistence module

For each $t \in \mathbb{R}$, we calculate $H_0(X_t)$, where $X_t := f^{-1}(]-\infty, t])$. For any $s \leq t$, the inclusion $X_s \to X_t$ gives a map $\iota_s^t : H_0(X_s) \to H_0(X_t)$.



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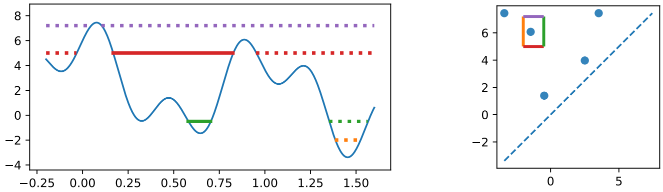
2. Rectangle measure

A measure m on rectangles of $\mathbb{R}^2$.

$$m(]a,b] \times [c,d[) = \dim\left(\frac{im(\iota_b^c) \cap \ker(\iota_c^d)}{im(\iota_a^c) \cap \ker(\iota_c^d)}\right),$$



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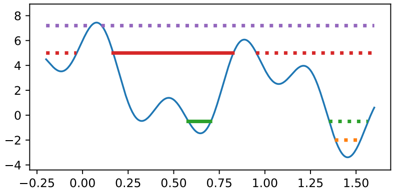
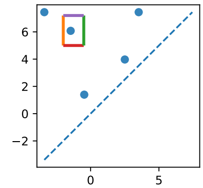
3. Persistence diagram

The persistence diagram $D(f)$ is a multi-set in $\mathbb{R}^2$, where $(s, t) \in \mathbb{R}^2$ has multiplicity

$$m(s, t) = \lim_{\delta \to 0^+} m(]s - \delta, s + \delta] \times [t - \delta, t + \delta[).$$





Proof of additivity of sub level sets: details

Proof.

Let $c := \inf\{x \in [0,1[| \ \phi(x) = \max \phi\}$, $N = \max\{n \in \mathbb{N} \mid c + n \leq R\}$ and denote by $\mathbb{X}_t := \phi^{-1}([-\infty, t[)$.
Step 1: For any $t < M$, $\mathbb{X}_t \cap [0, c] \cap [c, c+1] = \emptyset$, so

$$H_0(\mathbb{X}_t \cap [0, R]) \simeq H_0(\mathbb{X}_t \cap [0, c]) \oplus H_0(\mathbb{X}_t \cap [c, c+N]) \oplus H_0(\mathbb{X}_t \cap [c+N, R])), \tag{16}$$

Step 2: similarly,

$$H_0(\mathbb{X}_t \cap [c, c+N]) \simeq \bigsqcup_{n=1}^{N} H_0(\mathbb{X}_t \cap [c+(n-1), c+n]) \tag{17}$$

$$(x \mapsto x+n) \quad \simeq \bigsqcup_{n=1}^{N} H_0(\mathbb{X}_t \cap [c, c+1]) \tag{18}$$

Step 3: The inclusion $[0, c] \subset [c-1, c]$ induces an injective morphism

$$H_0(\mathbb{X}_t \cap [0, c]) \hookrightarrow H_0(\mathbb{X}_t \cap [c-1, c]).$$

$\square$

Stability: bottleneck distance (detailed)

Definition (Herbert Edelsbrunner and John Harer 2010, p. VIII.2)

We call a ϵ-matching between two persistence diagrams D and D' a bijection $\Gamma : A \to A'$ between some subsets of $A \subset D$ and $A' \subset D'$, considered with multiplicity, if

$$\begin{aligned} d_\infty(a, \Gamma(a)) &\leq \epsilon, && \text{for any } a \in A, \\ d_\infty(a, \Delta) &\leq \epsilon, && \text{for any } a \in (D \setminus A) \cup (D' \setminus A'). \end{aligned}$$

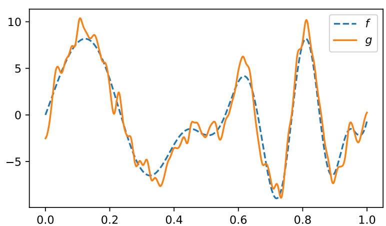
where $\Delta = \{(x, x) \in \mathbb{R}^2\}$ denotes the diagonal.

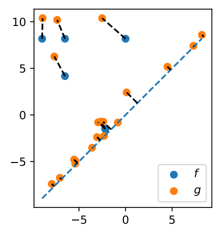
$$d_B(D, D') := \inf\{\epsilon > 0 \mid \Gamma \text{ is an } \epsilon\text{-matching between } D' \text{ and } D'\}.$$

Theorem (Bottleneck stability of diagrams)

Let $f, g : \mathbb{X} \to \mathbb{R}$ be two continuous functions on a compact space $\mathbb{X}$. Then,

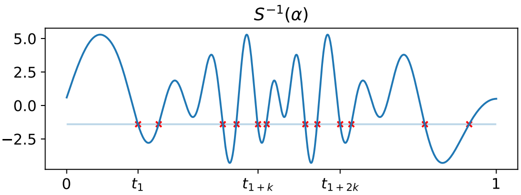
$$d_B(D(f), D(g)) \leq \|f - g\|_\infty.$$

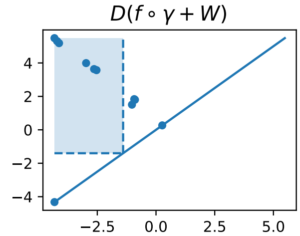




Zero-crossings from the persistence diagram

$$|S^{-1}(\alpha)| = 2 \lim_{\delta \to 0^+} |D(S) \cap (]-\infty, \alpha - \delta] \times [\alpha + \delta, \infty[)|.$$





Counting measure

The persistence diagram D is also a counting measure on rectangles $A \subset \Delta_+ = \{(b, d) \in \mathbb{R}^2 \mid x < y\}$. By (4),

$$|D(\phi \circ \gamma) \cap A| = N \, |D(\phi|_{[0,1]}) \cap A|$$

Application: magnetic odometry and speed estimation

Problem

Using the magnetic signal $\mathbf{B}$, recorded in a moving car, estimate the cars' trajectory. The angular position $t \mapsto \gamma(t)$ of a wheel in time is visible through $\mathbf{S}(t) = \mathbf{B}(\gamma(t), \gamma_h(t))$

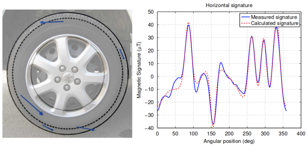


Fig. 2. Results of the optimization. Left: magnetic dipoles found after optimization. Right: measured and calculated signatures.

Proposed solution

1. $S := \langle \mathbf{S}, v \rangle$, project $\mathbf{S}$ along a suitable direction $v \in \mathbb{S}^2$
2. $\hat{N}_{c,\tau}(S)$, for an appropriate scale τ,
3. Derive an odometric sequence $t_1, \ldots, t_{\hat{N}_\tau(S)}$ from $\mathcal{C}_\tau$.
4. Construct $\hat{\gamma} : [0, T] \to \mathbb{R}$.

Results

Method	E_O $\mathbf{S}_{v_1}$	E_O $(\nabla \mathbf{S})_{v_1}$	E_I $\mathbf{S}_{v_1}$	E_I $(\nabla \mathbf{S})_{v_1}$
$\hat{N}_{c,\tau}$	15.75	16.66	3.02	3.01
$\hat{N}_{0,\tau}$	15.75	16.66	3.02	3.01
ZC	9.91	5.62	6.35	16.51

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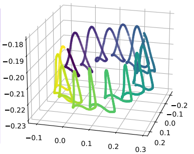

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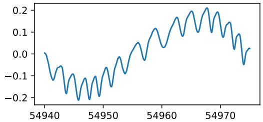

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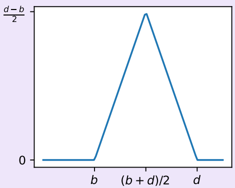
Normalized functionals of persistence

Functional representation

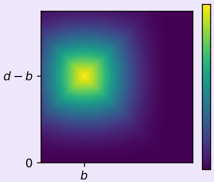
Let $\mathcal{H}$ be a functional Banach space

$$\begin{array}{rrcl} \kappa : & \mathbb{R}^2 & \to & \mathcal{H} \\ & (b,d) & \mapsto & \kappa_{(b,d)} : \begin{array}{rcl} \mathbb{T} & \to & \mathbb{R} \\ x & \mapsto & \kappa_{(b,d)}(x). \end{array} \end{array}$$

1. $\operatorname{supp}(\kappa_{(b,d)}) \subset K$, K bounded,
2. $x \mapsto \kappa_{(b,d)}(x)$ (uniformly) Lipschitz,
3. $\|\kappa_{(b,d)} - \kappa_{(b',d')}\|_{\mathcal{H}} \leq L_\kappa \|(b,d) - (b',d')\|$,
4. $\|\kappa_{(b,b)}\|_{\mathcal{H}} \leq C$.



Persistence silhouette[12]



Persistence image[13]

Normalized functionals of persistence diagrams

For some $p \geq 1$ and $\epsilon > 0$,

$$\bar{\rho}(D) := \frac{\sum_{(b,d)\in D} w(d-b)\kappa_{(b,d)}}{\sum_{(b,d)\in D} w(d-b)}, \qquad \text{where } w(d-b) = \max(d-b-\epsilon, 0)^p. \tag{19}$$

[12] Peter Bubenik (2015). "Statistical Topological Data Analysis Using Persistence Landscapes". In: *Journal of Machine Learning Research* 16.1, pp. 77–102

[13] Henry Adams et al. (2017). "Persistence Images: A Stable Vector Representation of Persistent Homology". In: *The Journal of Machine Learning Research* 18.1, pp. 218–252

Measures of dependence

Types of dependence

There are different ways to measure dependence in a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$:

- m-dependence,
- strong-mixing,
- weak-dependence,

Strong mixing

The β-mixing coefficient of a time series $(X_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ is

$$\beta_X(k) = \sup_{\mathcal{A},\mathcal{B}} \sum_{A\in\mathcal{A},B\in\mathcal{B}} |P(A\cap B) - P(A)P(B)|,$$

where $\mathcal{A} \subset \sigma^X_{-\infty,0}$, $\mathcal{B} \subset \sigma^X_{k,\infty}$ are finite partitions of the sample space and $\sigma^X_{a,b} := \sigma((X_n)_{a\leq n\leq b})$.

Example

1. If $(X_n)_n$ is m-dependent, then $\beta_X(k) = 0$ for $k \geq m$.
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Proposition

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