Dear Editor

We gratefully thank you for giving us the opportunity to submit a revised version of our article entitled *Topological signatures of periodic-like signals.*

We have revised our article, taking into account the comments made by the three anonymous reviewers and yours. We are grateful to the reviewers and to you for your thorough reading and your numerous comments, criticisms, and suggestions, which helped us to improve the manuscript.

We believe that this new version provides reasonable answers to the requirements and questions raised. In particular, we have partly reorganized the paper to improve its readability and clarity in response to the reviewers' comments.

A detailed response to the comments follows this letter. A copy of the revised article is also provided, with colors indicating the major changes.

We hope that this revised article will be suitable for publication in Bernoulli.

Sincerely yours,

Frédéric Chazal,
Bertrand Michel,
Wojciech Reise.

Description of the mains modifications of the paper

Following the recommendations of the Editor, we have partly reorganized the manuscript to make our contributions much more readable.

Section 2 is now totally dedicated to the properties of **normalized functionals of persistence**, in a general **deterministic** setting. We believe these objects are interesting for themselves and deserve to be presented in details.

The new section 2.1 takes up elements presented in Sections 2.1 and Section 4.2 of the previous manuscript. Section 2.1 starts with a new short introduction to persistent homology and persistence diagrams. A more formal presentation is now given in Appendix A1 (our previous Section 4.1). We then focus on the properties we need for the normalized persistence signatures defined further: invariance to reparametrization, stability, bounds on the (truncated) total persistence, continuity of this last, and additivity of persistence diagrams in the case of persistence homology of sublevel sets of periodic functions.

This additivity property of persistence diagrams motivates to introduce normalized versions of persistence-based signatures in Section 2.2 (which partially corresponds to previous Section 4.2). We derive stability, continuity and consistency results for these new signatures in a deterministic setting.

Section 3 introduces what we call the **signature** of a reparametrized periodic function, in a **probabilistic model** (Section 3.1). We provide robustness properties of the signature in Section 3.2 (Section 2.3 in the previous version). Section 3.3 is about the study of the signature for a **stationary time series** (Section 3 in the previous version). This section has been extensively rewritten to clarify the stationary assumptions, the construction of the signature for a time series and the bootstrap construction. We now also give results on the mixing properties of Model 2 in terms of ϕ-mixing. We have clarified the objects involved in the estimation of the signature in a stationary regime (Theorem 2.9 in this version, Theorem 3.2 in the previous version). A discussion has also been added on possible extensions in the non-stationary regime.

Section 4 is the **numerical illustration** corresponding to Section 5 in the previous version.

We have added or rephrased many remarks and discussions in the manuscript, particularly those concerning the paper's contributions. To improve readability, several technical results and proofs have also been postponed to appendices. Most of the proofs have been modified only on minor points and in response to reviewers' requests for clarification. The proofs of Corollary 2.12, Theorem 2.13 and Proposition 3.3 have been improved and slightly rewritten to fit in better with the new organization of the paper.

Following the reviewers' comments, again to improve the clarity of the document, we have made a few changes to the notations. The main modifications are the following:

- To simplify the presentation, the metric space $\mathbb{T}$ has been changed into an Euclidean space $\mathbb{U}$ (introduced in Section 2.2).
- We now have $u \in \mathbb{U}$ instead of $t \in \mathbb{T}$ to avoid confusion between the time t in Model (1) and the variable of $k(x) : \mathbb{U} \to \mathbb{R}$ (Section 2.2).

- The noise is denoted by g when it is not stochastic (it was denoted W in the previous version), in Propositions 2.3 and 2.14.

- $\pi_{L,U}$ has been changed by $\pi_{Q,U}$ to avoid confusion with the Lipschitz constants of k (Section 2.2).

- The time series is now denoted $(X_n)_{n\geq 1}$ in Model (17), it was $(S_n)_{n\geq 1}$in the previous version.

- The windows of observations is now denoted $\mathbf{X}_n$ in (20), it was X_n in the previous version.

Answer to the Associate Editor

- In the list of authors, delete the first comma and the space following it.
  Answer: Please note that we also modified the position of the authors to respect the alphabetical order.

The following points requested by the Associate Editor have also been taken into account for this new version:

- Order keywords alphabetically and use an uppercase letter to start the first keyword only.

- Refrain from using $\frac{a}{b}$ in inline text

- Adopt the Bernoulli format for references (all journal names should be abbreviated). Also remove DOIs.

- The revised manuscript should not exceed 25 pages. Please format the supplement in the same template.

Answer to the Reviewer 1

We gratefully thank Reviewer 1 for his or her help, advices and comments.

- (R1C1) The data generating procedure. In many classical statistical applications (in functional data analysis), we have functional samples $S_1, \ldots, S_N$ in order to estimate the function S, e.g., each S_i is sampled at several points in $[0, T]$, viz.,

$$(S_1(t_{1,1}), ..., S_1(t_{1,n_1})), \ldots, (S_N(t_{1,1}), ..., S_N(t_{1,n_N})) \tag{A}$$

  The authors should discuss or emphasize more (in an additional paragraph) how their underlying assumption of periodicity relates to this classical set-up. (Maybe the authors can also generalize their model to data of the type in (A) and drop the assumption of periodicity - however this is obviously a complex task which is beyond the scope of the paper.)

  [Answer:] Model (A) is very standard in FDA but the positioning of our work is a little different. If we want to recast our setting in this standard framework of FDA, we need to know how to segment the curve (or the time series) into successive periods, which is not trivial in practice when there is unknown reparametrization and phase variations in the signal, see for instance [Bonis et al., 2022][1]. The advantage of our approach is precisely that it avoids this additional step of segmentation, and thus we don't want to recast our model into Model (A). These comments have been added in Remarks 3.12.

- (R1C2) Is the noise process W random or deterministic. Currently, it is introduced as a deterministic map $W : [0, 1] \mapsto \mathbb{R}$ but this contradicts the time series model in section 3, does it? (Besides it is rather $[0, T]$ in the definition.) This is also a more general issue in the entire paper: You should describe more precisely the random objects.

  [Answer:] We apologize for any lack of clarity on the noise W. In the general Model (1), the noise W is a random process. We have made the assumptions about W more precise everywhere in the paper, and in particular in Section 3. Generally speaking, the paper has been reorganized to clarify the framework of the objects we describe (deterministic framework for Section 2, stochastic framework for Section 3).

- (R1C3) p.7 l-1. Can you clarify the relation between the distances between $\|\gamma_1^{-1} - \gamma_2^{-1}\|_\infty$ and $\|\gamma_1 - \gamma_2\|_\infty$.

  [Answer:] We now provide a sketch of proof for Theorem 3.4. A step-by-step derivation of this inequality is given in the proof in Section D.1 of the Supplementary Material.

- (R1C4) p.9 Proposition 2.7. What is the norm on the LHS and does the inequality hold with probability 1?

  [Answer:] This result is now Proposition 2.14. The result is deterministic: it is for fixed paths, as we now specify before the proposition. We corrected the norm on the LHS: it is the norm of the functional space, $\| \cdot \|_{\mathcal{H}}$. We have also corrected the norm in the proof (Appendix B.3).

- (R1C5) Section 3.2. You should mention somewhere on p.10 or 11, where you introduce the data $X_n = (S_n, \ldots, S_{n+M-1})$ that for $N \to \infty$ the data generation "guarantees

[1] accepted for publication in Adv. in Comp. Math.

enough data" for the statistical theory to work (like the laws of large numbers, ...) due to periodicity in the underlying mechanism.

[Answer:] Thank you for this suggestion, we have inserted the remark in the second sentence of Section 3.3.

- (R1C6) You should give the quantity $F^*_M(S)$ a name, like "bootstrap mean".

  [Answer:] We have followed your suggestion, we call it the bootstrap signature.

- (R1C7) Theorem 3.2. You should clarify the asymptotic role of M here. Is it also tending to infinity, like N ? Why do you mention d in G_d, is it necessary ?

  [Answer:] In Theorem 3.9 (Theorem 3.2 in the previous version), the window size M is constant, it does not vary with N. However, we believe that it would be interesting to make this quantity increases with N. For instance, when we use this signature to compare two time series, intuitively, increasing M may improve the discriminative power of the signature. But too large M will also decrease the size of the sample of windows $\mathbf{X}_n$, and thus the variance of the empirical signature will increase. Choosing M is a non-trivial issue, which moreover cannot be easily resolved in practice by a cross-validation approach. These comments are now given in Remarks 3.11.

  Regarding the asymptotic distribution, we have replaced G_d by G.

- (R1C8) Proposition 4.3. What is the range of ε ?

  [Answer:] This is now Proposition 2.4, which is valid for any $\epsilon > 0$. It shows that $\mathrm{pers}^p_{p,\epsilon}$ is locally Lipschitz, in a ball of radius $\epsilon/4$ around a diagram D_1. We have clarified this point in the statement of the proposition.

- (R1C9) Proposition 4.7. Give a reference or the definition of the "linear and normalized functionals" e.g. equation (3).

  [Answer:] We added a reference to the definitions of the linear and normalized functionals in Corollary 2.12.

- (R1 FC 1) Many citations are given in an usual style by simply appending them to the main phrase without interpunction or a conjuctions.

  [Answer:] We added parentheses whenever there was no conjunction with the sentence.

- (R1 FC 2) p.5 the paragraph above (4). Y and W are random functions with corresponding distributions μ and ν but what is the underlying function space is it something like the space of continuous functions endowed with the sup-norm $(C([0,T], \|\|_\infty)$ or is it some more general space?

  [Answer:] We thank the reviewer for pointing out this omission. Indeed, γ and W are random variables on $\Gamma_{v_{\min}}$ and $C([0,T],\mathbb{R})$, equipped with the Borel σ-algebra induced by $\|\cdot\|_\infty$. We now state it explicitly before introducing the definition of the signature in Section 3.1.

- (R1 FC 3) p.5 Theorem 2.3. You better introduce (20) and (21) before this theorem. Moreover, how is $\bar{\rho}(D(\phi|_{[0,R]}))$ related to the ρ given in (3) or is this a different quantity ?

  [Answer:] We now introduce all the assumptions on k before Proposition 2.10 and results which follow. The quantity $\bar{\rho}(D(\phi|_{[0,R]}))$ corresponds to $\bar{\rho}(D)$ when D is the persistence diagram of the sublevel sets of $\phi|_{[0,R]}$. This point is clarified after Definition 2.6, we also hope that the new organization of Section 2.2 makes easier to read this part of the paper.

- (R1 FC 4) [Answer:] Typo corrected.
- (R1 FC 5) right after (17): "We are interested exclusively ..." Maybe you give this statement in Section 2 as well.

  [Answer:] This point is now clarified much earlier in the paper, just after Definition 2.6
- (R1 FC 6) p.9 Prop 2.7 and p.15 Prop 4.4. What is C^α_Λ ?

  [Answer:] The set C^α_Λ refers to α-Hölder functions with Hölder constant Λ. However, instead of introducing yet another notation, we opted for including it explicitly in the statements of Propositions 2.3, 2.4 and 2.14.
- (R1 FC 7) [Answer:] Typo corrected.
- (R1 FC 8) [Answer:] Typo corrected.

Answer to the Reviewer 2.

We gratefully thank Reviewer 2 for his or her help, advises and comments.

- (R2 C1) In Section 2.1, the authors describe the construction of each pair in the persistence diagram through the tracking of connected components. Could the authors provide additional details on this process ?

  [Answer:] We now provide in Section 2.1 an introduction to persistence homology and persistence diagrams. A more formal presentation is also given in Appendix A1.

- (R2 C2) In the lemma 2.2, is there a typo in $\bigcup D_1$ in Equation (2)? Should it be D_k instead of D_1 ?

  [Answer:] While correct, the first form was indeed confusing. We opted for improving the readability of (2) in Lemma 2.5 (previously Lemma 2.2) by being more explicit about the terms in the sum and only later specifying that they are all equal.

- (R2 C3) In Equation (3), could the authors provide examples for k_{y_1,y_2} and $\mathbb{T}$ to improve clarity?

  [Answer:] The persistence silhouette kernel from Bubenik (2015) and the persistence image kernel from Adams et al. (2017) and now given just after introducing the kernel In Section 2.2. Note that the notation $\mathbb{T}$ has been replaced by $\mathbb{U}$ to avoid confusion with the time variable in the general Model (1).

- (R2 C4) More comprehensive details should be provided for $\bar{\rho}$. Should its input be S, $D(S)$, or are they interchangeable ?

  [Answer:] We apologize for the lack of clarity on this notation. We now clarify this point just after the definition of $\bar{\rho}$ (Definition 2.6).

- (R2 C5) Could the authors present some practical data applications for their proposed methods?

  [Answer:] The work is indeed motivated by applications to odometry and vehicle motion using magnetometer data. These analyses are developed in W. Reise's thesis [Reise, 2023] and in [Bonis et al., 2022]. However, describing these applications would take up a lot of space in the paper and seems beyond the scope of this more theoretical work. We have therefore decided not to include them in this article.

- (R2 C6) Different notation for vectors and scalars for improved readability, for example S is a function while $S_n \in \mathbb{R}$ a scalar.

  [Answer:] We have followed your recommendation by making several changes of the notation to improve the readability of the paper (see the main modifications of notation on p.2-3 of this letter).

- (R2 C7) In Theorem 3.2 for a fixed window size M ?

  [Answer:] About the fixed window size M, please refer to our answer R1 C7) to Reviewer 1.

- (R2 C8) Could the authors consider a comparative analysis with existing methods from the literature in numerical studies to demonstrate the effectiveness of their approach in recovering the signature ?

[Answer:] To our knowledge, this signature has never been introduced before and moreover very few works in general in TDA take into account temporal dependency. We fully understand your request but it is thus honestly difficult for us to answer it because this signature does not appear in other works.

Answer to the Reviewer 3.

We gratefully thank Reviewer 3 for his or her help, advices and comments.

- (R3 C1) Figure 1 may be removed. I find Figure 1 to be confusing since the whole paper does not use the principle components. On the other hand, Figure 2 is more relevant to the paper. So I will suggest to remove Figure 1 and use Figure 2 as an introduction.

  [Answer:] Following the Reviewers' suggestion, we removed Figure 1. However, note that we added a new introductory figure depicting the model.

- (R3 C2) When no noise in the time series, points have multiplicity: one thing that should be highlighted is that when there is no noise in the time-series (and we observe the whole curve), the persistence diagram will have multiple points at the same spot. The additive noises in the time series separate points in the persistence diagram.

  [Answer:] It is correct that in the absence of noise, the points have multiplicity and additive noise separates them (potentially creating also additional spurious points). This point is now highlighted just after Lemma 2.5.

- (R3 C3) Lemma 2. In the union, the diagram D_1 should be D_k. Also, the quantity D' is the residual part of the time series in the interval $[0, T]$. This should be noted.

  [Answer:] We meant to leave D_1 instead of D_k in Lemma 2.5 (previously Lemma 2.2). However, we realized that the first form of (2) was indeed confusing. We opted for improving the readability of Lemma 2.2 by being more explicit about the terms in the sum and only later specifying that they are all equal. Following the Reviewer's comment, we also added the interpretation of D' as a residual part.

- (R3 C4) Theorem 2.3. There should be a short description/high-level idea on equations (20) and (21).

  [Answer:] Following also (R1 FC 3), we now introduce all the assumptions (4-7) on k before Proposition 2.10 (and thus before Theorem 2.13). This allows us to discuss the assumptions before giving our results, which was not the case for the previous version.

- (R3 C5) Definition of V_k. It should be added that $V_k > 0$. Is there any particular reason why we set $h = T/N$? Are we requiring something like $\gamma_n \approx T$? If so, this will put some constraints in V_k.

  [Answer:] Following the Reviewers' suggestion, we changed the definition of I to $I := [v_{\min}, v_{\max}] \subset]0, \infty[$. The relation $h = T/N$ is not a requirement and is mentioned only in the context of the example we wanted to emphasize: V_k is the instantaneous speed, γ_k corresponds to a total displacement and h to a discrete integration step. We point out that the aforementioned relation only sets the scale for T/N, so that a time series of length N corresponds to a process sampled over $[0, T]$.

- (R3 C6) Example 3.1. It should be pointed out that the kernel here refers to the transition kernel in Model 2.

  [Answer:] Following the Reviewers' suggestion, "kernel" has been replaced by "transition kernel" in Example 3.8. (previously Example 3.1).

- (R3 C7) Section 3.2. One thing that should be noted is that we do NOT need to know the latent time γ_n. This is a feature of using the excursion set with persistence diagrams (PDs) that the PDs is invariant to the temporal changes.

  [Answer:] Thank you for this suggestion. We have included this remark in Section 3.1 (continuous model) and Section 3.3.1 (time series model).

- (R3 C8) and (R3 C9) Construction of the piecewise linear function from X_n and definition of $\bar{\rho}$ on the time series.

  [Answer:] Exactly as the Reviewer points out, we extend the definition of the signature F to compute it on an interpolation of $\mathbf{X}_n$. Starting from a kernel k, we first extend the definition of the normalized functional $\bar{\rho}$ to define it on a vector of length M by $\bar{\rho}(\mathbf{X}_1) := \bar{\rho}(\tilde{S}_M)$ where $\tilde{S}_M$ is a continuous process on $[0, T]$ which interpolates between entries of $\mathbf{X}_1$. Specifically, we define $\tilde{S}_M$ by prescribing its values on the set $((m-1)T/(M-1), X_m)_{m=1}^{M}$ and linearly interpolating in between. We then define the signature by integrating $\bar{\rho}(\mathbf{X}_1)$

  $$F_M(\mathbf{X}_1) = \mathbb{E}[\bar{\rho}(\mathbf{X}_1)].$$

  This construction is now presented in details in Section 3.3.3.

- (R3 C10) Theorem 3.2. This theorem should be polished. There are two conclusions–equations (13) and (15).

  [Answer:] We apologize for the misstatement of this result. We have reformulated the statement of Theorem 3.4 (previously Theorem 3.2). in particular the notation have been clarified and the stationary assumption is now explicit.

- (R3 C11) Variance of estimating $F_M(S)$ and the variance of V_k. I think the variance of V_k plays a key role in the variance of estimating F_M but this is not reflected in Theorem 3.2. One can see that if V_k has a very dispersed distribution, γ_k will be a very non-uniform set of points. This will make the same M consecutive observations to be from different period of the function.

  [Answer:] We thank the reviewer for this very interesting question. We agree that the distribution of V_k must have an impact on the quality of the signature estimate. We believe that further analysis of the limit covariance (23) could perhaps validate this conjecture. However, we have not investigated this interesting question further.

- (R3 C12) Interpretation for the quantity to be estimated. I was not very sure how do we interpret the quantity we are estimating, i.e., $F_M(S)$. It depends on the usual choice of the window size M . But due to the unobserved time, this window may contain different periods of the true signal Φ. So it is a bit hard for me to see why practitioners may be interested in this quantity.

  [Answer:] In the noiseless setting, we know that the signature is a proxy of $F(\phi_{[c,c+1]})$ for M very large (Theorem 2.13), if the sampling is dense enough so that the interpolation is close to the true signal. When the noise comes into the play, the invariance to reparametrization is lost, but we still have robustness (Theorem 3.4 in the continuous setting). We believe this signature is thus still interesting as a feature of the signal. We have added a discussion on he interest of introducing this signature at the end of Section 3.3.3.
  About the choice of M, please refer to our answer (R1C7) to Reviewer 1.

- (R3 C13) A characterization of $F_M(S)$. Following the previous point, perhaps a way to understand F_M is to think about how does this quantity associated with the generative model. Namely, how can we rewrite the signature function FM (S) as a mapping of Φ, V_k, W_k.

  [Answer:] We agree with the reviewer that $F_M(S)$ can be seen as a mapping. Considering the continuous version, the mapping depends on ϕ, the joint distribution of (μ, ν) but also on γ. With this "mapping " point of view, we show in the paper that the signature is not too sensitive to reparametrization and noise (Proposition 3.3 and Theorem 3.4). Showing that it characterizes ϕ and the distributions involved is indeed important for applications as change point detection. But this requires additional work and we have not investigated this question yet.

References

[Bonis et al., 2022] Bonis, T., Chazal, F., Michel, B., and Reise, W. (2022). Topological phase estimation method for reparameterized periodic functions. *Advances in Computational Mathematics*, page Accepted for publication.

[Reise, 2023] Reise, W. (2023). *Topological techniques for inference on periodic functions with phase variation*. PhD thesis, Université Paris-Saclay.