1 Problem 1

Prove that $C([a,b])$ equipped with the $L^2([a,b])$ norm is not a Banach space.

To show the space of continuous functions $C([a,b])$ quipped with the L^2-norm is not a Banach space, we need to show the vector space is not complete. We can accomplish this by constructing a Cauchy sequency that converges to a discontinuous function.

Let f_n be the sequence such that for some $c \in [a,b]$,

$$f_n(x) = \begin{cases} 0, & x \in [a, c - \frac{1}{n}] \\ n(x-c)+1, & x \in (c - \frac{1}{n}, c] \\ 1, & x \in (c, b] \end{cases}$$

Now, we want to show f_n is a Cauchy sequence, so we examine

$$\|f_j - f_k\|_{L^2} := (\int_a^b |f_j - f_k|^2 dx)^{\frac{1}{2}} \leq \epsilon \quad \forall \epsilon > 0$$

Observe how $\forall x \in [a,b]$ and $j, k \in \mathbb{N}$,

$$|f_j(x) - f_k(x)| \leq 1 \tag{1}$$

Now, say $n \geq m$. Then, $|f_n(x) - f_m(x)| = 0$ so that $f_n(x) = f_m(x)$ when $x \in [a,b] \setminus (c - \frac{1}{m}, c]$. Thus, we take

$$\begin{aligned} \|f_n - f_m\|_{L^2}^2 &= \int_a^b |f_n(x) - f_m(x)|^2 dx \\ &= \int_{c-\frac{1}{m}}^c |f_n(x) - f_m(x)|^2 dx \\ &\leq \int_{c-\frac{1}{m}}^c 1^2 dx && \text{(by expression (1))} \\ &= x|_{c-\frac{1}{m}}^c \\ &= c - (c - \frac{1}{m}) \\ &= \frac{1}{m} \end{aligned}$$

For the limit as $n, m \to \infty$, $\|f_n - f_m\|_{L^2} = \frac{1}{m} \to 0$. Thus, f_n is Cauchy in $L^2([a,b])$.

Following the conclusion that f_n is Cauchy with respect to the L^2-norm, let us assume there exists a function $f \in C([a,b])$ such that $f_n \to f$ in $L^2([a,b])$. Esentially, this means

$$\|f_n - f\|_{L^2}^2 = \int_a^b |f_n(x) - f(x)|^2 dx \to 0, \quad \text{as } n \to \infty.$$

Now, we can see that as $n \to \infty$, the function f_n converges to g, $f_n \to g$, where

$$g(x) = \begin{cases} 0, & x \in [a, c] \\ 1, & x \in (c, b] \end{cases}$$

for some $c \in [a,b]$. We can show that $f_n \to g$ in the L^2-norm similarly to how we showed f_n is Cauchy.

$$\|f_n - g\|_{L^2}^2 = \int_a^b |f_n(x) - g(x)|^2 dx = \int_{c-\frac{1}{n}}^{c} |f_n(x) - g(x)|^2 dx \leq \int_{c-\frac{1}{n}}^{c} 1^2 dx = \frac{1}{n} \to 0, \quad \text{as } n \to \infty$$

However, g is not a continuous function at c (i.e.: $f_n(x) \to g(x) \notin C([a,b])$). Now, we have constructed a Cauchy sequence from the L^2-norm that converges to a discontinuous function over the interval $[a,b]$. Thus, we have shown that $C([a,b])$ equipped with the L^2-norm is not a Banach space.

2 Problem 2

If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, show that the (Cartesian) product space $X = X_1 \times X_2$ becomes a normed space with the norm $\|x\| = \max(\|x_1\|_1, \|x_2\|_2)$ where $x \in X$ is defined as the tuple $x = (x_1, x_2)$ with addition and scalar multiplication operations: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.

We want to show that $X = X_1 \times X_2$ is a normed space. Therefore, we must verify that the product space satifies the 4 basic properties of vector addition and scalar multiplication. Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in X$ and $\alpha, \beta \in \mathbb{R}$.

(Vector Addition)

1. $x + y = y + x$

$$\begin{aligned} x + y &= (x_1, x_2) + (y_1, y_2) \\ &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) = y + x \end{aligned}$$

2. $x + (y + z) = (x + y) + z$

$$\begin{aligned} x + (y + z) &= (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] \\ &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, y_1) + (y_1, z_1)] + (z_1, z_2) = (x + y) + z \end{aligned}$$

3. $x + 0 = x$

$$\begin{aligned} x + 0 &= (x_1, x_2) + (0, 0) \\ &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2) = x \end{aligned}$$

4. $x + (-x) = 0$

$$\begin{aligned} x + (-x) &= (x_1, x_2) + (-1)(x_1, x_2) \\ &= (x_1, x_2) + (-x_1, -x_2) \\ &= (x_1 - x_1, x_2 - x_2) \\ &= (0, 0) = 0 \end{aligned}$$

(Scalar Multiplication)

1. $\alpha(\beta x) = (\alpha\beta)x$

$$\begin{aligned} \alpha(\beta x) &= \alpha(\beta(x_1, x_2)) \\ &= \alpha(\beta x_1, \beta x_2) \\ &= (\alpha\beta x_1, \alpha\beta x_2) \\ &= (\alpha\beta)(x_1, x_2) = (\alpha\beta)x \end{aligned}$$

2. $1x = x$

$$\begin{aligned} 1x &= (1)(x_1, x_2) \\ &= (1x_1, 1x_2) \\ &= (x_1, x_2) = x \end{aligned}$$

3. $\alpha(x+y) = \alpha x + \alpha y$

$$\begin{aligned} \alpha(x+y) &= \alpha((x_1, x_2) + (y_1, y_2)) \\ &= \alpha(x_1 + y_1, x_2 + y_2) \\ &= (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2) = \alpha x + \alpha y \end{aligned}$$

4. $(\alpha + \beta)x = \alpha x + \beta x$

$$\begin{aligned} (\alpha+\beta)x &= (\alpha+\beta)(x_1, x_2) \\ &= ((\alpha+\beta)x_1, (\alpha+\beta)x_2) \\ &= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \\ &= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2) \\ &= \alpha(x_1, x_2) + \beta(x_1, x_2) = \alpha x + \beta x \end{aligned}$$

(Norm) Furthermore, we can verify the four axioms of a norm for $\|x\| = \max(\|x_1\|_1, \|x_2\|_2)$ to show that the product space $X = X_1 \times X_2$ is a normed space.

1. $\|x\| \geq 0$

$$\begin{aligned} \|x\| &= \max(\|x_1\|_1, \|x_2\|_2) \\ &= \max(\sum_{i=1}^{n} |x_{1,i}|, (\sum_{i=1}^{n} |x_{2,i}|^2)^{\frac{1}{2}}) \\ &\geq \max(0, 0) = 0 \end{aligned}$$

since the sum of absolute values is always positive unless all elements are zero.

2. $\|x\| = 0 \Leftrightarrow x = 0$

$(\rightarrow)$

$$\begin{aligned} \|x\| &= 0, \text{then} \\ \max(\|x_1\|_1, \|x_2\|_2) &= \max(0, 0) \\ \max(\sum_{i=1}^{n} |x_{1,i}|, (\sum_{i=1}^{n} |x_{2,i}|^2)^{\frac{1}{2}}) &= \max(0, 0) \end{aligned}$$

giving us

$$\sum_{i=1}^{n} |x_{1,i}| = 0 \quad \text{and} \quad (\sum_{i=1}^{n} |x_{2,i}|^2)^{\frac{1}{2}} = 0$$

which implies that x_1, x_2 must both be zero, or the zero vector, and thus, $x = (x_1, x_2) = (0, 0) = 0$.

($\leftarrow$)

$$\begin{aligned} x &= (x_1, x_2) = (0,0) = 0, \text{then} \\ \|x\| &= \max(\|0\|_1, \|0\|_2) \\ &= \max(0, 0) \\ &= 0 \end{aligned}$$

3. $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{R}$

$$\begin{aligned} \|\alpha x\| &= \|\alpha(x_1, x_2)\| \\ &= \|(\alpha x_1, \alpha x_2)\| \\ &= \max(\|\alpha x_1\|_1, \|\alpha x_2\|_2) \\ &= \max(\sum_{i=1}^{n} |\alpha x_{1,i}|, (\sum_{i=1}^{n} |\alpha x_{2,i}|^2)^{\frac{1}{2}}) \\ &= \max(\sum_{i=1}^{n} |\alpha||x_{1,i}|, (\sum_{i=1}^{n} |\alpha|^2|x_{2,i}|^2)^{\frac{1}{2}}) \\ &= \max(|\alpha|\sum_{i=1}^{n} |x_{1,i}|, |\alpha|(\sum_{i=1}^{n} |x_{2,i}|^2)^{\frac{1}{2}}) \\ &= |\alpha| \max(\sum_{i=1}^{n} |x_{1,i}|, (\sum_{i=1}^{n} |x_{2,i}|^2)^{\frac{1}{2}}) \\ &= |\alpha| \max(\|x_1\|_1, \|x_2\|_2) \\ &= |\alpha|\|x\| \end{aligned}$$

4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

$$\begin{aligned} \|x + y\| &= \|(x_1, x_2) + (y_1, y_2)\| \\ &= \|(x_1 + y_1, x_2 + y_2)\| \\ &= \max(\|x_1 + y_1\|_1, \|x_2 + y_2\|_2) \\ &\leq \max(\|x_1\|_1 + \|y_1\|_1, \|x_2\|_2 + \|y_2\|_2) \\ &\leq \max(\|x_1\|_1, \|x_2\|_2) + \max(\|y_1\|_1, \|y_2\|_2) \\ &= \|(x_1, x_2)\| + \|(y_1, y_2)\| \\ &= \|x\| + \|y\| \end{aligned}$$

Thus, we have shown that the product space $X = X_1 \times X_2$ becomes a normed space with the norm $\|x\| = \max(\|x_1\|_1, \|x_2\|_2)$.

3 Problem 3

Show that the product (composition) of two linear operators, if it exists, is a linear operator.

Let $T : X \to Y$ and $U : Y \to Z$ be two linear operators. Then, $T[u+v] = T[u] + T[v]$, $T[\alpha u] = \alpha T[u]$ for all $u, v \in X$, $\alpha \in \mathbb{R}$ and $U[x+y] = U[x] + U[y]$, $U[\alpha x] = \alpha U[x]$ for all $x, y \in Y$, $\alpha \in \mathbb{R}$.

We can define the product, or composition, of these two operators $U \circ T = UT : X \to Z$ so that $\forall x \in \operatorname{dom}(UT) = M \subset X$, $(UT)[x] = U(T[x])$ where M is the largest subset of $\operatorname{dom}(T) = X$ whose image under T lies in $\operatorname{dom}(U)$. Now, for any $u, v \in X$ and $\alpha \in \mathbb{R}$, we must verify the addition and scalar multiplication properties of a linear operator. Then, we have

$$(U \circ T)[u+v] = U(T[u+v]) = U(T[u] + T[v]) = U(T[u]) + U(T[v]) = (U \circ T)[u] + (U \circ T)[v] \tag{2}$$

and

$$(U \circ T)[\alpha u] = U(T[\alpha u]) = U(\alpha T[u]) = \alpha U(T[u]) = \alpha (U \circ T)[u] \tag{3}$$

which are valid by the linearity of T and U. Thus, the product of two linear operators, $U \circ T$ is a linear operator.

4 Problem 4

Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < +\infty$. Show that $\operatorname{range}(T) = Y$ if and only if T^{-1} exists.

Let X, Y be vector spaces and $T : X \to Y$ be a linear operator where $\dim X = \dim Y = n < +\infty$.

(→) Let $\operatorname{range}(T) = Y$. Since T is linear and by the rank-nullity theorem (dimension theorem),

$$\dim \operatorname{range}(T) + \dim \operatorname{null}(T) = \dim X.$$

By the hypothesis, $\operatorname{range}(T) = Y$ and $\dim X = \dim Y = n$, so

$$0 = \dim Y - \dim X + \dim \operatorname{null}(T) = n - n + \dim \operatorname{null}(T) = \dim \operatorname{null}(T).$$

This implies that $\operatorname{null}(T)$ is the singleton set of the zero vector $\{\vec{0}\}$. Since T is a linear operator we know it is also an injection. Since $\operatorname{range}(T) = Y$ and T is one-to-one, we can conclude that T is bijective. Therefore, because bijectivity implies invertibility, T^{-1} must exist.

(←) Suppose T is invertible, then $\exists S : Y \to Z$ such that $S \circ T = T \circ S = I$.

$S \circ T = I \longrightarrow T$ is injective, and $T \circ S = I \longrightarrow T$ is surjective, so T is bijective.

Since T is surjective and $\dim X = \dim Y$, for any $y \in Y$ we have that $T(T^{-1}[y]) = y$ so that $\operatorname{range}(T) = Y$.

Therefore, given $T : X \to Y$ is a linear operator and $\dim X = \dim Y = n < +\infty$, $\operatorname{range}(T) = Y$ if and only if T^{-1} exists.

5 Problem 5

Let T be a bounded linear operator from a normed space X onto a normed space Y. Show that if there is a positive constant b such that $\|Tx\| \geq b\|x\|$ for all $x \in X$ then T^{-1} exists and is bounded.

Let X, Y be normed spaces such that $T : X \to Y$ is a bounded linear operator and $b > 0 \in \mathbb{R}$. Suppose that $\|Tx\| \geq b\|x\| \forall x \in X$. Let $Tx = 0$ for some $X \in X$. Then, we have

$$\begin{align} 0 &= \|Tx\| \geq b\|x\|, \tag{4} \\ \|x\| &= 0, \text{and} \tag{5} \\ x &= 0. \tag{6} \end{align}$$

Then, by the inverse operator theorem (2.6-10), we know that T^{-1} exists. Notice that since $\operatorname{range}(T) = Y$, we have that $T^{-1} : T \to X$. Now, let $y = Tx$ so that $T^{-1}y = x$. Finally, we conclude that T^{-1} is bounded by the relationship

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{b}\|Tx\| = \frac{1}{b}\|y\|.$$

6 Problem 6

Consider the functional $f(x) = \max_{t\in[a,b]} x(t)$ on $C([a,b])$ equipped with the sup norm. Is this functional linear? bounded?

Suppose we have a functional $f(x) = \max_{t\in[a,b]} x(t)$ on $C([a,b])$ equipped with the sup norm $\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1}$.

Recall, the function space $C([a,b])$ is the set of all real-valued, defined, continuous functions x,y of an independent variable t over the interval $J = [a,b]$. Also a Banach space with the supremum norm, every point of $C([a,b])$ is a function.

Continuous functions are guaranteed to reach a maximum and minimum on a closed interval. Let

$$x(t) = \frac{t-a}{b-a}, \qquad y(t) = \frac{-(t-a)}{b-a}$$

Then,

$$f(x+y) = \sup_{t\in[a,b]} (\frac{t-a}{b-a} + \frac{-(t-a)}{b-a}) = \sup_{t\in[a,b]} (0) = 0.$$

However,

$$f(x) + f(y) = \sup_{t\in[a,b]} (\frac{t-a}{b-a}) + \sup_{t\in[a,b]} (\frac{-(t-a)}{b-a}) = 1 + 0 = 1.$$

Therefore, f is not a linear functional since $f(x+y) \neq f(x) + f(y)$.

Though, such a function is in fact bounded since for any $x \in C([a,b])$,

$$|f(x)| = \sup_{t\in[a,b]} x(t) \leq \sup_{t\in[a,b]} |x(t)| = \|x\|.$$

7 Problem 7

Let X be a Banach space and denote its dual as X^*. Show that $\|\varphi\| : \varphi \to \sup_{\|x\|=1} |\varphi(x)|$ is a norm on X^*.

Suppose X is a Banach space with dual X^*. To show $\|\varphi\| : \varphi \to \sup_{\|x\|=1} |\varphi(x)|$ is a norm on X^*, we must prove the following properties:

1. Non-negativity:
   Clearly, $|\varphi| \geq 0$ for all $\varphi \in X^*$. Moreover, if $|\varphi| = 0$, then $\sup_{\|x\|=1} |\varphi(x)| = 0$. This means that $|\varphi(x)| = 0$ for all x with $\|x\| = 1$. But any $x \in X$ with $\|x\| = 1$ can be written as $\frac{\tilde{x}}{\|\tilde{x}\|}$ for some $\tilde{x} \in X$ with $\|\tilde{x}\| \neq 0$. Therefore, $\varphi(\tilde{x}) = 0$ for all $\tilde{x} \in X$, which implies that $\varphi = 0$.

2. Homogeneity:
   For any $\alpha \in \mathbb{C}$ and $\varphi \in X^*$, we have

$$\begin{aligned}
|\alpha\varphi| &= \sup_{\|x\|=1} |\alpha\varphi(x)| \\
&= \sup_{\|x\|=1} |\alpha| \cdot |\varphi(x)| \\
&= |\alpha| \sup_{|x|=1} |\varphi(x)| \\
&= |alpha||\varphi|.
\end{aligned}$$

3. Triangle inequality: For any $\varphi_1, \varphi_2 \in X^*$, we have

$$\begin{aligned}
|\varphi_1 + \varphi_2| &= \sup_{|x|=1} |\varphi_1(x) + \varphi_2(x)| \\
&\leq \sup_{|x|=1} |\varphi_1(x)| + \sup_{|x|=1} |\varphi_2(x)| \\
&= |\varphi_1| + |\varphi_2|.
\end{aligned}$$

8 Problem 8

Prove the Schwartz inequality on the inner product spaces: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$, $\forall x, y \in X$, where equality holds if and only if x, y are linearly dependent.

Let $x, y \in X$ be given. If we expand our scope to include the complex plane, w then define $f : \mathbb{C} \to \mathbb{C}$ by $f(t) = \langle x + ty, x + ty\rangle$, which is a continuous function. Since $f(t) \geq 0$ for all $t \in \mathbb{C}$, we have $0 \leq f(t) = |x|^2 + 2t\mathrm{Re}\langle x, y\rangle + t^2|y|^2$ for all $t \in \mathbb{C}$. This inequality holds for any complex number t, so we can choose t to be a complex number of the form $t = -\frac{\mathrm{Re}\langle x,y\rangle}{|y|^2}$, which yields

$$
\begin{aligned}
0 &\leq |x|^2 - 2\frac{|\mathrm{Re}\langle x, y\rangle|^2}{|y|^2} + \frac{|\mathrm{Re}\langle x, y\rangle|^2}{|y|^2} \\
&= |x|^2 - \frac{|\mathrm{Re}\langle x, y\rangle|^2}{|y|^2},
\end{aligned}
$$

where we have used the fact that $\mathrm{Re}\langle x, y\rangle^2 + \mathrm{Im}\langle x, y\rangle^2 = |\langle x, y\rangle|^2$ for any complex numbers x and y.

Rearranging the inequality above, we obtain

$$|\langle x, y\rangle|^2 \leq |x|^2|y|^2,$$

which is the desired Schwartz inequality. To obtain the inequality itself, we simply take the square root of both sides.

Finally, suppose that x and y are linearly dependent, so that there exists a nonzero complex number α such that $x = \alpha y$. Then we have

$$|\langle x, y\rangle| = |\langle \alpha y, y\rangle| \qquad = |\alpha||y|^2 \;\; = |x||y|,$$

so equality holds in the Schwartz inequality. Conversely, if equality holds in the Schwartz inequality, then we must have $|\mathrm{Re}\langle x, y\rangle| = \frac{1}{2}|\langle x + y, x + y\rangle - |x|^2 - |y|^2| = 0$, which implies that $\mathrm{Re}\langle x, y\rangle = 0$. This means that $\langle x, y\rangle$ is a purely imaginary number, and hence x and y are linearly dependent.