PEARSON'S Calculus Review, Single Variable

PEARSON

Limits

LIMIT LAWS

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$

   The limit of the sum of two functions is the sum of their limits.
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$

   The limit of the difference of two functions is the difference of their limits.
3. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L \cdot M$

   The limit of a product of two function is the product of their limits.
4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k \cdot L$

   The limit of a constant times a function is the constant times the limit of the function.
5. *Quotient Rule:* $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M},\ M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. *Power Rule:* If r and s are integers with no common factor $r, s \neq 0$, then

   $$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.
7. If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial then $\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$.
8. If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then the rational function $\dfrac{P(x)}{Q(x)}$ has $\lim_{x\to c}\dfrac{P(x)}{Q(x)} = \dfrac{P(c)}{Q(c)}$.

USEFUL LIMITS

1. $\lim_{x\to c} k = k,\ \lim_{x\to\pm\infty} k = k$, ($k$ constant)
2. For an integer $n > 0$, $\lim_{x\to\infty} x^n = \infty$,

   $$\lim_{x\to-\infty} x^n = \begin{cases}\infty\ (n \text{ even})\\ -\infty\ (n \text{ odd})\end{cases},\quad \lim_{x\to\pm\infty}\frac{1}{x^n} = 0$$
3. For integers $n, m > 0$,

   $$\lim_{x\to\pm\infty}\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m}\cdot\lim_{x\to\pm\infty} x^{n-m}$$

   (provided $a_n, b_m \neq 0$)
4. $$\lim_{x\to c^\pm}\frac{1}{(x-c)^n} = \begin{cases}+\infty\ (n \text{ even})\\ \pm\infty\ (n \text{ odd})\end{cases}$$
5. $\lim_{x\to 0}\dfrac{\sin x}{x} = 1,\ \lim_{x\to 0}\dfrac{\sin kx}{x} = k$ (k constant),

   $$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

Continuity

A function $f(x)$ is continuous at $x = c$ *if and only if* the following three conditions hold:

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

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Differentiation

DERIVATIVE

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h},$$

provided the limit exists; equivalently $f'(x) = \lim_{z\to x}\dfrac{f(z) - f(x)}{z - x}$.

FINDING THE TANGENT TO THE CURVE $y = f(x)$ AT (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope $f'(x_0) = m = \lim_{h\to 0}\dfrac{f(x_0+h) - f(x_0)}{h}$.
3. If the limit exists, the tangent line is $y = y_0 + m(x - x_0)$.

DIFFERENTIATION RULES

1. *Constant Rule:* If $f(x) = c$ (c constant), then $f'(x) = 0$.
2. *Power Rule:* If r is a real number, $\dfrac{d}{dx}x^r = rx^{r-1}$
3. *Constant Multiple Rule:* $\dfrac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$
4. *Sum Rule:* $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
5. *Product Rule:* $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$
6. *Quotient Rule:* $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
7. *Chain Rule:* $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x) = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

   if $y = f(u)$ and $u = g(x)$

USEFUL DERIVATIVES

1. $\dfrac{d}{dx}(\sin x) = \cos x$
2. $\dfrac{d}{dx}(\cos x) = -\sin x$
3. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
4. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
5. $\dfrac{d}{dx}(\sec x) = \sec x\ \tan x$
6. $\dfrac{d}{dx}(\csc x) = -\csc x\ \cot x$
7. $\dfrac{d}{dx}(e^x) = e^x$
8. $\dfrac{d}{dx}(a^x) = (\ln a)a^x$
9. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
10. $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x\ln a}$
11. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$
12. $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$
13. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$
14. $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$
15. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}$
16. $\dfrac{d}{dx}(\csc^{-1} x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$
17. $\dfrac{d}{dx}(\sinh x) = \cosh x$
18. $\dfrac{d}{dx}(\cosh x) = \sinh x$
19. $\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x$
20. $\dfrac{d}{dx}(\coth x) = -\text{csch}^2 x$
21. $\dfrac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x\tanh x$
22. $\dfrac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x\coth x$
23. $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$
24. $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$
25. $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$
26. $\dfrac{d}{dx}(\coth^{-1} x) = \dfrac{1}{1-x^2}$
27. $\dfrac{d}{dx}(\text{sech}^{-1} x) = \dfrac{-1}{x\sqrt{1-x^2}}$
28. $\dfrac{d}{dx}(\text{csch}^{-1} x) = \dfrac{-1}{|x|\sqrt{1+x^2}}$

Applications of Derivatives

FIRST DERIVATIVE TEST FOR MONOTONICITY

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each $x \in (a, b)$, then f is decreasing on $[a, b]$.

FIRST DERIVATIVE TEST FOR LOCAL EXTREMA

Suppose that c is a critical point ($f'(c) = 0$) of a continuous function f that is differentiable in some open interval containing c, except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

SECOND DERIVATIVE TEST FOR CONCAVITY

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

INFLECTION POINT

If $f''(c) = 0$ and the graph of $f(x)$ changes concavity across c then f has an inflection point at c.

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SECOND DERIVATIVE TEST FOR LOCAL EXTREMA

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

L'HÔPITAL'S RULE

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Integration

THE FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and

(1) $$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad a \le x \le b.$$

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

(2) $$\int_a^b f(x)\,dx = F(b) - F(a).$$

INTEGRATION BY PARTS FORMULA

★ $$\int u\,dv = uv - \int v\,du$$

USEFUL INTEGRATION FORMULAS

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u\tan u\,du = \sec u + C$
11. $\int \csc u\cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C$

    $= \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C$

    $= -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u du = \dfrac{a^u}{\ln a} + C,\ (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \quad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \quad (u > a > 0)$
23. $\int \sin^2 x\,dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$
24. $\int \cos^2 x\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$
25. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
26. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
27. $\int \sec^3 x\,dx = \dfrac{1}{2}\sec x\tan x + \dfrac{1}{2}\ln|\sec x + \tan x| + C$

Calculus Review, Single Variable

Volume of Solid of Revolution

DISK $V = \int_a^b \pi[f(x)]^2\,dx$

SHELL $V = \int_a^b 2\pi x f(x)\,dx$

LENGTH OF PARAMETRIC CURVE

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

where $x = f(t),\ y = g(t)$

LENGTH OF $y = f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

LENGTH OF $x = g(y)$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

Numerical Integration

TRAPEZOID RULE

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

where $\Delta x = \frac{b-a}{n}$ and $y_i = a + i\Delta x,\ y_0 = a,\ y_n = b$

SIMPSON'S RULE

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

where $\Delta x = \frac{b-a}{n}$, n is even and $y_i = a + i\Delta x,\ y_0 = a,\ y_n = b$

Polar Coordinates

EQUATIONS RELATING POLAR AND CARTESIAN COORDINATES

$$x = r\cos\theta \quad y = r\sin\theta, \quad x^2 + y^2 = r^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

SLOPE OF THE CURVE $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

CONCAVITY OF THE CURVE $r = f(\theta)$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\left.\frac{dy}{dx}\right|_{(r,\theta)}\right)}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $\frac{dx}{d\theta} \neq 0$ at (r, θ).

LENGTH OF POLAR CURVE

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

AREA OF REGION BETWEEN ORIGIN AND POLAR CURVE $r = f(\theta)$

$$A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$$

AREA OF A SURFACE OF REVOLUTION OF A POLAR CURVE (ABOUT x-AXIS)

$$S = \int_\alpha^\beta 2\pi r\sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Infinite Sequences and Series

FACTORIAL NOTATION

$$0! = 1,\quad 1! = 1,\quad 2! = 1\cdot 2,\quad 3! = 1\cdot 2\cdot 3,$$
$$n! = 1\cdot 2\cdot 3\cdot 4\cdots n$$

USEFUL CONVERGENT SEQUENCES

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \ (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \ (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

SEQUENCE OF PARTIAL SUMS

Let $s_n = a_1 + a_2 + \cdots + a_n$.

(a) $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} s_n$ exists.

(b) $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} s_n$ does not exist.

GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r},\ |r| < 1, \text{ and diverges if } |r| \geq 1.$$

THE nth-TERM TEST FOR DIVERGENCE

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

THE INTEGRAL TEST

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive decreasing function for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the improper integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

p-SERIES

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

COMPARISON TEST

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.

LIMIT COMPARISON TEST

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

THE RATIO TEST

$a_n \geq 0$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then $\sum a_n$ converges if $\rho < 1$, diverges if $\rho > 1$, and the test is inconclusive if $\rho = 1$.

THE ROOT TEST

$a_n \geq 0$, $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$, then $\sum a_n$ converges if $\rho < 1$, diverges if $\rho > 1$, and the test is inconclusive if $\rho = 1$.

ALTERNATING SERIES TEST

$a_n \geq 0$, $\sum(-1)^{n+1}a_n$ converges if a_n is monotone decreasing and $\lim_{n\to\infty} a_n = 0$.

ABSOLUTE CONVERGENCE TEST

If $\sum|a_n|$ converges, then $\sum a_n$ converges.

TAYLOR SERIES

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor Series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

If $a = 0$, we have a Maclaurin Series for $f(x)$.

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USEFUL SERIES

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ |x| < 1$
2. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,\ |x| < 1$
3. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, all x
4. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, all x
5. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, all x
6. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n},\ -1 < x \leq 1$
7. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1},\ |x| \leq 1$
8. (Binomial Series) $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,\ |x| < 1$

   where $\binom{m}{1} = m,\ \binom{m}{2} = \frac{m(m-1)}{2},$

   $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!},\ k \geq 3$

FOURIER SERIES

The Fourier Series for $f(x)$ is $a_0 + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$ where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$, $a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$,

$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx,\ k = 1, 2, 3, \ldots$

TESTS FOR CONVERGENCE OF INFINITE SERIES

1. **The nth-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. **Series with some negative terms:** Does $\sum|a_n|$ converge? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.