Continued Fractions: An exploration of the beauty of irrationality,
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1 Notes to the reader

Hello there! As the title suggests, this document will be an introduction to continued fractions. Of course, you may be wondering what we mean by a continued fraction. I'll save the formulas for a bit later, but for now you can think of a continued fraction as an expansion of a number into a fraction containing more fractions and sums nested within itself. I realize this explanation may not suffice and could be confusing, so let's look at a couple of examples to motivate this discussion.

$$\frac{214}{97} = 2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

If you don't believe me, please check this with a calculator! We will be going through the algorithm used for this example soon, but do feel free to start thinking about how this expansion may have been done. Not that this example is not pretty in its own right, but some of the true beauty of these continued fractions comes to light when we look at irrational numbers. That's right, the very numbers whose namesake indicates their lack of a rational representation have some of the most intriguing continued fractions. Take $\sqrt{2}$, for example:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

We will examine this example further as well, but this one is a little harder to put into a calculator, so we will rely on algebra and analysis to prove it.

To get the most you can from this crash course, some prerequisite knowledge is needed. You should be comfortable with some basic fraction arithmetic and know what a reciprocal is since a lot of our calculations use this idea. We'll briefly use the quadratic formula, so brush up on whatever pneumonic device you used to memorize it back in your earlier math days! It will also help to have some experience with the idea of infinite sequences and series, as well as familiarity with summation notation. We will briefly use some Taylor series, but the ones used will be provided again here to refresh your memory. In terms of more theoretical math prerequisites, it will help to have some experience with proofs, particularly with mathematical induction. To summarize the prerequisites in terms of courses, the knowledge you would get from Calculus II and an introductory proof course should more than suffice! Aside from that, any notation and concepts needed will be explained right here!

If at any point you feel lost or you would like to check your work, I will provide a link to an answer key at the end of the document. While we will be going into more detail and doing more examples, this was largely inspired by Dr. Paul Loya, so please do look into his work if you want to learn more. Now, let's get started!

2 Calculating Continued Fractions

It may be true that some of the more aesthetically pleasing continued fractions are those of some classic irrational numbers, but rational numbers do serve as a much easier starting point for learning some of the techniques we use to calculate these fractions. Let's look at a very simple example of the kind of algorithm we will be using. Hopefully, you will agree that if we have a rational number $\frac{a}{b}$, then we have

$$\frac{a}{b} = \frac{1}{\frac{b}{a}}.$$

For example, $\dfrac{1}{2} = \dfrac{1}{\frac{1}{2}}$. Rewriting fractions in this way is what allows us to expand numbers into continued fractions. Now let's look at some examples. I will walk you through the first one, and then provide some examples for you to do.

1. Express $\dfrac{214}{97}$ as a continued fraction.

   First, note that $\dfrac{214}{97} = 2 + \dfrac{20}{97}$. Then, note that we can write $\dfrac{20}{97}$ as $\dfrac{1}{\frac{97}{20}}$, which equals $\dfrac{1}{4+\frac{17}{20}}$. We can continue this algorithm of "inverting" fractions and splitting them into a sum of an integer and a rational number less than 1 to obtain the full continued fraction, as follows. Please note that this "inversion" is not truly multiplicative inversion, but rather multiplying the numerator and denominator by the reciprocal of the numerator. I'm struggling to find an appropriate word when referring to this action, so for now I have just been saying "inversion" when it comes up. I am open to ideas!

   $$\frac{214}{97} = 2+\frac{20}{97} = 2+\cfrac{1}{4+\cfrac{17}{20}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{3}{17}}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5+\cfrac{2}{3}}}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

   For the next examples, you can stop performing the algorithm when you reach a continued fraction that has a 1 in every numerator of every fraction (the integer summands will vary), and integers elsewhere. More information on this property will come later!

2. Express $\frac{14}{57}$ as a continued fraction. $\cfrac{1}{4+\cfrac{1}{14}}$

3. Express $\frac{89}{12}$ as a continued fraction. $7+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}$

4. Express $\frac{149}{84}$ as a continued fraction. $1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}$

As you have now seen and may guess, the structure of a general *finite* continued fraction is

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots \atop a_3 + \cfrac{}{a_{n-1} + \cfrac{b_n}{a_n}}}}}$$

where a_k and b_k are real numbers, and where this is well-defined (no dividing by zero).

Sometimes, instead of writing out the full nested fraction, we will write a continued fraction as

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{+ a_n}$$

which denotes the same continued fraction as above. The addition symbols line up with the denominators (aside from the first), which indicates that we are only adding to the denominator.

If it is the case that $b_k = 1$ and all $a_k \in \mathbb{Z}^+$ (positive integers) for $k \geq 1$, then the continued fraction is called *simple*, and we sometimes shorten the way we express it further, adopting the following notation:

$$\langle a_0; a_1, a_2, a_3, \ldots, a_n \rangle = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{+ a_n},$$

which can be again shortened to $\langle a_1, a_2, \ldots a_n \rangle$ if $a_0 = 0$. Returning to example 1 of this section, we would write $\dfrac{214}{97} = \langle 2; 4, 1, 5, 1, 2 \rangle$. As promised in the first example, there is a reason that this algorithm ends and results in a simple continued fraction. This property is one that only some numbers possess, described in the next theorem.

Theorem 1. A real number can be expressed as a finite simple continued fraction if and only if it is rational.

Now, these finite continued fractions are interesting, but what kinds of equivalences can we obtain when using *infinite* continued fractions?

3 The Golden Ratio

You may have already heard of the golden ratio before, given its fame both within and outside of the world of mathematics. Its discovery is credited to Euclid, and has been a marvel of mathematics since the time of the Ancient Greeks. It is found in nature as well as in things that humans have made, but we will of course be focusing on the mathematics of this famous number.

Now, let's try to express the golden ratio in two different ways starting with an exact calculation and then trying to find a continued fraction equivalent to it. Then we'll look at a more general case.

1. One way to define the golden ratio is as the positive solution to $x^2 - x - 1 = 0$. Use the quadratic formula to find this solution.

   $\dfrac{1+\sqrt{5}}{2}$

2. Great, now let's approach this a different way. Starting with the same equation, we can perform some simple algebra to obtain an expression for x in terms of itself, as follows:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ x^2 &= x + 1 \\ x &= 1 + \frac{1}{x}. \end{aligned}$$

   i. Using this expression, replace x on the right-hand side with the above expression.

   $x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}$

   ii. Repeat this process until the pattern becomes apparent to you.

From these examples, we see that

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots,$$

which is quite a pleasing formula for one of the world's most satisfying numbers!

4 Square Roots

We can think of the golden ratio as a special case of the equation $x^2 + ax - b = 0$, $a, b > 0$. Again we can perform some simple algebra to write an expression for x, as follows:

$$\begin{aligned} x^2 + ax - b &= 0 \\ x^2 + ax &= b \\ x(x+a) &= b \\ x &= \frac{b}{x+a}. \end{aligned}$$

1. As you did in the previous question, replace x in the denominator with this expression to write a formula for an infinite continued fraction equivalent to the positive solution x of the original equation in terms of a and b.

   $$x = \cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\ddots}}}}$$

2. For the next few examples, we're going to examine another case, but with more generality, by setting $a = 2$ and $b = c - 1$ where $c > 1$ (the reason why should become apparent soon). This gives us the equation $x^2 + 2x - (c-1) = 0$.

   i. Find an expression for the non-negative solution to $x^2 + 2x - (c-1) = 0$ in terms of c.

   $x = \sqrt{c} - 1$

ii. Keeping in mind that we set $a = 2$ and $b = c - 1$, write a formula for an infinite continued fraction equivalent to $\sqrt{c}$ (remember to add 1 to both sides to isolate $\sqrt{c}$).

$$\sqrt{c} = 1 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \ddots}}}}$$

Applying the process you just completed is exactly why we know that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}.$$

Now would be a great time to check the answer key if your formula seems inconsistent with this result, as we will be using this formula moving forward!

3. Using this process, find a continued fraction for $\sqrt{5}$.

$$\sqrt{5} = 1 + \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \ddots}}}}$$

4. Write an infinite continued fraction for 3.

$$3 = \sqrt{9} = 1 + \cfrac{8}{2 + \cfrac{8}{2 + \cfrac{8}{2 + \cfrac{8}{2 + \ddots}}}}$$

5. Now let's explore an example that shows a real number does not necessarily have a unique continued fraction representation. In some cases, we can simplify as we would with traditional fractions.

i. Write a continued fraction for $\sqrt{3}$.

$$\sqrt{3} = 1 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ddots}}}}$$

ii. Not that the above formula isn't beautiful as is, but we can simplify it further. Recall our fact earlier about "inverting" fractions, and simplify the expression

$$\frac{2}{2+\dfrac{2}{2+y}}$$

so that the numerator is 1.

$$\frac{1}{1+\dfrac{1}{2+y}}$$

iii. Now, simplify

$$\frac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+y}}}}$$

using the same process. I recommend doing this step by step so you can be careful about which numbers are being affected by each "inversion" step. (As a hint, this should require only *two* inversions)

$$\frac{1}{1+\dfrac{1}{2+\dfrac{1}{1+\dfrac{1}{2+y}}}}$$

iv. Finally, provide a formula for a *simple* infinite continued fraction equivalent to $\sqrt{3}$. Recall that a simple continued fraction satisfies $b_k = 1$ and $a_k \in \mathbb{Z}^+$ (all numerators are 1, and other parts are positive integers).

$$\sqrt{3} = 1+\frac{1}{1+\dfrac{1}{2+\dfrac{1}{1+\dfrac{1}{2+\ddots}}}} = \langle 1; \overline{1,2} \rangle$$

6. Let's take a look at another example that uses a similar process.

   i. First, provide a continued fraction for $\sqrt{8}$ using your formula from question 2(ii).

$$\sqrt{8} = 1 + \cfrac{7}{2 + \cfrac{7}{2 + \cfrac{7}{2 + \cfrac{7}{2 + \ddots}}}}$$

   ii. Now using the fact that $\sqrt{8} = 2\sqrt{2}$ and the continued fraction for $\sqrt{2}$, provide another continued fraction for $\sqrt{8}$ (no need to simplify yet).

$$\sqrt{8} = 2\sqrt{2} = 2 + \cfrac{2}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

   iii. Using the same trick we used in problem 5, multiply the numerator and denominator of the fraction by the reciprocal of the numerator so that the numerator is 1.

$$\sqrt{8} = 2\sqrt{2} = 2 + \cfrac{1}{1 + \cfrac{1/2}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

iv. To obtain a simple continued fraction, this isn't exactly what we want yet. Multiply the numerator and denominator of the red portion of

$$2+\cfrac{1}{1+\cfrac{1/2}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$

by the numerator of the same fraction and see what you obtain.

$$\sqrt{8}=2\sqrt{2}=2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{2}{2+\cfrac{1}{2+\ddots}}}}$$

v. Using all of the work you've done on this problem, provide a *simple* infinite continued fraction equivalent to $\sqrt{8}$.

$$\sqrt{8}=2\sqrt{2}=2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{4+\ddots}}}}=\langle 2;\overline{1,4}\rangle$$

7. If you feel confident, feel free to move on, but if you would like to test your understanding without guidance on these steps, provide a simple infinite continued fraction equivalent to $\sqrt{12}$.

$\sqrt{12}=2\sqrt{3}$