Continued Fractions: The Beauty of Irrationality,
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1 Notes to the reader

Hello there! As the title suggests, this document will be an introduction to continued fractions. Of course, you may be wondering what we mean by a continued fraction. I'll save the formulas for a bit later, but for now you can think of a continued fraction as an expansion of a number into a fraction containing more fractions and sums nested within itself. I realize this explanation may not suffice and could be confusing, so let's look at a couple of examples to motivate this discussion.

$$\frac{214}{97} = 2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

If you don't believe me, please check this with a calculator! We will be going through the algorithm used for this example soon, but do feel free to start thinking about how this expansion may have been done. Not that this example is not pretty in its own right, but some of the true beauty of these continued fractions comes to light when we look at irrational numbers. That's right, the very numbers whose namesake indicates their lack of a rational representation have some of the most intriguing continued fractions. Take $\sqrt{2}$, for example:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

We will examine this example further as well, but this one is a little harder to put into a calculator, so we will rely on algebra and analysis to prove it.

To get the most you can from this crash course, some prerequisite knowledge is needed. You should be comfortable with some basic fraction arithmetic and know what a reciprocal is since a lot of our calculations use this idea. We'll briefly use the quadratic formula, so brush up on whatever pneumonic device you used to memorize it back in your earlier math days! It will also help to have some experience with the idea of infinite sequences and series, as well as familiarity with summation notation. We will briefly use some Taylor series, but the ones used will be provided again here to refresh your memory. In terms of more theoretical math prerequisites, it will help to have some experience with proofs, particularly with mathematical induction. To summarize the prerequisites in terms of courses, the knowledge you would get from Calculus II and an introductory proof course should more than suffice! Aside from that, any notation and concepts needed will be explained right here!

If at any point you feel lost or you would like to check your work, I will provide a link to an answer key at the end of the document. While we will be going into more detail and doing more examples, this was largely inspired by Dr. Paul Loya, so please do look into his work if you want to learn more. Now, let's get started!

2 Calculating Continued Fractions

It may be true that some of the more aesthetically pleasing continued fractions are those of some classic irrational numbers, but rational numbers do serve as a much easier starting point for learning some of the techniques we use to calculate these fractions. Let's look at a very simple example of the kind of algorithm we will be using. Hopefully, you will agree that if we have a rational number $\frac{a}{b}$, then we have

$$\frac{a}{b} = \frac{1}{\frac{b}{a}}.$$

For example, $\dfrac{1}{2} = \dfrac{1}{\frac{1}{\frac{1}{2}}}$. Rewriting fractions in this way is what allows us to expand numbers into continued fractions. Now let's look at some examples. I will walk you through the first one, and then provide some examples for you to do.

1. Express $\dfrac{214}{97}$ as a continued fraction.

   First, note that $\dfrac{214}{97} = 2 + \dfrac{20}{97}$. Then, note that we can write $\dfrac{20}{97}$ as $\dfrac{1}{\frac{97}{20}}$, which equals $\dfrac{1}{4+\frac{17}{20}}$. We can continue this algorithm of "inverting" fractions and splitting them into a sum of an integer and a rational number less than 1 to obtain the full continued fraction, as follows. Please note that this "inversion" is not truly multiplicative inversion, but rather multiplying the numerator and denominator by the reciprocal of the numerator.

   $$\frac{214}{97} = 2+\frac{20}{97} = 2+\cfrac{1}{4+\cfrac{17}{20}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{3}{17}}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5+\cfrac{2}{3}}}} = 2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}}}$$

   For the next examples, you can stop performing the algorithm when you reach a continued fraction that has a 1 in every numerator of every fraction (the integer summands will vary), and integers elsewhere. More information on this property will come later!

2. Express $\frac{14}{57}$ as a continued fraction.

$$\cfrac{1}{4+\cfrac{1}{14}}$$

3. Express $\frac{89}{12}$ as a continued fraction.

$$7+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}$$

4. Express $\frac{149}{84}$ as a continued fraction.

$$1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}$$

As you have now seen and may guess, the structure of a general *finite* continued fraction is

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{\ddots}{a_{n-1} + \cfrac{b_n}{a_n}}}}}$$

where a_k and b_k are real numbers, and where this is well-defined (no dividing by zero).

Sometimes, instead of writing out the full nested fraction, we will write a continued fraction as

$$a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\cdots\frac{b_n}{+\, a_n}$$

which denotes the same continued fraction as above. The addition symbols line up with the denominators (aside from the first), which indicates that we are only adding to the denominator.

If it is the case that $b_k = 1$ and all $a_k \in \mathbb{Z}^+$ (positive integers) for $k \geq 1$, then the continued fraction is called *simple*, and we sometimes shorten the way we express it further, adopting the following notation:

$$\langle a_0; a_1, a_2, a_3, \ldots, a_n\rangle = a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots\frac{1}{+\, a_n},$$

which can be again shortened to $\langle a_1, a_2, \ldots a_n\rangle$ if $a_0 = 0$. Returning to example 1 of this section, we would write $\dfrac{214}{97} = \langle 2; 4, 1, 5, 1, 2\rangle$. As promised in the first example, there is a reason that this algorithm ends and results in a simple continued fraction. This property is one that only some numbers possess, described in the next theorem.

Theorem 1. A real number can be expressed as a finite simple continued fraction if and only if it is rational.

Now, these finite continued fractions are interesting, but what kinds of equivalences can we obtain when using *infinite* continued fractions?

3 Convergents

Before we jump into infinite continued fractions, we need to briefly go through some terminology and the foundational mathematics that allows us to actually consider infinite continued fractions. This section will be quite short, but the rest of the ideas presented here rely on what we will discuss here. For sequences $\{a_n\}, = 0, 1, 2, \ldots$ and $\{b_n\}$, $n = 1, 2, \ldots$, we define

$$c_n := a_0 + \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\cdots\frac{b_n}{+\, a_n}.$$

So long as c_n is defined for all n, c_n is called the *nth convergent*. If the limit as n approaches infinity exists, we say that

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} = a_n + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots$$

converges. It is only in this case that we use either of those notations to denote the value of the limit $\lim_{n\to\infty} c_n$. If the continued fraction is simple ($b_n = 1$ for all n) then we may switch to bracket notation and write $\langle a_0; a_1, a_2, \dots \rangle$. If it happens to be the case that there is infinite repetition of particular a_n values for a simple infinite continued fraction, we will use an overline to denote this, much as you may have learned to do in school for infinite repeating decimals. As an example, $\langle 2; 3, \overline{4} \rangle$ denotes the following simple infinite continued fraction:

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}}}.$$

Though it is a bit beyond the scope of this text, convergents also happen to be very good rational approximations of irrational numbers. In regards to the previous example, $\langle 2; 3, \overline{4} \rangle = \dfrac{7 + \sqrt{5}}{4}$. If this feels odd right now, rest assured as you will get some practice with similar examples later on. Let's take a look at the convergents in this example.

1. Fill in the remaining convergents.

   0th convergent: 2

   1st convergent: $2 + \dfrac{1}{3} = \dfrac{7}{3} \approx 2.3333$

   2nd convergent: $2 + \cfrac{1}{3 + \cfrac{1}{4}} = \dfrac{30}{13} \approx 2.3077$

   3rd convergent: $2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{4}}} = \dfrac{127}{55} \approx 2.3091$

4th convergent: $2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4}}}} = \frac{538}{233} \approx 2.3090$

Notice that these values are getting closer to $\frac{7+\sqrt{5}}{4}$. We will not cover more on these rational approximations here, but I encourage you to look into rational approximations using continued fractions after this if it interests you!

Now that you know of convergents and know that the fractions in the next sections represent limits, you are ready to tackle the world of infinite continued fractions!

4 The Golden Ratio

You may have already heard of the golden ratio before, given its fame both within and outside of the world of mathematics. Its discovery is credited to Euclid, and has been a marvel of mathematics since the time of the Ancient Greeks. It is found in nature as well as in things that humans have made, but we will of course be focusing on the mathematics of this famous number.

Now, let's try to express the golden ratio in two different ways starting with an exact calculation and then trying to find a continued fraction equivalent to it. Then we'll look at a more general case.

1. One way to define the golden ratio is as the positive solution to $x^2 - x - 1 = 0$. Use the quadratic formula to find this solution.

$$\frac{1+\sqrt{5}}{2}$$

2. Great, now let's approach this a different way. Starting with the same equation, we can perform some simple algebra to obtain an expression for x in terms of itself, as follows:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ x^2 &= x + 1 \\ x &= 1 + \frac{1}{x}. \end{aligned}$$

i. Using this expression, replace x on the right-hand side with the above expression.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}$$

ii. Repeat this process until the pattern becomes apparent to you.

From these examples, we see that

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots,$$

which is quite a pleasing formula for one of the world's most satisfying numbers!

5 Simple Square Roots

We can think of the golden ratio as a special case of the equation $x^2 + ax - b = 0$, $a, b > 0$. Again we can perform some simple algebra to write an expression for x, as follows:

$$\begin{aligned} x^2 + ax - b &= 0 \\ x^2 + ax &= b \\ x(x+a) &= b \\ x &= \frac{b}{x+a}. \end{aligned}$$

1. As you did in the previous question, replace x in the denominator with this expression to write a formula for an infinite continued fraction equivalent to the positive solution x of the original equation in terms of a and b.

$$x = \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ddots}}}}$$

2. For the next few examples, we're going to examine another case, but with more generality, by setting $a = 2$ and $b = c - 1$ where $c > 1$ (the reason why should become apparent soon). This gives us the equation $x^2 + 2x - (c-1) = 0$.

   i. Find an expression for the non-negative solution to $x^2 + 2x - (c-1) = 0$ in terms of c.

$$x = \sqrt{c} - 1$$

ii. Keeping in mind that we set $a = 2$ and $b = c - 1$, write a formula for an infinite continued fraction equivalent to $\sqrt{c}$ (remember to add 1 to both sides to isolate $\sqrt{c}$).

$$\sqrt{c} = 1 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \cfrac{c-1}{2 + \ddots}}}}$$

Applying the process you just completed is exactly why we know that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}.$$

Now would be a great time to check the answer key if your formula seems inconsistent with this result, as we will be using this formula moving forward!

3. Using your formula, find an infinite continued fraction for $\sqrt{5}$.

$$\sqrt{5} = 1 + \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \cfrac{4}{2 + \ddots}}}}$$

4. Using your formula, find an infinite continued fraction for $\sqrt{12}$.

$$\sqrt{12} = 1 + \cfrac{11}{2 + \cfrac{11}{2 + \cfrac{11}{2 + \cfrac{11}{2 + \ddots}}}}$$

5. Now let's explore an example that shows a real number does not necessarily have a unique continued fraction representation. In some cases, we can simplify as we would with traditional fractions.

i. Write a continued fraction for $\sqrt{3}$.

$$\sqrt{3} = 1 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \ddots}}}}$$

ii. Not that the above formula isn't beautiful as is, but we can simplify it further. Recall our fact earlier about "inverting" fractions, and simplify the expression

$$\cfrac{2}{2 + \cfrac{2}{2 + y}}$$

so that the numerator is 1.

$$\cfrac{1}{1 + \cfrac{1}{2 + y}}$$

iii. Now, simplify

$$\cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + y}}}}$$

using the same process. I recommend doing this step by step so you can be careful about which numbers are being affected by each "inversion" step. (As a hint, this should require only *two* inversions)

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + y}}}}$$

iv. Finally, provide a formula for a *simple* infinite continued fraction equivalent to $\sqrt{3}$. Recall that a simple continued fraction satisfies $b_k = 1$ and $a_k \in \mathbb{Z}^+$ (all numerators are 1, and other parts are positive integers).

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}} = \langle 1; \overline{1, 2} \rangle$$

6. Let's take a look at another example that uses a similar process.

i. First, provide a continued fraction for $\sqrt{8}$ using your formula from question 2(ii).

$$\sqrt{8} = 1 + \cfrac{7}{2 + \cfrac{7}{2 + \cfrac{7}{2 + \cfrac{7}{2 + \ddots}}}}$$

ii. Now using the fact that $\sqrt{8} = 2\sqrt{2}$ and the continued fraction for $\sqrt{2}$, provide another continued fraction for $\sqrt{8}$ (no need to simplify yet).

$$\sqrt{8} = 2\sqrt{2} = 2 + \cfrac{2}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

iii. Using the same trick we used in problem 5, multiply the numerator and denominator of the fraction by the reciprocal of the numerator so that the numerator is 1.

$$\sqrt{8} = 2\sqrt{2} = 2 + \cfrac{1}{1 + \cfrac{1/2}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}$$

iv. To obtain a simple continued fraction, this isn't exactly what we want yet. Multiply the numerator and denominator of the pink portion of

$$2+\cfrac{1}{1+\cfrac{1/2}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$

by the numerator of the same fraction and see what you obtain.

$$\sqrt{8}=2\sqrt{2}=2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{2}{2+\cfrac{1}{2+\ddots}}}}$$

v. Using all of the work you've done on this problem, provide a *simple* infinite continued fraction equivalent to $\sqrt{8}$.

$$\sqrt{8}=2\sqrt{2}=2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{4+\ddots}}}}=\langle 2;\overline{1,4}\rangle$$

6 Continued Fractions of Quadratic Irrationals

Fascinating as our formula for a continued fraction for $\sqrt{c}$ may be, we can run into trouble if we seek a *simple* continued fraction representation. I would encourage you to briefly explore what other examples there might be for using the formula we derived to obtain a simple continued fraction for $\sqrt{c}$ before moving on.

1. Use the simple continued fraction you found for $\sqrt{3}$ in the previous section to provide a continued fraction for $\sqrt{12}$.

$$\sqrt{12} = 2\sqrt{3} = 2 + \cfrac{2}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}}$$

Note your continued fraction for $\sqrt{12}$ from question 4 of the last section. We see that you can find a variety of continued fractions for $\sqrt{12}$, but it is more difficult to find one that is *simple*. Instead, it is often easier to rely on more direct calculations to derive the desired continued fraction.

Let's take a look at $\sqrt{12}$ in more detail. Though it may be slightly more complicated, there is no reason we can't use the same type of algorithm we used to find a simple continued fraction for a rational number. First, take a moment to note that since $\sqrt{12} = 3 + (\sqrt{12} - 3)$. You can think of this as separating off the integral part or using a floor function on $\sqrt{12}$. Since that is true, then

$$\sqrt{12} = 3 + \cfrac{1}{\cfrac{1}{\sqrt{12} - 3}}.$$

Now let $a_1 = \dfrac{1}{\sqrt{12} - 3}$. Using a calculator, we find that $\lfloor a_1 \rfloor = 2$, and $a_1 = 2 + (a_1 - 2)$. Thus, we have

$$\sqrt{12} = 3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{1}{a_1 - 2}}}.$$

Now let $a_2 = \dfrac{1}{a_1 - 2}$. The integral part of a_2 is 6 (feel free to check this yourself), and this pattern continues, giving us

$$\sqrt{12} = 3 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{6 + \ddots}}}} = \langle 3; \overline{2, 6} \rangle.$$

In general, this algorithm is what provides us with a (*unique*) simple continued fraction for any real number. Please feel free to utilize a calculator for this set of questions!

2. Find a simple continued fraction for $\sqrt{18}$. Remember, you could find a continued fraction using your formula for $\sqrt{2}$ (which is great practice), but we want a *simple* continued fraction now.

$$\sqrt{18} = 4 + \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{4 + \cfrac{1}{8 + \ddots}}}} = \langle 4; \overline{4, 8} \rangle$$

3. Find a simple continued fraction for $\dfrac{1 + \sqrt{2}}{3}$.

$$\frac{1 + \sqrt{2}}{3} = \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{4 + \cfrac{1}{8 + \ddots}}}}} = \langle 0; 1, \overline{4, 8} \rangle$$

You may have noticed a nice trait about the solutions to those problems as well as the simple continued fraction for $\sqrt{12}$. There seems to always be repetition at some point in the integral parts of the denominators (the a_i's in the fractional representation), which allows us to use an overline to denote the numbers that repeat. This is no accident, and is a special property unique to *quadratic irrationals*, which are irrational roots to the equation

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{Z}$, $a \neq 0$. You are likely extremely familiar with this equation and what roots you can obtain, but to formalize it more, a number is a quadratic rational if and only if it can be written in the form

$$\frac{p + \sqrt{q}}{r},$$

where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and with $q > 0$ not a perfect square. It may seem odd to consider only the irrational roots, but recall that a rational number always has a finite simple continued fraction, so there will never be infinite repetition.

If a continued fraction does possess this property of repetition, we call it *periodic*. A periodic continued fraction has the form

$$\langle a_0; a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_n} \rangle.$$

We will go through some of the supporting work required to prove that quadratic irrationals have periodic continued fractions, but some work, including the proof of the main theorem of this section, will be left out to keep this more on the introductory side. If you would like an in-depth discussion on this topic, please see the References section. We will go through three results that demonstrate some of the larger ideas at play here, which you will prove to yourself!

Theorem 2. If x is a quadratic irrational and $a_0, a_1, a_2, \dots \in \mathbb{Z}$, then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}}$$

is a quadratic irrational.

4. Prove Theorem 3. I will start you off with the base case, and leave the inductive step for you.

   Proof. Let x be a quadratic rational, so $x = \dfrac{a + \sqrt{b}}{c}$ for some $a, b, c \in \mathbb{Z}$, $c \neq 0$, and

$b > 0$ not a perfect square. Then for $n = 0$,

$$\begin{aligned} a_0 + \frac{1}{x} &= a_0 + \frac{1}{\frac{a+\sqrt{b}}{c}} = a_0 + \frac{c}{a+\sqrt{b}} \\ &= a_0 + \frac{c(a-\sqrt{b}}{a^2-b} = \frac{a_0a^2 - a_0b + ac - c\sqrt{b}}{a^2-b} \\ &= \frac{(a_0a^2 - a_0b + ac) - \sqrt{c^2b}}{a^2-b}. \end{aligned}$$

Since b is not a perfect square $a^2 - b \neq 0$ and $\sqrt{c^2b}$ is not a perfect square, so we have a quadratic irrational.

Assume the result holds for $n-1$, and consider

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}}.$$

By our inductive hypothesis,

$$a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}$$

is a quadratic irrational, say y. Then the larger fraction simply equals $a_0 + \frac{1}{y}$, which we know to be a quadratic irrational by our base case. So, the result holds by induction.

□

This next result guarantees that continued fractions of the form we have been discussing in this section can be rewritten in another form.

Theorem 3. Let

$$y = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}}$$

for some $a_0, a_1, a_2, \dots \in \mathbb{Z}$. Then $y = \dfrac{ax+b}{cx+d}$ for some $a, b, c, d \in \mathbb{Z}$.

5. Prove Theorem 4. It should feel similar in structure to the proof for Theorem 3!

Proof. We prove this result by induction. For $n = 0$, we see that $a_0 + \dfrac{1}{x} = \dfrac{a_0 x + 1}{x}$, which is of the correct form. Now assume the result holds for $n - 1$. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}} = a_0 + \cfrac{1}{\cfrac{ax+b}{cx+d}} = a_0 + \frac{cx+d}{ax+b}.$$

Writing this as a single fraction, we obtain $\dfrac{(a_0 a + c)x + (a_0 b + d)}{ax+b}$, which is of the correct form. So, the result holds by induction. $\square$

We aren't quite where we need to be, though, as we want this to apply to infinite continued fractions. Fortunately for us, it does!

Theorem 4. Let $a_0, a_1, a_2, \dots \in \mathbb{Z}$. Then $x = \langle \overline{a_0; a_1, a_2, \dots, a_n} \rangle$ is a quadratic irrational.

Notice here that we can write $x = \langle \overline{a_0; a_1, a_2, \dots, a_n} \rangle$ as $x = \langle a_0; a_1, a_2, \dots, a_n, x \rangle$ since we have infinite repetition (note that the continued fraction being infinite means x is irrational). Theorem 3 tells us that $\langle a_0; a_1, a_2, \dots, a_n, x \rangle = \dfrac{ax+b}{cx+d}$ for some $a, b, c, d \in \mathbb{Z}$, which is exactly what we want!

6. Why does $x = \langle a_0; a_1, a_2, \ldots, a_n, x\rangle = \dfrac{ax+b}{cx+d}$ ensure that x is a quadratic rational?

   $x = \dfrac{ax+b}{cx+d} \implies cx^2 + (d-a)x - b = 0$, so x is an irrational root to a quadratic equation of the appropriate form.

Recall that the general form of a periodic continued fraction is

$$\langle a_0; a_1, \ldots, a_{k-1}, \overline{a_k, a_{k+1}, \ldots, a_n}\rangle,$$

so we are close to what we need. Since the repeating part is guaranteed to be a quadratic rational, you can then consider a fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{a_n + \cfrac{1}{x}}}}$$

where x is the quadratic rational the repeating part converges to. Then you can apply Theorem 2 to obtain a more general result for periodic continued fractions. What we have shown is that periodic continued fractions converge to quadratic irrationals. For the converse, a hefty amount of number theory is required that will not be included here. That work, along with what we have done, culminates in the following theorem.

Theorem 5 (Lagrange). The continued fraction for a quadratic irrational is periodic.

What a remarkable result! If you would like some more practice with periodic continued fractions, you now know exactly how to make your own examples. Each periodic continued fraction corresponds with exactly one quadratic irrational! There are a multitude of sleek online calculators you can use to generate more of these and check your work as well, if this topic interests you. Next, we step into some use of sequences to find continued fractions for some of the most famous transcendental numbers.

7 Formulas for Continued Fractions

In this final topic, we are going to look at some formulas for infinite continued fractions, utilizing sequences of real numbers to help us. We will use some finite sequences to get adjusted to the formulas, but now is a great time to review what you know about sequences, limits, and the principle of mathematical induction! This section may be a little bit more difficult than the others, but yields some incredible results. It is through the use of these formulas that we can express numbers like π and $\ln 2$ as infinite continued fractions. I will provide inductive steps to the proofs for the first two theorems of this section, and the proof for the third will be in the answer key.

Before stating the first theorem, let's examine some algebra that, when generalized, will be the theorem. For two real numbers, a_1 and a_2, note that

$$\frac{1}{a_1} - \frac{1}{a_2} = \frac{a_2 - a_1}{a_1 a_2} = \cfrac{1}{\cfrac{a_1 a_2}{a_2 - a_1}}$$

by obtaining a common denominator and then inverting the fraction. Now let's look at the denominator of the final fraction above. We see that

$$\frac{a_1 a_2}{a_2 - a_1} = \frac{a_1 a_2 - a_1^2 + a_1^2}{a_2 - a_1} = \frac{a_1(a_2 - a_1) + a_1^2}{a_2 - a_1} = a_1 + \frac{a_1^2}{a_2 - a_1}.$$

Plugging this result in, we have that

$$\frac{1}{a_1} - \frac{1}{a_2} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1}},$$

which is starting to look like a continued fraction, albeit a small one. Generalizing this fact leads us to the following theorem.

Theorem 6. Let $a_1, a_2, a_3, \ldots$ be a sequence of nonzero real numbers with $a_k \neq a_{k-1}$ for all k. Then for all natural numbers n,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cfrac{\ddots}{\cfrac{a_{n-1}^2}{a_n - a_{n-1}}}}}}.$$

If the limit of this sum as n approaches infinity exists, then

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{a_k} = \frac{1}{a_1 +} \, \frac{a_1^2}{a_2 - a_1 +} \, \frac{a_2^2}{a_3 - a_2 +} \, \frac{a_3^2}{a_4 - a_3 +} \cdots.$$

Proof. We assume the theorem is true for a sequence $a_1, a_2, ..., a_n$ of real numbers, and show that it holds for a sequence of $n + 1$ numbers. For a sequence $a_1, a_2, ..., a_n, a_{n+1}$ of real numbers, note the following:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_k} &= \frac{1}{a_1} - \frac{1}{a_2} + \cdots + \frac{(-1)^{n-1}}{a_n} + \frac{(-1)^n}{a_{n+1}} \\
&= \frac{1}{a_1} - \frac{1}{a_2} + \cdots + (-1)^{n-1}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right) \\
&= \frac{1}{a_1} - \frac{1}{a_2} + \cdots + (-1)^{n-1}\left(\frac{a_{n+1}-a_n}{a_n a_{n+1}}\right) \\
&= \frac{1}{a_1} - \frac{1}{a_2} + \cdots + (-1)^{n-1}\left(\frac{1}{\dfrac{a_n a_{n+1}}{a_{n+1}-a_n}}\right).
\end{aligned}$$

Note that this is now a sum of n terms, so we can use the inductive hypothesis to rewrite this sum as

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_k} = \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cdots + \cfrac{a_{n-1}^2}{\dfrac{a_n a_{n+1}}{a_{n+1}-a_n} - a_{n-1}}}}},$$

as the number $\dfrac{a_n a_{n+1}}{a_{n+1}-a_n}$ is now the last term in the sequence $\{a_n\}$ of n numbers. Now we can use similar algebra that led to the formation of this theorem to note that

$$\begin{aligned}
\frac{a_n a_{n+1}}{a_{n+1}-a_n} - a_{n-1} &= \frac{a_n(a_{n+1}-a_n) + a_n^2}{a_{n+1}-a_n} - a_{n-1} \\
&= a_n + -a_{n-1}\frac{a_n^2}{a_{n+1}-a_n}.
\end{aligned}$$

We can rewrite our sum once again to show that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_k} &= \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cdots \cfrac{a_{n-1}^2}{\dfrac{a_n a_{n+1}}{a_{n+1}-a_n} - a_{n-1}}}}} \\
&= \cfrac{1}{a_1 + \cfrac{a_1^2}{a_2 - a_1 + \cfrac{a_2^2}{a_3 - a_2 + \cdots + \cfrac{a_{n-1}^2}{a_n - a_{n-1} + \cfrac{a_n^2}{a_{n+1}-a_n}}}}},
\end{aligned}$$

so the result holds for $n+1$ numbers. $\square$

Do note that this formula is not guaranteed to produce a *simple* continued fraction, but rather gives us a way to express more numbers as a general continued fraction. To get a feel for how to use this theorem, let's do an example with a finite sequence.

1. Express
$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8}$$
as a continued fraction. (Tip: If you calculate the sum, you will have a way to check if your continued fraction is equivalent!)

$$\cfrac{1}{2 + \cfrac{4}{2 + \cfrac{16}{2 + \cfrac{32}{2}}}} = \frac{7}{24}$$

Now, given that this number is certainly rational as a sum of rational numbers, we know there is a finite simple continued fraction for it. Though it is a nice representation, this theorem is capable of more than just representing rational numbers in more lengthy forms. For the next example, recall the following Taylor series:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}.$$

2. Using the Taylor series formula for $\ln(1+x)$, write out the first six terms of the Taylor series for $\ln 2$.

$$\ln 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

3. Using Theorem 6 and the result from question 2, find an infinite continued fraction for $\ln 2$.

$$\ln 2 = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cfrac{4^2}{1 + \ddots}}}}}$$

For this next problem, consider the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k+1}.$$

4. Using the Leibniz formula for π, represent $\frac{\pi}{4}$ as an infinite continued fraction.

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

Of course, we can multiply this fraction by 4 to obtain

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

as a continued fraction for our beloved number π. We can also find a quite pleasing continued fraction for e, but for this, we will need another theorem. This time, you will help build up the algebra that we want to generalize.

5. Consider the difference $\dfrac{1}{a_1} - \dfrac{1}{a_1a_2}$, where a_1 and a_2 are real numbers. Find a common denominator to write this as one fraction, then "invert" the result so that you have 1 in the numerator.

$$\frac{1}{a_1} - \frac{1}{a_1a_2} = \frac{a_2-1}{a_1a_2} = \frac{1}{\dfrac{a_1a_2}{a_2-1}}$$

6. I claim that $\dfrac{a_1a_2}{a_2-1} = a_1 + \dfrac{a_1}{a_2-1}$. Can you find the intermediary step that shows why?

$$\frac{a_1a_2}{a_2-1} = \frac{a_1(a_2-1)+a_1}{a_2-1} = a_1 + \frac{a_1}{a_2-1}$$

Generalizing this result, we will obtain the next theorem.

Theorem 7. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers with $a_k \neq 0, 1$. Then for each k,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{a_1a_2\cdots a_k} = \cfrac{1}{a_1 + \cfrac{a_1}{a_2 - 1 + \cfrac{a_2}{a_3 - 1 + \cfrac{\ddots}{a_{n-1} - 1 + \cfrac{a_{n-1}}{a_n - 1}}}}}.$$

If the limit of this sum as n approaches infinity exists, then

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{a_1a_2\cdots a_k} = \frac{1}{a_1+}\,\frac{a_1}{a_2-1+}\,\frac{a_2}{a_3-1+}\cdots\frac{a_{n-1}}{a_n-1+}\cdots.$$

Proof. Assume the result holds for n numbers. We will show the result holds for $n+1$ numbers. Note that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} &= \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n-1}}{a_1 \cdots a_n} + \frac{(-1)^n}{a_1 \cdots a_{n+1}} \\
&= \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + (-1)^{n-1} \left(\frac{1}{a_1 \cdots a_n} - \frac{1}{a_1 \cdots a_{n+1}} \right) \\
&= \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + (-1)^{n-1} \left(\frac{a_{n+1} - 1}{a_1 \cdots a_{n+1}} \right) \\
&= \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + (-1)^{n-1} \left(\frac{1}{\dfrac{a_1 \cdots a_{n+1}}{a_{n+1} - 1}} \right) \\
&= \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + (-1)^{n-1} \left(\frac{1}{(a_1 \cdots a_{n-1}) \left(\dfrac{a_n a_{n+1}}{a_{n+1} - 1} \right)} \right).
\end{aligned}$$

This sum now has n terms, so we can use the inductive hypothesis to write this as a continued fraction, with the last term in the sequence now being $\dfrac{a_n a_{n+1}}{a_{n+1} - 1}$. So,

$$\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} = \frac{1}{a_1 +} \frac{a_1}{a_2 - 1 +} \frac{a_2}{a_3 - 1 +} \cdots \frac{a_{n-1}}{+ \dfrac{a_n a_{n+1}}{a_{n+1} - 1} - 1}.$$

Now we can direct our attention to the denominator of the last term, and note that

$$\frac{a_n a_{n+1}}{a_{n+1} - 1} - 1 = \frac{a_n (a_{n+1} - 1) + a_n}{a_{n+1} - 1} - 1 = a_n - 1 + \frac{a_n}{a_{n+1} - 1}.$$

Finally, we have that

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{a_1 a_2 \cdots a_k} &= \frac{1}{a_1 +} \frac{a_1}{a_2 - 1 +} \frac{a_2}{a_3 - 1 +} \cdots \frac{a_{n-1}}{+ a_n - 1 + \dfrac{a_n}{a_{n+1} - 1}} \\
&= \frac{1}{a_1 +} \frac{a_1}{a_2 - 1 +} \frac{a_2}{a_3 - 1 +} \cdots \frac{a_{n-1}}{+ a_n - 1 +} \frac{a_n}{a_{n+1} - 1},
\end{aligned}$$

so the result holds. □

As we did before, let's look at an example where the sequence is finite to get a handle on what this means.

7. Represent the sum

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}$$

as a continued fraction using Theorem 7. Again note that this sum is rational and thus has a simple continued fraction, which is often preferred over this method.

$$\cfrac{1}{2+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}}}=\frac{11}{32}$$

Great! Now, it's time to find a continued fraction for e, but there are a few steps we need to take to get there. For this, we will start with the fact that

$$e^{x}=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}$$

and use algebra to eventually arrive at the desired continued fraction. It may be a good idea to check the answer key along the way for the next few problems.

8. Keep in mind the Taylor series for e^x and Theorem 7 for this question.

   i. Find a series for $\frac{1}{e}$ that starts at $k=1$, and write out the first five terms of the expansion.

$$\frac{1}{e}=e^{-1}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}$$

$$e^{-1}=\frac{1}{1}-\frac{1}{1}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\dots$$

ii. Represent $1 - e^{-1}$ as a series using summation notation. This should still start at $k = 1$.

Since we know that

$$e^{-1} = \frac{1}{1} - \left(\frac{1}{1} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots \right),$$

then

$$1 - e^{-1} = \frac{1}{1} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}$$

iii. Note that $1 - e^{-1} = \dfrac{e-1}{e}$. Represent $\dfrac{e-1}{e}$ as a continued fraction using Theorem 7.

$$\frac{e-1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}}}$$

iv. Represent $\dfrac{e}{e-1}$ as a continued fraction. (Hint: How would you accomplish this if the right side was simply a positive rational number?)

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \ddots}}}}$$

v. Represent $\dfrac{1}{e-1}$ as a continued fraction. (Hint: Note that $\dfrac{1-e}{e-1}+\dfrac{e-1}{e-1}=0$).

We know that

$$\frac{e}{e-1}=\frac{e}{e-1}+\frac{1-e}{e-1}+\frac{e-1}{e-1},$$

so we can subtract $\dfrac{e-1}{e-1}=1$ from both sides to obtain

$$\frac{1}{e-1}=\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\ddots}}}}.$$

vi. Almost there! Find a continued fraction for e. (Hint: Only two steps left!)

By inverting, we see that

$$e-1=1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\ddots}}},$$

so we can add 1 to both sides to obtain

$$e=2+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}.$$

If you prefer the shorthand notation, then

$$e=2+\frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\,\frac{5}{5+}\cdots,$$

which is again quite a pleasing representation of an irrational number.

This last result is similar to the previous, but I feel it warrants mentioning, if only to put a nice bow on this topic.

Theorem 8 (Euler's Continued Fraction Formula). Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Then

$$\sum_{k=1}^{n} a_1 a_2 \cdots a_k = \frac{a_1}{1\,+}\,\frac{-a_2}{1+a_2\,+}\,\frac{-a_3}{1+a_3\,+}\cdots\frac{-a_n}{+\,1+a_n}.$$

If the limit of this sum as n approaches infinity exists, then

$$\sum_{k=1}^{\infty} a_1 a_2 \cdots a_k = \frac{a_1}{1\,+}\,\frac{-a_2}{1+a_2\,+}\,\frac{-a_3}{1+a_3\,+}\cdots\frac{-a_n}{+\,1+a_n\,+}\cdots.$$

9. Prove Theorem 8.

Proof. Assume the result holds for a sequence of n numbers. We will show it holds for $n+1$ numbers. For a sequence of $n+1$ numbers,

$$\begin{aligned}\sum_{k=1}^{n+1} a_1 a_2 \cdots a_k &= a_1 + a_1 a_2 + \cdots + a_1 \cdots a_{n+1} \\ &= a_1 + a_1 a_2 + \cdots + (a_1 \cdots a_{n-1}) + (a_1 \cdots a_{n-1}(a_n + a_n a_{n+1})).\end{aligned}$$

This sum now has n terms, so we can use the inductive hypothesis, treating $a_n + a_n a_{n+1}$ as the last number in the sequence. Thus, we have

$$\sum_{k=1}^{n+1} a_1 a_2 \cdots a_k = \frac{a_1}{1\,+}\,\frac{-a_2}{1+a_2\,+}\cdots\frac{-(a_n + a_n a_{n+1})}{+\,1+a_n+a_n a_{n+1}}.$$

In regards to the last term here, note the following equivalences:

$$\begin{aligned}\frac{-(a_n + a_n a_{n+1})}{1+a_n+a_n a_{n+1}} &= \frac{-a_n(1+a_{n+1}}{1+a_n+a_n a_{n+1}} = \frac{-\,a_n}{\dfrac{1+a_n+a_n a_{n+1}}{1+a_{n+1}}} \\ &= \frac{-\,a_n}{\dfrac{1+a_n(1+a_{n+1})}{1+a_{n+1}}} = \frac{-\,a_n}{a_n + \dfrac{1}{1+a_{n+1}}} \\ &= \frac{-\,a_n}{a_n + \dfrac{1+(1+a_{n+1})-(a+a_{n+1})}{1+a_{n+1}}} = \frac{-\,a_n}{1+a_n+\dfrac{-\,a_{n+1}}{1+a_{n+1}}}.\end{aligned}$$

Returning to our sum, we see that

$$\sum_{k=1}^{n+1} a_1 a_2 \cdots a_k = \frac{a_1}{1\,+}\,\frac{-a_2}{1+a_2\,+}\cdots+\frac{-a_n}{1+a_n\,+}\,\frac{-a_{n+1}}{1+a_{n+1}},$$

exactly as desired. ∎

Now we are straying a little further away from applying this in a *useful* way for this next problem, but sometimes it's beneficial to discover what is possible given the tools we have at our disposal.

10. Find a continued fraction formula for $\sum_{k=1}^{n} n!$ where n is any natural number using Theorem 8.

$$\sum_{k=1}^{n} n! = \frac{1}{1+}\,\frac{-2}{3+}\,\frac{-3}{4+}\,\frac{-4}{5+}\cdots\frac{-n}{+(n+1)}$$

With this result, we can show, for example, that

$$153 = \sum_{k=1}^{5} k! = \cfrac{1}{1 - \cfrac{2}{3 - \cfrac{3}{4 - \cfrac{4}{5 - \cfrac{5}{6}}}}}.$$

Now we may be getting ahead of ourselves by expanding something as simple as an integer to such a bulky form, but is nonetheless interesting to consider. Before we leave this last example, take a moment to consider the limits of this formula we made.

11. Why *can't* we extend this formula to find a formula for $\sum_{k=1}^{\infty} k!$ using Theorem 8?

The sum

$$\sum_{k=1}^{\infty} k!$$

does not converge, so we can't apply the theorem result to it.

8 Concluding Thoughts

You've done it! If you made it to this point, pat yourself on the back for the amount of work you have done. Hopefully, you found some of these examples as beautiful as I have, and see the allure of irrational numbers. Despite them often being thought of as more difficult to work with, they have a deep beauty within them. At the very least, you got some review of calculus topics and proof techniques! As is the case with any introductory text, I could not include every topic related to continued fractions. A myriad of other examples and results remain for you to discover should the desire be present (the next page lists some great options to start with). Whether or not you decide to pursue this further, I hope you have enjoyed this text, and have learned a great deal from it. Thank you for reading!

9 References

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