Assignment C1

This assignment should be submitted on Learn by Wednesday of Week 13 (due at **16.00 on Wednesday 15th April**).

Your submission should consist of one `.m` file, of the form of the file `C1.m` provided on Learn. I will download this file, and click 'Run' (under the 'Editor' panel) on MATLAB. When I do this, the code should run cleanly, everything I have asked you to display should be visible in the Command Window, and all plots should appear.

This assignment contributes 10% of the course mark. It will be marked out of 15, comprising the following:

5 marks for correctness of Question 1.

6 marks for correctness of Question 2.

4 marks for presentation of the MATLAB code. This means the code should be easy to follow, well commented, plots clearly labelled and well presented, and all of the instructions on this sheet are followed.

Question 1 Crank–Nicolson method for the heat equation

Consider the partial differential equation

$$u_t = \kappa u_{xx} \quad \text{for } x \in (-1, 1)\,, t \in (0, T)\,,$$

where $\kappa > 0$ and $T > 0$, subject to an initial condition

$$u\,(x, t = 0) = \sin(\pi x) - 3\sin(3\pi x),$$

and homogeneous Dirichlet boundary conditions

$$u\,(x = -1, t) = u\,(x = 1, t) = 0.$$

The exact solution to this problem is given by

$$u(x, t) = e^{-\pi^2 \kappa t}\sin(\pi x) - 3\,e^{-9\pi^2\kappa t}\sin(3\pi x).$$

Discretising this problem using the Crank–Nicolson method gives that

$$\frac{U_m^{n+1} - U_m^n}{\Delta t} = \frac{1}{2}\left[\kappa\,\frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{(\Delta x)^2} + \kappa\,\frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}}{(\Delta x)^2}\right]$$
$$\text{for } m \in \{1, \dots, M-1\}\,, n \in \{0, \dots, N-1\}\,,$$

with

$$\begin{aligned} &U_m^0 = \sin(\pi x_m) - 3\sin(3\pi x_m) \quad \text{for } m \in \{1, \dots, M-1\}\,,\\ &U_0^n = U_M^n = 0 \quad \text{for } n \in \{0, \dots, N\}\,, \end{aligned}$$

and where U_m^n is the fully discrete approximation for $u\,(x = x_m = -1 + m\Delta x, t = t_n = n\Delta t)$, with $\Delta t = \frac{T}{N}$, $\Delta x = \frac{2}{M}$ for some positive integers M, N.

Problem Implement the Crank–Nicolson method for this problem. Take $\kappa = 0.5$ and $T = 0.25$, and set $\Delta t = 0.25\Delta x$, taking care to ensure M and N are integers.

For $\Delta x = 0.01$ and $\Delta t = 0.25\Delta x = 0.0025$, plot the evolution of the solution $u(x, t)$ over the time interval. Call this `figure(1)` in MATLAB. [You may make use of the code in Labs 2 and 3 to do this.] For the same values of Δx and Δt, display the following two error norms:

$$\max_{n\in\{0,\dots N\}} \|\underline{e}^n\|_2 = \max_{n\in\{0,\dots N\}} \sqrt{\sum_{m=0}^{M} e_m^n e_m^n}, \qquad \max_{n\in\{0,\dots N\}} \|\underline{e}^n\|_{2,\Delta x} = \max_{n\in\{0,\dots N\}} \sqrt{\Delta x \sum_{m=0}^{M} e_m^n e_m^n},$$

where
$$e_m^n = U_m^n - u\left(x = x_m = -1 + m\Delta x, t = t_n = n\Delta t\right).$$

In a comment on the MATLAB script, state the maximum integers a and b such that
$$\max_{n\in\{0,\dots N\}} \|\underline{e}^n\|_2 = \mathcal{O}\left((\Delta x)^{a/2}\right), \qquad \max_{n\in\{0,\dots N\}} \|\underline{e}^n\|_{2,\Delta x} = \mathcal{O}\left((\Delta x)^{b/2}\right),$$
for this problem, when $\Delta t = 0.25\Delta x$.

Question 2 Finite difference solution of Poisson's equation

Consider the partial differential equation
$$-u_{xx} - u_{yy} = -2x(x-1) - 2y(y-1) + 2\pi^2 \sin(\pi x)\sin(\pi y) \quad \text{for } (x,y) \in (0,1)^2,$$
with homogeneous Dirichlet boundary conditions
$$u\left(x=0,y\right) = u\left(x=1,y\right) = u\left(x,y=0\right) = u\left(x,y=1\right) = 0.$$

The exact solution to this problem is given by
$$u(x,y) = x(x-1)y(y-1) + \sin(\pi x)\sin(\pi y).$$

Discretising this problem using the 5-point Laplacian scheme gives that
$$-\frac{U_{m-1,p} + U_{m+1,p} - 4U_{m,p} + U_{m,p-1} + U_{m,p+1}}{(\Delta x)^2} = -2x_m(x_m-1) - 2y_p(y_p-1) + 2\pi^2\sin(\pi x_m)\sin(\pi y_p)$$
$$\text{for } m,p \in \{1,\dots,M-1\},$$
where $U_{m,p}$ is the discrete solution at $x = x_m = m\Delta x$ and $y = y_p = p\Delta x$, where $\Delta x = \frac{1}{M}$ with M a positive integer. The boundary conditions are applied via
$$U_{0,p} = U_{M,p} = U_{m,0} = U_{m,M} = 0 \qquad \text{for } m,p \in \{0,\dots,M\}.$$

It can be shown that this discretisation leads to the matrix system:
$$A\underline{u} = \underline{b},$$
where $A \in \mathbb{R}^{(M-1)^2\times(M-1)^2}$, $\underline{u} \in \mathbb{R}^{(M-1)^2}$, $\underline{b} \in \mathbb{R}^{(M-1)^2}$ are defined by
$$A = \frac{1}{(\Delta x)^2}\begin{bmatrix} B & -I & \underline{0} & \dots & \dots & \underline{0} & \underline{0} \\ -I & B & -I & \underline{0} & & & \underline{0} \\ \underline{0} & -I & B & -I & \underline{0} & & \vdots \\ \vdots & \underline{0} & -I & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -I & \underline{0} \\ \underline{0} & & & \underline{0} & -I & B & -I \\ \underline{0} & \underline{0} & \dots & \dots & \underline{0} & -I & B \end{bmatrix},$$
$$\underline{u} = \begin{bmatrix} U_{1,1} & U_{1,2} & \dots & U_{1,M-1} & U_{2,1} & \dots & U_{2,M-1} & U_{3,1} & \dots & \dots & U_{M-2,M-1} & U_{M-1,1} & \dots & U_{M-1,M-1} \end{bmatrix}^T,$$
$$\underline{b} = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,M-1} & b_{2,1} & \dots & b_{2,M-1} & b_{3,1} & \dots & \dots & b_{M-2,M-1} & b_{M-1,1} & \dots & b_{M-1,M-1} \end{bmatrix}^T.$$
Here, $I \in \mathbb{R}^{(M-1)\times(M-1)}$ is the identity matrix, $\underline{0} \in \mathbb{R}^{(M-1)\times(M-1)}$ denotes the zero matrix, $b_{m,p} = -2x_m(x_m-1) - 2y_p(y_p-1) + 2\pi^2\sin(\pi x_m)\sin(\pi y_p)$, and $B \in \mathbb{R}^{(M-1)\times(M-1)}$ is given by
$$B = \begin{bmatrix} 4 & -1 & 0 & \dots & \dots & 0 \\ -1 & 4 & -1 & \ddots & & \vdots \\ 0 & -1 & 4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 4 & -1 \\ 0 & \dots & \dots & 0 & -1 & 4 \end{bmatrix}.$$

Problem Use this formulation to compute the numerical solution of Poisson's equation. A starting point is provided in the file `C1.m`. Allow Δx to be specified by the user, taking care to ensure M is an integer whenever you select Δx.

Using the `surf` command in MATLAB, plot the numerical solution using $\Delta x = 0.005$. Call this `figure(2)` in MATLAB. [You may find the command `shading interp` helpful when presenting the figure.] For the same value of Δx, display the following two error norms:

$$\|\underline{e}\|_2 = \sqrt{\sum_{m=0}^{M}\sum_{p=0}^{M} e_{m,p}^2}, \qquad \|\underline{e}\|_{2,\Delta x} = \sqrt{(\Delta x)^2 \sum_{m=0}^{M}\sum_{p=0}^{M} e_{m,p}^2},$$

where

$$e_{m,p} = U_{m,p} - u\left(x = x_m = m\Delta x, y = y_p = p\Delta y\right).$$

<u>In a comment</u> on the MATLAB script, state the maximum integers c and d such that

$$\|\underline{e}\|_2 = \mathcal{O}\left((\Delta x)^{c/2}\right), \qquad \|\underline{e}\|_{2,\Delta x} = \mathcal{O}\left((\Delta x)^{d/2}\right).$$