Bifilar Suspension and Moment of Inertia

Physics

gzn364

Abstract

In this investigation, I explore how the separation of the strings of a bifilar bar pendulum relates to the time period of the pendulum's oscillation. I find that the the separation is inversely proportional to the time period. I use this fact to determine the moment of inertia of the bar, and compare that value to the theoretical moment of inertia predicted by a mathematical model of the bar. I conclude that the two values are extremely similar, and that the mathematical model is therefore a good model of reality.

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1 Introduction

Why does a tightrope walker use a long pole? The pole distributes mass further away from the rope, thereby increasing the funambulist's **moment of inertia**. This means that the same torque results in a smaller rotational acceleration - making it easier for them to balance. From ice skaters to spacecraft, moment of inertia is integral to how objects rotate.

I was inspired to investigate moments of inertia when I saw a video about how cats right themselves mid-air in order to land on their feet [1]. I realised that although I had learned a lot in class about kinematics, I didn't fully understand how rotational motion fitted into the picture.

The moment of inertia of an object can be calculated theoretically, or can be determined experimentally. Determining an object's moment of inertia using mathematical modelling is important for computer simulations, especially in engineering. For example, given the densities of the constituent materials, the moment of inertia of a satellite design could be calculated, and used to simulate how the satellite would rotate when its thrusters are fired.

The purpose of this investigation is therefore to determine how accurately theoretical models of moment of inertia describe reality. I use a bifilar suspension setup to experimentally determine the moment of inertia of a metal bar, and then compare this value to the moment of inertia as predicted by mathematical modelling.

My research questions were **How does the period of oscillation of a bifilar bar pendulum vary with the string separation?** and **How accurately does the mathematical model of the bar predict its moment of inertia?**

2 Background

2.1 Torque

Torque, τ, is the measure of a force that can cause angular acceleration. It is equal to the force applied perpendicular to the moment arm, $F_{\perp}$, multiplied by the distance from the pivot point, r, at which the force is applied:

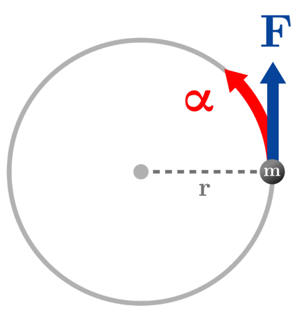
$$\begin{aligned}\tau &= F_{\perp} r \\ &= F r \sin\theta,\end{aligned}$$

where θ is the angle between the direction of the force applied and moment arm.

2.2 Moment of Inertia

The moment of inertia (or *rotational inertia*) of an object determines the amount of torque required to cause the object a given angular acceleration.

Say a force F is applied to a point mass, m, that pivots around a central point at a distance of r. What angular acceleration, α, does the mass experience?



The force causes linear acceleration at a tangent to the motion of the point mass, such that

$$F = ma_{\tan}.$$

Also, the resultant torque is given as

$$\tau = rF,$$

as the force is a applied perpendicular to the moment arm. Therefore,

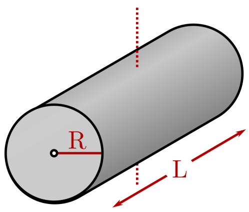
$$\begin{aligned} \tau &= rma_{\tan} && (F = ra_{\tan}) \\ &= \alpha mr^2 && (a_{\tan} = r\alpha) \\ &= \alpha I, \end{aligned}$$

$$\begin{aligned} \text{where} \quad I &= \text{moment of inertia (kg m}^2) \\ &= \frac{\tau}{\alpha} \\ &= mr^2. \end{aligned}$$

More, generally, the moment of inertia is given as the sum of each point mass multiplied by the square of its distance from the pivot,

$$I = \sum_i m_i r_i^2.$$

For a bar of radius R, mass M, and length L, rotating around its central axis,



the moment of inertia, I, can be derived using calculus (see Appendix A) to be

$$I = \frac{1}{12} M(3R^2 + L^2).$$

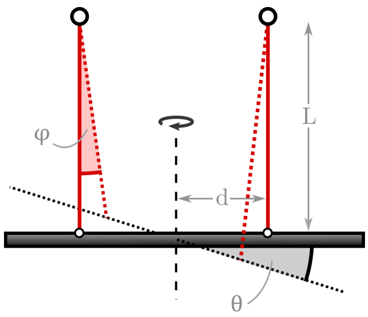
This is the mathematical model of the bar's moment of inertia.

2.3 Bifilar Suspension

Bifilar suspension is a method for determining the moment of inertia of an object. In my investigation, I determine the moment of inertia of a metal bar of mass M

The bar is suspended by two strings of length L, such that the bar is horizontal to the ground, and the two strings are perpendicular to the bar, parallel to each other, and each at a distance of d from the centre of the bar.

The bar is then rotated around its central axis, to an angle of 10°. The displacement angle, θ, is the angle between the bar and its equilibrium position. The angle between each string and its equilibrium position is φ.



If θ and φ are small, then

$$
\begin{aligned}
d\sin\theta &\approx L\sin\varphi. \\
d\theta &\approx L\varphi. \\
\varphi &\approx \frac{d}{L}\theta.
\end{aligned}
$$

The vertical component, F_v of the tension in each string is $\frac{Mg}{2}$, half the weight of the bar. The horizontal component of this tension is the restoring force, F_x:

$$
\begin{aligned}
\tan\phi &= \frac{F_x}{F_v}. \\
F_x &\approx F_v\phi && (\phi \text{ is small}) \\
&= \frac{Mg}{2}\phi \\
&= \frac{Mgd}{2L}\theta.
\end{aligned}
$$

The torque, τ, exerted on the bar by each string is

$$
\begin{aligned}
\tau &= -dF_x \\
&= -\frac{Mgd^2}{2L}\theta.
\end{aligned}
$$

Hence, the net torque on the bar, τ_{net} is

$$
\begin{aligned}
\tau_{\text{net}} &= 2\tau \\
&= -\frac{Mgd^2}{L}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Also,}\quad \tau_{\text{net}} &= I\alpha. \\
\omega &= \frac{d\theta}{dt} = \dot{\theta}. \\
\alpha &= \frac{d^2\theta}{dt^2} = \ddot{\theta}.
\end{aligned}
$$

$$
\begin{aligned}
\implies \tau_{\text{net}} &= I\ddot{\theta} \\
&= -\frac{Mgd^2}{L}\theta.
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\theta} &= -\frac{Mgd^2}{IL}\theta. \\
\ddot{\theta} &\propto -\theta.
\end{aligned}
$$

The angular acceleration, $\ddot{\theta}$, is proportional to the displacement angle, θ and acts in the opposite direction. Thus, the oscillation of the bar is simple harmonic motion, and θ can be written as a sinusoidal function,

$$\theta = A\cos(\omega t + \phi),$$

$$\begin{gathered}\text{where,}\\ A = \text{maximum displacement,}\\ \omega = \text{rotational frequency,}\\ \phi = \text{initial phase.}\end{gathered}$$

Now, by differentiation with respect to time,

$$\begin{aligned}\dot{\theta} &= -A\omega\sin(\omega t + \phi).\\ \ddot{\theta} &= -A\omega^2\cos(\omega t + \phi).\end{aligned}$$

$$\begin{aligned}\ddot{\theta} &= -\frac{Mgd^2}{IL}\theta.\\ -A\omega^2\cos(\omega t + \phi) &= -\frac{Mgd^2}{IL}(A\cos(\omega t + \phi)).\end{aligned}$$

$$\omega^2 = \frac{Mgd^2}{IL}.$$

$$I = \frac{Mgd^2}{L\omega^2}.$$

$$\omega T = 2\pi. \qquad (T = \text{time period})$$

$$\omega^2 = \frac{4\pi^2}{T^2}.$$

$$\boxed{I = \frac{Mgd^2T^2}{4\pi^2 L}.}$$

The moment of inertia of the bar is equal to its mass, multiplied by g, multiplied by half the separation of the strings suspending it squared, multiplied by the time period of oscillation squared, all divided by $4\pi^2$ multiplied by the length of the strings.

3 Hypothesis

From the equation above,

$$\begin{aligned}T^2 &= \frac{4\pi^2 IL}{Mgd^2}.\\ T &= 2\pi\sqrt{\frac{IL}{Mg}}\left(\frac{1}{d}\right).\\ T &\propto \frac{1}{d}.\end{aligned}$$

I expect the time period of oscillation of the bar to be inversely proportional to half the separation distance of the strings suspending it.

Also, the theoretical moment of inertia of the bar, I_0, is given as:

$$I_0 = \frac{1}{12}M(3R^2 + L^2)$$

where, (See Method)

$$M = 591.48 \text{ g} \pm 0.01 \text{ g}.$$
$$2R = 12.69 \text{ mm} \pm 0.01 \text{ mm}.$$
$$L = 600 \text{ mm} \pm 5 \text{ mm}.$$

$$\begin{aligned} I_0 &= \frac{1}{12} \cdot 0.59148 \cdot (3\left(\frac{0.01269}{2}\right)^2 + (0.600)^2) \\ &= 0.0177 \text{ kg m}^2 \pm 0.0003 \text{ kg m}^2. \end{aligned}$$

When T is graphed against $\frac{1}{d}$, the moment of inertia of the bar can be derived from the gradient of the graph. I expect that the value obtained for rotational inertia will fall within ± 0.0003 kg m^2 of 0.0177 kg m^2, that of the model.

4 Variables

Independent Variable	Separation of the strings, $2d$.
Dependent Variable	Time period of oscillation, T.
Control Variables	**String Length, L:** string length affects the time period of oscillation
	String Angle: the bifilar suspension assumes that the strings are parallel to each other and perpendicular to the floor at equilibrium.
	Bar: bar mass, material, radius, and length may affect the bar moment of inertia or time period of oscillation.

5 Methodology

Firstly, I measured the mass of the bar using a balance. I took several measurements, and calculated the average mass.

Mass (g) ± 0.01 g
591.49
591.47
591.47
591.48
591.47
Average
591.48

$$M = 591.48 \text{ g} \pm 0.01 \text{ g}.$$

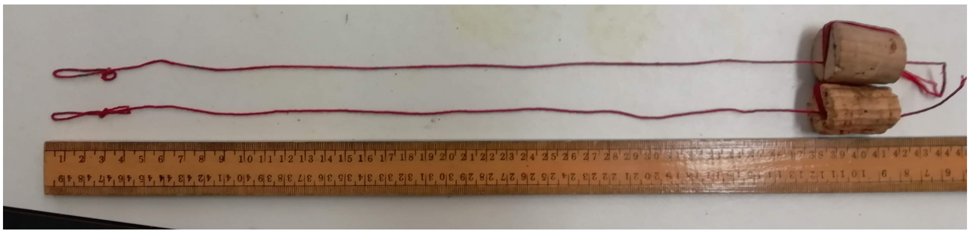
I then measured the diameter of the bar using a micrometer screw gauge, and the length of the bar using a meter rule. I realised the shape of the bar was not entirely cylindrical - the ends were slightly rounded and a small section was threaded. These factors would cause the actual moment of inertia to deviate from the theoretical value, so I factored them into the precision of my length measurement.

$$2R = 12.69 \text{ mm} \pm 0.01 \text{ mm}.$$
$$L = 600 \text{ mm} \pm 5 \text{ mm}.$$

5.1 Bifilar Pendulum

Firstly, I tied a loop in the end of each string (to hold the bar), and wound each around a cork, such that the distance between the end of the cork and the start of the loop was the same for each, when extended. I measured the length of the string when extended, L, making sure that both strings were the same length:

$$L = 380 \text{ mm} \pm 2 \text{ mm}.$$



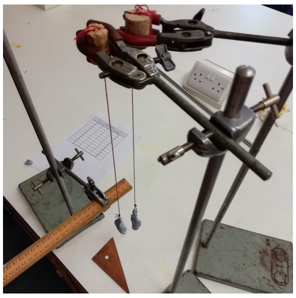
I then set up two clamp and stands to hold the corks at the same height. I set up another clamp stand to clamp a meter rule parallel to the ground, at the same height as the end of the strings.

Having measured the length of the bar to be 600 mm $\pm$ 5 mm, I marked the centre of the bar using a pencil.

Now, for each separation distance, $2d$:

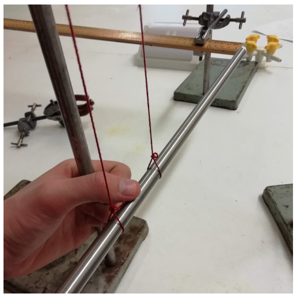
1. I used the pencil and a ruler to mark the bar at a distance of $d \pm 1$ mm either side of the centre point.
2. I attached a piece of BluTack to the ends of the strings, to make sure they were fully extended and perpendicular to the ground. I then used the clamped rule to position the string clamps such that the separation of the strings when at rest was $2d$.
3. I inserted the bar into the string loops, lining the loops up with the correct pencil markings.
4. I placed my thumb on the centre of the bar, using a fourth clamp stand to hold my thumb steady at the correct position. My thumb was perfect, as it provided enough friction to keep the centre in place while allowing the bar to rotate.
5. I positioned the clamped rule at the end of the suspended bar such that it was perpendicular to the bar in its equilibrium position, and the bar pointed to '0' on the rule at equilibrium.
6. I used the clamped rule to offset the bar by an angle of $\theta = 10°$. Given that the distance from the centre to the end of the bar was 300 mm, I made sure the end of bar pointed to $300 \sin 10° =$ 53.0 mm on the rule.
7. I released the end of the bar, and once the bar was oscillating around the central axis constrained by my thumb, I released my thumb.
8. Using a stopwatch I measured the amount of time that it took for the bar pendulum to complete 10 full oscillations (**for $2d \geq 42$, I measured the time for 20 oscillations, to maintain precision**).

I repeated this process for 10 separation values - starting at 6 mm, and increasing in 6 mm intervals.



(2)

(3)



(6)

6 Results

The uncertainty in $\frac{1}{d}$ can be calculated using the percentage uncertainty for the string separation. I estimated the uncertainty of the string separation, $\Delta(2d)$, to be

$$\begin{aligned}\Delta(2d) &= \pm 0.1 \text{ cm} \\ &= \pm\left(\frac{100 \cdot 0.1}{2d}\right)\% \\ &= \Delta\left(\frac{1}{d}\right)\%. && \text{(Percentage uncertainty)} \\ \Delta\left(\frac{1}{d}\right) &= \pm\left(\frac{\frac{1}{d}\cdot\Delta(\frac{1}{d})\%}{100}\right). && \text{(Absolute uncertainty)}\end{aligned}$$

Now, the time period T, is calculated as

$$T = \frac{t_{\text{avg}}}{n}.$$

So, the uncertainty in T, ΔT, can be determined using the percentage uncertainty in t_{avg}.

Average human reaction time is around 200ms (less when the event is expected). I therefore estimated the uncertainty in t_{avg} to be

$$\Delta(t_{\text{avg}}) = \pm 0.2 \text{ s}.$$

For each t_{avg} value, I also calculated an alternative uncertainty - half the range of the data:

$$\Delta(t_{\text{avg}}) = \pm\left(\frac{t_{\text{max}} - t_{\text{min}}}{2}\right),$$

and I took the uncertainty to be the greater of the two.

Then, I calculated the percentage uncertainty of t_{avg}, which is the same as the percentage uncertainty of T. Hence, I converted the percentage uncertainty for T to absolute uncertainty, as I did above for d.

Table 1: Bifilar Pendulum Experiment.

$2d$ (cm)	$\frac{1}{d}$ (m^{-1})	$\Delta\frac{1}{d}$ (m^{-1})	time, t, for n oscillations (s)				n	T (s)	ΔT (s)
			t_1	t_2	t_3	t_{avg}			
6.0	33.33	0.5556	71.85	72.15	71.64	71.88	10	7.188	0.0255
12.0	16.67	0.1389	35.13	35.21	35.01	35.12	10	3.512	0.0200
18.0	11.11	0.06713	24.08	23.92	24.06	24.02	10	2.402	0.0200
24.0	8.333	0.03472	18.05	18.04	18.34	18.04	10	1.814	0.0200
30.0	6.667	0.02222	14.32	14.42	14.39	14.38	10	1.438	0.0200
35.0	5.714	0.01633	12.17	12.31	12.27	12.25	10	1.225	0.0200
42.0	4.762	0.01134	20.50	20.72	20.84	20.69	20	1.034	0.0100
48.0	4.167	0.008631	18.05	18.16	18.03	18.08	20	0.9040	0.0100
54.0	3.704	0.006859	16.12	16.12	16.19	16.14	20	0.8072	0.0100
60.0	3.333	0.005556	14.58	14.61	14.64	14.61	20	0.7305	0.0100

7 Analysis

Recall that

$$T = 2\pi\sqrt{\frac{IL}{Mg}}\left(\frac{1}{d}\right).$$

Therefore, if time period, T, is graphed against $\frac{1}{d}$, the gradient of the graph, ∇, is

$$\nabla = 2\pi\sqrt{\frac{LI}{Mg}},$$
$$I = \frac{Mg\nabla^2}{4\pi^2 L}.$$

The gradient of the graph can be used to calculate the moment of inertia of the bar. See Figure 1 below (a graph of T against $\frac{1}{d}$).

The error bars were too small to draw by hand - the largest percentage error was 1.6% - so I used MS Excel's LINEST function to calculate the gradient of the graph, and the uncertainty in the gradient:

$$\begin{aligned}\nabla &= 0.21466 \pm 0.00104\\ &= 0.21466 \pm 0.4845\%.\\ \nabla^2 &= 0.0460789 \pm 0.9690\%.\end{aligned}$$

The rest of the equation for I in terms of ∇ only involves multiplication and division, so I just have to add up the percentage uncertainties.

$$I = \frac{Mg\nabla^2}{4\pi^2 L}.$$

$$\begin{aligned}M &= 0.59148 \text{ kg} \pm 0.00001 \text{ kg}\\ &= 0.59148 \text{ kg} \pm 0.00169067\%.\end{aligned}$$

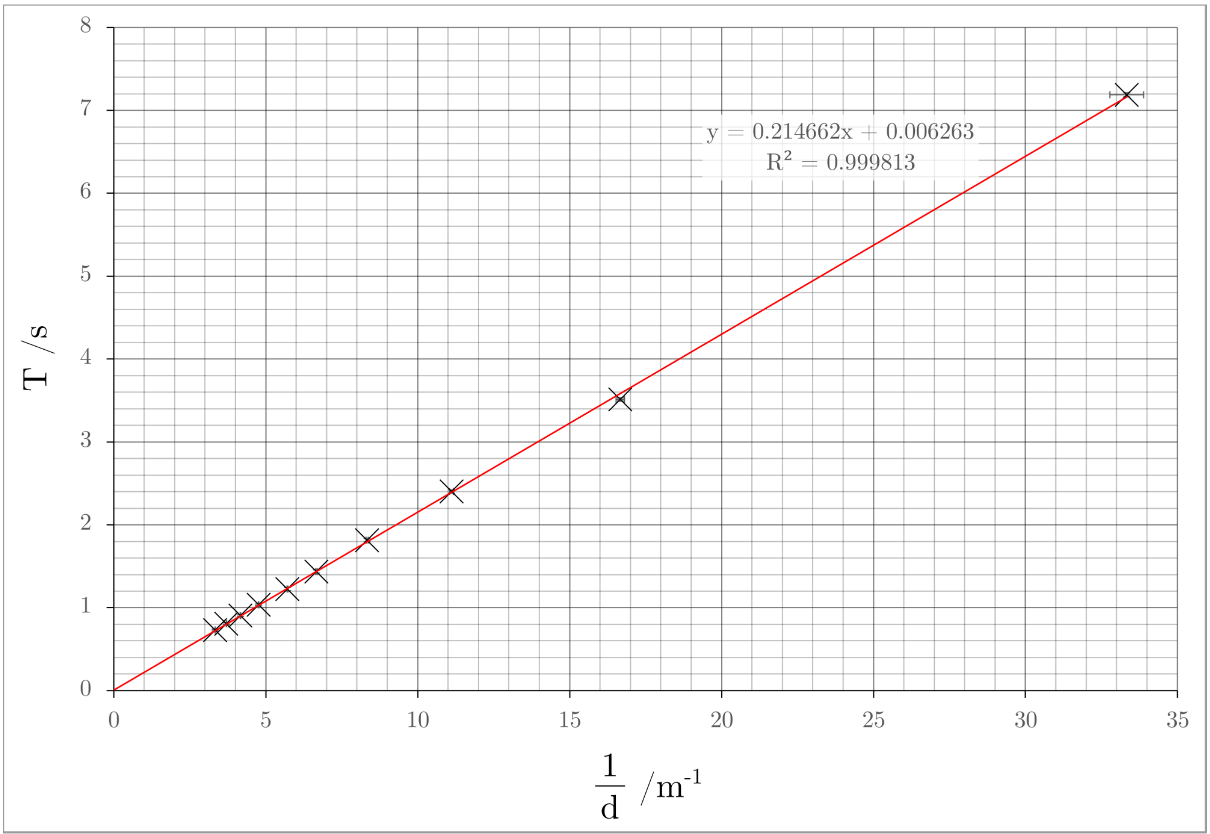
$$\begin{aligned}L &= 380 \text{ mm} \pm 2 \text{ mm}\\ &= 0.380 \text{ m} \pm 0.5263\%.\end{aligned}$$

$$g \approx 9.80665 \text{ ms}^2 \text{ [2]}.$$

$$I = \frac{(0.59148)(9.80665)(0.0460789)}{4\pi^2(0.6)} \pm (0.9690 + 0.00169067 + 0.5263)\%.$$

$$I = 0.0178 \text{ kg m}^2 \pm 1.49699\%$$
$$= 0.0178 \text{ kg m}^2 \pm 0.0003 \text{ kg m}^2.$$

Figure 1: Time Period of Oscillation vs. The Reciprocal of Half the String Separation



8 Conclusion

I hypothesised that $T \propto \frac{1}{d}$. The graph of T against $\frac{1}{d}$ is a straight line through the origin, with an R^2 value of 0.9998 (meaning that the data follows the line of best fit extremely closely), and therefore strongly supports the hypothesis.

I also hypothesised that the measured moment of inertia, I, of the bar would be equal to the theoretical moment of inertia, I_0.

$$I_0 = 0.0177 \text{ kg m}^2 \pm 0.0003 \text{ kg m}^2.$$
$$I = 0.0178 \text{ kg m}^2 \pm 0.0003 \text{ kg m}^2.$$

The difference between I_0 and I is 0.0001 kg m^2 - just 0.5618% of I and less than the uncertainty of I. This constitutes strong evidence to support my hypothesis that the mathematical model that I used the for moment of inertia of the bar *does* accurately describe reality.

9 Evaluation

My investigation was limited in that I only investigated a simple shape - a rod. Moment of inertia can be determined for much more complex shapes using both mathematical analysis and the bifilar suspension method. An extension to my investigation would therefore be to conduct the same experiment on a complex shape - a model plane, for example.

I am pleased with the precision I was able to achieve in my results. However, improving my experiment would involve improving the precision (and accuracy) of my measurements. For example, the biggest source of error in my final result was the gradient, which combined time period (measured by hand with stopwatch) and separation distance (measured with a ruler).

Another weakness in my experiment was that the bar shape was not entirely cylindrical, which would affect the moment of inertia of the bar compared to that of the mathematical model. Also, strings extended to a different length under tension, making them hard to measure. See Systematic Errors and Random Errors, below, for more detail.

9.1 Systematic Errors

Error	Significance	Solution
Axis - Although I used my thumb to keep the centre of the bar in position, it didn't stay completely still during oscillation. Therefore, the axis of rotation was not constant.	Effect appears to be negligible, given that I obtained the correct I value (this may be because although the centre was not stationary, it was not moving very much).	A different method for setting the displacement angle, and releasing the bar. For example, a mechanism that holds both ends of the bar in place until releasing them simultaneously.
Bar shape was not strictly cylindrical - the ends were slightly rounded, and a section of it was threaded (inwards).	Because these factors change the distribution of mass, they affect the moment of inertia. However, the shape does not deviate enough from a cylinder to cause a noticeable error.	Either replace the bar with a perfect cylinder, or model the bar more precisely.
The strings had some elasticity: under tension they stretched out, increasing in length.	This made the strings hard to measure, reducing the accuracy of my L measurement. Also, the elasticity may have impacted on the pendulum mechanics.	Use strings of a different material, eg. lightweight metal wire.

9.2 Random Errors

Error	Significance	Solution
Length - the precision of the meter rulers and cm rulers are only precise to 1mm.	Measurements taken by ruler were the greatest source of uncertainty in my investigation (the length of the string and the displacement angle, most importantly).	Use Vernier calipers where possible (marking bar, measuring displacement angle).
Mass - the balance I used measured mass to the nearest 0.01g.	Negligible, other measurement were far less precise.	None
Stopwatch - the uncertainty of the time taken for n oscillations is down to human reaction times - which can be more than 200ms. This may have caused time periods to be measured as greater than their real value.	Seemlingly negligible, as I obtained the correct answer. However, improving the precision of the time period would be a priority if I were to redo the investigation.	Use a digital data logger to restring the displacement of the bar with time (eg. using laser sensor), and use this to calculate time period.

9.3 Risk Assessment

My experiment did not entail much risk.

Activity	Hazard	Control
Using clamp stands and metre rules	Eye injury.	Take extra caution when handling especially heavy or long equipment. Clean up any floor spills to prevent slips.
	Dropping clamps on experimenter's hands or feet.	
	Clamp setup falling over, onto experimenter	Ensure all clamps are stable.

References

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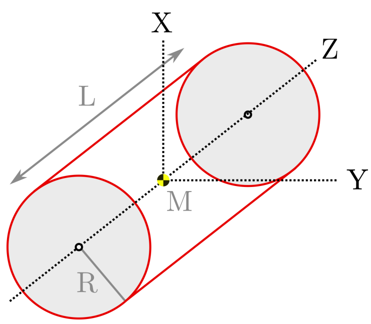
Appendices

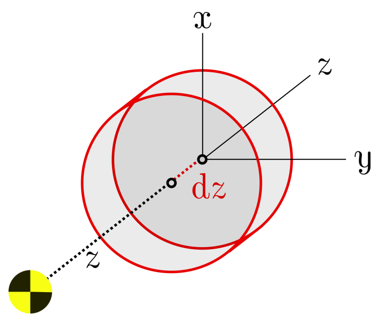
A Mathematical Modelling

The object in question is a rod of mass M, radius R, and length L. Let ρ be the object's density (per unit volume), where $\rho = \frac{M}{L\pi R^2}$.



The rod rotates around a central perpendicular axis (through the centre of mass), X, with a moment of inertia of I_X.

Consider a thin slice of the rod, of length $\mathrm{d}z$, at a distance of z from the X axis.



The mass of the slice, $\mathrm{d}m$, is

$$\begin{aligned} \mathrm{d}m &= \rho\,\mathrm{d}V \\ &= \left(\frac{M}{L\pi R^2}\right)(\pi R^2\,\mathrm{d}z\,) \\ &= \frac{M}{L}\,\mathrm{d}z\,. \end{aligned}$$

Now, the slice is a thin disk, meaning its moment of inertia around the z axis, $\mathrm{d}I_z$, is

$$\mathrm{d}I_z = \frac{1}{2}\,\mathrm{d}m\,R^2.$$

From the *Perpendicular Axis Theorem*,

$$\begin{aligned} \mathrm{d}I_z &= \mathrm{d}I_x + \mathrm{d}I_y \\ &= 2\,\mathrm{d}I_x\,. && \text{(By symmetry, } \mathrm{d}I_x = \mathrm{d}I_y\text{)} \\ \mathrm{d}I_x &= \frac{1}{2}\,\mathrm{d}I_z \\ &= \frac{1}{4}R^2\,\mathrm{d}m\,. \end{aligned}$$

Now, we have the disk's moment of inertia around its own x axis. We want the moment of inertia around X axis, which is parallel to x and intersects the centre of mass.

The *Parallel Axis Theorem* states that the moment of inertia I_{parellel} of a body (mass M) rotating around an axis that is parallel to the axis through the centre of mass (I), and displaced from it by a distance of d, is

$$I_{\text{parellel}} = I + Md^2.$$

Therefore, the rotational inertia of the disk around the centre of mass is.

$$\begin{aligned} \mathrm{d}I_X &= \mathrm{d}I_x + z^2\,\mathrm{d}m \\ &= \frac{1}{4}R^2\,\mathrm{d}m + z^2\,\mathrm{d}m \\ &= \frac{MR^2}{4L}\,\mathrm{d}z + z^2\frac{M}{L}\,\mathrm{d}z\,. && (\,\mathrm{d}m = \tfrac{M}{L}\,\mathrm{d}z\,) \end{aligned}$$

The moment of inertia of the bar is found by summing up the rotation inertia of all these infinitely small slices, between $\frac{L}{2}$ and $-\frac{L}{2}$:

$$
\begin{aligned}
I_X &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{MR^2}{4L}\ \mathrm{d}z\ + \int_{-\frac{L}{2}}^{\frac{L}{2}} z^2 \frac{M}{L}\ \mathrm{d}z \\
&= \frac{MR^2}{4L}\left[\ z\ \right]_{-\frac{L}{2}}^{\frac{L}{2}} + \frac{M}{L}\left[\frac{z^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\
&= \frac{MR^2}{4L}\left[\frac{L}{2} - -\frac{L}{2}\right] + \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] \\
&= \frac{MR^2}{4L}\left[\ L\ \right] + \frac{M}{3L}\left[\frac{2L^3}{8}\right] \\
&= \frac{MR^2}{4} + \frac{ML^2}{12} \\
&= \frac{1}{12}M(3R^2 + L^2).
\end{aligned}
$$