Logarithmic Spiral - Robot Problem

B12

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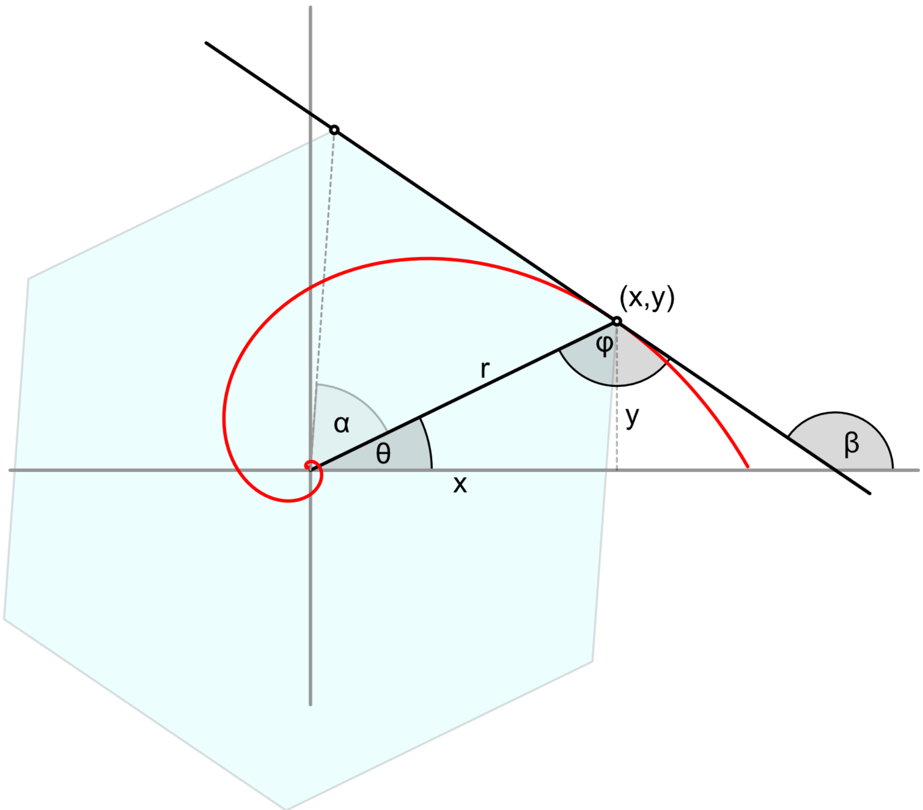
1 Pursuit Curve

This is based on `https://youtu.be/MP5F4O1hCFg`

We have n robots equally spaced around a circle radius R. Each robot follows the next robot round the circle anti-clockwise. We assume that the robots change direction simultaneously.

In the example below, $n = 6$. The path of one of the robots is shown. We see that this robot's instantaneous direction - the tangent to the curve - points towards the next robot around the polygon.



Let the coordinates of this robot be (x, y) in Cartesians and (r, θ) in polar coordinates.

Assume that the angle φ between the radial line (connecting to the robot to the circle centre) and the tangent line remains constant. *This is follows from the fact that the angle between the robots remains constant and are always the same distance from the origin as each other, which is true since they start equally spaced around the circle and are symmetrical in their motion.*

Since $\alpha = \frac{2\pi}{n}$,

$$\varphi = \pi - \frac{1}{2}(\pi - \alpha) = \frac{\pi}{2} + \frac{\pi}{n}.$$

Also,

$$\beta = \pi - (\pi - \theta - \varphi) = \theta + \varphi.$$

Now, by trigonometry, the slope of the tangent is given,

$$\frac{dy}{dx} = \tan\beta = \tan(\theta + \varphi).$$

However, we also have

$$\tan\theta = \frac{y}{x} \implies \theta = \arctan\left(\frac{y}{x}\right), \tag{1}$$

So

$$\begin{aligned}\frac{dy}{dx} &= \tan\left(\arctan\left(\frac{y}{x}\right) + \varphi\right) \\ &= \frac{\dfrac{y}{x} + \tan\varphi}{1 - \dfrac{y}{x}\tan\varphi},\end{aligned}$$

making use of the compound angle formula for tan.

Let $a = \tan\varphi$ Since φ is constant, a is constant. Then

$$\frac{dy}{dx} = \frac{\dfrac{y}{x} + a}{1 - \dfrac{y}{x}a} = F\left(\frac{y}{x}\right),$$

a first-order homogeneous differential equation[1]. We proceed using the substitution $t(x) = \frac{y}{x}$, so that

$$\frac{t + a}{1 - ta} = \frac{dy}{dx} = \frac{d}{dx}(xt) = x\frac{dt}{dx} + t,$$

by the product rule.

Now,

$$\begin{aligned} x\frac{dt}{dx} &= \frac{a + at^2}{1 - at} \\ \iff \frac{dt}{dx} &= \frac{1}{x} \cdot \frac{a + at^2}{1 - at} \\ &= f(x) \cdot g(t). \end{aligned}$$

[1]See `https://internal.ncl.ac.uk/ask/numeracy-maths-statistics/core-mathematics/calculus/homogeneous-first-order-differential-equations.html`.

Now, $\varphi \neq 0 \implies a \neq 0 \implies g(t) \not\equiv 0$. We thus proceed by separation of variables.

$$\begin{aligned}
f(x) &= \frac{1}{g(t)}\frac{dt}{dx} \\
\implies \int f(x)dx &= \int \frac{1}{g(t)}\frac{dt}{dx}dx \\
&= \int \frac{1}{g(t)}dt \\
\implies \int \frac{1}{x}dx &= \int \frac{1-at}{a+at^2}dt
\end{aligned}$$

$$\implies \ln|x| + c_1 = \frac{1}{a}\int \frac{1}{1+t^2}dt - \int \frac{t}{1+t^2}dt.$$

Using the substitution $u = 1+t^2$, we have

$$\int \frac{t}{1+t^2}dt = \int \frac{t}{u}\frac{du}{2t} = \int \frac{1}{2u}dt = \frac{1}{2}\ln|u| + c_2 = \frac{1}{2}\ln|1+t^2| + c_2.$$

Hence, using the standard integral $\int \frac{1}{1+t^2}dt = \arctan t$, we have

$$\ln|x| + c_1 = \frac{1}{a}\arctan t - \frac{1}{2}\ln|1+t^2| - c_2.$$

Recall that $t = \frac{y}{x}$ and that $\arctan(y/x) = \theta$ by (1). Furthermore, $x^2+y^2 = r^2$ by Pythagoras' theorem. So, combining constants c_1 and c_2,

$$\begin{aligned}
\ln|x| &= \frac{1}{a}\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| + c \\
&= \frac{1}{a}\theta - \frac{1}{2}\ln\left|\frac{x^2+y^2}{x^2}\right| + c \\
&= \frac{\theta}{a} - \frac{1}{2}\ln\left|\frac{r^2}{x^2}\right| + c \\
&= \frac{\theta}{a} - \ln\left|\frac{r}{x}\right| + c
\end{aligned}$$

$$\begin{aligned}
\implies \frac{\theta}{a} + c &= \ln|x| + \ln\left|\frac{r}{x}\right| \\
&= \ln\left|x\frac{r}{x}\right| \\
&= \ln(r).
\end{aligned}$$

Let $b = \frac{1}{a} = \frac{1}{\tan\varphi} = \cot\varphi$. Since a is constant, b is constant.
Then

$$r = \exp\left(\frac{\theta}{a} + c\right)$$

$$= ke^{b\theta},$$

where $k = e^c$.

Each robot starts on the edge of the circle, with $r = R$. Let λ be the starting angle of each robot (from positive x axis) so that $\theta = \lambda$ when $r = R$.

Then

$$\begin{aligned} R &= ke^{b\lambda} \\ \implies k &= Re^{-b\lambda}. \end{aligned}$$

Hence, the motion of the robot is describe by

$$r = ke^{b\theta} = Re^{-b\lambda}e^{b\theta} = Re^{b\theta - b\lambda}.$$

That is

$$\boxed{r = Re^{b\theta - b\lambda}.}$$

https://www.desmos.com/calculator/aqp22fmuie

2 Arc Length & Time

Based on https://www.math.usm.edu/lambers/mat169/book169.pdf. See Sections 3.2.1, 3.2.2, and 3.4.2.

The length L of the arc of the curve between angles θ_1 and θ_2 is derived in Section 3.4.2 above to be

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Since $r = Re^{b\theta - b\lambda}$,

$$\frac{dr}{d\theta} = Rbe^{b\theta - b\lambda} = br. \tag{2}$$

Then, remembering that b is constant,

$$\begin{aligned} L &= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (br)^2} d\theta \\ &= \int_{\theta_1}^{\theta_2} r\sqrt{1 + b^2} d\theta \\ &= \sqrt{1 + b^2} \int_{\theta_1}^{\theta_2} Re^{b\theta - b\lambda} d\theta. \end{aligned}$$

Observe that since $b = \cot \varphi$,

$$\sqrt{1 + b^2} = \sqrt{\csc^2 \varphi} = \frac{1}{\sin \varphi} \tag{3}$$

So,

$$L = \frac{R}{\sin \varphi} \int_{\theta_1}^{\theta_2} e^{b\theta - b\lambda} d\theta$$

$$
\begin{aligned}
&= \frac{R}{\sin\varphi}\left[\frac{1}{b}e^{b\theta-b\lambda}+C\right]_{\theta_1}^{\theta_2} \\
&= \frac{R}{\sin\varphi}\cdot\frac{1}{\cot\theta}\left[e^{b\theta_2-b\lambda}-e^{b\theta_1-b\lambda}\right] \\
&= \frac{R}{\cos\varphi}\left[e^{b\theta_2-b\lambda}-e^{b\theta_1-b\lambda}\right].
\end{aligned}
$$

Recall that $\varphi = \frac{\pi}{2}+\frac{\pi}{n}$. So,

$$
\begin{aligned}
\cos\varphi &= \cos\left(\frac{\pi}{2}+\frac{\pi}{n}\right) \\
&= \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{n}\right)-\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{n}\right) \\
&= -\sin\left(\frac{\pi}{n}\right). \qquad (4)
\end{aligned}
$$

So

$$
L = \frac{R}{\sin(\pi/n)}\left[e^{b\theta_1-b\lambda}-e^{b\theta_2-b\lambda}\right].
$$

Let's jump back a bit for a sec. Observe that $\pi \le \varphi \le 2\pi$ for all $n \ge 2$, so $b \le 0$. Thus,

$$
\lim_{\theta\to\infty} Re^{b\theta-b\lambda} = 0.
$$

As theta increases, we get closer and closer to the middle.

So, we want the length of the curve between the angle at the start of the circle, λ and ∞. Taking $\theta_1 = \lambda$ and $\theta_2 = \infty$ (we should really use a limit but cba),

$$
\begin{aligned}
L &= \frac{R}{\sin(\pi/n)}\left[e^{b\lambda-b\lambda}-0\right] \\
&= \frac{R}{\sin(\pi/n)}[1-0] \\
&= \frac{R}{\sin(\pi/n)}.
\end{aligned}
$$

If the robots move at constant speed u, they meet after time T_0, given

$$
T_0 = \frac{R}{u\sin(\pi/n)}
$$

3 Time Parametric

We have $r(\theta)$. Can parameterise both in terms of a time parameter t to obtain $r(t)$ and $\theta(t)$? Yes! And we can use it to find the time they meet *without the arc length*.

Let $t \in [0,\infty]$.

Note: this t has nothing to do with the substitution $t(x)$ from earlier!!

Let u be the constant speed of the robot, and let $\boldsymbol{p}$ be the position vector of the robot so that

$$\boldsymbol{p} = (x, y).$$

Then the vector velocity of the robot, $\dot{\boldsymbol{p}}$, is given

$$\dot{\boldsymbol{p}} = \frac{d}{dt}(\boldsymbol{p}) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Since u is the speed, we also have $|\dot{\boldsymbol{p}}| = u$. So

$$u^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Using the same polar coordinate system from the previous section, we have $x = r\cos\theta$. That means by the product rule

$$\frac{dx}{dt} = \frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}.$$

By the chain rule on $r(\theta)$, and using (2),

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = br\frac{d\theta}{dt}.$$

So,

$$\frac{dx}{dt} = br\frac{d\theta}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}.$$

Similarly, we have $y = r\sin\theta$, so

$$\begin{aligned}\frac{dy}{dt} &= \frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\\ &= br\frac{d\theta}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}.\end{aligned}$$

Combining the two, we obtain

$$\begin{aligned}u^2 &= \left(br\frac{d\theta}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt}\right)^2 + \left(br\frac{d\theta}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt}\right)^2\\ &= \left(r\frac{d\theta}{dt}\right)^2\left[(b\cos\theta - \sin\theta)^2 + (b\sin\theta + \cos\theta)^2\right]\\ &= \left(r\frac{d\theta}{dt}\right)^2\left[b^2+1\right].\end{aligned}$$

The last equality is obtained by expanding and using trig identities. Then, by (3),

$$\begin{aligned}u &= \left(r\frac{d\theta}{dt}\right)\sqrt{b^2+1}\\ &= \left(r\frac{d\theta}{dt}\right)\frac{1}{\sin\varphi}\end{aligned}$$

$$
\begin{aligned}
\implies \quad u\sin\varphi &= r\frac{d\theta}{dt} \\
\implies \quad \int u\sin\varphi \; dt &= \int r\frac{d\theta}{dt} \; dt \\
\implies \quad u\sin\varphi(t + c_1) &= \int r \; d\theta \\
&= \frac{r}{b} + c_2
\end{aligned}
$$

So, we deduce that

$$
\begin{aligned}
r &= (ub\sin\varphi)t + C \\
&= ut\cot(\varphi)\sin(\varphi) + C \\
&= ut\frac{\cos(\varphi)}{\sin(\varphi)}\sin(\varphi) + C \\
&= ut\cos\varphi + C \\
&= -ut\sin(\pi/n) + C.
\end{aligned}
$$

For the final equation we used (4).

Now, the robots start on the edge of the circle. That is, when $t = 0$, $r = R$. This holds if and only if $C = R$. We conclude that the time-parameterised form of r is

$$\boxed{r(t) = R - ut\sin(\pi/n).}$$

Suppose the robots meet at $t = T_0$. At this point, $r = 0$, so

$$0 = -uT_0\sin(\pi/n) + R \qquad \iff \qquad \boxed{T_0 = \frac{R}{u\sin(\pi/n)}}.$$

We don't need to parameterise θ to work out the time they meet, but it allows us to plot the position of the particle with time.

Recall that $r = Re^{b\theta - b\lambda}$. So,

$$
\begin{aligned}
Re^{b\theta - b\lambda} &= R - ut\sin(\pi/n) \\
\implies \quad b(\theta - \lambda) &= \ln\left(1 - \frac{u\sin(\pi/n)}{R}t\right) \\
\implies \quad \theta(t) &= \frac{1}{b}\ln\left(1 - \frac{u\sin(\pi/n)}{R}t\right) + \lambda.
\end{aligned}
$$