Pipeline - MMPS Project 1

B12

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Contents

1 Variables

All of these variables might have different values for overground and underground pipes. Dont have to use them all

Variables	**Symbol**
Cost of replacing entire pipeline	C_R
Cost of fixing leak	C_F
Cost of checking (looking at) each section	C_L
Number of sections in pipeline	N
Probability of finding leak in a section containing one	p
Hourly cost - labour, equipment renting, etc.	C_H
Time required to replace entire pipeline	T_R
Time required to fix leak	T_F
Time required to check each section	T_S
Length of each section	L
Probability that leak is in any given section (uniform?)	$p_s = \frac{1}{N}$?

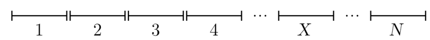

Figure 1: Model setup.

2 Model Idea

Model pipeline as N sections of pipe, uniquely labelled 1 to N. See Figure 1.

Let sections be represented by the set $\Omega = \{1, 2, 3, \ldots, N\}$.

2.1 Assumptions

1. There is exactly one leak somewhere along the pipeline.
2. Leak is equally likely to occur in each section.
3. Configuration doesn't matter (eg. t-shape).
4. We check sections for a leak in order (1 first, 2 second, ...)
5. If a leak is found, it is fixed, solving the problem.
6. If no leak is found after checking all sections, we replace the pipeline.
7. We want to minimise the general cost.

Let $\mathbb{P}$ be a uniform distribution on Ω, so that for each $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{N}.$$

Furthermore, for each $A \subseteq \Omega$,

$$\mathbb{P}(A) = \frac{|A|}{N}.$$

2.2 Prob 1

Two approaches give the same answer. First, let $\omega \in \Omega$ and let $\Pr(\omega)$ denote the probability of finding a leak in section ω. To find a leak we must have

1. The leak is in section, probability $\mathbb{P}(\omega) = \frac{1}{N}$.
2. The leak is successfully found, probability p.

Hence, $\Pr(\omega) = \frac{p}{N}$.

2.3 Prob 2

Maybe we want the probability given previous success/failure.

1. What is probability of finding leak in first section?
2. *Given no leak was found in the first section*, whats the probability of finding a leak in the second section?
3. *Given no leak was found in the first or second section*, whats the probability of finding a leak in the third section?

More generally, let $n \in \mathbb{N}$. Given no leak was found in sections $1, 2, 3, \ldots, n$, whats the probability of finding the leak in section $n+1$? To find a leak in section $n+1$ we must have the following.

1. We didn't miss the leak in the previous sections i.e. there wasn't a leak in any previous section, probability
$$1 - \mathbb{P}(\{1, 2, \ldots, n\}) = 1 - \frac{n}{N}.$$
2. The leak is actually in section $n+1$ out of remaining $N-n$, probability
$$\mathbb{P}(n+1 \mid \{n+1, n+2, \ldots, N\}) = \frac{1}{N-n}$$
3. We successfully find the leak, probability p.

Hence, we have that for each $n \in \{1, 2, \ldots, N-1\}$,

$$\begin{aligned}
\Pr(n+1) &= \left(1 - \frac{n}{N}\right)\left(\frac{1}{N-n}\right)p \\
&= \left(\frac{N-n}{N}\right)\left(\frac{1}{N-n}\right)p \\
&= \frac{p}{N}.
\end{aligned}$$

Also, $\Pr(1) = \frac{p}{N}$ since no section precedes it. So we have the same probability as before.

2.3.1 Validation

I wrote a python program [1] to test this formula experimentally. Given N and p, and a number T of tests, the program performs the following T times:

1. Randomly choose a leak position.
2. Iterate through sections, checking for leaks as described above.

[1] see `https://repl.it/@tomnwright/Project1-Simulation?lite=1&outputonly=1`

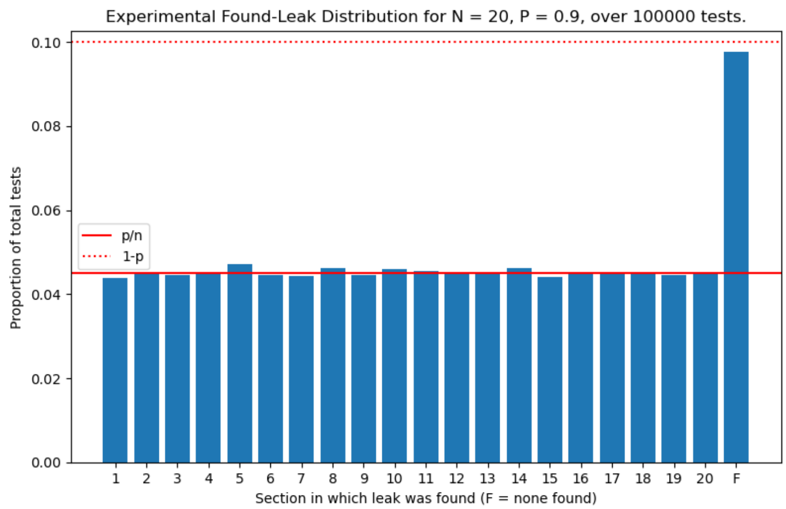

Figure 2: Model setup.

3. If a leak is found, exit and record which section the leak was found in.
4. If no leak is found, record $N+1$.
5. Graph the resulting distribution.

In Figure 2, we see the output of such a process. The proportion of leaks found in each section is the same across sections and matches the expected value. Also, the proportion of tests in which the leak wasn't found at all matches the probability predicted by this model: $1-p$.

Also,

$$1-p+\sum_{i=1}^{N}\Pr(i)=1-p+\sum_{i=1}^{N}\frac{p}{N}=1-p+N\frac{p}{N}=1,$$

as we would expect.

2.4 Prob 3

It seems like the probability should depend on previous checks. But it really depends only on where the leak is (uniformly distributed) and p which is constant. Can you argue that way?

2.5 Cost

Our strategy is to check $x \in \Omega$ sections before giving up and replacing the entire pipe. We want to find the optimal x value so that the expected cost is minimal.

Checking Firstly, it costs C_L to check each section, including labour, equipment, etc. So, we want the expected number of checks, $E(X)$, that need to be performed before we find the leak or check all x sections.

$$\begin{aligned} E(X) &= 1 \cdot \Pr(1) + 2 \cdot \Pr(2) + \ldots + x \cdot \Pr(x) \\ &= \sum_{i=1}^{x} i \cdot \Pr(i) \\ &= \sum_{i=1}^{x} i \cdot \frac{p}{N} \\ &= \frac{p}{N} \sum_{i=1}^{x} i \\ &= \frac{p}{N} \cdot \frac{x(x+1)}{2} = \frac{p}{2N}(x^2 + x). \end{aligned}$$

So the expected cost of checking is $\frac{p}{2N}(x^2 + x)C_L$.

Fixing We fix the leak for cost C_F if it is found in the first x sections. The probability of this happening is $\sum_{i=1}^{x} \Pr(i) = \frac{xp}{N}$, so the expected cost of repair is

$$\frac{xp}{N} C_F.$$

Replacing If we don't find the leak, probability $1 - \frac{xp}{N} = \frac{N-xp}{N}$, we replace the whole pipeline, at cost C_R. So the expected cost of replacing is

$$C_R \left(1 - \frac{xp}{N}\right)$$

Thus the total expected cost, $f(x)$, is given

$$\begin{aligned} f(x) &= \frac{p}{2N}(x^2 + x)C_L + \frac{xp}{N} C_F + C_R \left(1 - \frac{xp}{N}\right) \\ &= \frac{1}{2N}\left(Ax^2 + Bx + C\right), \end{aligned}$$

where

$$A = pC_L, \quad B = p(C_L + 2C_F - 2C_R), \quad C = 2NC_R.$$

Differentiating, we obtain

$$
\begin{aligned}
f'(x) &= \frac{1}{2N}(2Ax + B) \\
&= 0 \\
\iff \quad x &= -\frac{B}{2A} \\
&= -\frac{p(C_L + 2C_F - 2C_R)}{2pC_L} \\
&= -\left(\frac{1}{2} + \frac{C_F - C_R}{C_L}\right) \\
&= \frac{C_R - C_F}{C_L} - \frac{1}{2}.
\end{aligned}
$$

There is a turning point at $x = \frac{C_R - C_F}{C_L} - \frac{1}{2} =: P$.
Differentiating again,

$$
f''(x) = \frac{1}{2N}(2A) = \frac{pC_L}{N} > 0 \quad \forall x \in \mathbb{R},
$$

since all variables positive. Thus, $x = P$ is a minimum point for f(x) (from RA) - the point of minimal expected cost.

Thoughts

- p and N cancel from optimal solution (!)
- $f(0) = C_R$, what you get if you replace pipeline without doing any checks
- How to find optimal INTEGER solution?
- what if turning point is not within between 0 and N?
- Cost variables are not just money - they can represent any cost: time cost, monetary cost, risk, etc. (?)

2.6 Desmos

see desmos: `https://www.desmos.com/calculator/busqfklpwq`