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On the convergence of the Newton/log-barrier method

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Abstract. In the Newton/log-barrier method, Newton steps are taken for the log-barrier function for a fixed value of the barrier parameter until a certain convergence criterion is satisfied. The barrier parameter is then decreased and the Newton process is repeated. A naive analysis indicates that Newton's method does not exhibit superlinear convergence to the minimizer of each instance of the log-barrier function until it reaches a very small neighborhood, namely within $O(\mu^2)$ of the minimizer, where μ is the barrier parameter. By analyzing the structure of the barrier Hessian and gradient in terms of the subspace of active constraint gradients and the associated null space, we show that this neighborhood is in fact much larger – $O(\mu^\sigma)$ for any $\sigma \in (1, 2]$ – thus explaining why reasonably fast local convergence can be attained in practice. Moreover, we show that the overall convergence rate of the Newton/log-barrier algorithm is superlinear in the number of function/derivative evaluations, provided that the nonlinear program is formulated with a linear objective and that the schedule for decreasing the barrier parameter is related in a certain way to the step length and convergence criteria for each Newton process.

1. Introduction

We consider the nonlinear programming problem

$$\min f(x) \quad \text{subject to} \quad c(x) \geq 0, \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are smooth (twice Lipschitz continuously differentiable) functions. The logarithmic barrier function for (1) is

$$P(x; \mu) = f(x) - \mu \sum_{i=1}^m \log c_i(x). \quad (2)$$

We denote by $x(\mu)$ a minimizer of $P(\cdot; \mu)$ for $\mu > 0$ and assume that $x(\mu)$ exists for all sufficiently small μ . Methods based on (2) approximate $x(\mu)$ for a sequence of small, decreasing values of $\mu > 0$. Under certain conditions (see Fiacco and McCormick [10]), we have $\lim_{\mu \downarrow 0} x(\mu) = x^*$, where x^* is a local minimizer of (1).

The Newton/log-barrier method proceeds by fixing μ at a certain value and applying Newton's method to the unconstrained problem

$$\min_x P(x; \mu), \quad (3)$$

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stopping the Newton iterations when some tolerance is satisfied, giving an approximation to $x(\mu)$. (Typically, the size of the Newton step or of the gradient $P_x(x; \mu)$ is required to fall below a certain threshold that depends on μ .) The barrier parameter μ is then decreased (to μ_+ , say), and Newton's method is applied to $P(\cdot; \mu_+)$. In some algorithms, a different type of step such as a primal-dual or extrapolation step is used at the first iteration after reduction to μ_+ .

When the final approximation to $x(\mu)$ is used as a starting point for Newton's method applied to $P(\cdot; \mu_+)$, it is well known that the first Newton step usually is a poor step, and that a step length α considerably smaller than 1 usually is needed to remain feasible at this iteration (see Conn, Gould, and Toint [5], M. Wright [26], and S. Wright and Jarre [29]). Often, however, subsequent iterations of Newton's method converge rapidly to the minimizer $x(\mu_+)$. Although the Hessian $P_{xx}(x; \mu)$ is positive definite near the minimizer $x = x(\mu)$ (see the proof of [10, Theorem 12]), the observed rate of convergence of Newton's method is better than we might expect from a naive application of the standard local convergence theory, which implies that the largest Euclidean ball within which quadratic convergence can be expected has radius $O(\mu^2)$. In fact, as we show in Sect. 4, Newton's method with unit steps converges from all points in a Euclidean neighborhood of radius $O(\mu^\sigma)$, for any $\sigma > 1$ and all μ sufficiently small. In a sense, this radius is almost as large as can be expected for *any* minimization algorithm applied to $P(\cdot; \mu)$, because Euclidean balls with radii proportional to smaller powers of μ (μ^1 , for instance) may contain infeasible points for (1), which lie outside the domain of $P(\cdot; \mu)$.

In Sect. 5, we discuss a line-search variant of Newton's method and specify typical conditions that are required of the line-search parameter. We show that when the iterates lie in neighborhoods of the type discussed in Sect. 4, the unit step length satisfies these conditions for all sufficiently small values of μ .

In Sect. 6, we use earlier results of S. Wright and Jarre [29] to show that, when the objective function f is linear and the line search and stopping criteria for the Newton iterations are defined in a certain reasonable way, then the first Newton step taken after each substantial reduction of μ produces an iterate that is within the quadratic convergence domain of Sect. 4. Compiling all our results, we specify a particular line-search Newton/log-barrier method – Algorithm NLB – and show that for particular choices of the parameters, this algorithm converges at an overall superlinear rate.

In Sect. 2, we discuss related work on log-barrier and interior-point methods and put our results and Algorithm NLB into perspective. Section 3 outlines our assumptions and a basic lemma.

We use the following notation in the rest of the paper. For related positive quantities α and β , we say $\beta = O(\alpha)$ if there is a constant M such that $\beta \leq M\alpha$ for all α either sufficiently small or sufficiently large (according to context). We say that $\beta = o(\alpha)$ if $\beta/\alpha \rightarrow 0$ as $\alpha \rightarrow 0$; $\beta = \Omega(\alpha)$ if $\alpha = O(\beta)$, and $\beta = \Theta(\alpha)$ if $\beta = O(\alpha)$ and $\alpha = O(\beta)$. The expression $\beta = O(1)$ means that $\beta \leq M$ for some constant M .

2. Motivation

A good deal of literature exists on variants of the Newton/log-barrier method, starting with the book of Fiacco and McCormick [10]. Our focus in this paper is on variants of the

method in which all the steps are line-search Newton steps for the primal barrier function $P(\cdot; \mu)$, for various values of μ . We show that, provided the problem is formulated with a linear objective function, and provided the schedule for decreasing the barrier parameter is related in a certain way to the step length acceptance criterion and the convergence criterion for Newton's method, there is no need to use primal-dual or extrapolation steps. We are motivated partly by a desire to keep the algorithm simple, freeing it from devices that improve local convergence rate without having an intuitive motivation or relevance at iterates far from the solution. Another reason for our continued investigation of the primal log-barrier approach is that most of the effort in designing and implementing primal-dual algorithms (which appear to have greater practical potential than primal methods) has focused on global convergence issues, and the merit functions that are used to ensure global convergence often are modifications of the purely primal function (2). See, for example, the merit functions proposed by Forsgren and Gill [12], Byrd, Hribar, and Nocedal [3], Gay, Overton, and M. Wright [13], and Conn et al. [4]. A third reason for our interest in primal methods is that they continue to be relevant in the context of convex optimization via self-concordant barrier functions [23], discussed further below.

The approach of using only Newton steps for $P(\cdot; \mu)$ in the algorithm – in particular, of using the approximate minimizer of $P(\cdot; \mu)$ for one value of μ as the starting point for Newton's method at the next value of μ – is consistent with the approach specified in Fiacco and McCormick [10, p. 42] and also with approaches described and analyzed by other authors, notably, M. Wright [24, p. 389], [26, Sect. 2.2] and Conn et al. [4]. Nash and Sofer [22, Sect. 7] indicate that when default parameter settings were used, the algorithm implemented in the SUMT software (produced by Fiacco, McCormick, and their collaborators) is also consistent with the type of method that we analyze (with the exception that the Newton method can be replaced by quasi-Newton or steepest-descent in the local search.)

Methods based on the modern theory of self-concordant barrier functions are also consistent with the approach we analyze here, in that they take only line-search Newton steps for the barrier function, for a monotonic sequence of barrier values. Nesterov and Nemirovskii [23] discuss this approach for various types of convex programming problems, considering both short-step methods (in which each modest reduction in μ is followed by a single unit-length step of Newton's method) and long-step methods (in which more substantial reductions in μ are made, and a number of line-search Newton steps taken for $P(\cdot; \mu)$ for each value of μ).

The literature on self-concordant barrier methods typically assumes a linear objective function (for example, [23, p. 57]), as we do also in Sect. 6 of this paper. A simple reformulation can be used to express any nonlinear program (1) in this form: we can introduce the artificial variable η and the new constraint $\eta - f(x) \geq 0$ and replace the objective in (1) by η . When the objective is linear, the second-order and higher derivatives of the barrier function for the optimization problem do not contain any contribution from the objective and are linear in the barrier parameter. As a result, the self-concordant property for the barrier function becomes independent both of the barrier parameter and of the objective, depending only on the feasible set and possibly its algebraic representation. Our reasons for requiring a linear objective in Sect. 6 are similar, in that by doing so we make the Newton step linear in μ^{-1} and consequently much easier to analyze. S. Wright and Jarre [29] focus on the linear-objective case.

We believe our results in Sect. 6 to be novel in that rapid local convergence of primal barrier algorithms that use only Newton search directions for $P(\cdot; \mu)$ apparently has not been described before, either in the context of nonlinear programming or in convex programming applications of the type considered by Nesterov and Nemirovskii [23]. However, rapid convergence of methods for (1) based on the log-barrier function (2) that take other kinds of steps has been described in a number of papers. In the main, these papers address the poor performance of Newton's method at the step taken immediately after a reduction of the barrier parameter by using some other method to generate this step. Conn, Gould, and Toint [5] advocate taking a primal-dual step, obtained by applying Newton's method for nonlinear equations to the system (9), treating x and λ as independent variables. Gould [15] applies a similar strategy to the quadratic penalty function for equality-constrained optimization. A related approach is to take an approximate extrapolation step along the path of minimizers $\{x(\mu) \mid \mu > 0\}$, so that the first step taken after reducing μ to μ_{k+1} , say, is obtained by fitting a polynomial through the approximate minimizers x_k, x_{k-1}, \dots at the barrier parameter values μ_k, μ_{k-1}, \dots or by computing information about the tangent $\dot{x}(\mu)$ at x_k . Techniques of this class were suggested by Fiacco and McCormick [10]; see also Nash and Sofer [21], Benchakroun, Dussault, and Mansouri [1], and Dussault [9]. The papers [1] and [9] show that "superlinear" reductions in μ , and hence rapid convergence to x^* , can be attained. In fact, the method of Benchakroun, Dussault, and Mansouri [1] generates iterates identical to Algorithm NLB, for a particular choice of the initial step length along the first Newton step taken after each reduction of μ . In a slightly different vein, Fiacco and McCormick [10] and Jittorntrum and Osborne [17], show that extrapolation can also be used to obtain successive estimates of x^* , but their techniques assume that the minimizations of $P(x; \mu)$ are carried out accurately for each value of μ .

A related class of methods, known as analytic center methods, uses Newton's method to find the successive minima of a certain logarithmic potential function that is parametrized by an upper bound on the optimal value of f and possibly by a weighting term. The resulting function is similar (but not identical) to one that would be obtained by applying the linear-objective reformulation discussed above. The convergence of Newton's method in this context has been examined by Nesterov and Nemirovskii [23], Jarre [16], and Den Hertog, Roos, and Terlaky [6], and others. The cited papers consider a special class of problems in which the objective function and constraints are convex and satisfy a certain scaled Lipschitz or self-concordancy condition, whereas our analysis applies locally to general functions under smoothness assumptions that are standard in the local analysis of nonlinear programming methods. To the best of our knowledge, superlinear convergence has not been considered for analytic center methods.

In Sect. 7, we discuss some of these methods and results further and relate them to the results of this paper.

3. Assumptions

In this section, we specify the optimality conditions for the nonlinear program (1), outline our assumptions on its solution x^* , and introduce some notation.

The Lagrangian function for (1) is

$$\mathcal{L}(x, \lambda) = f(x) - \lambda^T c(x), \quad (4)$$

where λ is the vector of Lagrange multipliers. The solution x^* of (1) satisfies the first-order conditions for optimality, which are that there exists a Lagrange multiplier vector λ^* such that

$$c(x^*) \geq 0, \quad \lambda^* \geq 0, \quad (\lambda^*)^T c(x^*) = 0, \quad \nabla f(x^*) = \sum_{i=1}^m \lambda_i^* \nabla c_i(x^*). \quad (5)$$

The active constraints are the components of c for which $c_i(x^*) = 0$. Without loss of generality we assume these to be the first q components of c . We define the Jacobian of c and its submatrix of active columns to be

$$\begin{aligned} A(x) &= [\nabla c_1(x) \mid \cdots \mid \nabla c_m(x)], \\ \hat{A}(x) &= [\nabla c_1(x) \mid \cdots \mid \nabla c_q(x)]. \end{aligned}$$

We also assume that the solution is *nondegenerate*, that is,

$$\hat{A}(x^*) \text{ has rank } q. \quad (6)$$

(Nondegeneracy implies uniqueness of λ^* .) We also assume *strict complementarity*, that is,

$$\lambda_i^* + c_i(x^*) > 0, \quad i = 1, 2, \dots, m. \quad (7)$$

Finally, we assume that second-order sufficient conditions for optimality are satisfied at (x^*, λ^*) , that is,

$$y^T \mathcal{L}_{xx}(x^*, \lambda^*) y > 0 \quad \text{for all } y \neq 0 \text{ such that } \hat{A}(x^*)^T y = 0. \quad (8)$$

It is easy to see that (x^*, λ^*) is a root of the function $F(x, \lambda)$ defined by

$$F(x, \lambda) = \begin{bmatrix} \mathcal{L}_x(x, \lambda) \\ \Lambda c(x) \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A(x)\lambda \\ \Lambda c(x) \end{bmatrix}, \quad (9)$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$. The Jacobian of F is

$$\nabla F(x, \lambda) = \begin{bmatrix} \mathcal{L}_{xx}(x, \lambda) & -A(x) \\ \Lambda A(x)^T & C(x) \end{bmatrix}, \quad (10)$$

where $C(x) = \text{diag}(c_1(x), c_2(x), \dots, c_m(x))$. When the nondegeneracy, strict complementarity, and second-order sufficient conditions hold, $\nabla F(x^*, \lambda^*)$ is nonsingular. It follows from the assumed smoothness of f and c that the Jacobian $\nabla F(x, \lambda)$ is nonsingular for all (x, λ) close to (x^*, λ^*) (see Fiacco and McCormick [10, Theorem 14]).

Given any strictly feasible point x and any positive value of the barrier parameter μ in (2), we define a vector of Lagrange multiplier estimates $\lambda(x, \mu)$ by

$$\lambda(x, \mu) = \mu C(x)^{-1} e = \left[\frac{\mu}{c_1(x)}, \dots, \frac{\mu}{c_m(x)} \right]^T. \quad (11)$$

If x is the exact minimizer $x(\mu)$ of $P(\cdot; \mu)$, we define

$$\lambda(\mu) \triangleq \lambda(x(\mu), \mu). \quad (12)$$

For future reference, we note that the derivatives of the barrier function (2) are

$$P_x(x; \mu) = \nabla f(x) - \sum_{i=1}^m \frac{\mu}{c_i(x)} \nabla c_i(x), \quad (13a)$$

$$P_{xx}(x; \mu) = \nabla^2 f(x) + \mu \sum_{i=1}^m \left[\frac{1}{c_i^2(x)} \nabla c_i(x) \nabla c_i(x)^T - \frac{1}{c_i(x)} \nabla^2 c_i(x) \right]. \quad (13b)$$

4. Convergence of Newton's method with unit steps to the log-barrier minimizer

We now analyze the local convergence properties of Newton's method with unit steps to the minimizer $x(\mu)$ of the barrier function $P(x; \mu)$, for a fixed value of μ . From the current iterate w , the Newton step s is

$$s = -P_{xx}(w; \mu)^{-1} P_x(w; \mu), \quad (14)$$

and the next iterate is $w_+ = w + s$. In this section, we identify a Euclidean ball around the minimizer of $P(\cdot; \mu)$ such that when the unit-step Newton method is started from any point in this ball, it converges rapidly to the minimizer.

It is well known that under the second-order assumptions discussed above, and for μ sufficiently small, the barrier function $P(x; \mu)$ has a minimizer $x(\mu)$ at which the Hessian is positive definite, though ill conditioned. Moreover, since the objective $f(\cdot)$ and constraint functions $c_i(\cdot)$, $i = 1, 2, \dots, m$ are twice Lipschitz continuously differentiable, then $P(x; \mu)$ is also twice Lipschitz continuously differentiable near $x(\mu)$. Hence, quadratic convergence follows from a standard result (see, for example, Theorem 5.2.1 of Dennis and Schnabel [8]). If w_1, w_2, w_3, \dots are the Newton iterates, the standard theory yields the estimate

$$\|w_{t+1} - x(\mu)\| \leq L_1(\mu) L_2(\mu) \|w_t - x(\mu)\|^2, \quad (15)$$

where $L_1(\mu)$ is a Lipschitz constant for $P_{xx}(x; \mu)$ in the vicinity of $x(\mu)$ and $L_2(\mu)$ is a bound on $\|P_{xx}(x; \mu)^{-1}\|$ near $x(\mu)$. In fact, we have that

$$L_1(\mu) = O(\mu^{-2}), \quad L_2(\mu) = O(1). \quad (16)$$

(The estimate of L_2 is a consequence of the analysis in Murray [19] and the second-order sufficient conditions (8).) Therefore, (15) reduces to

$$\|w_{t+1} - x(\mu)\| = O(\mu^{-2}) \|w_t - x(\mu)\|^2.$$

This expression does not even imply convergence of the iteration sequence unless w_1 is in a very small neighborhood of the solution, specifically,

$$\|w_1 - x(\mu)\| = O(\mu^2). \quad (17)$$

S. Wright and Jarre [29] investigated the use of a reformulation of (1) in which the objective function is linear. They show that if the final approximation to $x(\mu)$ is reasonably accurate, then the first Newton step for P for the new, reduced value of μ passes quite close to the new minimizer. Even in this case, however, the point resulting from the Newton step will not usually lie in the small neighborhood (17), even if we use a line search to restrict the step length.

In this section, we show that the expressions (15) and (17) are unduly pessimistic, and that given any $\sigma \in (1, 2]$, there are constants $\bar{\mu}_0 > 0$ and $C_0 > 0$ such that quadratic convergence of Newton's method with unit steps can be obtained from any point w that satisfies

$$\|w - x(\mu)\| \leq C_0 \mu^\sigma, \quad \text{for all } \mu \in (0, \bar{\mu}_0]. \quad (18)$$

That is, the domain of quadratic convergence for $P(\cdot; \mu)$ shrinks as $\mu \downarrow 0$, but the rate of shrinkage is not especially severe. Key to our analysis is a partitioning of the space \mathbb{R}^n into the range space of the active constraint Jacobian and its complement. A decomposition of this type has been used previously (for example, by M. Wright [25–27]) to analyze the properties of the gradient and Hessian of $P(\cdot; \mu)$; we mention some specific connections below.

We assume a priori that w lies in the neighborhood

$$\|w - x(\mu)\| \leq C \mu^\sigma, \quad (19)$$

where $C > 0$ and $\sigma > 1$ are given constants. We assume too that

$$\mu \in (0, \bar{\mu}], \quad (20)$$

where in the analysis below, $\bar{\mu} > 0$ is chosen sufficiently small to make some of our basic results hold. We show later in the section how C_0 and $\bar{\mu}_0$ in (18) depend on C and $\bar{\mu}$.

The analysis is based on the following consequence of Taylor's theorem. If w is the current iterate and s is the Newton step, as in (14), we have that

$$\begin{aligned} P_x(w + s; \mu) &= P_x(w; \mu) + P_{xx}(w; \mu)s + \int_0^1 [P_{xx}(w + \tau s; \mu) - P_{xx}(w; \mu)] s \, d\tau \\ &= \int_0^1 [P_{xx}(w + \tau s; \mu) - P_{xx}(w; \mu)] s \, d\tau. \end{aligned} \quad (21)$$

By analyzing the integral term, we obtain estimates for the projection of $P_x(w + s; \mu)$ into two complementary subspaces induced by the Jacobian of the active constraint matrix. We then investigate the two-sided projection of $P_{xx}(w + s; \mu)$ in these same subspaces to obtain a bound on the size of the next Newton step – the one taken from the point $w + s$.

We assume without loss of generality that $\bar{\mu}$ is small enough that the neighborhood (19) excludes local minimizers of $P(\cdot; \mu)$ other than $x(\mu)$. Since $\sigma > 1$ in (19), it follows that the ratio $\|w - x(\mu)\|/\mu$ approaches zero as $\mu \downarrow 0$.

It follows immediately from the existence and boundedness of $(\dot{x}(\mu), \dot{\lambda}(\mu))$ for all sufficiently small μ (see Fiacco and McCormick [10, Theorem 15]) that

$$\left\| \begin{bmatrix} x(\mu) - x^* \\ \lambda(\mu) - \lambda^* \end{bmatrix} \right\| = O(\mu). \quad (22)$$

Hence, by using (11), (12), the strict complementarity assumption, and (19), we have for all active indices $i = 1, 2, \dots, q$ that

$$\begin{aligned} c_i(w) &= c_i(x(\mu)) + O(\|w - x(\mu)\|) \\ &= \frac{\mu}{\lambda_i(\mu)} + O(\mu^\sigma) = \frac{\mu}{\lambda_i^*} + O(\mu^{\min(2, \sigma)}) = \Theta(\mu). \quad i = 1, 2, \dots, q, \end{aligned} \quad (23)$$

for all μ sufficiently small.

Using this observation, we examine the relevant properties of $P_{xx}(w; \mu)$. The nature of the ill conditioning in this matrix at minimizing points $w = x(\mu)$ was examined by Lootsma [18] and Murray [19]. M. Wright [25] further examined the eigenstructure of this matrix in a strictly feasible neighborhood of the solution x^* , for small μ , and essentially showed in [25, Lemmas 3.1, 3.2, Theorem 3.1] that this matrix has q eigenvalues of size $\Theta(\mu^{-1})$ and $n - q$ eigenvalues of size $O(1)$. Drawing on Murray [19], M. Wright [25, Theorem 3.3] shows further that the $n - q$ small eigenvalues are of size $\Theta(1)$ at $x = x(\mu)$. We now show that similar eigenvalue estimates hold on the neighborhood (19), and we derive explicit expressions for the Hessian and its inverse based on the two-sided projection induced by the active constraint Jacobian $[\nabla c_i(w)]_{i=1}^q$.

From (13b), we obtain by partitioning the sums into active and inactive indices that

$$\begin{aligned} P_{xx}(w; \mu) &= \sum_{i=1}^q \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T + \left\{ \nabla^2 f(w) - \sum_{i=1}^q \frac{\mu}{c_i(w)} \nabla^2 c_i(w) \right\} \\ &\quad + \sum_{i=q+1}^m \left\{ \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T - \frac{\mu}{c_i(w)} \nabla^2 c_i(w) \right\}. \end{aligned} \quad (24)$$

We deal with the three terms on the right-hand side in turn. Because of (23), we have

$$\frac{\mu}{c_i^2(w)} = \Theta(\mu^{-1}), \quad i = 1, 2, \dots, q,$$

so by the nondegeneracy assumption, the first sum in (24) is a rank- q matrix, whose q nonzero eigenvalues are all positive with size $\Theta(\mu^{-1})$. (This claim follows from [25, Lemma 3.1].) Using a standard factorization, we can write

$$\sum_{i=1}^q \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T = \hat{U}(w) \hat{D}(w) \hat{U}(w)^T, \quad (25)$$

where $\hat{D}(w)$ is a $q \times q$ diagonal matrix whose diagonal elements all have size $\Theta(\mu^{-1})$ and $\hat{U}(w)$ is an $n \times q$ orthonormal matrix whose columns span the range space of $[\nabla c_i(w)]_{i=1}^q$.

It follows from (23) that $\mu/c_i(w) \approx \lambda_i^*$, so the second term in (24) is a small perturbation of the Lagrangian Hessian $\mathcal{L}_{xx}(x^*, \lambda^*)$, which by our second-order assumption (8) is positive definite on the null space of $\hat{A}(x^*)^T$. Hence, if we define $\tilde{U}(w)$ to be an $n \times (n - q)$ orthonormal matrix that spans the nearby null space of $\hat{A}(w)^T$ (so that $[\hat{U}(w) \mid \tilde{U}(w)]$ is orthogonal), straightforward arguments show that the $(n - q) \times (n - q)$ matrix $G_{22}(w)$ defined by

$$\begin{aligned} \tilde{G}_{22}(w) &\triangleq \tilde{U}(w)^T \left\{ \nabla^2 f(w) - \sum_{i=1}^q \frac{\mu}{c_i(w)} \nabla^2 c_i(w) \right\} \tilde{U}(w) \\ &= \tilde{U}(w)^T \mathcal{L}_{xx}(x^*, \lambda^*) \tilde{U}(w) + o(1) \end{aligned} \quad (26)$$

is positive definite, with all eigenvalues of size $\Theta(1)$.

Since $c_i(w) = \Omega(1)$ for $i = q + 1, \dots, m$ and all μ sufficiently small, the third term in (24) is $O(\mu)$. We see therefore that $P_{xx}(w; \mu)$ is an $O(1)$ perturbation of the matrix in (25), so that $P_{xx}(w; \mu)$ has q eigenvalues of size $\Theta(\mu^{-1})$ and $n - q$ eigenvalues of size $O(1)$. We now go on to show with the aid of a technical result that in fact the small eigenvalues are all positive with size $\Theta(1)$, when w lies in the neighborhood (19).

By combining all these observations about the three terms in (24), we find that

$$P_{xx}(w; \mu) = \begin{bmatrix} \hat{U}(w) & \tilde{U}(w) \end{bmatrix} \begin{bmatrix} G_{11}(w) & G_{12}(w) \\ G_{12}^T(w) & G_{22}(w) \end{bmatrix} \begin{bmatrix} \hat{U}(w)^T \\ \tilde{U}(w)^T \end{bmatrix}, \quad (27)$$

where

$$G_{11}(w) \triangleq \hat{U}(w)^T P_{xx}(w; \mu) \hat{U}(w) = \hat{D}(w) + O(1), \quad (28a)$$

$$G_{22}(w) \triangleq \tilde{U}(w)^T P_{xx}(w; \mu) \tilde{U}(w) = \tilde{G}_{22}(w) + O(\mu), \quad (28b)$$

$$G_{12}(w) \triangleq \hat{U}(w)^T P_{xx}(w; \mu) \tilde{U}(w) = O(1), \quad (28c)$$

where all the matrices in these expressions are defined in (25), (26), and the discussion above.

It follows immediately from (28) that

$$\|P_{xx}(w; \mu)\| = O(\mu^{-1}). \quad (29)$$

To uncover the properties of the inverse $P_{xx}(w; \mu)^{-1}$, we use the following technical result about the inverse of a block 2×2 matrix.

Lemma 1. *Let G be a symmetric matrix partitioned as*

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix},$$

where G_{11} and G_{22} are square. Suppose that G_{11} and $G_{22} - G_{12}^T G_{11}^{-1} G_{12}$ are nonsingular. Then G is nonsingular and G^{-1} has the form

$$G^{-1} = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}, \quad (30)$$

where

$$\begin{aligned} H_{11} &= G_{11}^{-1} + G_{11}^{-1} G_{12} (G_{22} - G_{12}^T G_{11}^{-1} G_{12})^{-1} G_{12}^T G_{11}^{-1} \\ H_{12} &= -G_{11}^{-1} G_{12} (G_{22} - G_{12}^T G_{11}^{-1} G_{12})^{-1} \\ H_{22} &= (G_{22} - G_{12}^T G_{11}^{-1} G_{12})^{-1}. \end{aligned}$$

In our case – the 2×2 block matrix in (28) – we have from the fact that the diagonals of $\hat{D}(w)$ have size $\Theta(\mu^{-1})$ that

$$G_{11}(w)^{-1} = \hat{D}(w)^{-1} (I + O(\mu))^{-1} = \hat{D}(w)^{-1} + O(\mu^2) = O(\mu).$$

By using this estimate, we further obtain from (28) and (26) that

$$\begin{aligned} H_{22}(w) &= [G_{22}(w) - G_{12}(w)^T G_{11}(w)^{-1} G_{12}(w)]^{-1} \\ &= [\tilde{G}_{22}(w) + O(\mu)]^{-1} = \tilde{G}_{22}(w)^{-1} + O(\mu) = O(1) \end{aligned} \quad (31)$$

and that all the $m - q$ eigenvalues of this matrix are positive with size $\Theta(1)$. Hence, from (27) and (30), we obtain

$$P_{xx}(w; \mu)^{-1} = \begin{bmatrix} \hat{U}(w) & \tilde{U}(w) \end{bmatrix} \begin{bmatrix} H_{11}(w) & H_{12}(w) \\ H_{12}^T(w) & H_{22}(w) \end{bmatrix} \begin{bmatrix} \hat{U}(w)^T \\ \tilde{U}(w)^T \end{bmatrix}, \quad (32)$$

where

$$H_{11}(w) = O(\mu), \quad H_{12}(w) = O(\mu), \quad H_{22}(w) = O(1). \quad (33)$$

From (31) and (33), we have that for sufficiently small μ , the largest $n - q$ eigenvalues of the matrix $H(w)$ are within $O(\mu)$ of the $n - q$ eigenvalues of $\tilde{G}_{22}(w)^{-1}$, which by (26) are all of size $\Theta(1)$ and positive. Hence the smallest $n - q$ eigenvalues of $P_{xx}(w; \mu)$ are all of size $\Theta(1)$ and positive.

We summarize the analysis of the paragraphs above as a lemma.

Lemma 2. *Let the constants $C > 0$ and $\sigma > 1$ in (19) be given. Then there is a $\bar{\mu} > 0$ such that for all w and μ satisfying (19) and (20), the Hessian $P_{xx}(w; \mu)$ is positive definite, and this matrix and its inverse have the structure shown in (27), (28) and (32), (33), respectively.*

We now examine the properties of $P_x(w; \mu)$. From (19) and (23) we have (by decreasing $\bar{\mu}$ if necessary) that

$$\|w - x(\mu)\|/c_i(w) = O(\mu^{\sigma-1}) \ll 1, \quad i = 1, 2, \dots, q, \quad (34)$$

for all μ satisfying (20). We can use this expression to estimate the difference between the reciprocals $c_i^{-1}(w)$ and $c_i^{-1}(x(\mu))$. We have

$$\begin{aligned} c_i^{-1}(x(\mu)) &= [c_i(w) + O(\|w - x(\mu)\|)]^{-1} \\ &= c_i^{-1}(w) [1 + O(c_i(w)^{-1} \|w - x(\mu)\|)]^{-1} \\ &= c_i^{-1}(w) + c_i^{-2}(w) O(\|w - x(\mu)\|), \quad i = 1, 2, \dots, q. \end{aligned} \quad (35)$$

From (13a), noting that $P_x(x(\mu); \mu) = 0$ and partitioning active and inactive indices, we have that

$$\begin{aligned} P_x(w; \mu) &= P_x(w; \mu) - P_x(x(\mu); \mu) \\ &= - \sum_{i=1}^q \left[\frac{\mu}{c_i(w)} \nabla c_i(w) - \frac{\mu}{c_i(x(\mu))} \nabla c_i(x(\mu)) \right] \\ &\quad + \nabla f(w) - \nabla f(x(\mu)) \\ &\quad - \sum_{i=q+1}^m \left[\frac{\mu}{c_i(w)} \nabla c_i(w) - \frac{\mu}{c_i(x(\mu))} \nabla c_i(x(\mu)) \right]. \end{aligned} \quad (36)$$

For the second term on the right-hand side of (36), we have by smoothness of f that $\nabla f(w) - \nabla f(x(\mu)) = O(\|w - x(\mu)\|)$. In the third term, we have for each index $i = q + 1, \dots, m$ that $c_i(w) = \Omega(1)$ and $c_i(x(\mu)) = \Omega(1)$. Hence, by smoothness of ∇c_i , this term has size $O(\mu \|w - x(\mu)\|)$.

In the first term, we have for each active index $i = 1, 2, \dots, q$, using the smoothness of ∇c_i together with (23), (34), and (35), that

$$\begin{aligned} &\frac{\mu}{c_i(x(\mu))} \nabla c_i(x(\mu)) \\ &= \left[\frac{\mu}{c_i(w)} + \frac{\mu}{c_i^2(w)} O(\|w - x(\mu)\|) \right] [\nabla c_i(w) + O(\|w - x(\mu)\|)] \\ &= \frac{\mu}{c_i(w)} \nabla c_i(w) + O(\mu^{-1} \|w - x(\mu)\|) \nabla c_i(w) + O(\|w - x(\mu)\|). \end{aligned} \quad (37)$$

By collecting these observations and substituting into (36), we obtain

$$P_x(w; \mu) = \sum_{i=1}^q O(\mu^{-1} \|w - x(\mu)\|) \nabla c_i(w) + O(\|w - x(\mu)\|). \quad (38)$$

From the definitions of $\hat{U}(w)$ and $\tilde{U}(w)$, it follows immediately that

$$\hat{U}(w)^T P_x(w; \mu) = O(\mu^{-1} \|w - x(\mu)\|), \quad (39a)$$

$$\tilde{U}(w)^T P_x(w; \mu) = O(\|w - x(\mu)\|). \quad (39b)$$

Meanwhile from (32) and (33), we have for the Newton step from w that

$$\begin{aligned} s &= -P_{xx}(w; \mu)^{-1} P_x(w; \mu) \\ &= [\hat{U}(w) \ \tilde{U}(w)] \begin{bmatrix} H_{11}(w) & H_{12}(w) \\ H_{12}^T(w) & H_{22}(w) \end{bmatrix} \begin{bmatrix} \hat{U}(w)^T P_x(w; \mu) \\ \tilde{U}(w)^T P_x(w; \mu) \end{bmatrix} \\ &= [\hat{U}(w) \ \tilde{U}(w)] \begin{bmatrix} O(\mu) & O(\mu) \\ O(\mu) & O(1) \end{bmatrix} \begin{bmatrix} \hat{U}(w)^T P_x(w; \mu) \\ \tilde{U}(w)^T P_x(w; \mu) \end{bmatrix}. \end{aligned}$$

Hence, by combining with (39), we find that we can choose $\bar{\mu}$ in (20) such that for all w and μ satisfying (19) and (20), we have

$$\begin{aligned} \|s\| &\leq O(\mu \|\hat{U}(w)^T P_x(w; \mu)\| + \|\tilde{U}(w)^T P_x(w; \mu)\|) \\ &\leq C_3 \left[\mu \|\hat{U}(w)^T P_x(w; \mu)\| + \|\tilde{U}(w)^T P_x(w; \mu)\| \right] \end{aligned} \quad (40a)$$

$$\leq C_1 \|w - x(\mu)\| = O(\mu^\sigma), \quad (40b)$$

for some positive numbers C_1 and C_3 independent of μ .

Note that the naive estimate of $\|s\|$ obtained by ignoring the structure of $P_x(w; \mu)$ and $P_{xx}(w; \mu)$ would be simply $\|s\| \leq \|P_{xx}(w; \mu)^{-1}\| \|P_x(w; \mu)\| = O(\mu^{\sigma-1})$, which is too pessimistic for our purposes.

We now examine the integrand in (21), partitioning it into the subspaces defined by the active constraint gradients at the next Newton iterate $w + s$. We start by writing

$$[P_{xx}(w + \tau s; \mu) - P_{xx}(w; \mu)]s = r_1 + r_2 + r_3 + r_4, \quad (41)$$

where $\tau \in [0, 1]$ and

$$\begin{aligned} r_1 &= \sum_{i=1}^q \left\{ \frac{\mu}{c_i^2(w + \tau s)} \nabla c_i(w + \tau s) \nabla c_i(w + \tau s)^T s - \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T s \right\} \\ r_2 &= - \sum_{i=1}^q \left\{ \frac{\mu}{c_i(w + \tau s)} \nabla^2 c_i(w + \tau s) s - \frac{\mu}{c_i(w)} \nabla^2 c_i(w) s \right\} \\ r_3 &= \left[\nabla^2 f(w + \tau s) - \nabla^2 f(w) \right] s, \\ r_4 &= \sum_{i=q+1}^m \left\{ \frac{\mu}{c_i^2(w + \tau s)} \nabla c_i(w + \tau s) \nabla c_i(w + \tau s)^T s - \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T s \right. \\ &\quad \left. - \frac{\mu}{c_i(w + \tau s)} \nabla^2 c_i(w + \tau s) s + \frac{\mu}{c_i(w)} \nabla^2 c_i(w) s \right\}. \end{aligned}$$

To estimate the first term r_1 , we note from (40b) that $\mu^{-1} \|s\| = O(\mu^{\sigma-1}) \ll 1$. Hence, as in (35), we have

$$\begin{aligned} \frac{1}{c_i(w + \tau s)} &= \frac{1}{c_i(w)} + \frac{1}{c_i^2(w)} O(\|\tau s\|) \\ &= \frac{1}{c_i(w)} + O(\mu^{-2} \|s\|), \quad i = 1, 2, \dots, q, \end{aligned} \quad (42a)$$

$$\begin{aligned} \frac{1}{c_i^2(w + \tau s)} &= \frac{1}{c_i^2(w)} \left[1 + \frac{1}{c_i(w)} O(\|\tau s\|) \right]^{-2} \\ &= \frac{1}{c_i^2(w)} + O(\mu^{-3} \|s\|), \quad i = 1, 2, \dots, q. \end{aligned} \quad (42b)$$

By smoothness of ∇c_i , we obtain for $i = 1, 2, \dots, q$ that

$$\begin{aligned} & \frac{\mu}{c_i^2(w + \tau s)} \nabla c_i(w + \tau s) \nabla c_i(w + \tau s)^T s \\ &= \left[\frac{\mu}{c_i^2(w)} + O(\mu^{-2} \|s\|) \right] [\nabla c_i(w) + O(\|s\|)] [\nabla c_i(w)^T s + O(\|s\|^2)] \\ &= \frac{\mu}{c_i^2(w)} \nabla c_i(w) \nabla c_i(w)^T s + O(\mu^{-2} \|s\| \|\hat{U}(w)^T s\| \nabla c_i(w) + O(\mu^{-1} \|s\|^2)), \end{aligned}$$

where we have used $|\nabla c_i(w)^T s| = O(\|\hat{U}(w)^T s\|)$, for $i = 1, 2, \dots, q$ (which follows from the definition of $\hat{U}(w)$ in (25)) and the fact that $\mu^{-1} \|s\| \ll 1$ (see (40b)). By substituting into the definition of r_1 , we obtain

$$r_1 = \sum_{i=1}^q O(\mu^{-2} \|s\| \|\hat{U}(w)^T s\| \nabla c_i(w) + O(\mu^{-1} \|s\|^2)) \quad (43a)$$

$$= \sum_{i=1}^q O(\mu^{-2} \|s\|^2 \nabla c_i(w + s) + O(\mu^{-1} \|s\|^2)), \quad (43b)$$

where we have used $\|\hat{U}(w)^T s\| \leq \|s\|$, noting that the change of argument from w to $w + s$ in the first term causes a perturbation that can be absorbed in the second term. (A minor variant of this analysis shows that the Lipschitz constant for $P_{xx}(\cdot; \mu)$ is $O(\mu^{-2})$, as claimed in (16).)

For the second term r_2 in (41), we have from (42a) that

$$\begin{aligned} \frac{\mu}{c_i(w + \tau s)} \nabla^2 c_i(w + \tau s) s &= \left[\frac{\mu}{c_i(w)} + O(\mu^{-1} \|s\|) \right] [\nabla^2 c_i(w) s + O(\|s\|^2)] \\ &= \frac{\mu}{c_i(w)} \nabla^2 c_i(w) s + O(\mu^{-1} \|s\|^2), \quad i = 1, 2, \dots, q. \end{aligned} \quad (44)$$

The remaining terms r_3 and r_4 are less significant. By Lipschitz continuity of ∇f , we have $r_3 = O(\|s\|^2)$. In r_4 , the denominators all have size $\Omega(1)$, so it is easy to show that $r_4 = O(\mu \|s\|^2)$.

By collecting all these estimates into (41), performing the integration, and substituting into (21), we obtain

$$P_x(w + s; \mu) = \sum_{i=1}^q O(\mu^{-2} \|s\| \|\hat{U}(w)^T s\| \nabla c_i(w) + O(\mu^{-1} \|s\|^2)) \quad (45a)$$

$$= \sum_{i=1}^q O(\mu^{-2} \|s\|^2 \nabla c_i(w + s) + O(\mu^{-1} \|s\|^2)). \quad (45b)$$

Hence, after a possible adjustment in $\bar{\mu}$ in (20), we have from (45b) that there is a positive number C_2 independent of μ (assumed without loss of generality to satisfy $C_2 \geq 1$) such that

$$\hat{U}(w + s)^T P_x(w + s; \mu) \leq C_2 \mu^{-2} \|s\|^2, \quad (46a)$$

$$\tilde{U}(w + s)^T P_x(w + s; \mu) \leq C_2 \mu^{-1} \|s\|^2, \quad (46b)$$

for any orthonormal matrix $\hat{U}(w + s)$ that spans the column space of $\hat{A}(w + s)$ and for any orthonormal matrix $\tilde{U}(w + s)$ such that $[\hat{U}(w + s) \mid \tilde{U}(w + s)]$ is orthogonal.

We summarize the observations above in the following result.

Lemma 3. *Let the constants $C > 0$ and $\sigma > 1$ in (19) be given. Then there exist a $\bar{\mu} > 0$ and constants C_1, C_2, C_3 such that for all w and μ satisfying (19) and (20), the relationships (40) and (46) are satisfied, where s is the Newton step (14).*

An important consequence of this lemma is that if the next Newton iterate $w + s$ also lies in the neighborhood (19), then we have from (40a) and (46) that the Newton step s_+ calculated from $w + s$ satisfies

$$\|s_+\| \leq 2C_2C_3\mu^{-1}\|s\|^2. \quad (47)$$

We now use all these estimates to show that if we choose the starting point w_1 for the Newton iteration in a slightly more restrictive neighborhood than (19), then *all* Newton iterates will remain inside the full neighborhood (19), and quadratic convergence of the Newton sequence to $x(\mu)$ will be observed. We state the result formally as a theorem.

Theorem 1. *Let the constants $C > 0$ and $\sigma > 1$ be given, and let C_1, C_2, C_3 , and $\bar{\mu}$ be defined as in Lemma 3. Let the constants $C_0 > 0$ and $\bar{\mu}_0$ be chosen in such a way that the following inequalities are satisfied:*

$$(1 + 2C_1)C_0 \leq C, \quad 2C_0C_1C_2C_3\bar{\mu}_0^{\sigma-1} \leq 1/4. \quad (48)$$

Then if $\mu \in (0, \bar{\mu}_0]$ and w_1 is any point that satisfies

$$\|w_1 - x(\mu)\| \leq C_0\mu^\sigma, \quad (49)$$

then Newton's method with unit steps, applied to the function $P(\cdot; \mu)$ and starting from w_1 , generates a sequence of steps $\{s_t\}_{t=1,2,3,\dots}$ and iterates $\{w_t\}_{t=1,2,3,\dots}$ converging to $x(\mu)$, such that

$$\|w_{t+1} - x(\mu)\| \leq 6C_2C_3\mu^{-1}\|w_t - x(\mu)\|^2, \quad t = 1, 2, 3, \dots \quad (50)$$

Therefore the convergence to $x(\mu)$ is Q -quadratic.

Proof. By (40b), we have that the first Newton step s_1 satisfies

$$\|s_1\| \leq C_1\|w_1 - x(\mu)\|, \quad (51)$$

and so, because of the definition of C_0 in (48), the next iterate $w_2 = w_1 + s_1$ satisfies

$$\begin{aligned} \|w_2 - x(\mu)\| &\leq \|w_1 - x(\mu)\| + \|s_1\| \leq (1 + C_1)\|w_1 - x(\mu)\| \\ &\leq C_0(1 + C_1)\mu^\sigma < C\mu^\sigma. \end{aligned}$$

Hence, w_2 also lies in the neighborhood (19), so we can apply (47) to obtain the following estimate for the next Newton step s_2 :

$$\|s_2\| \leq 2C_2C_3\mu^{-1}\|s_1\|^2. \quad (52)$$

From (49) and (51), we have that

$$\mu^{-1} \|s_1\| \leq C_0 C_1 \mu^{\sigma-1},$$

so by substituting (52) and using the definition (48), we obtain

$$\|s_2\| \leq 2C_0 C_1 C_2 C_3 \mu^{\sigma-1} \|s_1\| \leq (1/4) \|s_1\|.$$

Hence, for the next Newton iterate $w_3 = w_2 + s_2$, we have

$$\begin{aligned} \|w_3 - x(\mu)\| &\leq \|w_1 - x(\mu)\| + \|s_1\| + \|s_2\| \\ &\leq \|w_1 - x(\mu)\| + (5/4) \|s_1\| \leq C_0(1 + (5/4)C_1)\mu^\sigma < C\mu^\sigma, \end{aligned}$$

so that w_3 also lies in the neighborhood defined by (19).

The argument continues inductively. We find in general that for all $t = 1, 2, 3, \dots$, we have that

$$\|s_{t+1}\| \leq 2C_2 C_3 \mu^{-1} \|s_t\|^2 \leq (2C_2 C_3 \mu^{-1} \|s_1\|) \|s_t\| \leq (1/4) \|s_t\| \quad (53)$$

and that

$$\begin{aligned} \|w_{t+1} - x(\mu)\| &\leq \|w_1 - x(\mu)\| + \sum_{j=1}^t \|s_j\| \\ &\leq \|w_1 - x(\mu)\| + \sum_{j=1}^t 4^{-(j-1)} \|s_1\| \\ &\leq C_0(1 + (4/3)C_1)\mu^\sigma < C\mu^\sigma, \end{aligned}$$

so that all Newton iterates w_1, w_2, w_3, \dots belong to the neighborhood (19).

From (53), we have that $\|s_t\|$, $t = 1, 2, \dots$ decreases geometrically (in fact, quadratically) to zero. Therefore, $\{w_t\}$ is a Cauchy sequence, so it converges, say to a point $w_*(\mu)$. It follows from (46) that this limit point must satisfy

$$P_x(w_*(\mu); \mu) = 0.$$

Moreover, by the second-order condition (8), we have by the choice of $\bar{\mu}$ and the discussion about the Hessian $P_{xx}(\cdot; \mu)$ and its inverse that $P_{xx}(w; \mu)$ is positive definite for all w satisfying (19). Hence, $w_*(\mu)$ is a local minimizer of $P(\cdot; \mu)$. Since $x(\mu)$ is the only local minimizer of this function in the neighborhood (19) by assumption, we must have $w_*(\mu) = x(\mu)$.

To prove that the convergence of $\{w_t\}$ to $x(\mu)$ is quadratic, we estimate the error $\|w_t - x(\mu)\|$ in terms of $\|s_t\|$. By using (53), we have for all $t = 1, 2, 3, \dots$ that

$$\|w_t - x(\mu)\| = \left\| \sum_{j=t}^{\infty} s_j \right\| \leq \sum_{j=t}^{\infty} \|s_j\| \leq \sum_{j=t}^{\infty} 4^{-(j-1)} \|s_t\| \leq (4/3) \|s_t\|.$$

Similarly, we have that

$$\|w_t - x(\mu)\| \geq \|s_t\| - \sum_{j=t+1}^{\infty} \|s_j\| \geq (2/3)\|s_t\|.$$

Hence, from (47), we have

$$\begin{aligned} & \|w_{t+1} - x(\mu)\| \\ & \leq (4/3)\|s_{t+1}\| \leq (8/3)C_2C_3\mu^{-1}\|s_t\|^2 \leq 6C_2C_3\mu^{-1}\|w_t - x(\mu)\|^2, \end{aligned}$$

completing the proof. \square

5. Consistency with a Newton line-search strategy

Theorem 1 describes the classic locally convergent version of Newton's method, in which the unit step is taken at each iteration. Practical versions of Newton's method use a line search or trust region to ensure convergence to a stationary point or a local minimizer. In this section, we consider a line search strategy. Under standard conditions for acceptance of the step length α , and assuming that the assumptions of Theorem 1 are satisfied, we show that $\alpha = 1$ is an acceptable step length, for all μ sufficiently small. In fact, the Newton/log-barrier algorithm to be described in the next section will use a modification of the line search conditions below, but they reduce to these conditions within the neighborhood (19).

The line search conditions on α are as follows:

$$P(w + \alpha s; \mu) \leq P(w; \mu) + \underline{\gamma}\alpha s^T P_x(w; \mu) \quad (54a)$$

$$|s^T P_x(w + \alpha s; \mu)| \leq \overline{\gamma} s^T P_x(w; \mu), \quad (54b)$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are parameters that satisfy

$$0 < \underline{\gamma} < 1/2, \quad \underline{\gamma} < \overline{\gamma} < 1. \quad (55)$$

These conditions, which ensure that α is an approximate local minimizer of the function $P(\cdot; \mu)$ along the direction s , can be found in Fletcher [11, (2.5.1), (2.5.6)], who shows that there exists a range of acceptable points for which they are satisfied. In our present context, we assume too that $w + \alpha s$ is strictly feasible for the problem (1), since the function $P(\cdot; \mu)$ is not defined otherwise.

A number of general techniques are available for choosing α to satisfy conditions (54); see, for example, Fletcher [11, Sect. 2.6] and the references therein. A technique that is specialized to the characteristics of the log-barrier function is described by Murray and Wright [20].

In accordance with standard practice, we accept the step $\alpha = 1$ whenever this choice satisfies the conditions (54). We now verify that such is indeed the case for all μ sufficiently small, whenever w satisfies (19).

We verify first condition (54b). From (45a), (40b), and using $\nabla c_i(w)^T s = O(\|\hat{U}(w)^T s\|)$ for $i = 1, 2, \dots, q$, we have that

$$\begin{aligned}
s^T P_x(w + s; \mu) &= \sum_{i=1}^q O(\mu^{-2} \|s\| \|\hat{U}(w)^T s\| \nabla c_i(w)^T s + O(\mu^{-1} \|s\|^3)) \\
&= O(\mu^{-2} \|s\| \|\hat{U}(w)^T s\|^2) + O(\mu^{-1} \|s\|^3) \\
&= O(\mu^{-1} \|s\|) \left[\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2 \right] \\
&= O(\mu^{\sigma-1}) \left[\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2 \right]. \tag{56}
\end{aligned}$$

For the directional derivative at $\alpha = 0$, we use (14), (27), (28), and the properties of $\hat{D}(w)$ and $G_{22}(w)$ to deduce that there exist constants $C_5 > 0$ and $C_6 \geq 0$ such that the following chain of inequalities holds, for all μ sufficiently small:

$$\begin{aligned}
&-s^T P_x(w; s) \\
&= s^T P_{xx}(w; \mu) s \\
&= [\hat{U}(w)^T s]^T G_{11}(w) [\hat{U}(w)^T s] + 2[\hat{U}(w)^T s]^T G_{12}(w) [\tilde{U}(w)^T s] \\
&\quad + [\tilde{U}(w)^T s]^T G_{22}(w) [\tilde{U}(w)^T s] \\
&\geq C_5 \mu^{-1} \|\hat{U}(w)^T s\|^2 - 2C_6 \|\hat{U}(w)^T s\| \|\tilde{U}(w)^T s\| + C_5 \|\tilde{U}(w)^T s\|^2 \\
&= (C_5/2) \mu^{-1} \|\hat{U}(w)^T s\|^2 + (C_5 - 2\mu C_6^2/C_5) \|\tilde{U}(w)^T s\|^2 \\
&\quad + [(C_5/(2\mu))^{1/2} \|\hat{U}(w)^T s\| - (2\mu/C_5)^{1/2} C_6 \|\tilde{U}(w)^T s\|]^2 \\
&\geq (C_5/2) [\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2], \tag{57}
\end{aligned}$$

where the final inequality holds for $\mu \leq \min(\bar{\mu}_0, \frac{1}{4} C_5^2/C_6^2)$. By comparing this final expression with (56), we see that

$$|s^T P_x(w + s; \mu)| = O(\mu^{\sigma-1}) [-s^T P_x(w; \mu)], \tag{58}$$

so that for any fixed $\bar{\gamma} \in (0, 1)$, the condition (54b) will be satisfied by $\alpha = 1$ whenever μ is sufficiently small.

We now verify that (54a) holds for $\alpha = 1$. Since by Taylor's theorem we have

$$P(w + s; \mu) = P(w; \mu) + s^T P_x(w; \mu) + \frac{1}{2} s^T P_{xx}(w + \tau s; \mu) s,$$

for some $\tau \in (0, 1)$, a sufficient condition for (54a) is that

$$\frac{1}{2} s^T P_{xx}(w + \tau s; \mu) s \leq -(1 - \underline{\gamma}) s^T P_x(w; \mu),$$

for each $\tau \in (0, 1)$. Because of (14), the expression is equivalent to

$$\frac{1}{2} s^T [P_{xx}(w + \tau s; \mu) - P_{xx}(w; \mu)] s \leq -(1/2 - \underline{\gamma}) s^T P_x(w; \mu).$$

By (57), a sufficient condition for this inequality to be satisfied is that

$$\begin{aligned}
&|s^T [P_{xx}(w + \tau s; \mu) - P_{xx}(w; \mu)] s| \\
&\leq C_5 (1/2 - \underline{\gamma}) [\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2]. \tag{59}
\end{aligned}$$

We bound the left-hand side of (59) by taking the inner product of (41) with the vector s and estimating the four terms $s^T r_1$, $s^T r_2$, $s^T r_3$, and $s^T r_4$ individually. By (43a), and using again the fact that $\nabla c_i(w)^T s = O(\|\hat{U}(w)^T s\|)$, we obtain

$$\begin{aligned} s^T r_1 &= O(\mu^{-2} \|s\|) \|\hat{U}(w)^T s\|^2 + O(\mu^{-1} \|s\|^3) \\ &= O(\mu^{-1} \|s\|) [\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2]. \end{aligned} \quad (60)$$

From the definition of r_2 and (44), we have that

$$s^T r_2 = O(\mu^{-1} \|s\|^3) = O(\mu^{-1} \|s\|) \|s\|^2. \quad (61)$$

For the other two terms, we have from $r_3 = O(\|s\|^2)$ and $r_4 = O(\mu \|s\|^2)$ that

$$s^T r_3 = O(\|s\|^3), \quad s^T r_4 = O(\mu \|s\|^3). \quad (62)$$

Using (40), we have by combining (60), (61), and (62) with (41) that

$$|s^T [P_{xx}(w + \tau s; \mu) - P_x(w; \mu)] s| = O(\mu^{\sigma-1}) [\mu^{-1} \|\hat{U}(w)^T s\|^2 + \|\tilde{U}(w)^T s\|^2].$$

Because $\sigma - 1 > 0$ and $(1/2 - \underline{\gamma}) > 0$, we see that (59) and therefore (54a) is satisfied for all sufficiently small μ .

We now summarize the results of this section.

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. Then given any constants $\underline{\gamma}$ and $\overline{\gamma}$ satisfying (55), there exists $\bar{\mu}_1 \in (0, \bar{\mu}_0]$ such that when w_1 is any point satisfying (49) with $\mu \in (0, \bar{\mu}_1]$, the full step $\alpha = 1$ along the Newton direction s_1 will satisfy the tests (54).*

6. Rapid convergence when the objective function is linear

We now discuss the case in which the objective function f in (1) is linear, that is,

$$\nabla f(x) \equiv g, \quad (63)$$

for some $g \in \mathbb{R}^n$. We noted in Sect. 2 that any nonlinear program can be reformulated in this fashion. We outline a version of the Newton/log-barrier method and show that it converges rapidly for certain choices of the parameters and certain line search strategies. We assume throughout that there is at least one active constraint (that is, $q \geq 1$), since the other case is trivial in the current context.

We start with a result that describes the first Newton step taken immediately after a reduction in the barrier parameter. In particular, we examine the point obtained by applying a line search along the Newton direction together with a stopping criterion based on the directional derivative, similar to (54b). For convenience, we use a scaled version of the line search parameter, replacing α in (54) by $\bar{\alpha}(\mu/\mu_-)$.

Theorem 3. Suppose that f is linear and that the barrier parameter values μ_- and μ satisfy the condition

$$\mu \in [\rho_0 \mu_-^{\tilde{\sigma}}, \rho_1 \mu_-], \quad (64)$$

where $\rho_0 > 0$, $\rho_1 \in (0, 1)$, and $\tilde{\sigma} \in (1, 2]$ are constants. Suppose too that the bound

$$\|P_x(x; \mu_-)\| \leq \mu_- \quad (65)$$

is satisfied at the current value of x . Then if s is the Newton direction for $P(\cdot; \mu)$ from x , we have that

$$\begin{aligned} & -s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu) \\ &= \frac{\mu_-^2}{\mu} (1 - \mu/\mu_-) \sum_{i=1}^q \frac{c_i(x)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} [(1 - \bar{\alpha})(1 - \mu/\mu_-) + O(\mu_-)] \\ & \quad + O(\mu_-^2) \end{aligned} \quad (66)$$

$$\begin{aligned} &= \frac{\mu_-^2}{\mu} (1 - \mu/\mu_-) \sum_{i=1}^q \left[\frac{(1 - \bar{\alpha})(1 - \mu/\mu_-) + O(\mu_-)}{(1 - \bar{\alpha})(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)} \right] \\ & \quad + O(\mu_-^2). \end{aligned} \quad (67)$$

Proof. The proof follows from the analysis in the proof of Theorem 2 in S. Wright and Jarre [29], allowing for the changes of notation (s in our analysis replaces p in [29], μ_- replaces μ , μ replaces μ_+ , and $\bar{\alpha}$ replaces τ). Note that our condition (65) is obtained by setting $\sigma = 2$ in [29, eq. (30)]; this choice of σ is assumed below in all references to [29].

From (13a), using the notation (63), we have that

$$-s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu) = -s^T g + \mu \sum_{i=1}^m \frac{s^T \nabla c_i(x + \bar{\alpha}(\mu/\mu_-)s)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)}. \quad (68)$$

From [29, eq. (47)], we have

$$\begin{aligned} s^T g &= -q(\mu_-^2/\mu)[1 - \mu/\mu_-] + O(\mu_-^3/\mu) \\ &= -q(\mu_-^2/\mu)[1 - \mu/\mu_-](1 + O(\mu_-)), \end{aligned} \quad (69)$$

where the second inequality uses the following implication of (64):

$$1 - \mu/\mu_- \geq 1 - \rho_1 > 0. \quad (70)$$

We have from the display equation following [29, eq. (47)] that

$$\begin{aligned} & s^T \nabla c_i(x + \bar{\alpha}(\mu/\mu_-)s) \\ &= -(\mu_-^2/\mu) \lambda_i(x, \mu_-)^{-1} [1 - \mu/\mu_-] + O(\mu_-^3/\mu) \\ &= -(\mu_-^2/\mu) \lambda_i(x, \mu_-)^{-1} [1 - \mu/\mu_-] (1 + O(\mu_-)) \\ &= -(\mu_-/\mu) c_i(x) [1 - \mu/\mu_-] (1 + O(\mu_-)), \quad i = 1, 2, \dots, q, \end{aligned} \quad (71)$$

where $\lambda_i(x, \mu_-) = \mu_-/c_i(x)$ as defined in (11), and we use (70) and the fact that the $\lambda_i(x, \mu_-)$ are bounded away from zero, by [29, eq. (35)]. Similarly, we have from [29, eq. (41)] and (11) that

$$\begin{aligned} c_i(x + \bar{\alpha}(\mu/\mu_-)s) &= \mu_- \lambda_i(x, \mu_-)^{-1} [1 - \bar{\alpha}(1 - \mu/\mu_-) + O(\mu_-)] \\ &= c_i(x) [1 - \bar{\alpha}(1 - \mu/\mu_-) + O(\mu_-)], \quad i = 1, 2, \dots, q. \end{aligned} \quad (72)$$

For an estimate of $\|s\|$, we have from [29, eq. (33)] that

$$\|s\| = O(\mu_-^2/\mu). \quad (73)$$

For the inactive indices $i = q+1, \dots, m$, we note that $c_i(x + \bar{\alpha}(\mu/\mu_-)s)$ is uniformly bounded away from zero, so from (73) we have

$$\frac{s^T \nabla c_i(x + \bar{\alpha}(\mu/\mu_-)s)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} = O(\|s\|) = O(\mu_-^2/\mu), \quad i = q+1, \dots, m. \quad (74)$$

We now substitute (69), (71), (72), (73), and (74) into (68) to obtain

$$\begin{aligned} &-s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu) \\ &= \frac{\mu_-^2}{\mu} [1 - \mu/\mu_-] \sum_{i=1}^q \left[1 + O(\mu_-) - \frac{(\mu/\mu_-)c_i(x)(1 + O(\mu_-))}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} \right] + O(\mu_-^2) \\ &= \frac{\mu_-^2}{\mu} [1 - \mu/\mu_-] \sum_{i=1}^q \frac{c_i(x + \bar{\alpha}(\mu/\mu_-)s)(1 + O(\mu_-)) - (\mu/\mu_-)c_i(x)(1 + O(\mu_-))}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} \\ &\quad + O(\mu_-^2). \end{aligned} \quad (75)$$

For the numerator of the i th term in the summation, we have from (72) that

$$\begin{aligned} &c_i(x + \bar{\alpha}(\mu/\mu_-)s)(1 + O(\mu_-)) - (\mu/\mu_-)c_i(x)(1 + O(\mu_-)) \\ &= c_i(x)[1 - \bar{\alpha}(1 - \mu/\mu_-) + O(\mu_-)](1 + O(\mu_-)) - (\mu/\mu_-)c_i(x)(1 + O(\mu_-)) \\ &= c_i(x)[(1 - \bar{\alpha})(1 - \mu/\mu_-) + O(\mu_-)]. \end{aligned}$$

By substitution in (75), we obtain (66).

To prove the other estimate (67), we have from (72) that

$$\begin{aligned} \frac{c_i(x)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} &= \frac{1}{1 - \bar{\alpha}(1 - \mu/\mu_-) + O(\mu_-)} \\ &= \frac{1}{(1 - \bar{\alpha})(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)}. \end{aligned}$$

The result follows by substitution into (66). (Note that (67) is also an immediate consequence of [29, eq. (48)].) □

We next show that when we apply a condition similar to (54b) to the first Newton step immediately after reduction from μ_- to μ , the step lengths satisfying this condition form a small interval around $\bar{\alpha} = 1$.

Theorem 4. Suppose that the conditions of Theorem 3 are satisfied, in particular that x , s , μ , and μ_- have the properties specified therein. Let $\check{\sigma}$ be a constant satisfying $1 < \check{\sigma} < \check{\sigma} < 2$. We then have

$$-s^T P_x(x; \mu) = \Theta(\mu_-^2/\mu). \quad (76)$$

Moreover, a directional derivative condition of the form

$$|s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu)| \leq -C_{10}(\mu_-^{\check{\sigma}}/\mu)s^T P_x(x; \mu), \quad (77)$$

or alternatively

$$|s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu)| \leq C_{11}\mu_-^{\check{\sigma}+2}/\mu^2, \quad (78)$$

where C_{10} and C_{11} are given positive constants, is satisfied if and only if

$$|1 - \bar{\alpha}| = O(\mu_-^{\check{\sigma}-1}). \quad (79)$$

Proof. We prove (76) by substituting $\bar{\alpha} = 0$ into (67). We obtain

$$\begin{aligned} -s^T P_x(x; \mu) &= \frac{\mu_-^2}{\mu}(1 - \mu/\mu_-) \sum_{i=1}^q \left[\frac{1 - \mu/\mu_- + O(\mu_-)}{1 + O(\mu_-)} \right] + O(\mu_-^2) \\ &= q \frac{\mu_-^2}{\mu}(1 - \mu/\mu_-)(1 - \mu/\mu_- + O(\mu_-)) + O(\mu_-^2) \\ &= q \frac{\mu_-^2}{\mu}(1 - \mu/\mu_-)^2 + O(\mu_-^3/\mu), \end{aligned} \quad (80)$$

where in the third equality we used the relation $0 < \mu < \mu_- \ll 1$. It follows directly that

$$-s^T P_x(x; \mu) \leq q \frac{\mu_-^2}{\mu} + O(\mu_-^3/\mu). \quad (81)$$

Because of (70), we have from (80) that

$$-s^T P_x(x; \mu) \geq q \frac{\mu_-^2}{\mu}(1 - \rho_1)^2 + O(\mu_-^3/\mu). \quad (82)$$

By taking μ_- sufficiently small, the claim (76) follows immediately from (81) and (82).

We now prove the second statement for (78); the other case involving (77) follows immediately from (76). We first prove the reverse implication; that is, we assume that (79) is satisfied and prove that (78) holds. We start by estimating the various terms that appear in (67). Because of (64), we have

$$\mu_- \leq (\mu/\rho_0)^{1/\check{\sigma}}. \quad (83)$$

Using this condition, together with $\check{\sigma} > \check{\sigma}$, we have from (79) that

$$\begin{aligned} |1 - \bar{\alpha}|(1 - \mu/\mu_-) &= O(\mu_-^{\check{\sigma}-1}) = O(\mu_-^{\check{\sigma}}/\mu)(\mu/\mu_-) \\ &= O(\mu_-^{\check{\sigma}/\check{\sigma}-1})(\mu/\mu_-) = o(1)(\mu/\mu_-). \end{aligned}$$

Using (83) again, we also have

$$\mu_- = (\mu_-^2/\mu)(\mu/\mu_-) = O(\mu^{2/\check{\sigma}-1})(\mu/\mu_-) = o(1)(\mu/\mu_-).$$

It follows from these estimates that each term in the summation in (67) satisfies

$$\frac{(1 - \bar{\alpha})(1 - \mu/\mu_-) + O(\mu_-)}{(1 - \bar{\alpha})(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)} = \frac{O(\mu_-^{\check{\sigma}-1})}{(1 + o(1))(\mu/\mu_-)} = O(\mu_-^{\check{\sigma}}/\mu).$$

Therefore we have from (67) that

$$\begin{aligned} -s^T P_x(x + \bar{\alpha}(\mu/\mu_-)s; \mu) &= (\mu_-^2/\mu) [1 - \mu/\mu_-] O(\mu_-^{\check{\sigma}}/\mu) + O(\mu_-^2) \\ &= O(\mu_-^{\check{\sigma}+2}/\mu^2) + O(\mu_-^2). \end{aligned}$$

The second term is dominated by the first term in this expression because from $\mu < \mu_-$ and $\check{\sigma} < 2$, we have

$$\mu_-^2 < \frac{\mu_-^4}{\mu^2} = \frac{\mu_-^{2+\check{\sigma}}}{\mu^2} \mu_-^{2-\check{\sigma}} \leq \frac{\mu_-^{2+\check{\sigma}}}{\mu^2},$$

for all μ_- (hence, μ) sufficiently small. We conclude that (79) implies (78).

Now assume that (78) holds. We prove the required implication (79) by using the estimates (66) and (67). When $|1 - \bar{\alpha}| = O(\mu_-)$, then (79) holds trivially, since $\check{\sigma} \in (1, 2)$, so we assume from here onward that this relation does not hold. We then have that

$$|1 - \bar{\alpha}|(1 - \mu/\mu_-) \gg \mu_-,$$

for all μ_- sufficiently small. This relation implies that each of the q terms in the summation in (66) has the same sign as has $1 - \bar{\alpha}$. Using (70), we obtain

$$|(1 - \bar{\alpha})(1 - \mu/\mu_-) + O(\mu_-)| \geq \frac{1}{2}|1 - \bar{\alpha}|(1 - \mu/\mu_-) \geq \frac{1}{2}(1 - \rho_1)|1 - \bar{\alpha}|.$$

In addition, we have $c_i(x) > 0$, and for all $\bar{\alpha}$ of interest we also have that $c_i(x + \bar{\alpha}(\mu/\mu_-)s) > 0$. Hence, from (66) and (78), and using (70) again, we have

$$\frac{\mu_-^2}{\mu}(1 - \rho_1) \sum_{i=1}^q \frac{c_i(x)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} \left[\frac{1}{2}(1 - \rho_1)|1 - \bar{\alpha}| \right] \leq C_{11} \frac{\mu_-^{\check{\sigma}+2}}{\mu^2} + O(\mu_-^2),$$

which implies that

$$|1 - \bar{\alpha}| \sum_{i=1}^q \frac{c_i(x)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} \leq \frac{2C_{11}}{(1 - \rho_1)^2} \frac{\mu_-^{\check{\sigma}}}{\mu} + O(\mu).$$

Since $\mu_-^{\check{\sigma}}/\mu \geq \mu^{\check{\sigma}-1} \geq \mu$ for all μ in the range of interest, the $O(\mu)$ term is dominated by the first term on the right-hand side of this expression. Since all terms in the summation on the left-hand side are positive, we have for each $i = 1, 2, \dots, q$ that

$$|1 - \bar{\alpha}| \frac{c_i(x)}{c_i(x + \bar{\alpha}(\mu/\mu_-)s)} \leq \bar{C}_{11} \frac{\mu_-^{\check{\sigma}}}{\mu},$$

for some $\bar{C}_{11} > 0$. By using (72), we obtain

$$|1 - \bar{\alpha}| \leq \bar{C}_{11} \frac{c_i(x + \bar{\alpha}(\mu/\mu_-)s)}{c_i(x)} \frac{\mu_-^{\check{\sigma}}}{\mu} = \bar{C}_{11} [1 - \bar{\alpha}(1 - \mu/\mu_-) + O(\mu_-)] \frac{\mu_-^{\check{\sigma}}}{\mu}.$$

By further rearrangement, we obtain

$$|1 - \bar{\alpha}| \left[1 - \bar{C}_{11} \frac{\mu_-^{\check{\sigma}}}{\mu} \right] \leq \bar{C}_{11} \bar{\alpha} \mu_-^{\check{\sigma}-1} + O(\mu_-^{\check{\sigma}+1}/\mu) \leq 3\bar{C}_{11} \mu_-^{\check{\sigma}-1}, \quad (84)$$

since it follows readily from (72) and $c_i > 0$ that $\bar{\alpha} \leq 2$, while the remainder term $O(\mu_-^{\check{\sigma}+1}/\mu)$ is dominated by $\bar{C}_{11} \mu_-^{\check{\sigma}-1}$ for all μ_- sufficiently small. From (83), and the assumption of $1 < \tilde{\sigma} < \check{\sigma}$, we have that

$$\bar{C}_{11} \frac{\mu_-^{\check{\sigma}}}{\mu} \leq \frac{\bar{C}_{11}}{\rho_0^{1/\tilde{\sigma}}} \mu_-^{\check{\sigma}/\tilde{\sigma}-1} \leq \frac{1}{2},$$

for all μ_- sufficiently small. By substituting this bound into (84) we obtain the required estimate (79). \square

It follows from Theorem 4 that the step $\bar{\alpha} = 1$, or equivalently $\alpha = \mu/\mu_-$, is nearly optimal at this first step after reduction of μ . We show now that all step lengths satisfying (79) yield iterates that lie within the neighborhood (19), for a certain choice of σ .

Theorem 5. *Suppose that the conditions of Theorem 4 are satisfied and that the step $\alpha = \bar{\alpha}(\mu/\mu_-)$ taken along the direction s satisfies (79). We then have that*

$$\|x + \bar{\alpha}(\mu/\mu_-)s - x(\mu)\| = O(\mu_-^{\check{\sigma}}) = O(\mu_-^{\check{\sigma}/\tilde{\sigma}}). \quad (85)$$

Proof. Note first that

$$\|x + \bar{\alpha}(\mu/\mu_-)s - x(\mu)\| \leq \|x + (\mu/\mu_-)s - x(\mu)\| + |\bar{\alpha} - 1| \|(\mu/\mu_-)s\|. \quad (86)$$

From Wright and Jarre [29, Theorem 1], and using (65), we have that

$$\|x + (\mu/\mu_-)s - x(\mu)\| = O(\mu_-^2). \quad (87)$$

From [29, eqs. (32), (33)], and setting use $\sigma = 2$ in [29, eq. (33)], we have that

$$\|(\mu/\mu_-)s\| = \Theta(\mu_-). \quad (88)$$

The first equality in (85) is obtained by substituting (87), (88), and (79) into (86). The second equality follows from (83). \square

We now specify Algorithm NLB, our variant of the Newton/log-barrier method. Since we are interested only in local behavior of the method, we assume that the initial value μ_0 is already small enough that the range of values in (64) is nonempty for all $\mu_- \leq \mu_0$. Simple modifications can be applied to make this method valid globally.

Algorithm NLB

Given $\mu_0 > 0$, $\rho_0 \in (0, 1)$, $\rho_1 \in (0, 1)$, and $\tilde{\sigma}, \check{\sigma}$ with $1 < \tilde{\sigma} < \check{\sigma} < 2$;

Set $k \leftarrow 0$, $x_{-1} \leftarrow$ strictly feasible initial value;

Set $\text{tol1} \leftarrow \infty$;

repeat

$x_k \leftarrow \text{Newton}(x_{k-1}, \mu_k, \text{tol1})$;

if μ_k sufficiently small

terminate;

Set $k \leftarrow k + 1$;

Choose $\mu_k \in [\rho_0 \mu_{k-1}^{\tilde{\sigma}}, \rho_1 \mu_{k-1}]$;

Set $\text{tol1} \leftarrow \mu_{k-1}^{\tilde{\sigma}+2} / \mu_k^2$;

end (repeat)

The Newton procedure with line search is defined as follows

Procedure Newton($x, \mu, \text{tol1}$);

Given $\underline{\gamma}$ and $\overline{\gamma}$ satisfying (55);

Set $t \leftarrow 0$, $w_0 \leftarrow x$;

while $\|P_x(w_t; \mu)\| > \mu$

Set $s_t \leftarrow -P_{xx}(w_t; \mu)^{-1} P_x(w_t; \mu)$;

Choose α such that (54a) and

$$|s_t^T P_x(w_t + \alpha s_t; \mu)| \leq \min(\text{tol1}, -\overline{\gamma} s_t^T P_x(w_t; \mu)), \quad (89)$$

are satisfied, accepting $\alpha = 1$ if it satisfies these conditions;

Set $w_{t+1} \leftarrow w_t + \alpha s_t$;

Set $t \leftarrow t + 1$;

end (while)

return w_t .

We now show that in the line-search criterion (89), the right-hand side is equal to tol1 at the first iteration of each call to Procedure Newton, while the alternative value $-\overline{\gamma} s_0^T P_x(w_0; \mu)$ is operative on subsequent calls.

Theorem 6. *Suppose that f is linear. Then if μ_0 is sufficiently small, the following properties are satisfied at all steps $k \geq 1$ in Algorithm NLB:*

- (i) *At the first step ($t = 0$) taken within Procedure Newton within iteration k of Algorithm NLB, we have that $\text{tol1} < -\overline{\gamma} s_0^T P_x(w_0; \mu)$, so the value $\alpha = \bar{\alpha}(\mu_k / \mu_{k-1})$ selected at this step will satisfy (79), for $\mu_- = \mu_{k-1}$.*
- (ii) *At the second and subsequent steps ($t = 1, 2, 3, \dots$) within Procedure Newton at iteration k , we have that $\text{tol1} > -\overline{\gamma} s_t^T P_x(w_t; \mu)$. Moreover, $\alpha_t = 1$ is the chosen step length at each of these iterations.*

Proof. Using the notation $\mu_- = \mu_{k-1}$ and $\mu = \mu_k$, we have from (83), (76), and $\check{\sigma} > \tilde{\sigma}$ that

$$\text{tol1} = \frac{\mu_-^{\check{\sigma}+2}}{\mu^2} = \frac{\mu_-^2}{\mu} \frac{\mu_-^{\check{\sigma}}}{\mu} \leq \frac{\mu_-^2}{\mu} \rho_0^{-\check{\sigma}/\tilde{\sigma}} \mu^{\check{\sigma}/\tilde{\sigma}-1} < -\overline{\gamma} s^T P_x(x; \mu),$$

for all μ sufficiently small. It follows from (89) that (78) is satisfied for $C_{11} = 1$. Because of the convergence criterion at the previous invocation of Procedure Newton, we have that $\|P_x(w_0; \mu_-)\| \leq \mu_-$, so that condition (65) is satisfied. The other conditions of Theorem 4 also hold, so by the forward implication in Theorem 4, we conclude that (79) holds, proving (i).

Since (79) is satisfied by the first step length in Procedure Newton, and the other conditions of Theorem 4 are satisfied, we have from Theorem 5 that the new iterate

$$w_1 \stackrel{\Delta}{=} x + \bar{\alpha}(\mu/\mu_-)s$$

will satisfy

$$\|w_1 - x(\mu)\| = O(\mu^{\check{\sigma}}) = O(\mu^{\check{\sigma}/\check{\sigma}}). \quad (90)$$

By choosing some value

$$\sigma \in (1, \check{\sigma}/\check{\sigma})$$

and choosing μ_0 sufficiently small, we have that w_1 lies in the neighborhood (49), so that the results of Theorems 1 and 2 can be applied to the unit-step Newton's method starting at w_1 . From (40b) and (90), we have

$$\|s_1\| = O(\|w_1 - x(\mu)\|) = O(\mu^{\check{\sigma}}),$$

so by using (14) and (29), we have

$$\begin{aligned} & -\bar{\gamma}s_1^T P_x(w_1; \mu) \\ & = \bar{\gamma}s_1^T P_{xx}(w_1; \mu)s_1 \leq O(\|s_1\|^2)O(\|P_{xx}(w_1; \mu)\|) = O(\mu_-^{2\check{\sigma}}\mu^{-1}). \end{aligned}$$

Since

$$\frac{\mu_-^{2\check{\sigma}}}{\mu} \leq \frac{\mu_-^{2\check{\sigma}+1}}{\mu^2} \ll \frac{\mu_-^{\check{\sigma}+2}}{\mu^2} = \tau_{011},$$

we have for μ_0 sufficiently small that $\tau_{011} > -\bar{\gamma}s_1^T P_x(w_1; \mu)$. Therefore, Procedure Newton applies the step length conditions (54) to its second iteration (with $t = 1$). By Theorem 2, the step length $\alpha_1 = 1$ will be accepted at this iteration.

Moving to the next iteration ($t = 2$), we have from Theorem 1 and (90) that

$$\|w_2 - x(\mu)\| = O(\|w_1 - x(\mu)\|) = O(\mu^{\check{\sigma}}).$$

Moreover, we have from (53) that $\|s_2\| \leq (1/4)\|s_1\|$. The argument above can be repeated to deduce that $\tau_{011} > -\bar{\gamma}s_2^T P_x(w_2; \mu)$ and that the unit step is accepted at this iteration too. By repeating the argument inductively, we complete the proof of (ii). \square

Part (i) of Theorem 6 is predicated on the *existence* of an α satisfying both (54a) and (89). We now prove existence by showing that the particular choice $\bar{\alpha} = 1$ (corresponding to $\alpha = \mu_k/\mu_{k-1}$) satisfies these two line-search conditions. It follows that Procedure Newton is well defined for all μ_k sufficiently small.

Theorem 7. Suppose that f is linear. If μ_0 is sufficiently small, then at the first step taken at each invocation of Procedure Newton ($t = 0$) at iteration $k \geq 1$ of Algorithm NLB, there exists a value of α that satisfies (54a) and (89).

Proof. Theorem 4 shows that there exists a range of values of $\bar{\alpha}$ satisfying (79) for which (78) is satisfied, for any given $C_{11} > 0$. Noting that Theorem 6 (i) shows that the right-hand side of (89) is $\mu_{k-1}^{\bar{\sigma}+2}/\mu_k^2$ at the first step ($t = 0$) of Procedure Newton, and taking $C_{11} = 1$, we have that (89) is satisfied for all $\bar{\alpha}$ sufficiently close to 1. Therefore, to prove the result, it is sufficient to show that the other line-search condition (54a) is satisfied for $\bar{\alpha} = 1$.

As usual, we denote μ_{k-1} by μ_- and μ_k by μ . By Taylor's theorem, we have

$$P(x + (\mu/\mu_-)s; \mu) - P(x; \mu) = \frac{\mu}{\mu_-} \int_0^1 s^T P_x(x + \hat{\alpha}(\mu/\mu_-)s; \mu) d\hat{\alpha}. \quad (91)$$

Since $\hat{\alpha} \in [0, 1]$, we have from (67) that

$$\begin{aligned} & (\mu/\mu_-)s^T P_x(x + \hat{\alpha}(\mu/\mu_-)s; \mu) \\ &= -\mu_-(1 - \mu/\mu_-) \sum_{i=1}^q \frac{(1 - \hat{\alpha})(1 - \mu/\mu_-) + O(\mu_-)}{(1 - \hat{\alpha})(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)} + O(\mu\mu_-) \\ &= -\mu_-(1 - \mu/\mu_-) \sum_{i=1}^q \frac{(1 - \hat{\alpha})(1 - \mu/\mu_-)}{(1 - \hat{\alpha})(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)} \\ &\quad + \frac{O(\mu_-^2)}{\mu/\mu_- + O(\mu_-)} + O(\mu\mu_-). \end{aligned}$$

Because of (64) and $\tilde{\sigma} < 2$, we can show that

$$\frac{\mu}{\mu_-} \gg \mu_-, \quad \frac{O(\mu_-^2)}{\mu/\mu_- + O(\mu_-)} = O(\mu_-^3/\mu) = o(\mu_-),$$

for all μ_- sufficiently small, so that the last two terms in the expression above are both $o(\mu_-)$. Using $\hat{\alpha} \in [0, 1]$ again, we have that

$$\begin{aligned} & (\mu/\mu_-)s^T P_x(x + \hat{\alpha}(\mu/\mu_-)s; \mu) \\ &\leq -\mu_-(1 - \mu/\mu_-) \sum_{i=1}^q \frac{(1 - \hat{\alpha})(1 - \mu/\mu_-)}{(1 - \mu/\mu_-) + \mu/\mu_- + O(\mu_-)} + o(\mu_-) \\ &= -\mu_-(1 - \mu/\mu_-)^2 q(1 - \hat{\alpha}) + o(\mu_-). \end{aligned}$$

By integrating we therefore obtain

$$\frac{\mu}{\mu_-} \int_0^1 s^T P_x(x + \hat{\alpha}(\mu/\mu_-)s; \mu) d\hat{\alpha} \leq -\frac{1}{2} \mu_-(1 - \mu/\mu_-)^2 q + o(\mu_-). \quad (92)$$

By substituting $\bar{\alpha} = 0$ into (67), we obtain as in (80) that

$$\frac{\mu}{\mu_-} s^T P_x(x; \mu) = -\mu_- \left(1 - \frac{\mu}{\mu_-}\right)^2 q + o(\mu_-). \quad (93)$$

Since the first terms on the right-hand sides of both (92) and (93) have magnitude $\Theta(\mu_-)$, we have

$$\frac{\mu}{\mu_-} \int_0^1 s^T P_x(x + \hat{\alpha}(\mu/\mu_-)s; \mu) d\hat{\alpha} \leq \underline{\gamma} \frac{\mu}{\mu_-} s^T P_x(x; \mu),$$

for any $\underline{\gamma} \in (0, 1/2)$ and all μ_- sufficiently small. By combining this bound with (91), we conclude that (54a) holds for $\tilde{\alpha} = 1$ (that is, $\alpha = \mu/\mu_-$), as required. \square

The proofs of Theorems 6 and 7 suggest a version of Algorithm NLB that achieves superlinear convergence. Suppose we take $\mu_k = \rho_0 \mu_{k-1}^{\tilde{\sigma}}$ for all k sufficiently large and make the particular choice $\alpha = \mu_k/\mu_{k-1}$ at the first step taken at each invocation of Procedure Newton. Then the proofs of these theorems indicate that this value of α is accepted by the line search conditions at the first iteration of Procedure Newton, that $\alpha = 1$ is accepted on subsequent steps, and that the method of Procedure Newton converges quadratically to $x(\mu_k)$. In fact, Procedure Newton will satisfy the condition $\|P_x(w_t; \mu_k)\| \leq \mu_k$ and therefore will terminate after a very modest number of Newton iterations. As an example, using (50), (83) and (87), we have that

$$\|w_2 - x(\mu_k)\| = O(\mu_k^{-1}) \|w_1 - x(\mu_k)\|^2 = O(\mu_k^{-1} \mu_{k-1}^4) = O(\mu_k^{4/\tilde{\sigma}-1}).$$

If $\tilde{\sigma} \in (1, 4/3)$, we have $\|w_2 - x(\mu_k)\| = o(\mu_k^2)$, and therefore from (38) it follows that $\|P_x(w_2; \mu_k)\| = o(\mu_k)$, so that for this choice of $\tilde{\sigma}$, the convergence tolerance is satisfied after just two Newton steps. (Values $\tilde{\sigma} \geq 4/3$ can be accommodated by using additional Newton steps.) Overall superlinear convergence of Algorithm NLB to x^* follows from the fact that the work per invocation of Procedure Newton is bounded, and the sequence $\{\mu_k\}$ converges superlinearly to zero.

As pointed out by a referee, the search direction and special step length μ_k/μ_{k-1} used by this variant of Algorithm NLB produce an identical result to the extrapolation step of Benchakroun, Dussault, and Mansouri [1]. The fact that one additional Newton step suffices for each value of μ , provided that $\tilde{\sigma} \in (1, 4/3)$ is also proved by these authors. Conn, Gould, and Toint [5] have a similar result for their acceleration technique. What is perhaps surprising about the approach described in this paper is that the same result can be obtained by using the usual Newton search direction, provided that a linear-objective reformulation and a shortened step length are used.

Even if we do not make the special choice $\alpha = \mu_k/\mu_{k-1}$ for the first iteration of Procedure Newton, but instead rely on the line search procedure and the conditions (54a) and (89) to identify an appropriate value of α at this step, the overall rate will remain superlinear provided that the number of function and derivative evaluations required to identify this α is bounded. The latter requirement can be expected to hold for all reasonable line-search algorithms. For instance, the specialized line search procedure for log-barrier functions, such as that of Murray and Wright [20] can be expected to find an appropriate α with just a couple of evaluations of $P(\cdot; \mu_k)$ and $P_x(\cdot; \mu_k)$.

7. Discussion

We conclude by expanding the discussion of Sect. 2 to note a few more points about the context of this work.

A consequence of the ill conditioning of $P_{xx}(x; \mu)$ is that standard methods of linear algebra for obtaining s , which solve the system

$$P_{xx}(x; \mu)s = -P_x(x; \mu),$$

may be inefficient or inaccurate. A number of authors have proposed alternative formulations of systems such as this (Gould [14]) or modified factorization schemes (Broyden and Attia [2], M. Wright [25]) or have shown that the inaccuracy introduced by the Cholesky factorization may not be especially detrimental (M. Wright [27], S. Wright [28]). Since these numerical considerations are outside the scope of this paper, we omit further discussion of this point.

Another line of investigation in the literature that is relevant to this paper concerns the analytic center method for certain types of convex programming problems. In this method, a logarithmic potential function different from $P(\cdot; \mu)$ is used, typically having the form

$$\phi(x; \eta) = -\omega \log(\eta - f(x)) - \sum_{i=1}^m \log c_i(x).$$

where $\omega > 0$ is some constant and η is an upper bound on the optimal value $f(x^*)$. It can be shown that for $\eta > f(x^*)$, the minimizer of $\phi(x; \eta)$ coincides with the minimizer $x(\mu)$ of $P(x; \mu)$ for some value of μ . The algorithms apply Newton's method to find approximate minima of $\phi(x; \eta)$ for successively decreasing values of η . Variants of this approach are analyzed by Jarre [16] and Den Hertog, Roos, and Terlaky [6], while in [7] the latter authors outline an extension to the log-barrier function $P(x; \mu)$ of this paper.

Relevant results include those of Jarre [16, Lemma 4] and Nesterov and Nemirovskii [23, Theorem 2.2.3], who prove that Newton's method applied to $\phi(x; \eta)$ generates a sequence of steps s_1, s_2, \dots and iterates w_1, w_2, \dots (with $w_{j+1} = w_j + s_j$) such that the quantity

$$\left[s_j^T \phi_{xx}(w_j; \eta) s_j \right]^{1/2} = \left[\phi_x(w_j; \eta)^T \phi_{xx}(w_j; \eta)^{-1} \phi_x(w_j; \eta) \right]^{1/2}$$

converges quadratically to zero. This behavior requires the initial estimate w_1 to be close enough to the minimizer of $\phi(x; \eta)$ that the bound

$$\left[\phi_x(w_j; \eta)^T \phi_{xx}(w_j; \eta)^{-1} \phi_x(w_j; \eta) \right]^{1/2} < \beta \quad (94)$$

is satisfied, for some constant β that does not depend on η . Without embarking on a detailed discussion of the relationship of these results to Sect. 4, we note that near a solution x^* , the eigenvalues of $\phi_{xx}(x; \eta)$ can be partitioned into two size classes, in a similar way to those of $P_{xx}(x; \mu)$, and that weighting of the norm by $\phi_{xx}(w; \eta)$ has a similar purpose to the partitioning into subspaces spanned by the orthonormal matrices \hat{U} and \tilde{U} that we use in Sect. 4. The neighborhood defined by (94) has

a similar extent in the range space of \hat{U} (the active constraint range space) as does our neighborhood (19), for values of σ close to 1. However, the results for analytic center methods require the objective and constraint functions to satisfy scaled Lipschitz or self-concordance conditions, whereas this paper assumes the typical smoothness and second-order sufficient conditions that are used in most local analyses of nonlinear programming algorithms. Moreover, the literature on analytic center methods does not, to the best of our knowledge, address the possibility of superlinear convergence.

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