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IDENTIFIABLE SURFACES IN CONSTRAINED OPTIMIZATION*

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Abstract. The concept of a "class-C^p identifiable surface" of a convex set in Euclidean space is introduced. The paper shows how the smoothness of these surfaces is related to the smoothness of the projection operator and presents finite identification results for certain algorithms for minimization of a function over this set. The work uses a partially geometric view of constrained optimization to generalize previous finite identification results.

Key words. constrained optimization, active set identification

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1. Introduction. Here, we investigate the problem

$$\min_{x\in\Omega} F(x), \tag{1}$$

where F is continuously differentiable and $\Omega \subset \mathbb{R}^n$ is closed and convex. In particular, we are interested in finding subsets of Ω that can be identified by an optimization algorithm after a finite number of iterations. That is, if the solution x^* lies in one such subset, the iterates generated by the algorithm should eventually enter and remain within that subset. In the case in which Ω is defined by a set of algebraic inequalities, this property of the iterates corresponds to identifying the active constraints, and when Ω is a polyhedron, it means identifying the face, edge, or vertex, upon which the solution x^* lies.

The first-order conditions for x^* to be a solution of (1) are

$$-\nabla F(x^*) \in N(x^*),$$

where $N(x^*)$ is the normal cone to Ω at x^*. To prove the finite identification (capture) results, we assume a nondegeneracy condition due to Dunn [4]. This is stated simply as

$$-\nabla F(x^*) \in \mathrm{ri}(N(x^*)), \tag{2}$$

where $\Lambda \subset \mathbb{R}^n$ and $\mathrm{ri}(\Lambda)$ is the interior of Λ relative to $\mathrm{aff}(\Lambda)$, the affine hull of Λ. A condition equivalent to (2) was assumed by Gafni and Bertsekas [7] for the case of polyhedral sets. The condition (2), which is a geometric generalization of the strict complementarity condition of nonlinear programming, has been used in the convergence analysis of Dunn [4] and Burke and Moré [1]. Both these papers specify similar classes of subsets of Ω that are finitely identifiable by gradient projection and Newton-like algorithms. We define these *open facets* as in [4].

DEFINITION 1.

(a) For any closed convex cone $K \subset \mathbb{R}^n$, we use K° to denote the polar of K, and define the lineality $\mathrm{lin}(K)$ to be $(K^\circ)^\perp$;

(b) Let $T(x)$ be the tangent cone to Ω at x as defined in Clarke [2, Thm. 2.4.5]; the normal cone is $N(x) = T(x)^\circ$. A nonempty subset $S \subset \Omega$ is an *open facet* if the

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set $V = x + \text{lin}(T(x))$ is independent of $x \in S$, and $S = \text{int}_V(\Omega \cap V)$, where $\text{int}_V(.)$ denotes interior with respect to V.

It is easy to show that open facets are convex. When Ω is polyhedral, it can be partitioned into open facets, but when Ω has some curved boundaries, this is not the case. As an example, consider the set defined in [1, Eq. (2.2)]:

$$\Omega_1 = \left\{ (\xi_1, \xi_2) \mid \xi_2 \le \sqrt{1-\xi_1^2},\ 0 \le \xi_1 \le 1 \right\}.$$

The open facets in this set are its interior, the point $(0,1)$ and the edges $\{(0,\xi_2) \mid \xi_2 < 1\}$ and $\{(1,\xi_2) \mid \xi_2 < 0\}$. No subset of the curved face $\{(\xi_1,\xi_2) \mid \xi_1^2 + \xi_2^2 = 1,\ 0 < \xi_1 \le 1\}$ satisfies Definition 1.

When Ω is defined by algebraic inequalities, that is,

$$\Omega = \{\xi \mid g_i(\xi) \le 0,\ i = 1, \dots, m\}, \tag{3}$$

it is often assumed that the g_i are C^2 and that the set

$$\{\nabla g_i(x) \mid i \in \mathcal{A}(x)\}, \qquad \text{where} \qquad \mathcal{A}(x) = \{i \mid 1 \le i \le m,\ g_i(x) = 0\} \tag{4}$$

is linearly independent. In this case, the nondegeneracy condition (2) (which reduces to the standard strict complementarity condition) ensures that surfaces defined by a particular active index set $\mathcal{A} \subset \{1, 2, \dots, m\}$ are finitely identifiable by a number of standard algorithms. Note that Ω_1 is not definable in the form (3),(4) for $g_i \in C^2$, since there is a curvature discontinuity in the boundary at $(1,0)$. If we allow g_i to be only C^1, then Ω_1 *is* definable as (3),(4), but then the curved surface is indistinguishable from the face $\{(1,\xi_2) \mid \xi_2 \le 0\}$.

The next section defines the concept of a "class-C^p identifiable surface." Loosely speaking, such a surface S is usually a connected "patch" on $\partial\Omega$, which is locally parametrizable by a collection of C^p functions, for some integer $p \ge 1$. (The interior of Ω is defined to be a class-C^∞ surface.) Moreover, these functions can be defined so that their gradients can enclose any given ray in the relative interior of $N(x)$, where x is a given point in S. We prove that open facets and subsets of (3) that are defined by particular choices of $\mathcal{A}$ are identifiable surfaces. (For the set Ω_1, the curved boundary, with its two endpoints excluded, is also an identifiable surface.) We show that class-C^p identifiable surfaces generate connected open regions in the exterior of Ω, within which the operation of projection onto Ω is $p-1$ times continuously differentiable. In §3, we prove finite identification results for gradient projection and Newton-like algorithms.

In the remainder of the paper, $\|.\|$ denotes the Euclidean norm, B denotes the open unit ball $\{\xi \in \mathbb{R}^n \mid \|\xi\| < 1\}$, and $\text{co}\{\cdot\}$ denotes the convex hull of a set of vectors. The projection operator and distance function are defined as follows, with reference to any closed subset A of $\mathbb{R}^n$:

$$P_A(y) = \min_{\bar{y} \in A} \tfrac{1}{2}\|\bar{y} - y\|, \qquad d_A(y) = \arg\min_{\bar{y} \in A} \tfrac{1}{2}\|\bar{y} - y\|.$$

We use $P(\cdot)$ as shorthand for $P_\Omega(\cdot)$. Given a collection of functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, r$, we frequently use the notation

$$g(x) = [g_1(x), g_2(x), \dots, g_r(x)]^T, \qquad \nabla g(x) = [\nabla g_1(x) | \cdots | \nabla g_r(x)].$$

2. Identifiable surfaces and smoothness of the projection operator. Throughout the remainder of this paper, we make this following assumption.

Assumption 1. Ω is closed and convex and has an interior in $\mathbb{R}^n$.

The last part of this assumption is made for convenience. If it does not hold, the results of this section can be recovered by restricting attention to $\mathrm{aff}(\Omega)$.

DEFINITION 2. A connected set $S \subset \Omega$ is a *class-C^p identifiable surface*, p a positive integer, if either

(a) S is an open subset of $\mathrm{int}(\Omega)$, or

(b) $S \subset \partial\Omega$, and for any $y \in \mathbb{R}^n \backslash \Omega$ such that $\bar{y} = P(y) \in S$ and $y - \bar{y} \in \mathrm{ri}(N(\bar{y}))$, there exist functions g_i, $i = 1, \ldots, r = r(y)$, and a constant $\epsilon = \epsilon(y) > 0$, such that

(i) $g_i \in C^p(\bar{y} + \epsilon B)$, $i = 1, \ldots, r$;

(ii) $\{\nabla g_i(\bar{z}), i = 1, \ldots, r\}$ is linearly independent for all $\bar{z} \in S(\bar{y}; \epsilon) \triangleq (\bar{y} + \epsilon B) \cap S$;

(iii) $\mathrm{co}\{\nabla g_i(\bar{z}), i = 1, \ldots, r\} \subset N(\bar{z})$ for all $\bar{z} \in S(\bar{y}; \epsilon)$;

(iv) $y - \bar{y} \in \mathrm{ri}(\mathrm{co}\{\nabla g_i(\bar{y}), i = 1, \ldots, r\})$; and

(v) $S(\bar{y}; \epsilon) = \{\bar{z} \mid \|\bar{z} - \bar{y}\| < \epsilon,\ g_i(\bar{z}) = 0,\ i = 1, \ldots, r\}$.

Obviously, if S is a class-C^p identifiable surface, then it is also a class-C^q identifiable surface, for any q with $1 \leq q < p$.

There are some significant differences between the functions g_i that are used in Definition 2 and the ones used in the algebraic parametrization (3). First, we are now only seeking a *local* parametrization. Second, and more importantly, we are trying to parametrize the surface S and a piece of the normal cone to Ω at points on S, rather than the set Ω itself. This latter property allows a wider range of sets to be decomposed into an intuitively reasonable collection of surfaces than was possible in the case of open facets or (3).

Before proceeding, to give a sense of how this definition differs from that of open facets and from (3), we review the example Ω_1 from §1, and give two more examples. The interior of Ω_1, the point $(0, 1)$ and the edge $\{(0, \xi_2) \mid \xi_2 < 1\}$ are class-C^∞ identifiable surfaces. The remaining surface defined by

$$\left\{(\xi_1, \xi_2) \mid \xi_2 = \sqrt{1 - \xi_1^2},\ 0 \leq \xi_1 < 1\right\} \cup \{(1, \xi_2) \mid \xi_2 < 0\} \cup \{(1, 0)\}$$

is class-C^1 identifiable. Each of the first two component subsets is class-C^∞ identifiable.

A second example is an inverted cone in $\mathbb{R}^3$, whose apex is at the origin:

$$\Omega_2 = \left\{(\xi_1, \xi_2, \xi_3) \mid \xi_3 \geq \sqrt{\xi_1^2 + \xi_2^2},\ 0 \leq \xi_3 \leq 1\right\}.$$

Ω_2 has just three open facets: the point $(0, 0, 0)$, the circular face $\{(\xi_1, \xi_2, 1) \mid \xi_1^2 + \xi_2^2 < 1\}$, and the interior. A finite algebraic parametrization (3),(4) is apparently not possible, even if we allow $g_i \in C^1$ (the difficulty is, of course, at the apex). However, the whole set *can* be partitioned into five maximal class-C^∞ identifiable surfaces. They are the three open facets just mentioned, the circle $\{(\xi_1, \xi_2, 1) \mid \xi_1^2 + \xi_2^2 = 1\}$, and the curved face $\{(\xi_1, \xi_2, \xi_3) \mid \xi_3 = \sqrt{\xi_1^2 + \xi_2^2},\ 0 < \xi_3 < 1\}$. To show how the definition is satisfied in the case of $(0, 0, 0)$, take some $y \in \mathrm{int}(N(0, 0, 0))$. Then $y = (y_1, y_2, y_3)$, with $y_3 < -\sqrt{y_1^2 + y_2^2}$. Clearly, we can choose $\gamma > 0$ such that

$y + \gamma B \subset \mathrm{int}(N(0,0,0))$. Define three vectors as follows:

$$\begin{aligned} y^{(1)} &= (y_1 + \gamma, y_2, y_3) \\ y^{(2)} &= (y_1 - (1/2)\gamma, y_2 + (\sqrt{3}/2)\gamma, y_3) \\ y^{(3)} &= (y_1 - (1/2)\gamma, y_2 - (\sqrt{3}/2)\gamma, y_3). \end{aligned}$$

Elementary manipulation shows that these are linearly independent and that $y = (1/3)(y^{(1)} + y^{(2)} + y^{(3)})$. If we define $g_i(z) = z^T y^{(i)}$, the five conditions in Definition 2 are easily verified.

A third example is the set

$$\Omega_3 = \{(\xi_1, \xi_2, \xi_3) \mid \xi_3 \geq \xi_1^2 + |\xi_2| + \xi_2^{\sqrt{2}}\}.$$

This set is representable in the form (3),(4) by splitting the inequality into two (for the two possibilities $|\xi_2| = \pm\xi_2$), but the g_i are only C^1. An algebraic parametrization that uses C^p functions, $p \geq 2$, is not possible. There are no open facets, except the interior. However, the set can be partitioned into four class-C^∞ identifiable surfaces. These are the interior, the face defined by

$$\{(\xi_1, \xi_2, \xi_3) \mid \xi_2 > 0,\ \xi_3 = \xi_1^2 + \xi_2 + \xi_2^{\sqrt{2}}\}$$

and its counterpart

$$\{(\xi_1, \xi_2, \xi_3) \mid \xi_2 < 0,\ \xi_3 = \xi_1^2 - \xi_2 + \xi_2^{\sqrt{2}}\},$$

and the ridge

$$\{(\xi_1, 0, \xi_3) \mid \xi_3 = \xi_1^2\}.$$

The ridge can be made to fit the definition by taking

$$\begin{aligned} g_1(\xi) &= \xi_1^2 + \xi_2 - \xi_3 \\ g_2(\xi) &= \xi_1^2 - \xi_2 - \xi_3, \end{aligned}$$

independently of the choice of $y \in \mathrm{ri}(N(\xi))$.

The concept of a class-C^p identifiable surface is, in a certain sense, a generalization of the concept of a class-$C^{p,\alpha}$ boundary of a bounded domain $\Omega \subset I\!R^n$, as used extensively in the theory of partial differential equations (see, for example, the definition on page 94 of Gilbarg and Trudinger [9]). In fact, if Ω is convex, closed, and bounded, and its boundary $\partial\Omega$ is of class-$C^{p,0}$ according to the latter definition, then it can be partitioned into a class-C^∞ identifiable surface ($\mathrm{int}(\Omega)$) and a class-C^p identifiable surface ($\partial\Omega$). Such sets have no "edges" or "corners"—the value of r corresponding to each $y \in I\!R^n \backslash \Omega$ is 1—and hence, they are not very interesting from the viewpoint of this paper.

We now derive some elementary properties of identifiable surfaces and the functions g_i that are used to describe them. We focus on the case $S \subset \partial\Omega$, since the corresponding results for $S \subset \mathrm{int}(\Omega)$ are trivial.

LEMMA 2.1. *Let S be a class-C^p identifiable surface with $S \subset \partial\Omega$ and $p \geq 1$, and let $y \in I\!R^n \backslash \Omega$ be such that $\bar{y} = P(y) \in S$ and $y - \bar{y} \in \mathrm{ri}(N(\bar{y}))$. Suppose that $r = r(y)$, $\epsilon = \epsilon(y)$, and g_i, $i = 1, \ldots, r$ are chosen as in Definition 2. Then, for all $\bar{z} \in S(\bar{y}; \epsilon)$,*

(i) *$T_S(\bar z) \subset T(\bar z)$, where $T_S(\cdot)$ is the tangent cone with respect to S, as defined in Clarke* [2, *Thm.* 2.4.5];

(ii) *$T_S(\bar z) = \{s \mid s^T \nabla g_i(\bar z) = 0,\ i = 1, \ldots, r\}$;*

(iii) *$\mathrm{lin}(T(\bar z))^\perp = \mathrm{aff}(N(\bar z)) = \mathrm{span}\{\nabla g_i(\bar z),\ i = 1, \ldots, r\} = T_S(\bar z)^\perp$;*

(iv) *$\mathrm{ri}(\mathrm{co}\{\nabla g_i(\bar z),\ i = 1, \ldots, r\}) \subset \mathrm{ri}(N(\bar z))$.*

(v) *if $p \geq 2$, the projection of $\nabla^2 g_i(\bar z)$, $i = 1, \ldots, r$, onto $T_S(\bar z)$ is positive semidefinite.*

Proof.

(i) If $v \in T_S(\bar z)$, it follows from the definition of tangent cone that for any sequence $\{t_j\}$ with $t_j \downarrow 0$ there is a sequence v_j such that $\bar z + t_j v_j \in S$ and $v_j \to v$. Since $S \subset \Omega$ and $\bar z + t_j v_j \in S$,

$$0 \leq d_\Omega(\bar z + t_j v) \leq d_{\bar S}(\bar z + t_j v) = d_{\bar S}(\bar z + t_j v_j + t_j(v - v_j)) \leq t_j \|v - v_j\|.$$

Hence,

$$0 \leq \lim_{j \to \infty} \frac{d_\Omega(\bar z + t_j v) - d_\Omega(\bar z)}{t_j} \leq \lim_{j \to \infty} \|v - v_j\| = 0.$$

Since t_j is an arbitrary decreasing sequence,

$$d'_\Omega(\bar z; v) = \lim_{t \downarrow 0} \frac{d_\Omega(\bar z + tv) - d_\Omega(\bar z)}{t} = 0,$$

and so, by a result of Clarke [2, p. 53], $v \in T(\bar z)$.

(ii) This is a standard result, which follows from Definition 2(v).

(iii) We prove the second equality. By Definition 2(iii), $\mathrm{span}\{\nabla g_i(\bar z),\ i = 1, \ldots, r\} \subset \mathrm{aff}(N(\bar z))$. Since both sets are subspaces, the containment can be strict only if there is some $v \in \mathrm{aff}(N(\bar z))$ with $v \neq 0$ such that $v^T \nabla g_i(\bar z) = 0$, $i = 1, \ldots, r$, that is, $v \in T_S(\bar z)$. Clearly, also, $-v \in T_S(\bar z)$. Part (i) of this Lemma implies that v and $-v$ are in $T(\bar z)$, and hence, $v \in \mathrm{lin}(T(\bar z)) = N(\bar z)^\perp$. Hence, $0 \neq v \in \mathrm{aff}(N(\bar z)) \cap N(\bar z)^\perp$, giving a contradiction. The remaining equalities follow from Part (i) of the Theorem and

$$\mathrm{lin}(T(\bar z)) = N(\bar z)^\perp = \mathrm{aff}(N(\bar z))^\perp = T_S(\bar z).$$

(iv) From (iii), we have that the affine hulls of $\mathrm{co}\{\nabla g_i(\bar z),\ i = 1, \ldots, r\}$ and $N(\bar z)$ are identical. The result follows from the definition of $\mathrm{ri}(\cdot)$ and Definition 2(iii).

(v) Let $v \in T_S(\bar z)$, and suppose for contradiction that $v^T \nabla^2 g_i(\bar z) v < 0$. There are sequences $v_j \to v$ and $\{t_j\}$ with $0 < t_j \in \mathbb{R}$, $t_j \to 0$ such that $\bar z + t_j v_j \in S \subset \Omega$, so $g_i(\bar z + t_j v_j) = 0$, $i = 1, \ldots, r$. Since $\nabla g_i(\bar z) \in N(\bar z)$, we have $\nabla g_i(\bar z)^T t_j v_j \leq 0$, and so

$$0 = g_i(\bar z + t_j v_j) = g_i(\bar z) + \nabla g_i(\bar z)^T (t_j v_j) + \tfrac{1}{2} t_j^2 v_j \nabla^2 g_i(\hat v_j) v_j,$$

where $\hat v_j \in [\bar z, \bar z + t_j v_j]$. Hence,

$$v_j^T \nabla^2 g_i(\hat v_j) v_j = -\frac{2}{t_j} v_j^T \nabla g_i(\bar z) \geq 0.$$

For j sufficiently large,

$$0 > \tfrac{1}{2} v^T \nabla^2 g_i(\bar z) v \geq v_j^T \nabla^2 g_i(\hat v_j) v_j \geq 0,$$

giving a contradiction. □

The next result, which will be useful when we come to prove finite identification properties for constrained optimization algorithms, shows that the direct sum of an identifiable surface S and the relative interior of the normal cones along S, is a set that is open in $\mathbb{R}^n$. This property is analogous to that described for open facets in Theorem 2.8 of Burke and Moré [1].

LEMMA 2.2. *Suppose that S is as in Lemma 2.1 with $p \geq 2$. Define the set*

$$K = \{x + w \mid x \in S,\ w \in \mathrm{ri}(N(x))\}.$$

For each $y \in K$, there is a $\delta \in (0, \epsilon(y))$ such that $y + \delta B \subset K$, that is, K is open in $\mathbb{R}^n$.

Proof. We start by finding $\delta > 0$ such that $u \in y + \delta B \Rightarrow P(u) \in S$. Let $\bar{y} = P(y)$, $r = r(y)$, $\epsilon = \epsilon(y)$ and g_i, $i = 1, \ldots, r$ be chosen as in Definition 2. Let $\delta_1 \in (0, \epsilon)$ have the property that $(y + \delta_1 B) \cap \Omega = \emptyset$. By Definition 2(ii),(iv),(v), we know that there is $\lambda \in \mathbb{R}^r$ with $\lambda > 0$ such that

$$\begin{aligned} y - \bar{y} - \nabla g(\bar{y})\lambda &= 0, \\ g(\bar{y}) &= 0. \end{aligned} \tag{5}$$

From Definition 2(i) and Lemma 2.1(iii),we know that the matrix

$$\begin{bmatrix} I + \sum_{i=1}^{r} \lambda_i \nabla^2 g_i(\bar{y}) & \nabla g(\bar{y}) \\ \nabla g(\bar{y})^T & 0 \end{bmatrix}$$

is nonsingular and continuous with respect to $\bar{y}$ and λ. We can now view $\bar{y}$ and λ as functions of y in (5), and apply the implicit function theorem to obtain the following result: There is $\delta \in (0, \delta_1]$ such that, if $\|u - y\| \leq \delta$, the solution $(\bar{u}, \lambda^u)$ of the system

$$\begin{aligned} u - \bar{u} - \nabla g(\bar{u})\lambda^u &= 0, \\ g_i(\bar{u}) &= 0 \end{aligned} \tag{6}$$

satisfies

$$\lambda^u > 0, \qquad \|\bar{u} - \bar{y}\| < \epsilon. \tag{7}$$

Now (6) and (7) imply that $(\bar{u}, \lambda^u)$ solves the problem

$$\min_{\bar{u}} \tfrac{1}{2}\|u - \bar{u}\|^2, \qquad g(\bar{u}) = 0, \qquad \|\bar{u} - \bar{y}\| < \epsilon$$

and so, by Definition 2(v), $\bar{u}$ is the projection of u onto $S(\bar{y}; \epsilon)$. From (6), (7), and Definition 2(iii), we have that

$$u - \bar{u} \in \mathrm{co}\,\{\nabla g_i(\bar{u}),\ i = 1, \ldots, r\} \subset N(\bar{u}).$$

Now $\bar{u} \in \Omega$, $u - \bar{u} \in N(\bar{u})$ and uniqueness of the projection onto a convex set imply that $\bar{u} = P(u)$.

Finally, since $\lambda^u > 0$, we have

$$u - \bar{u} \in \mathrm{ri}(\mathrm{co}\,\{\nabla g_i(\bar{u}),\ i = 1, \ldots, r\}).$$

Hence, by Lemma 2.1(iv),

$$u - \bar{u} \in \mathrm{ri}(N(\bar{u})),$$

and so $u \in K$, as required. $\square$

In the following two results, we show how open facets and active index sets relate to identifiable surfaces.

THEOREM 2.3. *Let S be an open facet in Ω. Then S is a class-C^∞ identifiable surface.*

Proof. The case $S = \mathrm{int}(\Omega)$ is trivially true. Consider $S \subset \partial\Omega$. Burke and Moré [1] show that any open facet S is the relative interior of a quasipolyhedral face. Hence, $N(\bar{y})$ and $T(\bar{y})$ are the same for all $\bar{y} \in S$, and

$$\mathrm{aff}(S) = \bar{y} + \mathrm{lin}(T(\bar{y})) \tag{8}$$

for all $\bar{y} \in S$.

Suppose, as in Definition 2, that we are given some y such that $\bar{y} = P(y) \in S$ and $y - \bar{y} \in \mathrm{ri}(N(\bar{y}))$. Then there is a constant $\gamma > 0$ such that $(y-\bar{y}) + \gamma v \in \mathrm{ri}(N(\bar{y}))$ for all $v \in \mathrm{aff}(N(\bar{y}))$ with $\|v\| = 1$. Supposing that $\mathrm{aff}(N(\bar{y}))$ has dimension r, we can choose unit vectors $v_1, \ldots, v_{r-1}$, such that $\{v_1, \ldots, v_{r-1}, y-\bar{y}\}$ is linearly independent in $\mathrm{aff}(N(\bar{y}))$, and hence a spanning set. Now set

$$v_r = -\frac{1}{r-1}(v_1 + \cdots + v_{r-1})$$

and

$$\hat{v}_i = y - \bar{y} + \gamma v_i, \qquad i = 1, \ldots, r.$$

Clearly, $\hat{v}_i \in \mathrm{aff}(N(\bar{y}))$ and $\|\hat{v}_i - (y-\bar{y})\| \le \gamma\|v_i\| \le \gamma$, so $\hat{v}_i \in \mathrm{ri}(N(\bar{y}))$, $i = 1, \ldots, r$. Moreover, we can show that $\{\hat{v}_1, \ldots, \hat{v}_r\}$ is linearly independent by the following argument: Suppose there are real coefficients $\mu_1, \ldots, \mu_r$ such that $\sum \mu_i \hat{v}_i = 0$. Then

$$\begin{aligned} 0 = \sum_{i=1}^{r} \mu_i \hat{v}_i &= \left(\sum_{i=1}^{r} \mu_i\right)(y-\bar{y}) + \gamma \sum_{i=1}^{r} \mu_i v_i \\ &= \left(\sum_{i=1}^{r} \mu_i\right)(y-\bar{y}) + \gamma \sum_{i=1}^{r-1} [\mu_i - \mu_r/(r-1)] v_i. \end{aligned}$$

By the original choice of $v_1, \ldots, v_{r-1}$, we must have

$$\sum_{i=1}^{r} \mu_i = 0, \quad \mu_i = \mu_r/(r-1), \quad i = 1, \ldots, r-1,$$

and it follows that $\mu_1 = \cdots = \mu_r = 0$, as desired. Now define $g_i(z) = (z-\bar{y})^T \hat{v}_i$, $i = 1, \ldots, r$. Conditions (i) and (ii) of Definition 2 are readily verified. Condition (iii) follows since $N(\bar{z})$ is constant for $\bar{z} \in S$, and $\hat{v}_i \in \mathrm{ri}(N(\bar{z}))$, $i = 1, \ldots, r$. Condition (iv) is verified by noting that

$$y - \bar{y} = \sum_{i=1}^{r-1} \frac{\hat{v}_i}{2(r-1)} + \frac{\hat{v}_r}{2} \in \mathrm{co}\{\hat{v}_i,\ i = 1, \cdots, r\}.$$

To prove condition (v), we first take $V = \mathrm{aff}(S)$ in Definition 1 and note that, if $\bar{y} \in S$, there is $\epsilon > 0$ such that $\bar{z} \in \mathrm{aff}(S) \cap (\bar{y} + \epsilon B) \Rightarrow \bar{z} \in S$. That is,

$$S(\bar{y}; \epsilon) = \{\bar{z} \mid \|\bar{z} - \bar{y}\| \le \epsilon,\ \bar{z} \in \mathrm{aff}(S)\} = \mathrm{aff}(S) \cap (\bar{y} + \epsilon B).$$

However, by (8),

$$\mathrm{aff}(S) = \bar{y} + \mathrm{lin}(T(\bar{y})) = \bar{y} + N(\bar{y})^{\perp},$$

and so,

$$\bar{z} \in \mathrm{aff}(S) \Leftrightarrow (\bar{z} - \bar{y})^T \hat{v}_i = 0, \quad i = 1, \dots, r.$$

Hence,

$$S(\bar{y}, \epsilon) = \{\bar{z} \mid \|\bar{z} - \bar{y}\| \leq \epsilon,\ g_i(\bar{z}) = 0,\ i = 1, \dots, r\},$$

as required. □

THEOREM 2.4. *Suppose that Ω is defined by (3) and (4), where g_i, $i = 1, \dots, m$ are C^1 functions. Suppose that for some set $\mathcal{A} \subset \{1, \dots, m\}$, the surface S defined by*

$$S = \{z \mid g_i(z) = 0,\ i \in \mathcal{A},\ g_i(z) > 0,\ i \notin \mathcal{A}\}$$

is a connected subset of Ω. Then S is a class-C^1 identifiable surface. Moreover, if $g_i \in C^p$ for $i \in \mathcal{A}$ and $p \geq 2$, then S is a class-C^p identifiable surface.

Proof. This follows trivially, by identifying g_i, $i \in \mathcal{A}$ with g_i, $i = 1, \dots, r$, in Definition 2. □

We now consider smoothness of the projection operator $P(.)$. The motivation for this comes from the work of Holmes [10] and Fitzpatrick and Phelps [6], who consider closed convex sets with smooth boundaries. In these papers, smoothness of the boundary is defined in terms of smoothness of the gauge function

$$\rho_{\Omega}(x) = \inf\{t > 0 \mid x \in t(\Omega - x_0) + x_0\}, \qquad \text{for some } x_0 \in \mathrm{int}(\Omega),$$

and the boundary of Ω is said to be C^p if ρ_Ω is C^p in some neighborhood of $\partial\Omega$. By showing that this definition is equivalent to a local C^p parametrization of the boundary, Holmes [10] essentially shows that a C^p boundary (by the definition above) is the same as a class-$C^{p,0}$ boundary, as defined in [9]. Hence, as discussed earlier, $\partial\Omega$ is a class-C^p identifiable surface.

Holmes proves the following result.

THEOREM 2.5 ([10, Thm. 2]). *If Ω has a C^p boundary, for $p \geq 2$, then the projection operator $P(.)$ is C^{p-1} in $\mathbb{R}^n \backslash \Omega$, and $P'(y)$ is invertible in* $\mathrm{lin}(T(P(y)))$.
Fitzpatrick and Phelps [6, Thm. 3.10] prove the converse.

The case of $p = 2$ is the most interesting. It is a classical result [5, p. 216] that, since P is Lipschitz continuous, it is differentiable almost everywhere. Below, we extend Theorem 2.5 to sets with *piecewise* smooth boundaries, by showing that class-C^p identifiable surfaces generate open regions in $\mathbb{R}^n \backslash \Omega$ in which P is C^{p-1}.

THEOREM 2.6. *Let S and K be as defined in Lemma 2.2, with $p \geq 2$. Then $P(.)$ is C^{p-1} on K. Also, $P'(y)$ is invertible in* $\mathrm{lin}(T(P(y)))$.

Proof. For any $y \in K$, we can choose $\epsilon > 0$ and $\delta > 0$ as in Lemma 2.2 such that, when $u \in y + \epsilon B$, $P(u)$ is also the projection of u onto the set $\{\bar{z} \mid g_i(\bar{z}) = 0,\ i = 1, \dots, r,\ \|\bar{z} - \bar{y}\| \leq \epsilon\}$. Hence, we can differentiate the system (5) with respect to y to obtain

$$\begin{bmatrix} I + \sum_{i=1}^{r} \lambda_i \nabla^2 g_i(\bar{y}) & \nabla g(\bar{y}) \\ \nabla g(\bar{y})^T & 0 \end{bmatrix} \begin{bmatrix} \frac{d\bar{y}}{dy} \\ \frac{d\lambda}{dy} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{9}$$

where $P'(y) = d\bar{y}/dy$. The first result follows immediately from (5) and the implicit function theorem (see, for example, Lang [11, p. 125]) by noting that the coefficient matrix in (9) is nonsingular.

For the second result, let $Z \in \mathbb{R}^{n\times(n-r)}$ be a matrix of full rank such that $\nabla g(\bar{y})^T Z = 0$. By Lemma 2.1(iii), the columns of Z span $\operatorname{lin}(T(P(y)))$. The second equation in (9) implies that

$$P'(y) = \frac{d\bar{y}}{dy} = ZW^T$$

for some $W \in \mathbb{R}^{n\times(n-r)}$. Multiplying the first equation in (9) by Z^T, we find that

$$Z^T\left(I + \sum_{i=1}^{r} \lambda_i \nabla^2 g_i(\bar{y})\right) ZW^T = Z^T$$

and so

$$P'(y) = Z\left[Z^T\left(I + \sum_{i=1}^{r} \lambda_i \nabla^2 g_i(\bar{y})\right) Z\right]^{-1} Z^T. \tag{10}$$

It follows from (10) and Lemma 2.1(v) that $P'(y)$ has nonsingular projection onto $\operatorname{lin}(T(P(y)))$. $\square$

We conjecture that the converse of this theorem is also true; that is, if there is an open connected region $K \subset \mathbb{R}^n \setminus \Omega$ such that $P(.)$ is C^{p-1} on K, and $P'(y)$ is invertible in $\operatorname{lin}(T(P(y)))$ for each $y \in K$, then $P(K)$ is a class-C^p identifiable surface. The continuity condition alone is not sufficient, as an example from Fitzpatrick and Phelps [6, p. 496] illustrates. Define

$$\Omega_4 = \{(\xi_1, \xi_2) \mid \xi_2 \geq |\xi_1| + \xi_1^{4/3}\}.$$

There is a corner in Ω_4 at $(0,0)$, and the set has four maximal class-C^∞ identifiable surfaces: the corner, the interior, and the two edges. Tedious calculation shows that P is C^1 on $\mathbb{R}^n \setminus \Omega$, although $\partial\Omega$ is obviously not a class-C^2 surface. It can be shown that $P'(y) = 0$ along the lines $\{(\xi_1, \xi_2) \mid \xi_2 = -\xi_1,\ \xi_1 > 0\}$ and $\{(\xi_1, \xi_2) \mid \xi_2 = \xi_1,\ \xi_1 < 0\}$, and so the invertibility condition is not satisfied.

It is clear from (10) that the invertibility condition is related to the boundedness of the quantities $\lambda_i \nabla^2 g_i(\bar{y})$ on $\operatorname{lin}(T(\bar{y}))$. Note that these quantities are invariant under scaling of the g_is, that is, if g_i is replaced by αg_i, then λ_i becomes λ_i/α.

3. Finite identification in constrained optimization algorithms. We turn now to algorithms for solving the optimization problem (1).

In analyzing the gradient projection algorithm, we use the work of Dunn [4, §2], who provided a framework for proving capture results. Dunn states this algorithm as follows: Choosing constants γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < 1$, and an initial iterate $x_0 \in \Omega$, set

$$x_{k+1} = P(x_k - \sigma_k \nabla F(x_k)), \tag{11}$$

where σ_k is chosen to satisfy

$$F(x_k) - F(P(x_k - \nabla F(x_k))) \geq \gamma_1 \Rightarrow \sigma_k = 1, \tag{12}$$

$$F(x_k) - F(P(x_k - \nabla F(x_k))) < \gamma_1 \Rightarrow \sigma_k \in (0,1) \tag{13}$$

and

$$\gamma_1 \leq \frac{F(x_k) - F(P(x_k - \sigma_k \nabla F(x_k)))}{\nabla F(x_k)^T [x_k - P(x_k - \sigma_k \nabla F(x_k))]} \leq \gamma_2. \tag{14}$$

Gawande and Dunn [8] adapted Dunn's earlier work to prove capture and convergence results for *scaled* gradient projection algorithms and algebraic parametrizations (3) of the feasible set.

We start with a simple result that exploits openness of the set K of Lemma 2.2:

THEOREM 3.1. *Suppose that*

(i) *Assumption* 1 *and* (2) *hold at some point* x^*;
(ii) ∇F *is continuous at* x^*;
(iii) $x^* \in S$, *where* S *is a class-*C^p *identifiable surface of* Ω *with* $p \geq 1$;
(iv) *there is* $\bar{\sigma} > 0$ *such that* $\sigma_k \in [\bar{\sigma}, 1]$ *for all* k; *and*
(v) *the sequence* $\{x_k\}$ *generated by* (11)–(14) *converges to* x^*.

Then $x_k \in S$ *for all* k *sufficiently large.*

Proof. Define the set K as in Lemma 2.2 Setting $y = x^* - \nabla F(x^*)$, we can apply Definition 2 to find $\delta > 0$ such that

$$x^* - \nabla F(x^*) + \delta B \subset K.$$

By construction of K, this implies that

$$x^* - \sigma \nabla F(x^*) + \sigma\delta B \subset K \quad \text{for all} \quad \sigma \in [\bar{\sigma}, 1].$$

Now, choose $\bar{k}$ such that, for all $k \geq \bar{k}$,

$$\|x_k - x^*\| + \|\nabla F(x_k) - \nabla F(x^*)\| \leq \bar{\sigma}\delta.$$

Then

$$\|[x_k - \sigma_k \nabla F(x_k)] - [x^* - \sigma_k \nabla F(x^*)]\| \leq \bar{\sigma}\delta,$$

and so

$$x_k - \sigma_k \nabla F(x_k) \in x^* - \sigma_k \nabla F(x^*) + \bar{\sigma}\delta B \subset x^* - \sigma_k \nabla F(x^*) + \sigma_k \delta B \subset K.$$

Hence, $x_{k+1} \in P(K) = S$, for $k \geq \bar{k}$. $\square$

Before proving the next result, we state second-order conditions and define some terms:

Assumption 2. Suppose that Ω satisfies Assumption 1 and that there is x^* that satisfies (2), such that $x^* \in S$, where S is a class-C^p identifiable surface of Ω with $p \geq 2$. Suppose that F is twice continuously differentiable in a neighborhood of x^*, and let g_i, $i = 1, \ldots, r$ be as defined in Definition 2, for $y = x^* - \nabla F(x^*)$. Choose $\lambda^* \in \mathbb{R}^r$ such that $\lambda^* > 0$ and

$$-\nabla F(x^*) = \nabla g(x^*) \lambda^*, \tag{15}$$

and suppose that for all $h \in T_S(x^*)$,

$$h^T \left[\nabla^2 F(x^*) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^*) \right] h \geq \alpha \|h\|^2, \qquad \text{for some } \alpha > 0.$$

DEFINITION 3.

(i) x^* is a proper local minimizer of F in Ω if there is $\rho_1 > 0$ such that

$$x \in \Omega,\ 0 < \|x - x^*\| \leq \rho_1 \Rightarrow F(x) > F(x^*).$$

(ii) x^* is a stable fixed point for (11)–(14) if $-\nabla F(x^*) \in N(x^*)$, and there are $d_1 > 0$, $d_2 > 0$ such that

$$\|x_0 - x^*\| \leq d_1 \Rightarrow \|x_k - x^*\| \leq d_2 \qquad \text{for all } k \geq 0.$$

(iii) x^* is a stable local attractor for (11)–(14) if it is a stable fixed point, and $d_1 > 0$ can be chosen so that

$$\|x_0 - x^*\| \leq d_1 \Rightarrow \lim_{k \to \infty} x_k = x^*.$$

The following theorem contains capture and convergence results like those proved in Gawande and Dunn [8, §4]. Here, we prove these results for the gradient projection method on the identifiable surface containing x^*; in [8], the focus was on scaled gradient projection methods and active index sets for Ω defined by (3).

THEOREM 3.2. *Suppose that Assumption 2 holds. Then*

(i) *there are positive scalars ρ_1 and α_1 such that*

$$x \in \Omega,\ \|x - x^*\| \leq \rho_1 \Rightarrow F(x) - F(x^*) \geq \alpha_1 \|x - x^*\|^2;$$

(ii) *there are positive scalars ρ_2 and α_2 such that*

$$x \in S,\ \|x - x^*\| \leq \rho_2 \Rightarrow \|x - P(x - \nabla F(x))\| \geq \alpha_2 \|x - x^*\|,$$

that is, the defect $E(x) = x - P(x - \nabla F(x))$, restricted to S, has an isolated zero at x^*;

(iii) *given any $\bar{\sigma} > 0$, there is $\rho_3 = \rho_3(\bar{\sigma}) > 0$ such that*

$$\|x - x^*\| \leq \rho_3,\ \sigma \in [\bar{\sigma}, 1] \Rightarrow P(x - \sigma \nabla F(x)) \in S;$$

(iv) *x^* is a stable local attractor for the gradient projection algorithm, and the sequences $\{x_k\}$ that approach x^* eventually enter and remain in S.*

Proof. Throughout the proof, let ϵ denote $\epsilon(x^* - \nabla F(x^*))$.

(i) We show first that if $w = \nabla g(x^*)\mu$ with $\nabla g_i(x^*)^T w \leq c_1$ for $c_1 \geq 0$ and $i = 1, \ldots, r$, then there is some $\tau_1 > 0$ such that

$$\nabla F(x^*)^T w \geq \tau_1 \|w\| + O(c_1). \tag{16}$$

Using the definition of λ^* from Assumption 2, we have

$$\begin{aligned} \nabla F(x^*)^T w &= -\lambda^{*T} \nabla g(x^*)^T \nabla g(x^*) \mu \\ &\geq (\min_i \lambda_i^*) \|\nabla g(x^*)^T \nabla g(x^*) \mu\| + O(c_1). \end{aligned}$$

Since $\nabla g(x^*)$ has full rank, and $\|\mu\| \geq \|w\| / \|\nabla g(x^*)\|$, there is τ_2 such that

$$\|\nabla g(x^*)^T \nabla g(x^*) \mu\| \geq \tau_2 \|\mu\| \geq \frac{\tau_2}{\|\nabla g(x^*)\|} \|w\|.$$

By setting

$$\tau_1 = (\min_i \lambda_i^*) \frac{\tau_2}{\|\nabla g(x^*)\|},$$

we obtain (16).

Now, given x in the vicinity of x^*, we seek vectors $v \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^r$ such that

$$\begin{aligned} v + \nabla g(x^*)\mu &= x - x^* \\ g(x^* + v) &= 0. \end{aligned} \tag{17}$$

We can again apply the implicit function theorem to (17) to find $\bar{\rho}_1 > 0$ such that a solution v, μ exists for $\|x - x^*\| \leq \bar{\rho}_1$. Moreover, $\bar{\rho}_1$ can be chosen small enough that $\|v\| \leq \epsilon$, and hence, $x^* + v \in S \subset \Omega$. It follows that, since $\nabla g_i(x^* + v) \in N(x^* + v)$ for $i = 1, \ldots, r$, and since $x \in \Omega$,

$$\nabla g_i(x^* + v)^T [x - (x^* + v)] \leq 0, \quad i = 1, \ldots, r.$$

Writing $w = \nabla g(x^*)\mu = x - (x^* + v)$, we have

$$\nabla g_i(x^*)^T w = \nabla g_i(x^* + v)^T w + O(\|v\|\|w\|) \leq O(\|v\|\|w\|), \quad i = 1, \ldots, r.$$

Application of (16) shows that

$$\nabla F(x^*)^T w \geq \tau_1 \|w\| + O(\|v\|\|w\|). \tag{18}$$

Now consider the v component. Since $x^* + v \in S$, and since (15) holds,

$$\begin{aligned} F(x^* + v) - F(x^*) &= [F(x^* + v) + \lambda^{*T} g(x^* + v)] - [F(x^*) + \lambda^{*T} g(x^*)] \\ &= \tfrac{1}{2} v^T \left[\nabla^2 F(x^* + \beta_1 v) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^* + \beta_1 v) \right] v \end{aligned} \tag{19}$$

for some $\beta_1 \in (0, 1)$. Since $\nabla g_i(x^*)^T v = O(\|v\|^2)$, we can choose $\bar{\rho}_2 \in (0, \bar{\rho}_1]$ such that when $\|x - x^*\| \leq \bar{\rho}_2$, v is close enough to $T_S(x^*)$ and $\|v\|$ is small enough that

$$v^T \left[\nabla^2 F(x^* + \beta_1 v) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^* + \beta_1 v) \right] v \geq \frac{\alpha}{2} \|v\|^2,$$

for all $\beta_1 \in [0, 1]$. Hence, from (19),

$$F(x^* + v) - F(x^*) \geq \frac{\alpha}{4} \|v\|^2. \tag{20}$$

By using (18) and (20), we can now write that, for $x \in \Omega \cap (x^* + \bar{\rho}_2 B)$,

$$\begin{aligned} F(x) - F(x^*) &= F(x^* + v + w) - F(x^* + v) + F(x^* + v) - F(x^*) \\ &= \nabla F(x^*)^T w + O(\|v\|\|w\| + \|w\|^2) + F(x^* + v) - F(x^*) \\ &\geq \tau_1 \|w\| + \frac{\alpha}{4} \|v\|^2 + O(\|v\|\|w\| + \|w\|^2) \\ &\geq \tau_1 \|w\| + \frac{\alpha}{4} \|v\|^2 - c_2 (\|v\|\|w\| + \|w\|^2), \end{aligned} \tag{21}$$

where $c_2 > 0$ is some constant. Now, choose a constant $\delta_1 > 0$ such that

$$c_2(\delta_1^2 + \delta_1) \leq \frac{\alpha}{8}, \tag{22}$$

and define $\rho_1 \in (0, \bar{\rho}_2]$ such that both of the following conditions are satisfied:

$$x \in \Omega \cap (x^* + \rho_1 B) \Rightarrow \|w\| \le \frac{\tau_1 \delta_1}{2c_2(1+\delta_1)}, \tag{23}$$

$$\rho_1 \le \frac{4\tau_1\delta_1(1+\delta_1)}{\alpha}. \tag{24}$$

In the case $\|w\| \ge \delta_1 \|v\|$, we have

$$\|x - x^*\| \le \|w\| + \|v\| \le \left(1 + \frac{1}{\delta_1}\right)\|w\|. \tag{25}$$

Also, from (21),

$$F(x) - F(x^*) \ge \tau_1\|w\| - c_2\left(1 + \frac{1}{\delta_1}\right)\|w\|^2. \tag{26}$$

Now, from (23), (25), and (26), we have that

$$F(x) - F(x^*) \ge \frac{\tau_1}{2}\|w\| \ge \frac{\tau_1\delta_1}{2(1+\delta_1)}\|x - x^*\| \ge \frac{\tau_1\delta_1}{2(1+\delta_1)\rho_1}\|x - x^*\|^2,$$

for $x \in \Omega \cap (x^* + \rho_1 B)$. Application of (24) yields that

$$F(x) - F(x^*) \ge \frac{\alpha}{8(1+\delta_1)^2}\|x - x^*\|^2. \tag{27}$$

In the remaining case $\|w\| < \delta_1\|v\|$, we find from (21) and (22) that

$$F(x) - F(x^*) \ge \frac{\alpha}{4}\|v\|^2 - c_2(\delta_1^2 + \delta_1)\|v\|^2 \ge \frac{\alpha}{8}\|v\|^2.$$

Also,

$$\|x - x^*\| \le \|v\| + \|w\| < (1+\delta_1)\|v\|,$$

and hence, (27) still applies. The result follows by setting

$$\alpha_1 = \frac{\alpha}{8(1+\delta_1)^2}.$$

(ii) By setting $y = x^* - \nabla F(x^*)$ in Lemma 2.2, we can choose $\delta \in (0, \epsilon]$ such that $P(x^* - \nabla F(x^*) + \delta B) \subset S$. Now, there is a $\bar{\rho}_1 \in (0, \epsilon]$ such that

$$\|x - x^*\| \le \bar{\rho}_1 \Rightarrow \|[x - \nabla F(x)] - [x^* - \nabla F(x^*)]\| \le \delta,$$

and hence, $\hat{x} = P(x - \nabla F(x)) \in S$. By contractivity of $P(.)$, $\|\hat{x} - x^*\| = \|P(x - \nabla F(x)) - P(x^* - \nabla F(x^*))\| \le \delta \le \epsilon$. It therefore follows from Definition 2 (v) that $\hat{x}$ solves the projection subproblem

$$\min_{\hat{x}} \tfrac{1}{2}\|\hat{x} - (x - \nabla F(x))\|_2^2, \qquad g(\hat{x}) = 0. \tag{28}$$

When $x = x^*$, then $\hat{x} = x^*$, and (15) holds. By the implicit function theorem, $\bar{\rho}_2 \in (0, \bar{\rho}_1]$ can be chosen small enough that there is λ such that, in fact,

$$\|x - x^*\| \le \bar{\rho}_2 \Rightarrow [x - \nabla F(x)] - \hat{x} = \nabla g(\hat{x})\lambda, \tag{29}$$

with

$$\|\hat{x} - x^*\| = O(\|x - x^*\|), \qquad \|\lambda - \lambda^*\| = O(\|x - x^*\|), \;\; \lambda > 0.$$

Since $\|x - x^*\| \le \bar{\rho}_1 \le \epsilon$, we also have by Definition 2(v) that $g_i(x) = 0$, $i = 1, \ldots, r$.

Let $Z \in I\!R^{n \times (n-r)}$ be an orthonormal matrix whose columns span the subspace $T_S(x^*)$. By using a Taylor series expansion of g about x^*, it is easy to show that there are vectors $\eta, \hat{\eta} \in I\!R^{n-r}$ and $\zeta, \; \hat{\zeta} \in I\!R^r$ such that

$$x - x^* = Z\eta + \nabla g(x^*)\zeta, \tag{30}$$

$$x - \hat{x} = Z\hat{\eta} + \nabla g(x^*)\hat{\zeta}, \tag{31}$$

where $\|\zeta\| = O(\|x - x^*\|^2)$ and $\|\hat{\zeta}\| = O(\|x - x^*\|^2) + O(\|\hat{x} - x^*\|^2) = O(\|x - x^*\|^2)$. From the second-order conditions and boundedness of $\nabla^2 g_i$ in a neighborhood of x^* there exists a constant $c_2 > 0$ such that

$$\eta^T Z^T \left[\nabla^2 F(x^*) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^*) \right] Z\eta \ge \alpha \|\eta\|^2, \tag{32}$$

and

$$\left\| Z^T \left[I + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^*) \right] Z\eta \right\| \le c_2 \|\eta\|, \tag{33}$$

for all $\eta \in I\!R^{n-r}$. It follows trivially from (32) that

$$\|Z^T \left[\nabla^2 F(x^*) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^*) \right] Z\eta\| \ge \alpha \|\eta\|. \tag{34}$$

Now, from (29),

$$\begin{aligned}
& x - \hat{x} = \nabla F(x) + \nabla g(\hat{x})\lambda \\
\Rightarrow \;\; & x - \hat{x} = \nabla F(x) + \nabla g(x)\lambda^* + [\nabla g(\hat{x}) - \nabla g(x)]\lambda + \nabla g(x)[\lambda - \lambda^*] \\
\Rightarrow \;\; & x - \hat{x} = \left[\nabla^2 F(x^{(1)}) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^{(1)}) \right] (x - x^*) + \\
& \qquad \sum_{i=1}^r \lambda_i \nabla^2 g_i(x^{(2)})(\hat{x} - x) + \nabla g(x^*)[\lambda - \lambda^*] + O(\|x - x^*\|^2),
\end{aligned}$$

for $x^{(1)} \in [x, x^*]$ and $x^{(2)} \in [\hat{x}, x]$. Premultiplying this equation by Z^T, and using (30) and (31), we obtain

$$\begin{aligned}
& Z^T \left[I + \sum_{i=1}^r \lambda_i \nabla^2 g_i(x^{(2)}) \right] [Z\hat{\eta} + \nabla g(x^*)\hat{\zeta}] \\
&= Z^T \left[\nabla^2 F(x^{(1)}) + \sum_{i=1}^r \lambda_i^* \nabla^2 g_i(x^{(1)}) \right] [Z\eta + \nabla g(x^*)\zeta] + O(\|x - x^*\|^2).
\end{aligned}$$

Now, from (33) and (34), we can choose $\bar{\rho}_3 \in (0, \bar{\rho}_2]$ small enough that, for $\|x - x^*\| \leq \bar{\rho}_3$,

$$2c_2\|\hat{\eta}\| \geq \left\| Z^T \left[I + \sum_{i=1}^{r} \lambda_i \nabla^2 g_i(x^{(2)}) \right] Z\hat{\eta} \right\|$$
$$\geq O(\|\hat{\zeta}\|) + O(\|\zeta\|) + O(\|x - x^*\|^2) + \frac{\alpha}{2}\|\eta\|.$$

Since

$$\|\hat{\eta}\| = \|x - \hat{x}\| + O(\|x - x^*\|^2),$$
$$\|\eta\| = \|x - x^*\| + O(\|x - x^*\|^2),$$

there is a constant $c_3 > 0$ such that

$$2c_2\|x - \hat{x}\| \geq \frac{\alpha}{2}\|x - x^*\| - c_3\|x - x^*\|^2$$
$$\Rightarrow \quad \|x - \hat{x}\| \geq \frac{\alpha}{4c_2}\|x - x^*\| \left[1 - \frac{2c_3}{\alpha}\|x - x^*\| \right].$$

Now, choosing $\rho_2 = \min(\bar{\rho}_3, \alpha/(4c_3))$, the desired result follows, with $\alpha_2 = \alpha/(8c_2)$.

(iii) The proof of this part is identical to that of Theorem 3.1, and hence, is omitted.

(iv) This follows from Theorem 2.1 of Dunn [4], after we make the following observations. Part (i) of this theorem implies that x^* is a uniformly proper local minimizer of F in Ω. The fact that $F \in C^2$ in a neighborhood of x^* means that it is possible to choose a $\bar{\sigma} \in (0, 1)$ such that, for x_k in this neighborhood, any σ_k satisfying (12)–(14) lies in $[\bar{\sigma}, 1]$. □

We turn now to Newton-like methods for (1). Here, an initial iterate $x_0 \in \Omega$ is chosen, and for each $k \geq 0$, the following subproblem is solved to find a search direction p_k:

$$\min_{p_k} \nabla F(x_k)^T p_k + \tfrac{1}{2} p_k^T B_k p_k, \qquad x_k + p_k \in \Omega. \tag{35}$$

A steplength $\sigma_k \in [0, 1]$ is chosen, usually with the help of some "sufficient decrease" criterion, and the next iterate is obtained by setting

$$x_{k+1} = x_k + \sigma_k p_k. \tag{36}$$

A simple result, similar to Theorem 3.1, follows.

THEOREM 3.3. *Suppose that*

(i) *Assumption* 1 *and* (2) *hold at some point* x^*;
(ii) $\nabla F(x)$ *is continuous at* x^*;
(iii) $x^* \in S$, *where* S *is some class-*C^p *identifiable surface of* Ω *with* $p \geq 1$; *and*
(iv) $x_k \to x^*$ *and* $p_k \to 0$ *as* $k \to \infty$, *and* $\{\|B_k\|\}$ *is bounded.*

Then $x_k + p_k \in S$ *for all* k *sufficiently large.*

Proof. As in Lemma 2.2, we can find a set $K \subset \mathbb{R}^n \backslash \Omega$ with $P(K) \subset S$, and a scalar $\delta > 0$ such that

$$x^* - \nabla F(x^*) + \delta B \subset \mathrm{int}(K).$$

First-order conditions for (35) are that

$$-\nabla F(x_k) - B_k p_k \in N(x_k + p_k) \Leftrightarrow x_k + p_k = P(x_k + p_k - \nabla F(x_k) - B_k p_k).$$

Now,

$$\begin{aligned}
&\|(x_k + p_k - \nabla F(x_k) - B_k p_k) - (x^* - \nabla F(x^*))\| \\
&\leq \|x_k - x^*\| + \|\nabla F(x_k) - \nabla F(x^*)\| + (1 + \|B_k\|)\|p_k\|.
\end{aligned}$$

We can choose $\bar{k}$ large enough that, for $k \geq \bar{k}$, the right-hand side of the above inequality does not exceed δ. Then $x_k + p_k \in S$, as required. □

Finally, we prove a capture and convergence result for Newton's method, which makes use of the second-order conditions in Assumption 2.

THEOREM 3.4. *Suppose that Assumption* 2 *holds, and that, in addition,* $\nabla^2 F(x)$ *is Lipschitz continuous in a neighborhood of* x^*. *Let* $B_k = \nabla^2 F(x_k)$ *in* (35). *Then there are positive constants* ρ_4 *and* α_4 *such that, if* $x_0 \in \Omega \cap (x^* + \rho_4 B)$ *and* $\sigma_k \equiv 1$ *for all* $k \geq 0$, *then the algorithm* (35), (36) *generates a sequence* $\{x_k\}$ *such that*

$$\|x_{k+1} - x^*\| \leq \alpha_4 \|x_k - x^*\|^2 \qquad \text{for all } k \geq 0.$$

In addition, $x_k \in S$ *for all* k *sufficiently large.*

Proof. The proof of the first part follows from results of Dunn [3, Thm. 3.1, Note 3.1] provided that we find positive constants $\bar{\alpha}_1$ and $\bar{\rho}_1$ such that

$$\nabla F(x^*)^T(x - x^*) + \tfrac{1}{2}(x - x^*)^T \nabla^2 F(x^*)(x - x^*) \geq \bar{\alpha}_1 \|x - x^*\|^2 \tag{37}$$

for all $x \in \Omega \cup (x^* + \bar{\rho}_1 B)$.

Suppose we choose $\bar{\rho}_1$ small enough that F is twice Lipschitz continuously differentiable on the open ball $x^* + \bar{\rho}_1 B$, with Lipschitz constant L and, in addition, that

$$\bar{\rho}_1 \leq \min(\rho_1, \alpha_1 / L),$$

where α_1 and ρ_1 are the constants from Theorem 3.2(i). For $\|x - x^*\| \leq \bar{\rho}_1$, we have

$$\begin{aligned}
&\nabla F(x^*)^T(x - x^*) + \tfrac{1}{2}(x - x^*)^T \nabla^2 F(x^*)(x - x^*) \\
&\geq F(x) - F(x^*) - \tfrac{1}{2} L \|x - x^*\|^3 \\
&\geq \left[\alpha_1 - \tfrac{1}{2} L \, \|x - x^*\|\right] \|x - x^*\|^2 \\
&\geq \tfrac{1}{2} \alpha_1 \, \|x - x^*\|^2,
\end{aligned}$$

and so (37) is satisfied if we set $\bar{\alpha}_1 = \frac{1}{2}\alpha_1$.

The final statement in the theorem follows from Theorem 3.3. □

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