Summary of Vector Integration

Line Integrals

The scalar form: $\int_C f(x(t), y(t))\, ds$, where *C* is a path and ds is the arc-length element given by $ds = |\mathbf{r}'(t)|\, dt$, where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is the path on the *xy*-plane. This gives the area of the "sheet" above the path *C* on the *xy*-plane and below the surface $f(x, y)$. You need to parameterize your path in terms of *t*, and the whole integral will be in terms of *t*.

The vector (work/circulation) form: $\int_C \mathbf{F} \cdot d\mathbf{r}$, with vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$, and path *C* is parameterized by the v.v.f. $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Take care to parameterize the path so that *t* has easy bounds. The "expanded" form of this line integral is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy$. A good application of this line integral is to illustrate work done on a particle as it follows a path in a vector field, and if the vector field "helps" or "hinders" the particle's movement. This form is also known as the ***circulation*** form of the line integral, since it shows how the vector field will affect the movement of a particle along a path, especially if the path is a closed loop.

If $Q_x = P_y$, then the line integral is path-independent and the vector field **F** is said to be ***conservative***, and there exists a ***potential*** function $\varphi(x, y)$ such that its gradient $\nabla\varphi = \langle \varphi_x, \varphi_y\rangle = \langle P, Q\rangle$. Therefore, the line integral is simply the potential function evaluated at the start (*a*) and end (*b*) points: $\int_C \mathbf{F} \cdot d\mathbf{r} = [\varphi(x, y)]_a^b$. This is called the Fundamental Theorem of Line Integrals (FTLI). Conservative vector fields are ***irrotational***.

The line integral can also be defined in terms of the outward normal vector **n** along a path. This form is $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$. This form is called the ***flux*** form of the line integral. It makes most sense when the path is a closed loop. This form of the line integral will show if the vector field is "flowing" into or out of an enclosed region. If $P_x = -Q_y$, then the vector field is ***source-free*** and there exists a ***stream*** function $\psi(x, y)$ such that $\psi_y = P$ and $\psi_x = -Q$.

An ideal fluid can be represented by a vector field $\mathbf{F}(x, y)$ that is both irrotational (rotation-free) and incompressible (source-free). Gravity is an example of an irrotational field but not source-free, since the gravitational field wants to pull all objects "in" toward a theoretical center.

Green's Theorem

If the path *C* forms a closed loop and is traversed in the counter-clockwise direction, then the line integral around path *C* can be evaluated equivalently as a double integral over the region *R* within the path:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(Q_x - P_y\right) dA.$$

You still have the option to use rectangular or polar coordinates for these integrals.

The expression $Q_x - P_y$ is called the curl of **F** in the *xy*-plane. Technically, it is the curl of **F** through an axis normal to the *xy*-plane.

When calculating a line integral, you should check two things:

- Is the vector field conservative?
- Is the path a simple closed loop?

The following table will help you plan your calculation accordingly.

	F is conservative	**F** is not conservative
C is a simple closed loop	0	Use Green's Theorem
C is not a loop of any kind (i.e. it has different start and end points.)	Find the potential function $\varphi(x,y)$ and calculate the line integral by the "FTLI"	Parameterize the path(s) in variable t, and calculate the line integral "the long way".

The del operator:

The del operator is defined as: $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$. It is sometimes short-handed as $\langle \partial_x, \partial_y, \partial_z \rangle$.

Curl

Curl is defined only on a 3-dimensional vector field (although it can be applied to a 2-dimension field where the axis of rotation will be orthogonal to the *xy*-plane). If $\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ then curl **F** is defined as:

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

The curl is meant to describe the rotational quality of a fluid flow. Does it "spin" or cause things to turn as it flows? Note that the final element in the curl vector is the same as the integrand in Green's Theorem. Therefore, we can define a line integral using curl as follows:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(Q_x - P_y\right) dA = \iint_R (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA.$$

Curl also helps us to show if a 3-dimensional vector field is conservative. If **F** is conservative, then curl $\mathbf{F} = \mathbf{0}$, but the converse is not always true (usually involving singularity points in the region R). However, if you calculate curl **F** and it comes back **0** (the zero vector), then you take that as a hint and should try to determine the potential function $f(x,y,z)$ whose gradient is $\langle P, Q, R\rangle$.

Curl is a <u>vector</u>!

Divergence

The divergence of a vector field **F** is div **F** $= \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Using the outward normal definition for line integrals, the line integral can be defined in terms of divergence:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (\text{div}\, \mathbf{F})\, dA.$$

Divergence is meant to show the radial movement of a flow "from" a point-source (div is positive) or "into" a point-source (div is negative; this is called a sink). Divergence is a scalar function.

Please be aware of the difference between the gradient of a scalar function $\nabla f = \langle f_x, f_y, f_z \rangle$, which is a vector-valued function, and the divergence of a vector field, $\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, which is a scalar function. They are not identically the same. Remember, play the "is it a scalar or vector?" game. For instance, if you saw $\nabla \cdot f$, you'd know this is not possible. Why?

Surface Integrals

In the purely scalar sense, a surface integral is the integral of a function $w = f(x, y, z)$ over a surface $z = g(x, y)$, where the surface element *dS* must be considered. The definition is:

$$\iint_S f(x, y, z)\, dS = \iint_R f\big(x, y, g(x, y)\big)\sqrt{g_x^2 + g_y^2 + 1}\, dA.$$

It will be a double integral in terms of *x* and *y* (although you can make it in terms of the other pairs of variables as you see best). Note that $dS = \sqrt{g_x^2 + g_y^2 + 1}\; dA$, where $\mathbf{n} = \langle g_x, g_y, -1 \rangle$.

In the vector sense, a surface integral is defined by $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where the surface-area element *dS* is defined by $|\mathbf{r}_u \times \mathbf{r}_v|\, dA$. This means you have to parameterize the surface as a vector-valued function in two variables *u* and *v* as such: $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, then find the magnitude of the cross product of the partial derivatives of **r**.

A good way to view the vector form of the surface integral is as ***flux*** (flow) through a surface (membrane) per unit time. The value of $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ is literally the mass of the flow per unit time through the surface *S*.

The Divergence Theorem

Surface integrals in vector form can be very difficult to evaluate directly (mainly since the parametrization step can be difficult). The ***divergence theorem*** relates the flux through a closed surface *S* (one that encloses a finite volume with no "holes" or other peculiarities) with a triple integral over the volume *V* enclosed by *S*:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\text{div}\, \mathbf{F})\, dV$$

You "trade out" a flux integral for a triple integral! Trust me, this is actually a good thing most of the time. You save yourself the hassle of parameterizing the surface and instead simply evaluate a triple (scalar) integral over a volume *V*. Here, you can use rectangular, cylindrical or spherical approached to determine the value of the integral.

Stokes' Theorem

A simple regular surface S in 3-dimensions will have a boundary C. We can perform a line integral around this boundary or evaluate an equivalent double integral over S, where $\vec{n}$ is a unit normal vector with outward projection from the surface S:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS.$$

Green's Theorem is actually a corollary of Stokes' Theorem. In Green's Theorem, the region S is on the xy plane, so that $dS = dA$ and $\mathbf{n} = \mathbf{k}$. In Stokes' Theorem, region S is in 3 dimensions.

The left side of Stokes' Theorem is a line integral, while the right side is a surface integral. Thus, you can "trade" one for the other as is convenient. You may find it easier to work on the boundary, or work in the interior. Go with your heart.