第三章作业

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1 第 1 题

已阅。

2 第 2 题

2.1 $\{\mathbb{Z},+\}$ 是否为群？若是，验证其满足群定义；若不是，说明理由。

$\{\mathbb{Z},+\}$ 是群，验证如下：

1. 封闭性：$\forall x_1, x_2 \in \mathbb{Z}, \quad x_1 + x_2 \in \mathbb{Z}$
2. 结合律：$\forall x_1, x_2, x_3 \in \mathbb{Z}, \quad (x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$
3. 幺元：$\exists 0 \in \mathbb{Z}, \quad s.t. \quad \forall x \in Z, \quad 0 + x = x + 0 = x$
4. 逆: $\forall x \in \mathbb{Z}, \quad \exists x^{-1} = -x \in \mathbb{Z}, \quad s.t. \quad x + (-x) = 0$

综上所述，$\{\mathbb{Z},+\}$ 四个条件均满足，所以 $\{\mathbb{Z},+\}$ 是群。

2.2 $\{\mathbb{N},+\}$ 是否为群？若是，验证其满足群定义；若不是，说明理由。

$\{\mathbb{N},+\}$ 不是群，验证如下：

1. 封闭性：$\forall x_1,x_2\in\mathbb{N},\quad x_1+x_2\in\mathbb{N}$
2. 结合律：$\forall x_1,x_2,x_3\in\mathbb{N},\quad (x_1+x_2)+x_3=x_1+(x_2+x_3)$
3. 幺元：$\exists 0\in\mathbb{N},\quad s.t.\quad \forall x\in N,\quad 0+x=x+0=x$
4. 逆: $\forall x\in\mathbb{Z}$, 找不到 $x^{-1}\in\mathbb{N},\quad s.t.\quad x+(-x)=0$, 所以

综上所述，$\{\mathbb{N},+\}$ 不满足“逆”的条件，所以 $\{\mathbb{Z},+\}$ 不是群。

2.3 解释什么是阿贝尔群。并说明矩阵及乘法构成的群是否为阿贝尔群。

阿贝尔群是由集合 G 和运算 * 组成的一种代数结构，除了满足一般的群公理 (封结幺逆)，还需要满足交换律。

交换律：如果群 $<G,*>$ 中的运算 * 是可交换的，则满足 $\forall x_1,x_2\in\mathbb{G},\quad x_1*x_2=x_2*x_1$

对于矩阵及乘法构成的群，容易验证封闭性和结合律，幺元为 $\boldsymbol{I}$，逆为矩阵的逆，而对于矩阵乘法却不是可交换的，如下例，记 * 为矩阵乘法：对于 2 阶满秩矩阵 $\boldsymbol{A}$ 和 $\boldsymbol{B}$ 有 $\boldsymbol{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}e & f\\ g & h\end{bmatrix}$ 则

$$\boldsymbol{A}*\boldsymbol{B}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}*\begin{bmatrix}e & f\\ g & h\end{bmatrix}=\begin{bmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{bmatrix} \tag{2.1}$$

$$\boldsymbol{B}*\boldsymbol{A}=\begin{bmatrix}e & f\\ g & h\end{bmatrix}*\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}ae+cf & be+df\\ ag+ch & bg+dh\end{bmatrix} \tag{2.2}$$

显然 $(1)\neq(2)$，所以矩阵及乘法够成的群不是阿贝尔群。

3 第 3 题

设 $\forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathbb{V}$，其中 $\boldsymbol{X} = [x_1, x_2, x_3]^T, \boldsymbol{Y} = [y_1, y_2, y_3]^T, \boldsymbol{Z} = [z_1, z_2, z_3]^T$ 证明如下：

1. 封闭性：

$$
\begin{aligned}
[\boldsymbol{X}, \boldsymbol{Y}] &= \boldsymbol{X} \times \boldsymbol{Y} \\
&= \boldsymbol{X}^\wedge \boldsymbol{Y} \\
&= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix} \in \mathbb{V}
\end{aligned} \tag{3.1}
$$

2. 双线性：

$$
\begin{aligned}
[a\boldsymbol{X} + b\boldsymbol{Y}, \boldsymbol{Z}] &= \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -(ax_3 + by_3) & (ax_2 + by_2) \\ ax_3 + by_3 & 0 & -(ax_1 + by_1) \\ -(ax_2 + by_2) & ax_1 + by_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\
&= \begin{bmatrix} -(ax_3 + by_3) z_2 + (ax_2 + by_2) z_3 \\ (ax_3 + by_3) z_1 - (ax_1 + by_1) z_3 \\ -(ax_2 + by_2) z_1 + (ax_1 + by_1) z_2 \end{bmatrix}
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
a[\boldsymbol{X},\boldsymbol{Z}]+b[\boldsymbol{Y},\boldsymbol{Z}] &= a\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + b\begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\
&= a\begin{bmatrix} -x_3z_2+x_2z_3 \\ x_3z_1-x_1z_3 \\ -x_2z_1+x_1z_2 \end{bmatrix} + b\begin{bmatrix} -y_3z_2+y_2z_3 \\ y_3z_1-y_1z_3 \\ -y_2z_1+y_1z_2 \end{bmatrix} \\
&= \begin{bmatrix} z_2(-ax_3-by_3)+z_3(ax_2+by_2) \\ z_1(ax_3+by_3)+z_3(-ax_1-by_1) \\ z_1(-ax_2-by_2)+z_2(ax_1+by_1) \end{bmatrix} \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
[\boldsymbol{Z}, a\boldsymbol{X}+b\boldsymbol{Y}] = a[\boldsymbol{Z},\boldsymbol{X}]+b[\boldsymbol{Z},\boldsymbol{Y}] \\
&= \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \times \begin{bmatrix} ax_1+by_1 \\ zx_2+by_2 \\ zx_3+by_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}\begin{bmatrix} ax_1+by_1 \\ zx_2+by_2 \\ zx_3+by_3 \end{bmatrix} \\
&= \begin{bmatrix} -z_3(ax_2+by_2)+z_2(ax_3+by_3) \\ z_3(ax_1+by_1)-z_1(ax_3+by_3) \\ -z_2(ax_1+by_1)+z_1(ax_2+by_2) \end{bmatrix} \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
a[\boldsymbol{Z},\boldsymbol{X}]+b[\boldsymbol{Z},\boldsymbol{Y}] &= a\begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b\begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= \begin{bmatrix} z_3(-ax_2-by_2)+z_2(ax_3+by_3) \\ z_3(ax_1+by_1)+z_1(-ax_3-by_3) \\ z_2(-ax_1-by_1)+z_1(ax_2+by_2) \end{bmatrix} \qquad (3.5)
\end{aligned}
$$

对比 (3.2)(3.3)，(3.4)(3.5) 可得，双线性成立。

3. 自反性:

$$
\begin{aligned}
[\boldsymbol{X}, \boldsymbol{X}] &= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} -x_3x_2 + x_2x_3 \\ x_3x_1 - x_1x_3 \\ -x_2x_1 + x_1x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \mathbf{0}
\end{aligned} \tag{3.6}
$$

4. 雅克比等价

$$
\begin{aligned}
&[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] \\
&= \left[\boldsymbol{X}, \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right] + \left[\boldsymbol{Y}, \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right] \\
&\quad + \left[\boldsymbol{Z}, \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right] \\
&= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} -y_3z_2 + y_2z_3 \\ y_3z_1 - y_1z_3 \\ -y_2z_1 + y_1z_2 \end{bmatrix} + \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} -z_3x_2 + z_2x_3 \\ z_3x_1 - z_1x_3 \\ -z_2x_1 + z_1x_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix} \begin{bmatrix} -x_3y_2 + x_2y_3 \\ x_3y_1 - x_1y_3 \\ -x_2y_1 + x_1y_2 \end{bmatrix} \\
&= \begin{bmatrix} ① \\ ② \\ ③ \end{bmatrix}
\end{aligned} \tag{3.7}
$$

其中

$$
\begin{aligned}
① &= -x_3y_3z_1 + x_3y_1z_3 - x_2y_2z_1 + x_2y_1z_2 \\
&\quad + x_1y_3z_3 + x_3y_3z_1 - x_1y_2z_2 + x_2y_2z_1 \\
&\quad - x_3y_1z_3 - x_1y_3z_3 - x_2y_1z_2 + x_1y_2z_2 \\
&= 0
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
② &= -x_3y_3z_2 + x_3y_2z_3 - x_1y_2z_1 - x_1y_1z_2 \\
&\quad + x_1y_3z_3 + x_3y_3z_2 - x_1y_1z_2 - x_2y_1z_1 \\
&\quad - x_3y_2z_3 + x_2y_3z_3 + x_2y_1z_1 - x_1y_2z_1 \\
&= 0
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
③ &= x_1y_3z_2 - x_2y_2z_3 - x_1y_3z_1 - x_1y_1z_3 \\
&\quad + x_2y_2z_3 - x_3y_2z_2 + x_1y_1z_3 - x_3y_1z_1 \\
&\quad + x_3y_2z_2 - x_2y_3z_2 + x_3y_1z_1 - x_1y_3z_1 \\
&= 0
\end{aligned} \tag{3.10}
$$

故 (3.7) 式有

$$
[\boldsymbol{X},[\boldsymbol{Y},\boldsymbol{Z}]] + [\boldsymbol{Y},[\boldsymbol{Z},\boldsymbol{X}]] + [\boldsymbol{Z},[\boldsymbol{X},\boldsymbol{Y}]] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \tag{3.11}
$$

雅克比等价得证。

综上所述，$g = (\mathbb{R}^3, \mathbb{R}, \times)$ 构成李代数。

4 第 4 题

已知 $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\rho} \end{bmatrix}$， $\boldsymbol{\xi}^{\wedge} = \begin{bmatrix} \hat{\boldsymbol{\phi}} & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}$

且由于任何矩阵的 0 次幂都是单位阵, 故有

$$
\begin{aligned}
exp(\boldsymbol{\xi}^\wedge) &= \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{\xi}^\wedge)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n
\end{aligned} \tag{4.1}
$$

对于

$$
\sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n \tag{4.2}
$$

当 $n=2$ 时，(4.2) 有

$$
(4.2) \xRightarrow{n=2} \frac{1}{2!} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} = \frac{1}{2!} \begin{bmatrix} (\theta\boldsymbol{a}^\wedge)^2 & (\theta\boldsymbol{a}^\wedge)\boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \tag{4.3}
$$

当 $n=3$ 时，(4.2) 有

$$
(4.2) \xRightarrow{n=3} \frac{1}{3!} \begin{bmatrix} (\theta\boldsymbol{a}^\wedge)^2 & \theta\boldsymbol{a}^\wedge\boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} = \frac{1}{3!} \begin{bmatrix} (\theta\boldsymbol{a}^\wedge)^3 & (\theta\boldsymbol{a}^\wedge)^2\boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \tag{4.4}
$$

故对于 (4.2) 有

$$
\sum_{n=1}^{\infty} \frac{1}{n!} \begin{bmatrix} \theta\boldsymbol{a}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n = \begin{bmatrix} \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\theta\boldsymbol{a}^\wedge)^n & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\theta\boldsymbol{a}^\wedge)^n\boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \tag{4.5}
$$

将 (4.5) 带入 (4.1) 中得

$$\begin{aligned} exp(\boldsymbol{\xi}^\wedge) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\theta \boldsymbol{a}^\wedge)^n & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\theta \boldsymbol{a}^\wedge)^n \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\theta \boldsymbol{a}^\wedge)^n & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\theta \boldsymbol{a}^\wedge)^n \boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sum\limits_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{\phi}^\wedge)^n & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n \boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{R} & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n \boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \end{aligned} \tag{4.6}$$

(4.6) 即为 SE(3) 的指数映射，得证。对于 (4.6) 中 $\boldsymbol{\rho}$ 的系数 $\sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n$ 有，其中对于 $\sin x$ 和 $\cos x$ 的泰勒展开有

$\sin x = \sum\limits_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos x = \sum\limits_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\theta \boldsymbol{\phi}^\wedge)^n \\ &= \boldsymbol{I} + \frac{1}{2!}(\theta \boldsymbol{a}^\wedge) + \frac{1}{3!}(\theta \boldsymbol{a}^\wedge)^2 + \frac{1}{4!}(\theta \boldsymbol{a}^\wedge)^3 \\ &\quad + \frac{1}{5!}(\theta \boldsymbol{a}^\wedge)^4 + \frac{1}{6!}(\theta \boldsymbol{a}^\wedge)^5 + \frac{1}{7!}(\theta \boldsymbol{a}^\wedge)^6 \\ &= \boldsymbol{I} + (\frac{\theta}{2!} - \frac{\theta^3}{4!} + \frac{\theta^5}{6!} + \cdots)\boldsymbol{a}^\wedge \\ &\quad + (\frac{\theta^2}{3!} - \frac{\theta^4}{5!} + \frac{\theta^6}{7!} + \cdots)\boldsymbol{a}^\wedge \boldsymbol{a}^\wedge \\ &= \boldsymbol{I} + \frac{1}{\theta}(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots)\boldsymbol{a}^\wedge \\ &\quad + \frac{1}{\theta}(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots)\boldsymbol{a}^\wedge \boldsymbol{a}^\wedge \\ &= \boldsymbol{I} + \frac{1}{\theta}(1 - \cos\theta)\boldsymbol{a}^\wedge + \frac{1}{\theta}(\theta - \sin\theta)\boldsymbol{a}^\wedge \boldsymbol{a}^\wedge \\ &= \frac{\sin\theta}{\theta}\boldsymbol{I} + (1 - \frac{\sin\theta}{\theta})\boldsymbol{a}\boldsymbol{a}^T + (\frac{1 - \cos\theta}{\theta})\boldsymbol{a}^\wedge \\ &= J \end{aligned} \tag{4.7}$$

J 即为左雅克比矩阵，证毕。

5 第 5 题

首先证明 $(\boldsymbol{Rp})^\wedge = \boldsymbol{Rp}^\wedge\boldsymbol{R}^T$。

$\forall \boldsymbol{p}, \boldsymbol{v} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)$ 有

$$\begin{aligned}
(\boldsymbol{Rp})^\wedge\boldsymbol{v} &= (\boldsymbol{Rp}) \times \boldsymbol{v} \\
&= (\boldsymbol{Rp}) \times (\boldsymbol{RR}^{-1}\boldsymbol{v}) \\
&= \boldsymbol{R}(\boldsymbol{p} \times (\boldsymbol{R}^{-1}\boldsymbol{v})) \\
&= \boldsymbol{Rp}^\wedge\boldsymbol{R}^{-1}\boldsymbol{v} \\
&= \boldsymbol{Rp}^\wedge\boldsymbol{R}^T\boldsymbol{v}
\end{aligned} \tag{5.1}$$

即 $(\boldsymbol{Rp})^\wedge\boldsymbol{v} = (\boldsymbol{Rp}^\wedge\boldsymbol{R}^T)\boldsymbol{v}$ 故 $(\boldsymbol{Rp})^\wedge = \boldsymbol{Rp}^\wedge\boldsymbol{R}^T$ 得证。对于 $exp((\boldsymbol{Rp})^\wedge)$ 有

$$\begin{aligned}
exp((\boldsymbol{Rp})^\wedge) &= exp(\boldsymbol{Rp}^\wedge\boldsymbol{R}^T) \\
&= \sum_{n=0}^{\infty}\frac{(\boldsymbol{Rp}^\wedge\boldsymbol{R}^T)^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(\boldsymbol{Rp}^\wedge\boldsymbol{R^T}\boldsymbol{Rp}^\wedge\boldsymbol{R^T}\boldsymbol{Rp}^\wedge\boldsymbol{R^T}\cdots\boldsymbol{R}^T)^n}{n!} \\
&\overset{\boldsymbol{R}^T\boldsymbol{R}=\boldsymbol{I}}{=} \sum_{n=0}^{\infty}\frac{\boldsymbol{R}(\boldsymbol{p}^\wedge)^n\boldsymbol{R}^T}{n!} \\
&= \boldsymbol{R}(\sum_{n=0}^{\infty}\frac{(\boldsymbol{p}^\wedge)^n}{n!})\boldsymbol{R}^T \\
&= \boldsymbol{R}exp(\boldsymbol{p}^\wedge)\boldsymbol{R}^T
\end{aligned} \tag{5.2}$$

证毕。

6 第 6 题

6.1 旋转点对旋转的导数

1. 左扰动 φ:

$$\begin{aligned}\frac{\partial \boldsymbol{Rp}}{\partial \boldsymbol{\varphi}} &= \lim_{\boldsymbol{\varphi}\to 0}\frac{exp(\boldsymbol{\varphi}^{\wedge})exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p} - exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p}}{\boldsymbol{\varphi}} \\ &= \lim_{\boldsymbol{\varphi}\to 0}\frac{(\boldsymbol{I}+\boldsymbol{\varphi}^{\wedge})exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p} - exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p}}{\boldsymbol{\varphi}} \\ &= \lim_{\boldsymbol{\varphi}\to 0}\frac{\boldsymbol{\varphi}^{\wedge}\boldsymbol{Rp}}{\boldsymbol{\varphi}} = \lim_{\boldsymbol{\varphi}\to 0}\frac{-(\boldsymbol{Rp}^{\wedge})\boldsymbol{\varphi}}{\boldsymbol{\varphi}} = -(\boldsymbol{Rp})^{\wedge}\end{aligned} \tag{6.1}$$

2. 右扰动 φ:

$$\begin{aligned}\frac{\partial \boldsymbol{Rp}}{\partial \boldsymbol{\varphi}} &= \lim_{\boldsymbol{\varphi}\to 0}\frac{exp(\boldsymbol{\phi}^{\wedge})exp(\boldsymbol{\varphi}^{\wedge})\boldsymbol{p} - exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p}}{\boldsymbol{\varphi}} \\ &= \lim_{\boldsymbol{\varphi}\to 0}\frac{exp(\boldsymbol{\phi}^{\wedge})(\boldsymbol{I}+\boldsymbol{\varphi}^{\wedge})\boldsymbol{p} - exp(\boldsymbol{\phi}^{\wedge})\boldsymbol{p}}{\boldsymbol{\varphi}} \\ &= \lim_{\boldsymbol{\varphi}\to 0}\frac{\boldsymbol{R}\varphi\hat{}\boldsymbol{p}}{\boldsymbol{\varphi}} \\ &\overset{\boldsymbol{a}^{\wedge}\boldsymbol{b}=-\boldsymbol{b}^{\wedge}\boldsymbol{a}}{=} \lim_{\boldsymbol{\varphi}\to 0}\frac{\boldsymbol{R}(-\boldsymbol{p}^{\wedge}\boldsymbol{\varphi})}{\boldsymbol{\varphi}} \\ &= -\boldsymbol{Rp}^{\wedge}\end{aligned} \tag{6.2}$$

6.2 旋转的复合

1. 对 R_1 左扰动后对 R_1 求导:

$$\begin{aligned}\frac{\partial ln(\boldsymbol{R_1R_2})^{\vee}}{\partial \boldsymbol{R_1}} &= \lim_{\boldsymbol{\varphi_1}\to 0}\frac{ln(exp(\boldsymbol{\varphi}_1^{\wedge})\boldsymbol{R_1R_2})^{\vee} - ln(\boldsymbol{R_1R_2})^{\vee}}{\boldsymbol{\varphi}_1} \\ &\overset{BCH}{\approx} \lim_{\boldsymbol{\varphi_1}\to 0}\frac{\boldsymbol{J}_l(ln(\boldsymbol{R}_1\boldsymbol{R_2})^{\vee})^{-1}\boldsymbol{\varphi}_1 + ln(\boldsymbol{R}_1\boldsymbol{R}_2)^{\vee} - ln(\boldsymbol{R}_1\boldsymbol{R}_2)^{\vee}}{\boldsymbol{\varphi}_1} \\ &= \boldsymbol{J}_l(ln(\boldsymbol{R}_1\boldsymbol{R_2})^{\vee})^{-1}\end{aligned} \tag{6.3}$$

2. 对 R_1 右扰动后对 R_1 求导：

$$
\begin{aligned}
\frac{\partial ln(\boldsymbol{R_1R_2})^\vee}{\partial \boldsymbol{R_1}} &= \lim_{\boldsymbol{\varphi_1}\to 0}\frac{ln(\boldsymbol{R}_1exp(\boldsymbol{\varphi}_1^\wedge)\boldsymbol{R}_2)^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_1} \\
&= \lim_{\boldsymbol{\varphi_1}\to 0}\frac{ln(\boldsymbol{R}_1exp(\boldsymbol{\varphi}_1^\wedge)\boldsymbol{R}_1^T\boldsymbol{R}_1\boldsymbol{R}_2)^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_1} \\
&\overset{\boldsymbol{Ra}^\wedge\boldsymbol{R}^T=(\boldsymbol{Ra})^\wedge}{=} \lim_{\boldsymbol{\varphi_1}\to 0}\frac{ln((\boldsymbol{R}_1\boldsymbol{\varphi}_1)^\wedge\boldsymbol{R}_1\boldsymbol{R}_2)^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_1} \\
&\overset{BCH}{\approx} \lim_{\boldsymbol{\varphi_1}\to 0}\frac{\boldsymbol{J}_l(ln(\boldsymbol{R_1R_2})^\vee)^{-1}\boldsymbol{R}_1\boldsymbol{\varphi}_1 + ln(\boldsymbol{R_1R_2})^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_1} \\
&= \boldsymbol{J}_l(ln(\boldsymbol{R_1R_2})^\vee)^{-1}\boldsymbol{R}_1
\end{aligned} \tag{6.4}
$$

3. 对 R_2 左扰动后对 R_2 求导：

$$
\begin{aligned}
\frac{\partial ln(\boldsymbol{R_1R_2})^\vee}{\partial \boldsymbol{R_1}} &= \lim_{\boldsymbol{\varphi_2}\to 0}\frac{ln(\boldsymbol{R}_1exp(\boldsymbol{\varphi}_2^\wedge)\boldsymbol{R}_2)^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&= \lim_{\boldsymbol{\varphi_2}\to 0}\frac{ln(\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_2^T exp(\boldsymbol{\varphi}_1^\wedge)\boldsymbol{R}_2)^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&\overset{\boldsymbol{R}^T\boldsymbol{a}^\wedge\boldsymbol{R}=(\boldsymbol{R}^T\boldsymbol{a})^\wedge}{=} \lim_{\boldsymbol{\varphi_2}\to 0}\frac{ln(\boldsymbol{R}_1\boldsymbol{R}_2exp((\boldsymbol{R}_2^T\boldsymbol{\varphi}_2)^\wedge))^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&\overset{BCH}{\approx} \lim_{\boldsymbol{\varphi_2}\to 0}\frac{\boldsymbol{J}_r(ln(\boldsymbol{R_1R_2})^\vee)^{-1}\boldsymbol{R}_2^T\boldsymbol{\varphi}_2 + ln(\boldsymbol{R_1R_2})^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&= \boldsymbol{J}_r(ln(\boldsymbol{R_1R_2})^\vee)^{-1}\boldsymbol{R}_2^T
\end{aligned} \tag{6.5}
$$

4. 对 R_2 右扰动后对 R_2 求导：

$$
\begin{aligned}
\frac{\partial ln(\boldsymbol{R_1R_2})^\vee}{\partial \boldsymbol{R_2}} &= \lim_{\boldsymbol{\varphi_2}\to 0}\frac{ln(\boldsymbol{R}_1\boldsymbol{R}_2exp(\boldsymbol{\varphi}_2^\wedge))^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&\overset{BCH}{\approx} \lim_{\boldsymbol{\varphi_2}\to 0}\frac{\boldsymbol{J}_r(ln(\boldsymbol{R_1R_2})^\vee)^{-1}\boldsymbol{\varphi} + ln(\boldsymbol{R_1R_2})^\vee - ln(\boldsymbol{R_1R_2})^\vee}{\boldsymbol{\varphi}_2} \\
&= \boldsymbol{J}_r(ln(\boldsymbol{R_1R_2})^\vee)^{-1}
\end{aligned} \tag{6.6}
$$

7 第 7 题

7.1 $\boldsymbol{T}_{WC}$ 的平移部分即构成了机器人的轨迹。它的物理意义是什么？为何画出 $\boldsymbol{T}_{WC}$ 的平移部分就得到了机器人的轨迹？

$\boldsymbol{T}_{WC}$ 的旋转部分 $\boldsymbol{R}_{WC}$ 就是从机器人坐标系 C 变换到世界坐标系 W 的旋转矩阵，表示机器人的姿态；而平移部分 $\boldsymbol{t}_{WC}$ 就代表从 W 系的原点指

向 C 系的原点的向量，世界坐标系不随相机运动变化，因此可以认为 $\boldsymbol{t}_{WC}$ 是机器人相对于原点坐标在移动, 移动可视化在观察者眼中就是机器人的运动轨迹。

7.2 补充代码，画出轨迹

各部分代码如 Listing1-Listing2 所示，轨迹如图 7.1 所示

```cpp
int main(int argc, char **argv) {

    vector<Sophus::SE3d, Eigen::aligned_allocator<Sophus::SE3d>> poses;

    /// implement pose reading code
    // start your code here (5~10 lines)
    ifstream fin(trajectory_file);
    if (!fin) {
        cout << "cannot find trajectory file at " << trajectory_file << endl
    ;
        return 1;
    }

    while (!fin.eof()) {
        double time, tx, ty, tz, qx, qy, qz, qw;
        fin >> time >> tx >> ty >> tz >> qx >> qy >> qz >> qw;
        Eigen::Vector3d t(tx,ty,tz);                          // 平移部分
        Eigen::Quaterniond q(qw, qx, qy, qz);             // 用四元数构造SE3
    群
        Sophus::SE3d SE3_qt(q,t);                 // 从q,t构造SE(3)
        poses.push_back(SE3_qt);
    }
    cout << "read total " << poses.size() << " pose entries" << endl;
    // end your code here

    // draw trajectory in pangolin
    DrawTrajectory(poses);
    return 0;
}
```

Listing 1: 工程/draw_trajectory.cpp::main()

```cmake
project(ch3_T7)
cmake_minimum_required( VERSION 2.8 )

```

```
find_package(Pangolin REQUIRED)
include_directories(${Pangolin_INCLUDE_DIRS})

find_package( Sophus REQUIRED )

include_directories( "/usr/local/include/eigen3" )  #添加头文件
include_directories( "/usr/local/include/sophus" )

add_executable(draw_trajectory draw_trajectory.cpp)
target_link_libraries(draw_trajectory ${Pangolin_LIBRARIES} )
```

Listing 2: 工程/CMakeLists.txt

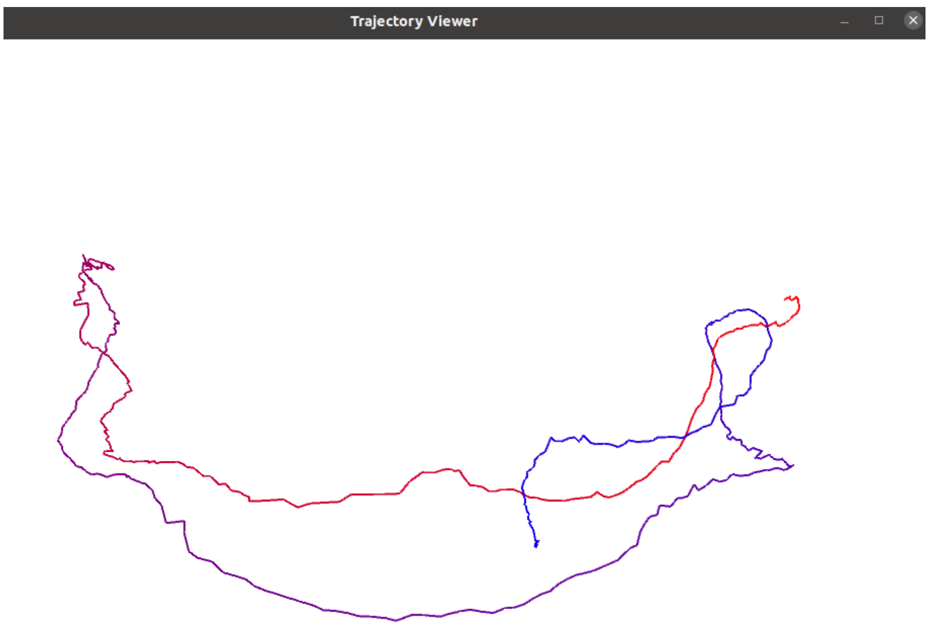

图 7.1 轨迹图

8 第 8 题

8.1 轨迹的误差

代码如下所示，运行结果如图 8.1 所示，RMSE=2.20727

```cpp
//
// Created by wrk on 2022/2/11.
//

#include <iostream>
#include <fstream>
#include <unistd.h>
#include <pangolin/pangolin.h>
#include <sophus/se3.hpp>

using namespace Sophus;
using namespace std;

string groundtruth_file = "../groundtruth.txt";
string estimated_file = "../estimated.txt";

typedef vector<Sophus::SE3d, Eigen::aligned_allocator<Sophus::SE3d>>
    TrajectoryType;  //是轨迹的类型

//函数声明
void DrawTrajectory(const TrajectoryType &gt, const TrajectoryType &esti);

TrajectoryType ReadTrajectory(const string &path);

int main(int argc, char **argv)
{
    //读取估计轨迹和真实轨迹
    TrajectoryType  groundtruth = ReadTrajectory(groundtruth_file);
    TrajectoryType estimated = ReadTrajectory(estimated_file);
    assert(!groundtruth.empty() && !estimated.empty());
    assert(groundtruth.size() == estimated.size());

    //计算误差RMSE
    double rmse = 0;
    for(size_t i=0;i<estimated.size();i++){
        Sophus::SE3d p1 = estimated[i], p2=groundtruth[i];
        double error = (p2.inverse()*p1).log().norm();
        rmse += error * error;
    }
    rmse = rmse/double(estimated.size());  //均值
    rmse = sqrt(rmse);   //开方
    cout<<"RMSE = "<<rmse<<endl;

    DrawTrajectory(groundtruth, estimated);
    return 0;
}
```

```cpp

//从文件中读取轨迹
TrajectoryType ReadTrajectory(const string &path){
    ifstream fin(path);
    TrajectoryType trajectory;
    if(!fin){
        cerr<<"trajectory "<<path<<"not found."<<endl;
        return trajectory;
    }
    while(!fin.eof()){
        double time, tx, ty, tz, qx, qy, qz, qw;
        fin>>time>>tx>>ty>>tz>>qx>>qy>>qz>>qw;
        //根据四元数和平移向量构造出SE(3)
        Sophus::SE3d p1(Eigen::Quaterniond(qw,qx,qy,qz), Eigen::Vector3d(tx,
    ty,tz));
        trajectory.push_back(p1);
    }
    return trajectory;
}

void DrawTrajectory(const TrajectoryType &gt, const TrajectoryType &esti){
    //创建Pangolin窗口并画出轨迹
    pangolin::CreateWindowAndBind("My Trajectory Viewer Name", 1024, 768);
    //下面3条看不懂
    glEnable(GL_DEPTH_TEST);
    glEnable(GL_BLEND);
    glBlendFunc(GL_SRC_ALPHA, GL_ONE_MINUS_SRC_ALPHA);

    //这个s_cam和d_cam分别有啥用?
    pangolin::OpenGlRenderState s_cam(
            pangolin::ProjectionMatrix(1024, 768, 500, 500,512,389,0.1,1000)
    ,
            pangolin::ModelViewLookAt(0, -0.1, -1.8, 0, 0, 0, 0.0, -1.0,
    0.0)
            );

    pangolin::View &d_cam = pangolin::CreateDisplay()
            .SetBounds(0.0, 1.0, pangolin::Attach::Pix(175), 1.0, -1024.0f
    /768.0f)
            .SetHandler(new pangolin::Handler3D(s_cam));

    //当还没有画完的时候继续画
    while(pangolin::ShouldQuit()==false){
        glClear(GL_COLOR_BUFFER_BIT| GL_DEPTH_BUFFER_BIT);

        d_cam.Activate(s_cam);
```

```cpp
        glClearColor(1.0f, 1.0f, 1.0f, 1.0f);   //RGB Flpha

        //绘制GroundTruth
        glLineWidth(2);  //线的粗细
        for(size_t i=0; i<gt.size()-1; i++){
            glColor3f(0.0f, 0.0f, 1.0f);   //GroundTruth用蓝线
            glBegin(GL_LINES);
            auto p1=gt[i],p2=gt[i+1];
            glVertex3d(p1.translation()[0], p1.translation()[1], p1.translation()[2]);
            glVertex3d(p2.translation()[0], p2.translation()[1], p2.translation()[2]);
            glEnd();
        }

        //绘制esti
        glLineWidth(2);  //线的粗细
        for(size_t i=0;i<esti.size()-1;i++){
            glColor3f(1.0f, 0.0f, 0.0f);   //估计的轨迹用红线
            glBegin(GL_LINES);
            auto p1=esti[i],p2=esti[i+1];
            glVertex3d(p1.translation()[0], p1.translation()[1], p1.translation()[2]);
            glVertex3d(p2.translation()[0], p2.translation()[1], p2.translation()[2]);
            glEnd();
        }
        pangolin::FinishFrame();
        usleep(5000);  //sleep 5s
    }
}
```

Listing 3: 工程/trajectoryError.cpp

```cmake
cmake_minimum_required(VERSION 3.0)
project(ch3_T8)
set(CMAKE_CXX_STANDARD 11)
find_package(Pangolin REQUIRED)
find_package(Sophus REQUIRED)
include_directories(${Pangolin_INCLUDE_DIRS})
add_executable(mytype mytype.cpp)
target_link_libraries(mytype ${Pangolin_LIBRARIES})
```

Listing 4: 工程/CMakeLists.txt

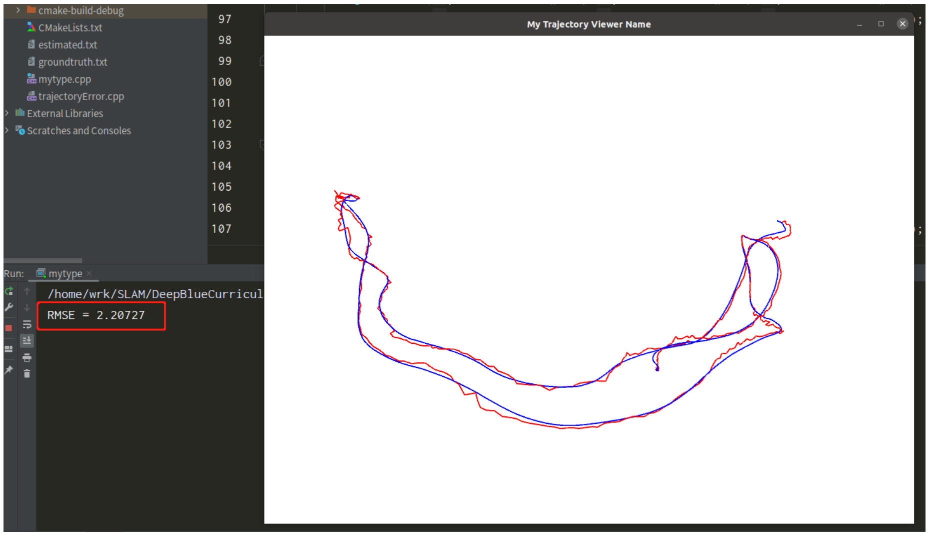
cmake-build-debug
CMakeLists.txt
estimated.txt
groundtruth.txt
mytype.cpp
trajectoryError.cpp
External Libraries
Scratches and Consoles
My Trajectory Viewer Name
Run: mytype
/home/wrk/SLAM/DeepBlueCurricul
RMSE = 2.20727

图 8.1 轨迹图和误差