SP25 - Math Structures - Final Project

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Due: Monday, June 9 at 9:30pm

Textbook Chapter 7.4: Partial Orders and Posets

As we learn in chapter 7.1, relations give us a way to describe the comparison of two elements in a set. In chapter 7.2, we also learned about various different properties of relations, including reflexivity, symmetry, and transitivity. If a relation has all of these properties, then it is considered an equivalence relation, and it partitions the set it acts on into distinct equivalence classes. Similarly, a partial order is another special kind of relation. Partial orders are similar to the equivalence relation, in that they fulfill the properties of reflexivity and transitivity. However, while equivalence relations are symmetric, partial orders are *antisymmetric*, a term which we will define in a moment. Due to these properties, when a partial order R is a applied to a set A, then we get a *partially ordered set*, or a *poset*, (A, R), which we will also strictly define in a moment. Posets are important, as they allow us to impose orderings on sets that otherwise would not have any kind of strict ordering.

We will begin by defining antisymmmetry, a concept essential to understanding partial orders. This definition is closely related to that of symmetry, however it has one important difference.

Definition 1: Antisymmetry

Let R be a relation on a set A. Then R is *antisymmetric* if $x \; R \; y$ and $y \; R \; x \implies x = y$.

Now that we have defined asymmetry, we have the necessary terminology to define a partial order.

Definition 2: Partial Order

A *partial order* is a relation R on a set A such that R is reflexive, transitive, and antisymmetric.

Extending our definition of partial orders, we define a total order as a partial order that can compare every element of a given set.

Definition 3: Total Order

An order $\precsim$ on a set A is a *total order* if $\forall x, y \in A, x \precsim y$ or $y \precsim x$.

Now that we have defined both how to put orders on sets, we need to establish terminology for how certain elements fulfill roles within these sets.

Definition 4: Minimal and Maximal

Let $\precsim$ be an order on some set A. Then, we say that $a \in A$ is *minimal* if $\forall b \in A$, if $b \precsim a$, then $b = a$. Likewise, we say that $a \in A$ is *maximal* if $\forall b \in A$, if $b \succsim a$, then $b = a$.

One relatively straightforward way to understand the concept of minimality and maximality is to apply these terms to finite sets. Consider Example 1.

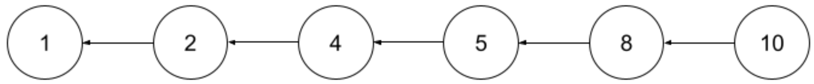

Figure 1: A totally ordered, finite poset, $(A, \leq)$

Example 1. Let $(A, \leq)$ be a finite, partially ordered set where $A = \{1, 2, 4, 5, 8, 10\}$. In this case, since every element in the set A can be compared to every other element, then $(A, \leq)$ is considered a total order. To find the minimal element, we have to go back to Definition 4. Since for every element $a \in A$, if $a \leq 1$, then $a = 1$, we know that 1 is a minimal element of the poset. Similarly, Since for every element $b \in A$, if $b \geq 10$, then $b = 10$, we know that 10 is a maximal element of the poset.

Additionally, we can represent finite posets as directed graphs (digraphs) to understand them in a more visual manner. As shown in Figure 1, given two elements $a, b \in A$, we make these elements nodes, and draw an edge from a to b if $b \leq a$. We do not draw edges between non-adjacent elements, however, as it is implied that they can be compared due to the fact that $\leq$ is a total order. This visual representation makes it clear that 1 is a minimal element in A, and 10 is a maximal element in A.

We can see that a simple finite set like Example 1 has both a minimal and a maximal element. However, this raises the question: does every finite set have to have at least one minimal or maximal element, or can finite sets exist that don't have any minimal or maximal element? This leads us to Theorem 1.

Theorem 1: Finite Poset Minimality and Maximality

Any finite, partially ordered set contains a minimal element and a maximal element.

To begin to think about how to prove this theorem, we first must think about what the definition of minimality really means. When an element of a set is minimal, that means that it is less than every other element that it can be compared to in a set. This, in some way, imposes a kind of "limit" on the size of the set. Since we are considering the definition of minimality on a finite set, this statement lends itself to be proven by contradiction, as we can prove that the existence of no minimal element will mean that there are no "limits" on the size of the set, meaning that the size of the set can possibly be infinite.

Proof. Let A be some finite, partially ordered set. Assume for contradiction that A does not have a minimal element. By the negation of Definition 4, this means that for every $a \in A$, there exists some $b \in A$ such that $b \precsim a$ and $b \neq a$. Now, select some arbitrary $a_0 \in A$. By the negation we just established, then we know that there exists some $a_1 \in A$ such that $a_1 < a_0$. Now, again by the negation, we know there exists some a_2 such that $a_2 < a_1 < a_0$. By continuing in this fashion, we can construct an infinite sequence of descending elements in A. However, we know that A is finite. This means that we eventually must reach some element a_i such that if $a_i' \precsim a_i$, then $a_i = a_i'$. Therefore, by contradiction, A must contain a minimal element. The existence of a maximal element can be proven in a completely symmetric manner by flipping the relation. Thus, any finite, partially ordered set must contain a minimal and a maximal element. $\square$

One key point about minimal and maximal elements of a set is that it is possible to have multiple minimal and multiple maximal elements in a set. This occurs when there are two elements that cannot be compared by the relation. This brings us to Example 2.

Example 2. Considering the finite set $B = \{1, 2, 3, 4, 5, 6, 9, 12, 15\}$, with the partial ordering of $R =$ divides. That is, for $a, b \in B$, then $(a, b) \in R$ when a divides b, denoted by $a \mid b$. Notice that not all elements of the set B can be compared; for example, 9 does not divide 15, so $(9, 15) \notin R$. Thus, while R is a partial ordering on B, it is not a total ordering.

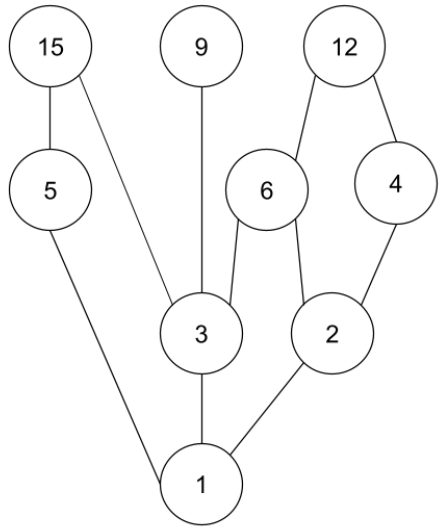

Figure 2: A partially ordered, finite poset, $(B, |)$

We can also draw digraphs for posets that are partially, but not totally, ordered. This can be seen in Figure 2. Here, the direction of the edges goes from the top of the graph downward. For example, there is an edge going from 12 to 6, because $6 \mid 12$. Notice here that while (B, R) only has only one minimal element, 1, it has multiple maximial elements. These are 15, 9, and 12. This is because, going back to Definition 4, if 15 divides some $b \in B$, then we know $b = 15$. This holds true for the other maximal elements in (B, R), 9 and 12.

So, how can we generalize the concept of minimal and maximal elements? This brings us to Definition 5.

Definition 5: Smallest and Largest

Again, let $\precsim$ be an order on some set A. We say $a \in A$ is *smallest* if $\forall b \in A, a \precsim b$. Likewise, we say $a \in A$ is *largest* if $\forall b \in A, a \succsim b$.

While the concepts of smallest and largest elements and minimal and maximal elements are very similar, they have slightly different definitions. Theorem 2 illustrates some of these differences.

Theorem 2: Minimality

If $(A, \leq)$ is a partially ordered set and x is the smallest element, then
(i) x is the only smallest element of A, and
(ii) x is the unique minimal element of A.

To prove (i), we have to assume that A has another smallest element, y, in addition to x. From here, we then work through the definition of smallest, to then show that x and y must be the same element, proving that A can only have one smallest element. Then, to prove (ii), we use a similar proof technique. We first assume that, while x is the smallest element of A, it is not minimal, and show how this leads to a contradiction within the definition of smallest element. Then, we assume that A has another minimal element, and show how this also leads to a contradiction, meaning that x must be the unique minimal element of A.

Proof. (i). We know that x is the smallest element of A. Lets assume that A has another smallest element, y. Then by Definition 5, we know that $\forall b \in A, x \leq b$ and $y \leq b$. Since $y \in A$ and $x \in A$, then again by Definition 5, we know that $x \leq y$ and $y \leq x$. Then, by Definition 1, we know that $x = y$, and therefore that x is the only smallest element of A. □

Proof. (ii). Assume, for contradiction, that x is not minimal. Then there exists some $a \in A$ such that $a \neq x$ and $a \leq x$. But x is smallest, so by Definition 5, we know that $x \leq a$ since $a \in A$. So by Definition 1, $a = x$. However, this contradicts our assumption that $a \neq x$, so x is minimal in A. Now, suppose another minimal element $m \in A$ exists such that $m \neq x$. Since x is smallest, we know that $x \leq m$. However, since m is minimal, this implies that $x = m$, which contradicts the assumption that $m \neq x$. Thus, a is the unique minimal element of A. □

Another concept that is very similar to smallest and largest elements of a poset is upper and lower bounds.

Definition 6: Upper and Lower Bounds

Let $(A, \leq)$ be a poset, and let $B \subseteq A$. A *lower bound* in A of B is some $a \in A$ such that $a \leq b, \forall b \in B$. Likewise, a *upper bound* in A of B is some $a \in A$ such that $a \geq b, \forall b \in B$.

According to this definition, there can be many possible upper and lower bounds in A of some $B \subseteq A$. To expand upon this, and specify this even further, we can define the *smallest upper bound* and the *greatest lower bound* of a subset.

Definition 7: Smallest Upper and Greatest Lower Bounds

The *greatest lower bound* in A of B is a largest element in the set of lower bounds of B in A. Likewise, the *smallest upper bound* in A of B is a smallest element in the set of upper bounds of B in A.

Now, you may be wondering how the concept of a lower bound interacts with definitions we have established previously, such as the smallest element in a set. If a lower bound of a set is itself an element of that set, then we know that that lower bound is the smallest element of that set. This brings us to our final theorem, Theorem 3.

Theorem 3: Lower Bound is the Smallest Element

Suppose $(A, \leq)$ is a partially ordered set and $B \subset A$. If B has a lower bound x and $x \in B$, then x is a smallest element of B.

For this proof, we really have to be familiar with the definition of lower bounds. By breaking down the definition of what a lower bound is, and then using the fact that $x \in B$, we can show that x is the smallest element of B by the definition of smallest.

Proof. We know that x is a lower bound of B. Then by the definition of a lower bound, we know that $x \leq b$ for all $b \in B$. However, we also know that $x \in B$. So, we know that $x \in B$, and for all $b \in A, x \leq b$. So, by Definition 5, we know that x is also the smallest element of B. □

0.1 Exercises

Exercise 1. Let $A = \{1, 2, 3\}$. Now, let $S = \mathcal{P}(A)$ be the power set of A with all nonempty elements of A. Define the partially ordered set $B = (S, \subseteq)$, where the relation is the subset relation. Draw a digraph for the poset B, identify all maximal and minimal elements, and smallest and largest elements (if they exist).

Solution. See Figure 3. As shown by the digraph, B only has one maximal element, $\{1, 2, 3\}$. B has three minimal elements, $\{1\}$, $\{2\}$, and $\{3\}$. Since B has multiple minimal elements, then B has no smallest element, as there exists minimal elements within B that cannot be compared by the relation $\subseteq$. Since every element in B is a subset of $\{1, 2, 3\}$, then it is the largest element in addition to being the only maximal element. □

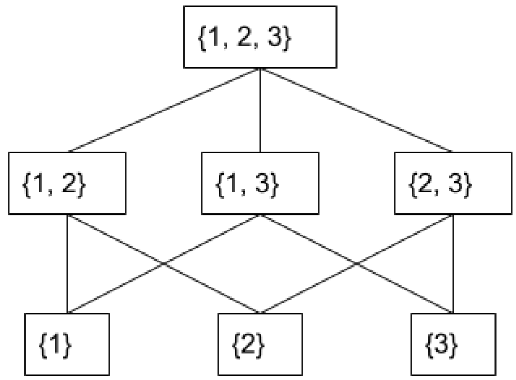

Figure 3: Solution to Exercise 1

Exercise 2. Let $(A, \precsim)$ be some finite, partially ordered set, and let $B \subseteq A$. Prove that if B has a greatest lower bound in A, then this greatest lower bound is unique.

Solution. Let $a \in A$ be the greatest lower bound of B. Then we know that for every $b \in B$, $a \precsim b$. Then, since a is the greatest lower bound, we also know that for any lower bound x, $x \precsim a$. Now, suppose that there is some other greatest lower bound of B in A, a'. Then we also know that for any other lower bound x, $x \precsim a'$. But, a is another lower bound of B in A. So $a \precsim a'$. Thus we know that $a \precsim a'$ and $a' \precsim a$, so by the property of antisymmetry, $a = a'$. Therefore, there is only one unique greatest lower bound of B in A. $\square$

1 Reflections

Say that a ruler is *well-marked* if for every $n \in \{1, 2, \ldots, 11, 12\}$ there are marks on the ruler which are n inches apart from each other. Note that the endpoints of the ruler are considered marks.

Theorem: Well-Marked Ruler

Any ruler of length 12 inches must have at least 4 marks in addition to the endpoints in order for the ruler to be well-marked.

Proof. Let the endpoints of a 12 inch ruler be at the positions of 0 and 12. Then, let M be the set of all marks on the ruler, where $0, 12 \in M$. Then, let k be the number of marks that are required in addition to the endpoints of the ruler for the ruler to be well marked. Thus $|M| = k + 2$. A ruler is considered well-marked if every distance $n \in \{1, 2, 3, \ldots, 12\}$ can be measured as the difference of two markings on the ruler. That is, there exists $x, y \in M$ such that $|x - y| = n$ for every n. Since M has $k + 2$ markings on it, we want to know how many different distances can be made by taking every possible pair of points. We can use the formula $\frac{n(n-1)}{2}$ (or n choose 2) to find the total number of pairs. Plugging $k + 2$ in, we get $\frac{(k+2)(k+1)}{2}$. Since the ruler is 12 inches long, there are 12 distinct distances we want to cover, so we say $\frac{(k+2)(k+1)}{2} \geq 12$. SimplifyinExercise 2. (Proof Exercise on Bounds)

Let (A,)(A,) be a finite partially ordered set, and let BABA.

(a) Prove that if BB has a greatest lower bound (glb) in AA, then this glb is unique.

(b) Prove that if BB has a smallest element in AA (that is, an element xBxB such that xbxb for all bBbB), then xx is both the smallest element and the greatest lower bound of BB.

(c) Give an example of a finite poset (A,)(A,) and subset BABA where BB has a greatest lower bound in AA that is not an element of BB.g, we get $(k + 2)(k + 1) \geq 24$. Now we can try some small values for k. When $k = 3$, then $(3 + 2)(3 + 1) = 20 < 24$, so 3 is too small. When $k = 4$, then $(4 + 2)(4 + 1) = 30 > 24$. This is the smallest k that satisfies the inequality. Therefore, any ruler of length 12 inches must have at least 4 marks in addition to the endpoints in order for the ruler to be well marked. □

Example. A well-marking of a ruler using only 4 marks in addition to the endpoints is a ruler with marks at 2, 3, 7, 8.

Reflection 1. Compare your write-up to the write-up you did for this problem at the beginning of class. How has your proof writing changed since the start of this class? Would you consider what you wrote at the beginning of class to be a 'proof'? Why or why not? *Suggested Length: 1-2 paragraphs*

My write up for this problem is noticibly more complete and mathematically sound than my write up at the beginning of the year. Although I used the same mathematical ideas that I did at the beginning of the year, such as the idea of it being a $k + 2$ choose 2 problem, I was able to express this much more clearly in my final write up than I did at the beginning of the year. Additionally, I was able to set up the definitions at the beginning of my proof much better in this version than in my initial version, as I had the knowledge about how to consider the markings a set with a cardinality. I honestly don't think I would consider what I wrote at the beginning of the year to be a proof, as it is way too simple, and it doesn't really get at the underlying problem that is being described in this question, but rather stays extremely surface-level.

Reflection 2. Choose 3-5 concepts from class that you think are the 'most important.' These may be a proof technique, a definition, a theorem,.... Explain why you chose each of the concepts as 'most' important. *Suggested Length: 1 paragraph per topic.*

1. **Direct Proofs.** While I had taken classes in the past that required me to write proofs, especially in the CS department here at Carleton, I had never really understood what made a proof thorough and well-written. I think that starting with learning how to write a clear direct proof is one of the most important and fundamental skills that I learned in this class, at is serves as the basis for pretty much the rest of the class. Most other proof techniqes derive their structure from a direct proof, and being

able to write a clear direct proof is a required prerequisite for being able to write any other kind of proof. After taking this class, I sort of want to go back and revisit the proofs I wrote in my Algorithms class to try to revise them and make them more rigorous.

2. **Functions.** I think that functions in general were definitely one of the most important things that we covered in this class. Functions are something that you begin learning about very early on in math education, and they are described in a very strict way where you never really have much room to question what it is that actually makes a function a function. Functions were very important throughout algebra, precalculus, and calculus, and this is the first time where I think I have ever really picked apart the definition of a function in a analytical way.

3. **The Cantor-Schroeder-Bernstein Theorem.** I thought that this theorem was really cool, and I think that it really incorporates many different topics that we have learned throughout the term. The proof for this theorem that we learned in class uses functions, set theory, definitions of injectivity and surjectivity, and many other areas. I thought that this theorem really helped tie lots of things together for me which was cool. I also really liked the way that the proof for this theorem defined ancestry of an element of a set. I had never really heard of a specific definition being created just for a single proof, which I really think shows the creativity that is required to make a mathematical argument.

Reflection 3. What topic from class did you find especially difficult to master, and why? What did you do to overcome that difficulty? *Suggested Length: 1 paragraph.*

One topic from this class that I found especially difficult to master at first was proofs by induction. I was first exposed to proofs by induction in CS 202: Math of CS, and I wrote quite a lot in CS 252: Algorithms. Throughout these two classes, I don't think that I could confidently say that I really understood what I was doing at all though. I knew that you had to write a base case, and then an inductive step, but I pretty much just followed the formula that my professor had given me for this and did not stop to think about why it worked. During Math Structures, I had to apply inductive proofs to completely different scenarios than I had to in the past, challenging me to actually understand why proofs like this work. To overcome this difficulty that I faced, I would say that the strategy that worked best for me was practice. After practicing induction during our weekly problem sets, and really trying to understand what I was doing rather than just filling out a template, I eventually got to the point where I can now say that I confidently understand why inductive proofs work.

Reflection 4. Think back to the start of class. What did you think 'doing math' meant at the start of the term, and what do you think it means now? Has your answer changed at all? You might consider the following aspects of 'doing math': collaboration, writing, problem solving, learning new concepts, the difference between 'figuring it out' and 'writing a proof',.... *Suggested Length: 2-3 paragraphs.*

Back at the start of the term, I think that I had a pretty strict vision of what I thought 'doing math' was. Up until this point, I had only really taken classes that required the memorization of formulas and the plugging in of numbers. I definitely had taken some difficult calculus classes in high school, and I had the concepts pretty much down, but I only really had this vague notion of what higher level math looked like in practice. While I had taken CS 202: Math of CS here at Carleton, which I thought seemed pretty similar to this class at the beginning, Math Structures certainly went way more into the conceptual side of math than CS 202 ever did.

Now that I have completed Math Structures, I think that I am really walking away with a much more complete picture of what higher level math looks like in general. While I know there is so much math out there, I really do feel like I have the understand and the fundamental mathematical techniqes necessary to begin to learn math at the higher level. Specifically, I feel like my definition of 'doing math' has shifted more toward collaborative problem solving and clear, analytical writing. Previously, I think that my definition of 'doing math' would be working to solve some problem to get a specific number as output, for example, calculating an integral or solving some complex system of equations. Now, while I think there definitely is a place for that, I think that the action of 'doing math' is instead understanding why and how we are doing what we are doing. Not only that, but I think my definition also includes being able to explain this to somebody else, using mathematical writing techniques and expressing thinking in extremely logical terms.