Homework 4: Orientation in Robotics (Exponential Coordinates, and Euler Angles)

Course Name: Modern Robotics I: Arm-type Manipulators

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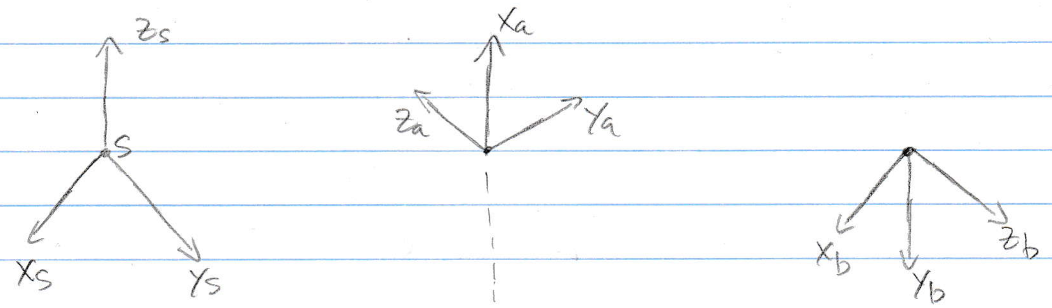
Structure of This Homework

This submission is divided into two parts:

1. Part I is Questions 1 - 4, written by hand. You will find these answers starting in the next page of this PDF file.
2. Part II is the code, videos, and images, used for the previously answered questions. You will find these files this GitHub repository.

Question 1

Frames:



$R_{ss} = I$ $\quad R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

a) $1 + 2\cos\theta = r_{11} + r_{22} + r_{33}$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) = \cos^{-1}(-1/2) \rightarrow \theta = 2\pi/3, \quad \theta = 4\pi/3$$

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$$

Option 1: $[\hat{\omega}] = \frac{1}{2\sin(2\pi/3)}\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\right)$

$$\boxed{[\hat{\omega}] = \begin{bmatrix} 0 & -0.5774 & -0.5774 \\ 0.5774 & 0 & -0.5774 \\ 0.5774 & 0.5774 & 0 \end{bmatrix} \text{ with } \theta = 2\pi/3}$$

Option 2: $[\hat{\omega}] = \frac{1}{2\sin(4\pi/3)}\left(\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\right)$

Drawings in repo!

$$\boxed{[\hat{\omega}] = \begin{bmatrix} 0 & 0.5774 & 0.5774 \\ -0.5774 & 0 & 0.5774 \\ -0.5774 & -0.5774 & 0 \end{bmatrix} \text{ with } \theta = 4\pi/3}$$

②

b) $\hat{\omega}\theta = (1, 2, 0) \leftarrow$ This is a vector! <u>NOT</u> a unit vector!

To obtain $\hat{\omega}$ (the unit vector) we obtain the norm of $\hat{\omega}\theta$:

$$\|\hat{\omega}\theta\| = \sqrt{1^2 + 2^2 + 0} = \sqrt{5}$$

$$\hat{\omega} = \left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}, 0\right) \longrightarrow [\hat{\omega}] = \begin{bmatrix} 0 & 0 & \tfrac{2}{\sqrt{5}} \\ 0 & 0 & -\tfrac{1}{\sqrt{5}} \\ -\tfrac{2}{\sqrt{5}} & \tfrac{1}{\sqrt{5}} & 0 \end{bmatrix}$$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}][\hat{\omega}]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 0 & \tfrac{2}{\sqrt{5}} \\ 0 & 0 & -\tfrac{1}{\sqrt{5}} \\ -\tfrac{2}{\sqrt{5}} & \tfrac{1}{\sqrt{5}} & 0 \end{bmatrix} +$$

$$\hookrightarrow (1-\cos\theta) \begin{bmatrix} 0 & 0 & \tfrac{2}{\sqrt{5}} \\ 0 & 0 & -\tfrac{1}{\sqrt{5}} \\ -\tfrac{2}{\sqrt{5}} & \tfrac{1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \tfrac{2}{\sqrt{5}} \\ 0 & 0 & -\tfrac{1}{\sqrt{5}} \\ -\tfrac{2}{\sqrt{5}} & \tfrac{1}{\sqrt{5}} & 0 \end{bmatrix}$$

Knowing that $\theta = \|\hat{\omega}\theta\| = \sqrt{5}$, we have:
(solving in Python):

$$\boxed{e^{[\hat{\omega}]\theta} = \begin{bmatrix} -0.2938 & 0.6469 & 0.7036 \\ 0.6469 & 0.6765 & -0.3518 \\ -0.7037 & 0.3518 & -0.6173 \end{bmatrix}}$$ Final Answer

Verification in question_1b.py!!

(3)

Question 2

a) $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \theta = \cos^{-1}\left(\frac{0-1+0-1}{2}\right) = \cos^{-1}(-1)$

$\theta = \pi \Rightarrow \underline{\underline{\theta = K\pi}}$ ← But we are limited to $[0, 2\pi)$ so: $\theta = \pi$

$R = I + 2[\hat{\omega}]^2 = \begin{bmatrix} 2\hat{\omega}_1^2 - 1 & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_2 - 1 & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_3 & 2\hat{\omega}_2\hat{\omega}_3 & 2\hat{\omega}_3^2 - 1 \end{bmatrix}$

$2\hat{\omega}_1^2 - 1 = 0$	$2\hat{\omega}_2 - 1 = -1$	$2\hat{\omega}_3^2 - 1 = 0$
$\hat{\omega}_1^2 = 1/2$	$2\hat{\omega}_2^2 = 0$	$\omega_3^2 = 1/2$
$\hat{\omega}_1 = \pm 1/\sqrt{2}$	$\hat{\omega}_2 = 0$	$\hat{\omega}_3 = \pm 1/\sqrt{2}$

Now to determine how they pair up:

$2\hat{\omega}_1\hat{\omega}_3 = 1 \Rightarrow$ both must have the same sign!!

So:

Option #1: $\hat{\omega} = (1/\sqrt{2}, 0, 1/\sqrt{2})$ with $\theta = \pi$

Option #2: $\hat{\omega} = (-1/\sqrt{2}, 0, -1/\sqrt{2})$ with $\theta = \pi$

This answer is verified with the code question1_a.py from the repo!

Question 2

b) $v_2 = R_{v_1} = e^{[\hat{\omega}]\theta} v_1$

$\hat{\omega} = (2/3, 2/3, 1/3) = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$

$v_1 = (1, 0, 1)$

$v_2 = (0, 1, 1)$

$$v_2 = Rv_1 \Longrightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We get: $r_{11} + r_{13} = 0$ // $\quad r_{21} + r_{23} = 1$ // $\quad r_{31} + r_{33} = 1$ //

We also have $Rv_1 = e^{[\hat{\omega}]\theta v_1} \Longrightarrow R = e^{[\hat{\omega}]\theta}$ AND $(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) = (2/3, 2/3, 1/3)$

SO:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + (4/9)(1-c_\theta) & (4/9)(1-c_\theta) - (1/3)s_\theta & (2/9)(1-c_\theta) + (2/3)s_\theta \\ (4/9)(1-c_\theta) + (1/3)s_\theta & c_\theta + (4/9)(1-c_\theta) & (2/9)(1-c_\theta) - (2/3)s_\theta \\ (2/9)(1-c_\theta) - (2/3)s_\theta & (2/9)(1-c_\theta) + (2/3)s_\theta & c_\theta + (1/9)(1-c_\theta) \end{bmatrix}$$

$r_{11} = c_\theta + (4/9)(1-c_\theta)$

$r_{13} = (2/9)(1-c_\theta) + (2/3)s_\theta$

$r_{11} + r_{13} = 0$

$c_\theta + (4/9)(1-c_\theta) + (2/9)(1-c_\theta) + (2/3)s_\theta = 0$

$c_\theta + 4/9 - (4/9)c_\theta + 2/9 - (2/9)c_\theta + (2/3)s_\theta =$

$(1 - 4/9 - 2/9)c_\theta + (2/3)s_\theta + 6/9 = 0$

$(1/\cancel{3})c_\theta + (2/\cancel{3})s_\theta = -2/\cancel{3}$

$c_\theta + 2s_\theta = -2$

$s_\theta = \dfrac{-2 - c_\theta}{2}$ // $\qquad$ I need another one!!

$r_{21} + r_{23} = 1$

$(4/9)(1-c_\theta) + (1/3)s_\theta + (2/9)(1-c_\theta) - (2/3)s_\theta = 1$

$(2/3)(1-c_\theta) - (1/3)s_\theta = 1$

$(2/3)(1-c_\theta) - 1 = (1/3)s_\theta$

$2(1-c_\theta) - 3 = s_\theta$ //

$\frac{-2-c_\theta}{2} = 2(1-c_\theta) - 3$

$-2 - c_\theta = 4(1-c_\theta) - 6$

$-2 - c_\theta = 4 - 4c_\theta - 6$

$-c_\theta + 4c_\theta = 4 - 6 + 2$

$3c_\theta = 0$

$\theta = \cos^{-1}(0) \rightarrow \theta = 90°$, $\theta = 270°$ Now I need to see if both work!!

$2(1-\overset{0}{\cos 90°}) - 3 \doteq \overset{1}{\sin(90°)}$ $2(1-\overset{0}{\cos 270°}) - 3 = \sin(270°)$

$2 - 3 = 1$ $2 - 3 = -1$

$-1 \neq 1$ X $-1 = -1$ // Only 270° works!!

$\sin 90° = \frac{-2-\cos 90°}{2}$ $\sin 270° = \frac{-2-\cos 270°}{2}$

$1 = \frac{-2-0}{2}$ $-1 = \frac{-2-0}{2}$

$1 \neq -1$ X $-1 = -1$ //

The only angle θ that satisfies the equation $v_2 = Rv_1 = e^{[\omega]\theta}v_1$ is $\theta = 270°$!

⑤

Question 3

a) XYZ - roll, pitch, yaw - γ, β, α

$$R(\hat{x},\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \quad R(\hat{y},\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R(\hat{z},\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All rotations are around the space frame, so:

$$Rot = R(\hat{z},\alpha)\,R(\hat{y},\beta)\,R(\hat{x},\gamma)\,I \leftarrow \text{Pre-multiplication!}$$

$$R(\gamma,\beta,\alpha) = \begin{bmatrix} C_\alpha & -S_\alpha & 0 \\ S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\gamma & -S_\gamma \\ 0 & S_\gamma & C_\gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{R(\gamma,\beta,\alpha) = \begin{bmatrix} C_\alpha C_\beta & -S_\alpha C_\gamma + S_\beta S_\gamma C_\alpha & S_\alpha S_\gamma + S_\beta C_\alpha C_\gamma \\ S_\alpha C_\beta & S_\alpha S_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta C_\gamma - S_\gamma C_\alpha \\ -S_\beta & S_\gamma C_\beta & C_\beta C_\gamma \end{bmatrix}}$$ (with python!)

b) Yes, this is the same as the matrix we discussed in Lesson 4! The orientations are the same, but the physical representations (and real-life movement considerations) are NOT the same!! The difference lies in the order of rotations AND the reference axis for each of those rotations:

- ZYX from lesson: $\overset{①}{R(\hat{z},\alpha)} \rightarrow \overset{②}{R(\hat{y},\beta)} \rightarrow \overset{③}{R(\hat{x},\alpha)}$ all w.r.t. <u>body</u> frame!
- XYZ from now: $\overset{①}{R(\hat{x},\gamma)} \rightarrow \overset{②}{R(\hat{y},\beta)} \rightarrow \overset{③}{R(\hat{z},\alpha)}$ all w.r.t. <u>space</u> frame!

c) $R = \begin{bmatrix} 0.6 & 0.79 & -0.01 \\ 0.47 & -0.34 & 0.81 \\ 0.64 & -0.5 & -0.58 \end{bmatrix}$

Using $R = R(\gamma, \beta, \alpha) \leftarrow$ (from before!)

$-\sin\beta = 0.64$

$\beta = \sin^{-1}(-0.64)$

$\beta = -39.79°$

$\beta = 320.21°$ OR $\beta = 219.79$

$\cos\alpha \cos\beta = 0.6$ $\quad$ $\sin\alpha \cos\beta = 0.47$

$\cos\beta = \dfrac{0.6}{\cos\alpha}$ $\quad$ $\cos\beta = \dfrac{0.47}{\sin\alpha}$

$$\frac{0.6}{\cos\alpha} = \frac{0.47}{\sin\alpha}$$

$\tan\alpha = 0.47/0.6$

$\alpha = \tan^{-1}(0.47/0.6)$ $\rightarrow \alpha = 38.07°$ $\rightarrow \alpha = 218.07°$

$\sin\gamma \cos\beta = -0.5$ $\quad$ $\cos\beta \cos\gamma = -0.58$

$\cos\beta = \dfrac{-0.5}{\sin\gamma}$ $\quad$ $\cos\beta = \dfrac{-0.58}{\cos\gamma}$

$$\frac{-0.5}{\sin\gamma} = \frac{-0.58}{\cos\gamma}$$

$\tan\gamma = \dfrac{-0.5}{-0.58}$

$\gamma = \tan^{-1}(0.5/0.58)$ $\rightarrow \gamma = 40.76°$ $\rightarrow \gamma = 220.76°$

⑥

Now we need to figure out who pairs with who!

$$\cos\beta = \frac{0.6}{\cos\alpha} \rightarrow \cos(320.21^\circ) = \frac{0.6}{\cos(38.07^\circ)} \Rightarrow 0.7684 = \frac{0.6}{0.7873}$$

$$0.7684 \approx 0.7621$$

$$\rightarrow \cos(320.21^\circ) = \frac{0.6}{\cos(218.07)} \Rightarrow 0.7684 \neq \frac{0.6}{-0.7873}$$

So $\beta = 320.21^\circ$ goes with $\alpha = 38.07^\circ$ //

$$\cos\beta = \frac{-0.5}{\sin\gamma} \rightarrow \cos(320.21^\circ) = \frac{-0.5}{\sin(40.76^\circ)} \Rightarrow 0.7684 = \frac{-0.5}{0.7575}$$

$$\rightarrow \cos(320.21^\circ) = \frac{-0.5}{\sin(220.76^\circ)} \Rightarrow 0.7684 = \frac{-0.5}{-0.7575}$$

$$0.7684 \approx 0.66$$

So $\beta = 320.21^\circ$ goes with $\gamma = 220.76^\circ$ //

These differences are due to rounding approximations!!

Having these two constraints, we get:

Option 1: $\alpha = 38.07^\circ$, $\beta = 320.21^\circ$, $\gamma = 220.76^\circ$
Option 2: $\alpha = 218.07^\circ$, $\beta = 219.79^\circ$, $\gamma = 40.76^\circ$

Question 4

a) For roll-pitch-yaw (from Question 3)

$$R(\gamma_{rpy}, \beta_{rpy}, \alpha_{rpy}) = \begin{bmatrix} C\alpha_{rpy} C\beta_{rpy} & -S\alpha_{rpy} C\gamma_{rpy} + S\beta_{rpy} S\gamma_{rpy} C\alpha_{rpy} & A_1 \\ S\alpha_{rpy} C\beta_{rpy} & S\alpha_{rpy} S\beta_{rpy} S\gamma_{rpy} + C\alpha_{rpy} C\gamma_{rpy} & A_2 \\ -S\beta_{rpy} & S\gamma_{rpy} C\beta_{rpy} & A_3 \end{bmatrix}$$

$$A_1 = S\alpha_{rpy} S\gamma_{rpy} + S\beta_{rpy} C\alpha_{rpy} C\gamma_{rpy}$$
$$A_2 = S\alpha_{rpy} S\beta_{rpy} C\gamma_{rpy} - S\gamma_{rpy} C\alpha_{rpy}$$
$$A_3 = C\beta_{rpy} C\gamma_{rpy}$$

For ZYZ Euler Angles:

$$R(\hat{z},\alpha) = \begin{bmatrix} C_\alpha & -S_\alpha & 0 \\ S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R(\hat{y},\beta) = \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \quad R(\hat{z},\gamma) = \begin{bmatrix} C_\gamma & -S_\gamma & 0 \\ S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} C_\alpha & -S_\alpha & 0 \\ S_\alpha & C_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix} \begin{bmatrix} C_\gamma & -S_\gamma & 0 \\ S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(solving symbollically in Python)

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} -S_\alpha S_\gamma + C_\alpha C_\beta C_\gamma & -S_\alpha C_\gamma - S_\gamma C_\alpha C_\beta & S_\beta C_\alpha \\ S_\alpha C_\beta C_\gamma + S_\gamma C_\alpha & -S_\alpha S_\gamma C_\beta + C_\alpha C_\gamma & S_\alpha S_\beta \\ -S_\beta C_\gamma & S_\beta S_\gamma & C_\beta \end{bmatrix}$$

Now we do $R(\gamma_{rpy}, \beta_{rpy}, \alpha_{rpy}) = R(\alpha,\beta,\gamma)$

(next page)

We get:

$$\cos\beta = \cos(\beta_{rpy})\cos(\gamma_{rpy})$$

$$\boxed{\beta = \cos^{-1}(\cos\beta_{rpy}\cos\gamma_{rpy})}$$

$$-\sin\beta\cos\gamma = -\sin\beta_{rpy} \qquad \sin\beta\sin\gamma = \sin\gamma_{rpy}\cos\beta_{rpy}$$

$$\underbrace{\sin\beta = \frac{\sin\beta_{rpy}}{\cos\gamma} \qquad \sin\beta = \frac{\sin\gamma_{rpy}\cos\beta_{rpy}}{\sin\gamma}}$$

$$\frac{\sin\beta_{rpy}}{\cos\gamma} = \frac{\sin\gamma_{rpy}\cos\beta_{rpy}}{\sin\gamma}$$

$$\tan\gamma = \frac{\sin\gamma_{rpy}\cos\beta_{rpy}}{\sin\beta_{rpy}}$$

$$\boxed{\gamma = \tan^{-1}\left(\frac{\sin\gamma_{rpy}\cos\beta_{rpy}}{\sin\beta_{rpy}}\right)}$$

$$\sin\alpha\sin\beta = \sin\alpha_{rpy}\sin\beta_{rpy}\cos\gamma_{rpy} - \sin\gamma_{rpy}\cos\alpha_{rpy}$$

$$\sin\alpha = \frac{\sin\alpha_{rpy}\sin\beta_{rpy}\cos\gamma_{rpy} - \sin\gamma_{rpy}\cos\alpha_{rpy}}{\sin\beta}$$

$$\boxed{\alpha = \sin^{-1}\left(\frac{\sin\alpha_{rpy}\sin\beta_{rpy}\cos\gamma_{rpy} - \sin\gamma_{rpy}\cos\alpha_{rpy}}{\sin(\cos^{-1}(\cos\beta_{rpy}\cos\gamma_{rpy}))}\right)}$$

Each of these will give two answers for each angle. Which angles will be paired up with each other will depend on the numerical example in question!!

b) roll $= \gamma_{rpy} = 30°$
pitch $= \beta_{rpy} = 60°$
yaw $= \alpha_{rpy} = 45°$

Their equivalents in ZYZ would be:

$$\beta = \cos^{-1}(\cos\beta_{rpy}\cos\gamma_{rpy}) = \cos^{-1}(\underbrace{\cos 60°}_{1/2}\ \underbrace{\cos 30°}_{\sqrt{3}/2}) \rightarrow \beta = 64.34°$$

$$\rightarrow \beta = -64.34 = 295.66°$$

$$\alpha = \sin^{-1}\left(\frac{\overbrace{\sin 45°}^{\sqrt{2}/2}\ \overbrace{\sin 60°}^{\sqrt{3}/2}\ \overbrace{\cos 30°}^{\sqrt{3}/2} - \overbrace{\sin 30°}^{1/2}\ \overbrace{\cos 45°}^{\sqrt{2}/2}}{\sin(\cos^{-1}(\underbrace{\cos 60°}_{1/2}\ \underbrace{\cos 30°}_{\sqrt{3}/2}))}\right) \rightarrow \alpha = 11.31°$$

$$\rightarrow \alpha = 191.31°$$ ← Because of β's sign!

$$\gamma = \tan^{-1}\left(\frac{\sin 30°\cos 60°}{\sin 60°}\right) \rightarrow \gamma = 16.10°$$

$$\rightarrow \gamma = 196.10°$$

Now, to determine who pairs with who:

$-\sin\beta_{rpy} = -\sin\beta\cos\gamma$
$\sin\beta_{rpy} = \sin\beta\cos\gamma$
$\sin 60° = 0.866 =$
$\rightarrow \sin 64.34° \cdot \cos 16.10° \approx 0.866$ ✓
$\rightarrow \sin 64.34° \cdot \cos 196.10° \approx -0.866$ ✗
$\sin 295.66° \cdot \cos 16.10° \approx -0.866$ ✗
$\sin 295.66° \cdot \cos 196.10° \approx 0.866$ ✓

This means $\beta = 64.34°$ must be paired with $\gamma = 16.10°$ AND $\beta = 295.66°$ must be paired with $\gamma = 196.10°$. This is Constraint 1!!

$\sin\alpha \sin\beta = \sin\alpha_{rpy} \sin\beta_{rpy} \cos\gamma_{rpy} - \sin\gamma_{rpy} \cos\alpha_{rpy}$

$\sin\alpha \sin\beta = \underbrace{\sin 45°}_{\sqrt{2}/2} \underbrace{\sin 60°}_{\sqrt{3}/2} \underbrace{\cos 30°}_{\sqrt{3}/2} - \underbrace{\sin 30°}_{1/2} \underbrace{\cos 45°}_{\sqrt{2}/2}$

$\sin\alpha \sin\beta = 0.1768$

$\rightarrow \sin 11.31° \cdot \sin 64.34° = 0.1768$ ✓

$\rightarrow \sin 11.31° \cdot \sin 295.66° = -0.1768$ ✗

$\rightarrow \sin 191.31 \cdot \sin 64.34° = -0.1768$ ✗

$\rightarrow \sin 191.31 \cdot \sin 295.66° = 0.1768$ ✓

This means that $\alpha = 11.31°$ must be paired with $\beta = 64.34°$ AND $\alpha = 191.31°$ must be paired with $\beta = 295.66°$. This is Constraint 2

Finally, using Constraints 1 and 2:

$$\boxed{\begin{array}{l} \text{Option \#1}: \alpha = 11.31°, \beta = 64.34°, \gamma = 16.10° \\ \text{Option \#2}: \alpha = 191.31°, \beta = 295.66°, \gamma = 196.10° \end{array}}$$ for ZYZ!!

RoboDK video shown in repo!