Multibody System Dynamics (2005) 14: 345–366

Kinematics of a Hybrid Manipulator by Means of Screw Theory

J. GALLARDO-ALVARADO
Instituto Tecnológico de Celaya, Department of Mechanical Engineering, Av. Tecnológico y A. García Cubas, 38010 Celaya, Gto., México

(Received: 19 May 2004; accepted in revised form: 18 March 2005)

Abstract. In this work the kinematics of a hybrid manipulator, namely a fully parallel-serial manipulator, with a particular topology is approached by means of the theory of screws. Given the length of the six independent limbs, the forward position analysis of the mechanism under study, indeed the computation of the resulting pose, position and orientation, of the end-platform with respect to the fixed platform, is carried out in closed-form solution. Therefore conveniently this initial analysis avoids the use of a numerical technique such as the Newton-Raphson method. Writing in screw form the reduced acceleration state of the translational platform, with respect to the fixed platform, a simple expression for the computation of the acceleration of the translational platform is derived by taking advantage of the properties of reciprocal screws, via the Klein form, a bilinear symmetric form of the Lie algebra $e(3)$. Following a similar procedure, a simple expression for the computation of the angular acceleration of the end-platform, with respect to the translational platform, is easily derived. Naturally, as an intermediate step, this contribution also provides the forward and inverse velocity analyses of the chosen parallel-serial manipulator. Finally, in order to prove the versatility of the expressions obtained via screw theory for solving the kinematics, up to the acceleration analysis, of the proposed spatial mechanism, a numerical example is solved with the help of commercial computer codes.

Keywords: hybrid manipulator, forward analysis, screw theory, Klein form, kinematics and dynamics

1. Introduction

The manipulators can be classified in three main types: serial, parallel or hybrid. A serial manipulator is a mechanism composed of an end-effector connected to a base link by means of a single serial chain, while a parallel manipulator is a mechanism composed of a moving platform connected to a fixed platform by at least two kinematic chains. A hybrid manipulator is a combination of parallel and serial manipulators, or it is an arrangement of two or more parallel manipulators connected in tandem.

In comparison with serial manipulators, most parallel manipulators have interesting merits such as higher stiffness and greater load capacity, among a better accuracy, but suffer from a limited workspace volume. Even though this critical drawback, the advantages of the parallel manipulators have not been ignored by the researchers, and therefore important applications such as air flight simulators, pointing devices and walking machines have been introduced. Furthermore, recently parallel manipulators have also been developed as high-speed multi-dof machining centers; detailed information of these applications can be found in consulting the Parallel Mic Center, http://www.parallemic.org/. As it is developed in [1], in order to simplify the analysis of a multiloop system, it is strongly advisable to gradually decompose it into simpler subsystems for which individually closed-form solutions, if any, can be generated. Then the individual solutions thus obtained are assembly back to yield the global solution.

Hybrid manipulators conjugate the main advantages of serial and parallel manipulators. Clearly, they have a considerable workspace in addition to a rigid architecture.

The study of parallel manipulators, and their possible applications in the industry, dates in reality from several years ago.

In fact, the first theoretical contribution dealing with the study of a parallel mechanism was published by Maxwell in 1890 [2], while the first industrial application of such a mechanism is attributed to Willard Pollard Jr., who in 1934 filed a patent for a spray painting machine [3]. In 1947 Gough introduced an octahedral hexapod as a universal tire-testing machine. In 1965 Stewart proposed a fully parallel mechanism as a flight simulator.

The Gough and Stewart contributions had, without doubt, a great impact on the subsequent development in the field of parallel mechanisms. Thus, nowadays mechanisms that employ the same architecture of the Gough and Stewart mechanisms are known universally as Gough-Stewart platforms. Since then, parallel manipulators have been extensively studied by many researchers, which is reflected in the appreciable amount of valuable contributions reported in the Literature, see for instance [4–11].

The most studied type of parallel manipulator is the so-called general Gough-Stewart platform, a fully parallel mechanism. The forward position analysis of a general Gough-Stewart platform is a challenging intensive task which yields forty possible solutions [12]. On the other hand, as it is shown in [13], the acceleration analysis of an in-parallel manipulator does not represent any difficulty when the theory of screws is employed.

This work deals with the kinematics of the hybrid manipulator showed in Figure 1, which is a sequence of two parallel manipulators.

The lower parallel manipulator, a variant of the translational in-parallel mechanism introduced in [14], is composed of a moving platform, namely a translational platform, connected to a fixed platform by means of three independent or actuator limbs, and two passive kinematic chains. Each independent limb is composed of a Universal-Prismatic-Spherical, or for brevity UPS-type, serial manipulator. On the other hand, the upper parallel manipulator is composed of a moving platform, namely the end-platform, connected to the translational platform of the lower parallel manipulator, by means of three independent limbs, also UPS-type. The architecture of the upper parallel manipulator is such that the end-platform

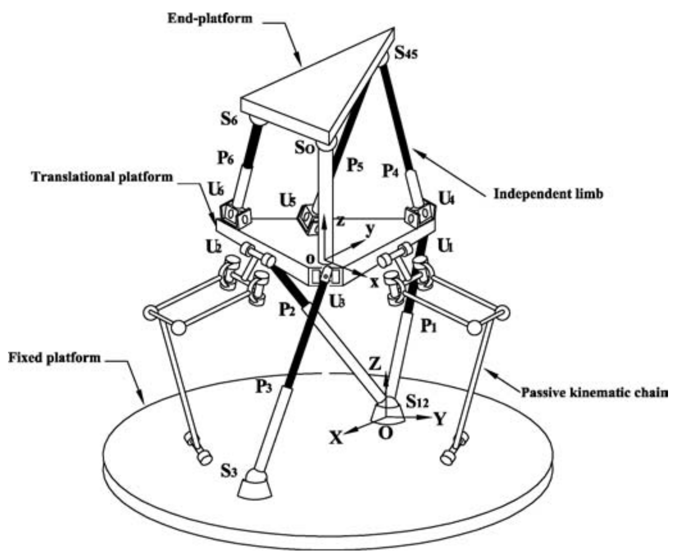

Figure 1. Architecture of the proposed hybrid manipulator.

is restricted to undergo only spherical motions with respect to the translational platform.

Owing its architecture, the forward position analysis of the proposed mechanism is subdivided into two different problems. The first one consists in computing the coordinates of the universal joints, attached to the inferior face of the translational platform, given the length of the three independent limbs of the lower parallel manipulator. The second one consists in computing the coordinates of the spherical joints attached to the end-platform, given the length of the three independent limbs of the upper parallel manipulator.

The acceleration analysis of the hybrid manipulator is easily approached by means of the theory of screws. To this end, the reduced acceleration state, a six-dimensional vector, of the translational platform is expressed in screw form, with respect to the fixed platform, through each one of the independent limbs. Then the application of the Klein form between the ith line, in Plücker coordinates, along the ith independent limb yields a simple expression for the computation of the joint rate acceleration of the ith actuated prismatic joint. Casting in matrix form the expressions thus obtained for the computation of the joint rate accelerations associated to the three actuated prismatic leads to a compact and simple expression for the computation of the acceleration of the translational platform. Following a similar procedure, a simple expression is derived for the computation of the angular acceleration of the end-platform. Of course, this contribution also provides the velocity analysis of the hybrid manipulator. Finally, in order to prove the versatility of the expressions obtained via screw theory, a numerical example is solved with the aid of commercial computer codes.

2. Preliminary Concepts

In order to provide a proper foundation of the next sections, a brief summary of well-known concepts dealing with the kinematics of open serial chains by means of screw theory is presented in this section, for a detailed explanation of it the reader is referred to [15, 16].

A screw is a straight line with which a definite linear magnitude termed the pitch is associated. This definition of a screw was given by Ball [17] at the beginning of the twentieth century.

A screw, $\$ = (\hat{s}, \mathbf{s}_O)$, can be considered as a six-dimensional vector composed of a primal part $\hat{s}$ and a dual part $\mathbf{s}_O$. The primal part, $\hat{s}$, is a unit vector along the direction of the axis of the screw, whereas $\mathbf{s}_O$ is the moment produced by $\hat{s}$, about a point O fixed to the reference frame, which it is calculated according to the pitch h of the screw and a vector $\mathbf{r}_O$, which denotes the position of the point O with respect to any point on the screw axis. The moment $\mathbf{s}_O$ is given by

$$\mathbf{s}_O = h\hat{s} + \hat{s} \times \mathbf{r}_O.$$

Any lower kinematic pair can be represented either by a screw or a group of screws. For example, if the pitch of a screw is equal to zero then the screw is given in Plücker coordinates by

$$\$ = \begin{bmatrix} \hat{s} \\ \mathbf{s} \times \mathbf{r}_O \end{bmatrix}.$$

whereas if the pitch h of a screw is infinity, then the screw pair reduces to a prismatic pair given by

$$\$ = \begin{bmatrix} \mathbf{0} \\ \hat{s} \end{bmatrix}.$$

It is interesting to say that the use of the term unitary screw, or normalized screw, still in our days is a common practice [18]. However, note that usually only the primal part of a screw is a normalized vector, and therefore the convenience of using the term unitary screw must be reconsidered.

The screw algebra, which is isomorphic to the Lie algebra $e(3)$ also referred as motor algebra, is the set of elements of the form $\$ = (\hat{s}, \mathbf{s}_O)$ with the following operations.

$$\forall \$_1 = (\hat{s}_1, \mathbf{s}_{O1}), \$_2 = (\hat{s}_2, \mathbf{s}_{O2}) \in e(3) \quad \text{and} \quad \lambda \in \Re$$

Addition

$$\$_1 + \$_2 = (\hat{s}_1 + \hat{s}_2, \mathbf{s}_{O1} + \mathbf{s}_{O2}).$$

Multiplication by a scalar

$$\lambda\$_1 = (\lambda\hat{s}_1, \lambda\mathbf{s}_{O1}).$$

Lie product or dual motor product

$$[\$_1 \quad \$_2] = (\hat{s}_1 \times \hat{s}_2, \hat{s}_1 \times \mathbf{s}_{O2} - \hat{s}_2 \times \mathbf{s}_{O1}).$$

Furthermore, the Lie algebra $e(3)$ is endowed with two bilinear symmetric forms.

The Killing form

$$Ki : e(3) \times e(3) \to \Re \quad Ki(\$_1, \$_2) = \hat{s}_1 \cdot \hat{s}_2, \quad \text{and}$$

The Klein form

$$KL : e(3) \times e(3) \to \Re \quad KL(\$_1, \$_2) = \hat{s}_1 \cdot \mathbf{s}_{O2} + \hat{s}_2 \cdot \mathbf{s}_{O1}.$$

The notion of motor, the union of the terms moment and rotor, was introduced by Clifford in 1873 [19], in his algebra of bi-quaternion. However Clifford did not apply this concept to the modelling of motion of a rigid body.

Half century later, von Mises published, in two parts, his "Motorrechnung, ein neues Hilfsmittel in Mechanik". In the first part von Mises [20], introduces the dual motor product and indicates its role as a measure of the instantaneous change of a motor associated to a rigid body by the action of a second motor.

In the second part von Mises [21], applied the computation of motors in diverse problems, including the derivation of the general form of the equations of motion of a rigid body. Without doubt, the investigations of von Mises constitute a remarkable contribution in the mechanical engineering. The importance of the algebra of motors was immediately recognized by the scientists of that time. However, for most engineers, the importance of the algebra of motors and their possible applications were practically ignored. Recently, the monumental work of von Mises was republished in [22].

It is prudent to emphasize that only five parameters are sufficient to define a screw, as it is correctly indicated by one of the reviewers. However, in this work the concept of screw has a direct connection with the definition of motor. The motors constitute a space of dimension 6, and by such reason the same treatment is given in this work to the concept of screw.

The initial work of the theory of screws is credited to Ball [17], and the fundamental concepts of this efficient mathematical resource in the modelling of the motion of rigid bodies was reviewed in [23].

The twist about a screw of a rigid body, also known as the velocity state of a rigid body, was defined as a six-dimensional vector $\mathbf{V}$ by Ball [17] as follows

$$\mathbf{V} = \begin{bmatrix} \omega \\ \mathbf{v}_O \end{bmatrix}, \tag{1}$$

where ω is the angular velocity of the rigid body, while $\mathbf{v}_O$ is the velocity of a point O, fixed to the rigid body, that is instantaneously coincident with a point fixed in the reference frame.

Note that while the angular velocity ω, a property of the rigid bodies, does not depend of the chosen point. On the other hand, the velocity $\mathbf{v}_O$ depends of the point O.

It is very tempting to assume that the reduced acceleration state of a rigid body, $\mathbf{A}_R$, can be defined in a simple way as follows

$$\mathbf{A}_R = \begin{bmatrix} \dot{\omega} \\ \mathbf{a}_O \end{bmatrix}. \tag{2}$$

where $\dot{\omega}$ is the angular acceleration of the rigid body, whereas $\mathbf{a}_O$ is the acceleration of the point O. However, since the velocity state of a rigid body was defined as a six-dimensional vector, then it is straightforward to demonstrate that expression (2) is a fallacious representation of the reduced acceleration state of a rigid body. In fact, expression (2) clearly does not satisfy the conditions of a helicoidal vectorial field.

With this in mind and with the purpose of correcting this deficiency, the reduced acceleration state must be considered as the time derivative of the velocity state, via a helicoidal field [24].

Taking into account that the acceleration of any other point Q of the rigid body can be calculated using elementary kinematics as follows

$$\mathbf{a}_Q = \mathbf{a}_O + \dot{\omega} \times \mathbf{r}_{Q/O} + \omega \times (\omega \times \mathbf{r}_{Q/O}). \tag{3}$$

The dual part of the reduced acceleration state, $D(\$)$, taking Q as the reference point and applying the concept of helicoidal vectorial field, according to point O, takes the form

$$\begin{aligned} D(\$) &= \mathbf{a}_O + \dot{\omega} \times \mathbf{r}_{Q/O} = \mathbf{a}_Q - \dot{\omega} \times \mathbf{r}_{Q/O} - \omega \times (\omega \times \mathbf{r}_{Q/O}) + \dot{\omega} \times \mathbf{r}_{Q/O} \\ &= \mathbf{a}_Q - \omega \times (\omega \times \mathbf{r}_{Q/O}) = \mathbf{a}_Q - \omega \times \mathbf{v}_Q. \end{aligned} \tag{4}$$

Since Q is an arbitrary point, then the reduced acceleration state is, in reality, a six-dimensional vector given by

$$\mathbf{A}_R = \begin{bmatrix} \dot{\omega} \\ \mathbf{a}_O - \omega \times \mathbf{v}_O \end{bmatrix}. \tag{5}$$

In a serial manipulator, the velocity state of the end-effector, body m, with respect to the base link, body 0, can be expressed as a linear combination of the infinitesimal screws representing the kinematic pairs of the serial manipulator as follows

$$ {}_0\omega_1{}^0\$^1 + {}_1\omega_2{}^1\$^2 + \cdots + {}_{m-2}\omega_{m-1}{}^{m-2}\$^{m-1} + {}_{m-1}\omega_m{}^{m-1}\$^m = {}^0\mathbf{V}^m. \tag{6}$$

On the other hand, the reduced acceleration state in screw form, a concept introduced in [15], of the end-effector, with respect to the base link, is given by

$$ {}_0\dot{\omega}_1{}^0\$^1 + {}_1\dot{\omega}_2{}^1\$^2 + \cdots + {}_{m-2}\dot{\omega}_{m-1}{}^{m-2}\$^{m-1} + {}_{m-1}\dot{\omega}_m{}^{m-1}\$^m + \$_L = \mathbf{A}_R, \tag{7}$$

where the *Lie screw* $\$_L$ is calculated as follows

$$\begin{aligned} \$_L = {} & \big[{}_0\omega_1{}^0\$^1 \quad {}_1\omega_2{}^1\$^2 + {}_2\omega_3{}^2\$^3 + \cdots + {}_{m-2}\omega_{m-1}{}^{m-2}\$^{m-1} + {}_{m-1}\omega_m{}^{m-1}\$^m\big] \\ & + \big[{}_1\omega_2{}^1\$^2 \quad {}_2\omega_3{}^2\$^3 + {}_3\omega_4{}^3\$^4 + \cdots + {}_{m-2}\omega_{m-1}{}^{m-2}\$^{m-1} \\ & + {}_{m-1}\omega_m{}^{m-1}\$^m\big] + \cdots + \big[{}_{m-2}\omega_{m-1}{}^{m-2}\$^{m-1} \quad {}_{m-1}\omega_m{}^{m-1}\$^m\big]. \end{aligned}$$

3. Kinematics of the Lower Parallel Manipulator

In this section the kinematics of the lower parallel manipulator is presented. Taking into account that the inverse position analysis, indeed the computation of the generalized coordinates when the pose of the moving platform is given, of most parallel manipulators is a trivial task, then this initial analysis is deliberately omitted here. On the other hand, the forward position analysis, indeed the computation of the pose of the moving platform when the length of the independent limbs is given, is carried out in closed-form solution. To this end, and considering that the moving platform is restricted to execute only translational displacements with respect to the fixed platform, a closed loop expression, in combination with three compatibility equations, leads to expressions for determining all the possible locations of the translational platform.

Afterwards, the velocity and acceleration analyses of the lower parallel manipulator are approached by means of the theory of screws.

3.1. POSITION ANALYSIS

The forward position analysis consists in computing the coordinates of the universal joints attached to the lower face of the translational platform, expressed in the reference frame XYZ, given the length of the independent limbs of the lower parallel manipulator.

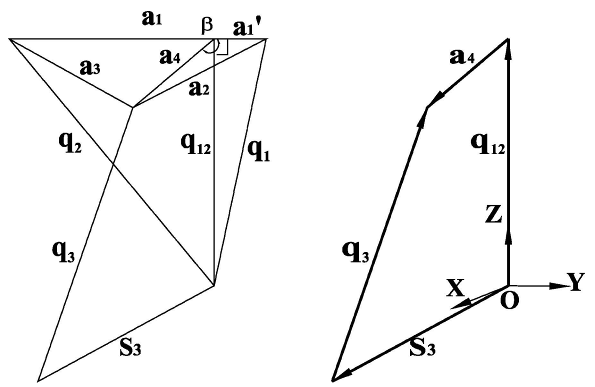

Figure 2. Geometric scheme of the lower parallel manipulator.

In order to simplify the forward position analysis, the reference frame XYZ is attached to the fixed platform in such a way that the line a_1, which denotes the separation between the universal joints U_1 and U_2, is parallel to the Y axis.

With reference to Figure 1, the corresponding geometric scheme of the lower parallel manipulator is showed in Figure 2. Thus immediately emerges that

$$\mathrm{a}_1' = \frac{q_1^2 - q_2^2 + \mathrm{a}_1^2}{2\mathrm{a}_1}, \tag{8}$$

and

$$q_{12} = \sqrt{q_1^2 - \mathrm{a}_1'^2}. \tag{9}$$

Furthermore, a closed loop expression can be written as follows

$$\mathbf{q}_3 + \mathbf{S}_3 = \mathbf{q}_{12} + \mathbf{a}_4. \tag{10}$$

Taking into account the following compatibility expressions

$$\left.\begin{array}{l}(\mathbf{q}_1 - \mathbf{q}_2) \cdot (\mathbf{q}_1 - \mathbf{q}_2) = \mathrm{a}_1^2,\\ (\mathbf{q}_1 - \mathbf{S}_3 - \mathbf{q}_3) \cdot (\mathbf{q}_1 - \mathbf{S}_3 - \mathbf{q}_3) = \mathrm{a}_2^2,\\ (\mathbf{q}_2 - \mathbf{S}_3 - \mathbf{q}_3) \cdot (\mathbf{q}_2 - \mathbf{S}_3 - \mathbf{q}_3) = \mathrm{a}_3^2.\end{array}\right\} \tag{11}$$

Then, after a few computations, the coordinates of the universal joints, attached to the translational platform, expressed in the reference frame XYZ, result in

$$\left.\begin{array}{l}\mathbf{U}_1 = (U_X, \mathrm{a}_1', U_Z),\\ \mathbf{U}_2 = (U_X, \mathrm{a}_1' - \mathrm{a}_1, U_Z),\\ \mathbf{U}_3 = (U_X + \mathrm{a}_4\cos(\pi/2 - \beta), \mathrm{a}_4\sin(\pi/2 - \beta), U_Z).\end{array}\right\} \tag{12}$$

where

$$U_X = \frac{2\mathrm{a}_4(\cos(\pi/2-\beta)S_{3X}+\sin(\pi/2-\beta)S_{3Y}))+q_3^2-\mathrm{a}_4^2-S_{3X}^2-S_{3Y}^2-q_{12}^2}{2(\mathrm{a}_4\cos(\pi/2-\beta)-S_{3X})},$$

and

$$U_Z = \sqrt{q_{12}^2 - U_X^2}.$$

3.2. VELOCITY ANALYSIS

The inverse velocity analysis is stated as follows. Given the velocity state of the translational platform, with respect to the fixed platform, compute the joint rate velocities associated to the kinematic pairs of the independent limbs. To this end, each independent limb is modelled as a UPS-type serial manipulator, see Figure 3.

According with expression (6) the velocity state of the translational platform, with respect to the fixed platform, can be written through any of the three independent limbs as follows

$$_0\omega_{1i}\,{}^0\$^{1i} + {}_1\omega_{2i}\,{}^1\$^{2i} + {}_2\omega_{3i}\,{}^2\$^{3i} + \dot{q}_i\,{}^3\$^{4i} + {}_4\omega_{5i}\,{}^4\$^{5i} + {}_5\omega_{6i}\,{}^5\$^{6i} = {}^0\mathbf{V}^6 \qquad i = 1, 2, 3. \quad (13)$$

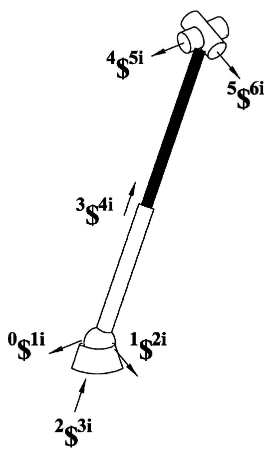

Figure 3. Kinematic scheme of an independent limb.

Therefore, the computation of the joint rate velocities of each independent limb results in

$$\begin{bmatrix} {}_0\omega_{1i} \\ {}_1\omega_{2i} \\ {}_2\omega_{3i} \\ \dot{q}_i \\ {}_4\omega_{5i} \\ {}_5\omega_{6i} \end{bmatrix} = \mathbf{J}_i^{-1} \quad {}^0\mathbf{V}^6 \quad i = 1, 2, 3. \tag{14}$$

where, $\mathbf{J}_i$ is the ith Jacobian matrix, a subspace generated by the set of screws representing the kinematic pairs of the i-th independent limb, given by

$$\mathbf{J}_i = [{}^0\$^{1i} \quad {}^1\$^{2i} \quad {}^2\$^{3i} \quad {}^3\$^{4i} \quad {}^4\$^{5i} \quad {}^5\$^{6i}].$$

Furthermore, according with expression (1) it is evident that the velocity state of the translational platform, free of rotational motions, is given by

$${}^0\mathbf{V}^6 = \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix}, \tag{15}$$

where $\mathbf{v} = (\mathrm{v}_X, \mathrm{v}_Y, \mathrm{v}_Z)$ is the velocity of any point fixed to the translational platform.

On the other hand, the forward velocity analysis is stated as follows. Given the joint rate velocities associated to the three actuated prismatic joints of the lower parallel manipulator, compute the resulting velocity of the translational platform with respect to the fixed platform.

In order to simplify the forward velocity analysis consider the ith line $\$_i = (\hat{s}_i, \mathbf{s}_{Oi}) = ((s_{iX}, s_{iY}, s_{iZ}), (s_{OiX}, s_{OiY}, s_{OiZ}))$ in Plücker coordinates along the ith independent limb. The application of the Klein form of this line with both sides of expression (13) leads to the cancellation of all the terms of the left side of expression (13), excepting the term associated to the actuated prismatic joint. Thus it is possible to write

$$\dot{q}_i = KL(\$_i, {}^0\mathbf{V}^6) \quad i = 1, 2, 3. \tag{16}$$

or

$$\dot{q}_i = s_{iX}\,\mathrm{v}_X + s_{iY}\,\mathrm{v}_Y + s_{iZ}\,\mathrm{v}_Z \quad i = 1, 2, 3 \tag{17}$$

Finally, casting in matrix form expressions (17) the velocity $\mathbf{v}$ of the translational platform results in

$$\mathbf{v} = \mathbf{S}_L^{-1} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}, \tag{18}$$

where, $\mathbf{S}_L$ is the subspace generated by the primal parts of the lines, $\$_i$ $(i = 1, 2, 3)$, along the three independent limbs

$$\mathbf{S}_L = [\hat{s}_1 \quad \hat{s}_2 \quad \hat{s}_3]^T.$$

Please note that expression (18) does not require the values of the passive joint rate velocities of the lower parallel manipulator. Furthermore, the relation input-output equations for determining the velocity of the translational platform is clearly specified.

3.3. ACCELERATION ANALYSIS

The inverse acceleration analysis is stated as follows. Given the reduced acceleration state of the translational platform, with respect to the fixed platform, compute the joint acceleration rates associated to the kinematic pairs of the three independent limbs of the lower parallel manipulator.

According with expression (7) the reduced acceleration state of the translational platform, with respect to the fixed platform, can be written through any of the independent limbs as follows

$$\begin{aligned} {}_0\dot{\omega}_{1i}{}^0\$^{1i} + {}_1\dot{\omega}_{2i}{}^1\$^{2i} + {}_2\dot{\omega}_{3i}{}^2\$^{3i} + \ddot{q}_i{}^3\$^{4i} + {}_4\dot{\omega}_{5i}{}^4\$^{5i} + {}_5\dot{\omega}_{6i}{}^5\$^{6i} + \$_{Li} = {}^0\mathbf{A}_R^6 \\ i = 1, 2, 3. \end{aligned} \tag{19}$$

where, the ith Lie screw is given by

$$\begin{aligned} \$_{Li} = {} & \left[{}_0\omega_{1i}{}^0\$^{1i} \quad {}_1\omega_{2i}{}^1\$^{2i} + {}_2\omega_{3i}{}^2\$^{3i} + \dot{q}_i{}^3\$^{4i} + {}_4\omega_{5i}{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[{}_1\omega_{2i}{}^1\$^{2i} \quad {}_2\omega_{3i}{}^2\$^{3i} + \dot{q}_i{}^3\$^{4i} + {}_4\omega_{5i}{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[{}_2\omega_{3i}{}^2\$^{3i} \quad \dot{q}_i{}^3\$^{4i} + {}_4\omega_{5i}{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[\dot{q}_i{}^3\$^{4i} \quad {}_4\omega_{5i}{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] + \left[{}_4\omega_{5i}{}^4\$^{5i} \quad {}_5\omega_{6i}{}^5\$^{6i}\right]. \end{aligned}$$

Therefore, the passive joint rate accelerations of the independent limbs can be calculated as follows

$$\begin{bmatrix} {}_0\dot{\omega}_{1i} \\ {}_1\dot{\omega}_{2i} \\ {}_2\dot{\omega}_{3i} \\ \ddot{q}_i \\ {}_4\dot{\omega}_{5i} \\ {}_5\dot{\omega}_{6i} \end{bmatrix} = \mathbf{J}_i^{-1}\left({}^0\mathbf{A}_R^6 - \$_{Li}\right) \quad i = 1, 2, 3. \tag{20}$$

Furthermore, according with expression (5) it is evident that the reduced acceleration state of the translational platform, free of rotational motions, is given by

$${}^0\mathbf{A}_R^6 = \begin{bmatrix} \mathbf{0} \\ \mathbf{a} \end{bmatrix}. \tag{21}$$

where $\mathbf{a} = (\mathrm{a}_X, \mathrm{a}_Y, \mathrm{a}_Z)$ is the acceleration of any point fixed to the translational platform.

On the other hand, the forward acceleration analysis is stated as follows. Given the joint acceleration rates associated to the actuated prismatic joints of the three independent limbs, compute the resulting acceleration of the translational platform.

The application of the Klein form of the i-th line $\$_i$ with both sides of expression (19), and with the corresponding reduction of terms, leads to

$$\ddot{q}_i + KL(\$_i, \$_{Li}) = KL\left(\$_i, {}^0\mathbf{A}_R^6\right) \quad i = 1, 2, 3. \tag{22}$$

or

$$\ddot{q}_i + KL(\$_i, \$_{Li}) = s_{iX}\,\mathrm{a}_X + s_{iY}\,\mathrm{a}_Y + s_{iZ}\,\mathrm{a}_Z \quad i = 1, 2, 3 \tag{23}$$

Finally, casting in matrix form expressions (23), the acceleration of the translational platform $\mathbf{a}$ is given by

$$\mathbf{a} = \mathbf{S}_L^{-1} \begin{bmatrix} \ddot{q}_1 + KL(\$_1, \$_{L1}) \\ \ddot{q}_2 + KL(\$_2, \$_{L2}) \\ \ddot{q}_3 + KL(\$_3, \$_{L3}) \end{bmatrix}. \tag{24}$$

Expression (24) does not require the computation of the passive joint rate accelerations of the lower parallel manipulator. Furthermore, the relation input-output equations for computing the acceleration of any point fixed to the translational platform is clearly indicated.

4. Kinematics of the Upper Parallel Manipulator

Owing the architecture of the upper parallel manipulator, the end-platform is restricted to undergo only rotational motions with respect to the translational platform. On the other hand, the solution of the kinematics of the upper parallel maninpulator is similar to the methodology of analysis applied to the lower parallel manipulator, excepting the forward position analysis. Thus only the forward position analysis and the most relevant results of the velocity and acceleration analyses of the upper parallel manipulator will be presented in this section.

4.1. POSITION ANALYSIS

The forward position analysis is stated as follows. Given the length of the three upper independent limbs, compute the coordinates of the upper spherical joints attached to the end-platform expressed in the reference frame xyz.

According with Figure 1, the geometric scheme of the upper parallel manipulator is showed in Figure 4. Thus immediately emerges that

$$\mathrm{a}_5' = \frac{q_4^2 - q_5^2 + \mathrm{a}_5^2}{2\mathrm{a}_5}, \tag{25}$$

and

$$q_{45} = \sqrt{q_4^2 - \mathrm{a}_5'^2}. \tag{26}$$

Furthermore, a closed loop can be written as follows

$$\mathbf{U}_4 + \mathbf{a}_5 + \mathbf{q}_{45} - \mathbf{a}_6 - \mathbf{h} = \mathbf{0}. \tag{27}$$

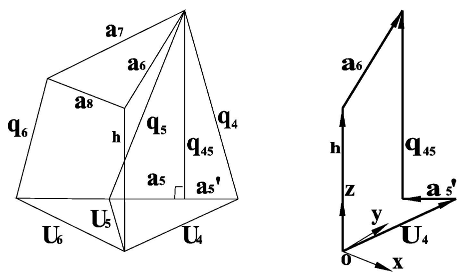

Figure 4. Geometric scheme of the upper parallel manipulator.

Then, after a few computations the coordinates of the spherical joint $\mathbf{S}_{45} = (S_{45x}, S_{45y}, S_{45z})$ are given by

$$\left.\begin{aligned} S_{45x} &= U_{4x} + \mathrm{a}_5', \\ S_{45z} &= h + (-B \pm \sqrt{B^2 - 4AC}/(2A), \\ S_{45y} &= \sqrt{\mathrm{a}_6^2 - S_{45x}^2 - S_{45z}^2}, \end{aligned}\right\} \tag{28}$$

where

$$\left.\begin{aligned} A &= 4\big(U_4^2 + h^2\big), \\ B &= 4h(C_1 - C_2), \\ C &= 4U_{4y}^2 C_2 - 2C_1 C_2 + C_2^2 + C_1^2, \\ C_1 &= U_{4y}^2 + h^2 - q_{45}^2, \\ C_2 &= U_{4x}^2 + 2U_{4x}\mathrm{a}_5 + \mathrm{a}_5'^2 - \mathrm{a}_6^2. \end{aligned}\right\}$$

In order to compute the coordinates of the spherical joint $\mathbf{S}_6 = (S_{6x}, S_{6y}, S_{6z})$ it is necessary to consider the following compatibility expressions

$$\left.\begin{aligned} (\mathbf{S}_6 - \mathbf{h}) \cdot (\mathbf{S}_6 - \mathbf{h}) &= \mathrm{a}_8^2, \\ (\mathbf{S}_6 - \mathbf{S}_{45}) \cdot (\mathbf{S}_6 - \mathbf{S}_{45}) &= \mathrm{a}_7^2, \\ (\mathbf{S}_{45} - \mathbf{h}) \cdot (\mathbf{S}_{45} - \mathbf{h}) &= \mathrm{a}_6^2. \end{aligned}\right\} \tag{29}$$

Afterwards, the coordinates of the spherical joint $\mathbf{S}_6$ result in

$$\left.\begin{aligned} S_{6z} &= (-B' \pm \sqrt{B'^2 - 4A'C'})/(2A'), \\ S_{6x} &= C_3 S_{6z} + C_4, \\ S_{6y} &= C_5 S_{6z} + C_6. \end{aligned}\right\} \tag{30}$$

where the coefficients are given by

$$\left.\begin{aligned} A' &= C_3^2 + C_5^2 + 1, \\ B' &= 2(C_5C_6 + C_3C_4 - h), \\ C' &= h^2 + C_6^2 + C_4^2 - \mathrm{a}_8^2, \\ C_3 &= (U_{6y}(S_{45z} - h) + S_{45y}h - S_{45y}U_{6z})/(S_{45y}U_{6x} - U_{6y}S_{45x}), \\ C_4 &= \big(S_{45y}\big(U6^2 - h^2 - q_3^2 + \mathrm{a}_8^2\big) + U_{6y}\big(\mathrm{a}_7^2 + h^2 - \mathrm{a}_8^2 - S_{45}^2\big)\big)/ \\ &\quad (2(-U_{6y}S_{45x} + S_{45y}U_{6x}), \\ C_5 &= (U_{6x}(S_{45z} - h) - S_{45x}(U_{6z} - h))/(U_{6y}S_{45x} - S_{45y}U_{6x}), \\ C_6 &= \big(U_{6x}\big(\mathrm{a}_7^2 + h^2 - \mathrm{a}_8^2 - S_{45}^2\big) + S_{45x}\big(U6^2 - h^2 - q_3^2 + \mathrm{a}_8^2\big)\big)/ \\ &\quad (2(U_{6y}S_{45x} - S_{45y}U_{6x})). \end{aligned}\right\}$$

4.2. VELOCITY AND ACCELERATION ANALYSES

The passive joint velocity rates of the independent limbs, namely the inverse velocity analysis, are given by

$$\begin{bmatrix} {}_0\omega_{1i} \\ {}_1\omega_{2i} \\ \dot{q}_i \\ {}_3\omega_{4i} \\ {}_4\omega i_5 \\ {}_5\omega_{6i} \end{bmatrix} = \mathbf{J}_i^{-1}\mathbf{V} \quad i = 4, 5, 6. \tag{31}$$

where

$$\mathbf{J}_i = [{}^0\$^{1i} \quad {}^1\$^{2i} \quad {}^2\$^{3i} \quad {}^3\$i^3 \quad {}^4\$^{5i} \quad {}^5\$^{6i}].$$

Of course, in order to compute the joint velocity rates, the Jacobian matrix $\mathbf{J}_i$ must be invertible. Otherwise, the upper UPS-type serial manipulator is at a singular configuration.

Furthermore, assuming that S_O is the reference point, then the velocity state of the end-platform, with respect to the translational platform, results in

$$\mathbf{V} = \begin{bmatrix} \omega \\ \mathbf{0} \end{bmatrix}, \tag{32}$$

where $\omega = (\omega_x, \omega_y, \omega_z)$ is the angular velocity of the end-platform.

Given the joint velocity rates associated to the actuated prismatic joints of the three upper independent limbs, the angular velocity of the end-platform takes the form

$$\omega = \mathbf{S}_U^{-1} \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}, \tag{33}$$

where the matrix $\mathbf{S}_U$ is the subspace generated by the dual parts of the lines $\$_i (i = 1, 2, 3)$ along the independent limbs and is given by

$$\mathbf{S}_U = [\mathbf{s}_{O4} \quad \mathbf{s}_{O5} \quad \mathbf{s}_{O6}]^T.$$

The joint acceleration rates associated to the independent limbs of the upper parallel manipulator, namely the inverse acceleration analysis, are given by

$$\begin{bmatrix} {}_0\dot{\omega}i_1 \\ {}_1\dot{\omega}i_2 \\ \ddot{q}_i \\ {}_3\dot{\omega}i_4 \\ {}_4\dot{\omega}i_5 \\ {}_5\dot{\omega}i_6 \end{bmatrix} = Ji^{-1}(\mathbf{A}_R - \$_L i) \quad i = 4, 5, 6. \tag{34}$$

where the ith Lie screw is calculated as follows

$$\begin{aligned} \$_L i = {} & \left[{}_0\omega_{1i}{}^0\$^{1i} \quad {}_1\omega_{2i}{}^1\$^{2i} + \dot{q}i^2\$^{3i} + {}_3\omega_{4i}{}^3\$^{4i} + {}_4\omega i_5{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[{}_1\omega_{2i}{}^1\$^{2i} \quad \dot{q}i^2\$^{3i} + {}_3\omega_{4i}{}^3\$^{4i} + {}_4\omega i_5{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[\dot{q}i^2\$^{3i} \quad {}_3\omega_{4i}{}^3\$^{4i} + {}_4\omega i_5{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[{}_3\omega_{4i}{}^3\$^{4i} \quad {}_4\omega i_5{}^4\$^{5i} + {}_5\omega_{6i}{}^5\$^{6i}\right] \\ & + \left[{}_4\omega i_5{}^4\$^{5i} \quad {}_5\omega_{6i}{}^5\$^{6i}\right] \quad i = 4, 5, 6. \end{aligned}$$

Furthermore, assuming that S_O is the reference point, then the reduced acceleration state of the end-platform, with respect to the translational platform, results in

$$\mathbf{A}_R = \begin{bmatrix} \dot{\omega} \\ \mathbf{0} \end{bmatrix}, \tag{35}$$

where $\dot{\omega} = (\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)$ is the angular acceleration of the end-platform.

Finally, given the joint acceleration rates associated to the actuated prismatic joints of the three independent limbs, the angular acceleration of the end platform can be calculated as follows

$$\dot{\omega} = \mathbf{S}_U^{-1} \begin{bmatrix} \ddot{q}_4 + KL(\$_4, \$_{L4}) \\ \ddot{q}_5 + KL(\$_5, \$_{L5}) \\ \ddot{q}_6 + KL(\$_6, \$_{L6}) \end{bmatrix}. \tag{36}$$

5. Numerical Example

In order to prove the versatility of the methodology of analysis indicated in previous sections, in this section a numerical example is solved.

The numerical problem is stated as follows. Firstly, given the instantaneous length of the three independent limbs of the lower parallel manipulator, and their time derivatives, compute the instantaneous pose, velocity, and acceleration of the translational platform with respect to the fixed platform. Finally, given the instantaneous length of the three independent limbs of the upper parallel manipulator, and their time derivatives, compute the instantaneous pose, angular velocity, and angular acceleration of the end-platform with respect to the translational platform.

The parameters of the hybrid manipulator are chosen as follows

$$\mathrm{a}_1 = \mathrm{a}_2 = 2.5,\ \mathrm{a}_3 = 3.5,\ \mathrm{a}_5 = 1.5,\ \mathrm{a}_6 = \mathrm{a}_7 = \mathrm{a}_8 = 1.0$$

where all values are in meters.

On the other hand, the coordinates of the spherical joints attached to the fixed platform, expressed in the reference frame XYZ, are given by

$$\mathbf{S}_1 = \mathbf{S}_2 = (0, 0, 0), \quad \mathbf{S}_3 = (4, 0, 0).$$

whereas the coordinates of the universal joints attached to the upper face of the translational platform, expressed in the local reference frame xyx, are chosen as follows

$$\mathbf{U}_4 = (-0.75, -1.0, 0), \quad \mathbf{U}_5 = (0.75, -1.0, 0), \quad \mathbf{U}_6 = (-1.0, -0.5, 0).$$

The instantaneous length of the independent limbs is governed by the periodical functions

$$\left.\begin{aligned} &q_1 = 2.5 + 0.25\sin(t), && q_2 = 3.5 + 0.5\sin(t), \\ &q_3 = 2.75 - 0.75\sin(t), && q_4 = 1.75 + 0.125\sin(t), \\ &q_5 = 1.75 - 0.125\sin(t), && q_6 = 1.25 - 0.125\sin(t), \quad t = 0, \ldots, 2\pi \end{aligned}\right\}$$

Thus, the hybrid manipulator begins its motion at the time $t = 0$ and 2π seconds later returns to its initial configuration. Finally, it is evident that the instantaneous joint rate velocities and accelerations of the actuated prismatic joints are obtained as simple time derivatives of the periodical functions assigned to the six independent limbs.

The numerical computations was carried out with the help of commercial computer codes. In what follows, the most representative numerical results of the analysis are presented in graphic form.

At the time $t = 0$, the forward position analysis, only one solution is reported here, of the hybrid manipulator is showed in Figure 5 by means of a wire frame obtained with the help of special software like AutoCad©.

The numerical solution of the velocity analyses, both forward and inverse, was carried out with the help of special software like Maple©. The numerical results obtained for the forward kinematics, up to the acceleration analysis, of the lower

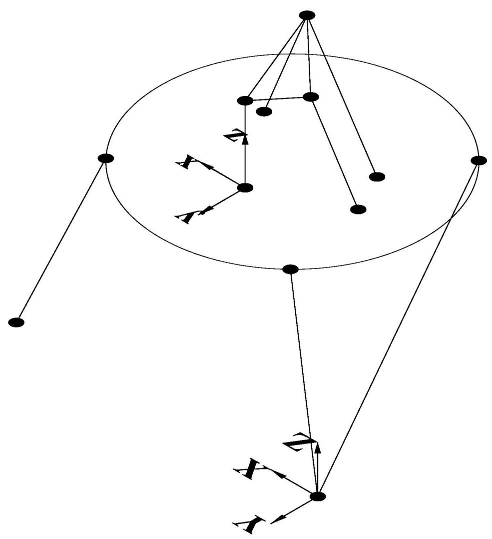

Figure 5. Initial configuration of the hybrid manipulator.

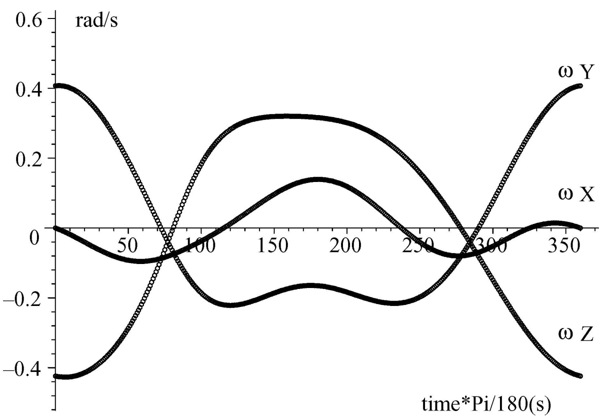

Figure 6. Angular velocity of the end-platform.

parallel manipulator are reported in Figures 6 and 7. With regards to the numerical results obtained for the forward kinematics of the upper parallel manipulator, Figures 8 and 9.

6. Conclusions

The search of efficient methods of analysis is a topic of permanent interest in the study of robot manipulators. In this work the kinematics of a hybrid manipulator, which is composed of a sequence of two parallel manipulators, was approached successfully by means of screw theory.

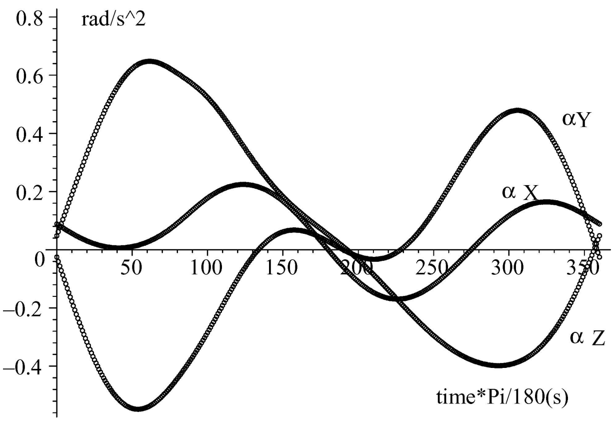

Figure 7. Angular acceleration of the end-platform.

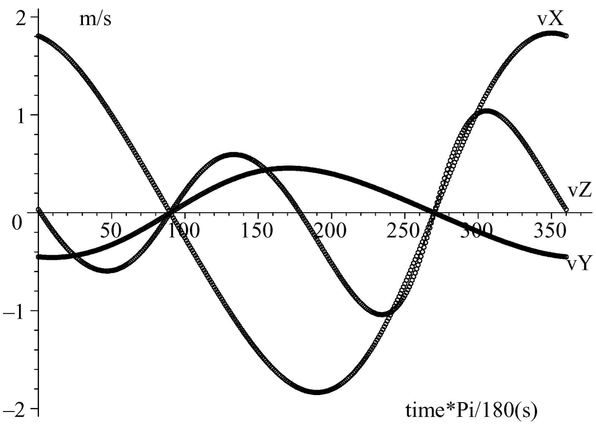

Figure 8. Velocity of the translational platform.

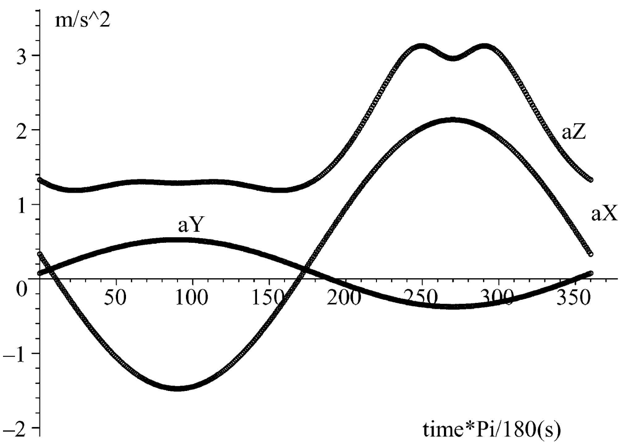

Figure 9. Acceleration of the translational platform.

The forward position analysis, a challenging intensive task of most parallel manipulators, was carried out in closed-form solution. To this end, the forward position analysis was subdivided into two individual problems. The first one dealing with the computation of the coordinates, expressed in a global reference frame, of the three universal joints attached to the lower face of the so-called translational platform, and the second one dealing with the computation of the coordinates of the three spherical joints attached to the end-platform expressed in a local reference frame attached to the translational platform.

It is important to emphasize the simplicity of the expressions derived in this contribution for determining all the possible solutions of the forward position analysis of the proposed mechanism. The expressions are compact and can be translated without considerable effort into computer code. In order to provide a comparison with other procedures reported in the Literature, consider for instance the forward position analysis of a general Gough-Stewart platform which demands the solution of a univariate 40th-order polynomial equation [25]. It is not surprising that the first non-iterative algorithm proposed to find the coefficients of such a polynomial equation dates from less than ten years ago [26]. On the other hand, the implementation of a numerical technique, such as the Newton-Raphson method, requires of an iterative process, upon initial values, that only yields one solution, if any.

The acceleration analysis of the mechanism under study was approached by means of screw theory. Simple and compact expressions for the computation of the acceleration of the translational platform, with respect to the fixed platform, and the computation of the angular acceleration of the end-platform, with respect to the translational platform, are derived by applying the concept of reciprocal screws via the Klein form of the Lie algebra $e(3)$. As an intermediate step, this contribution also provides the velocity analysis of the hybrid manipulator. It is interesting to mention that the expressions derived in this contribution for solving the forward acceleration analysis do not require the values of the passive joint rate accelerations of the proposed spatial mechanism. Furthermore, the relation input-output equations for determining the forward velocity analysis of the chosen parallel-serial manipulator is clearly specified, which is a condition for a comprehensive singularity analysis of hybrid-chain manipulators [27]. Finally, a numerical example is provided.

Acknowledgements

This work was supported by Consejo de Ciencia y Tecnología del Estado de Guanajuato, Concyteg, México.

References

1. Kecskeméthy, A., Krupp, T. and Hiller, M., 'Symbolic processing of multiloop mechanism dynamics using closed-form kinematics solutions', *Multibody System Dynamics* **1**, 1997, 23–45.

2. Tol, U. A., Clerc, J.-P. and Wiens, G. J., 'Micro/Macro approach for dexterity enhacement of PKM's', in *Proceedings of the WORKSHOP on Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators*, Toronto, Canada, 2002, pp. 34–39.
3. Bonev, I., 'The true origins of parallel robots', *The Parallel Mechanisms Information Center*, http://www.parallemic.org/, 2003.
4. Agrawal, S. K. and Roth, B., 'Statics of in-parallel manipulator systems', *ASME Journal of Mechanical Design* **114**, 1992, 564–568.
5. Gosselin, C. and Angeles, J., 'The optimum kinematic design of a planar three-degree-of-freedom parallel manipulator', *ASME Journal of Mechanisms, Transmissions and Automation in Design* **110**, 1988, 35–41.
6. Merlet, J.-P., *Parallel Robots*, Kluwer Academic Publishers, 1999.
7. Mohamed, M. G. and Duffy, J., 'A direct determination of the instantaneous kinematics of fully parallel manipulators', *ASME Journal of Mechanisms, Transmissions, and Automation in Design* **107**, 1985, 226–229.
8. Parenti-Castelli, V. and Innocenti, C., 'Forward displacements analysis of parallel mechanisms: Closed form solution of PRR-3S and PPR-3S structures', *ASME Journal of Mechanical Design* **114**, 1992, 68–73.
9. Tsai, L. W., *Robot Analysis, The Mechanics of Serial and Parallel Manipulators*, New York: John Wiley & Sons, Inc., 1999.
10. Tsai, L. W., 'Solving the inverse dynamics of a Stewart-Gough manipulator by the principle of virtual Work', *ASME Journal of Mechanical Design* **122**, 2000, 3–9.
11. Wang, J. and Gosselin, C. M., 'A new approach for the dynamic analysis of parallel manipulators', *Multibody System Dynamics* **2**, 1998, 317–334.
12. Raghavan, M., 'The Stewart platform of general geometry has 40 configurations', *ASME Journal of Mechanical Design* **115**, 1993, 277–282.
13. Rico, J. M. and Duffy, J., 'Forward and inverse acceleration analyses of in-parallel manipulators', *ASME Journal of Mechanical Design* **122**, 2000, 299–303.
14. Romdhane, L., Affi, Z. and Fayet, M., 'Design and singularity analysis of a 3-translational-DOF in-parallel manipulator', *ASME Journal of Mechanical Design* **124**, 2002, 419–426.
15. Rico, J. M. and Duffy, J., 'An application of screw algebra to the acceleration analysis of serial chains', *Mechanism and Machine Theory* **31**, 1996, 445–457.
16. Rico, J. M., Gallardo, J. and Duffy, J., 'Screw theory and higher order kinematic analysis of open serial and slosed shains', *Mechanism and Machine Theory* **34**, 1999, 559–586.
17. Ball, R. S., *A Treatise on the Theory of Screws*, Cambridge University Press, Cambridge, 1900 (Reprinted 1998).
18. Fang, Y. and Tsai, L. W., 'Enumeration of a class of overconstrained mechanisms using the theory of reciprocal screws', *Mechanism and Machine Theory* **39**, 2004, 1175–1187.
19. Bayro-Corrochano, E. and Kähler, D., 'Motor algebra approach for computing the kinematics of robot manipulators', *Journal of Robotic Systems* **17**, 2000, 495–516.
20. von Mises, R., 'Motorrechnung, ein neues hilfsmittel der mechanik', *Zeitschrift für Angewandte Mathematik und Mechanik* **4(2)**, 1924, 155–181.
21. von Mises, R., 'Anwendungen der motorrenchung', *Zeitschrift für Angewandte Mathematik und Mechanik* **4**(3), 1924, 193–213.
22. von Mises, R., E. J. Baker and K. Wohlhart (eds.), *Motor Calculus a New Theoretical Device for Mechanics*, Institute for Mechanics, University of Technology Graz, 1996.
23. Roth, B., 'Screws, motors, and wrenches that can not be bought in a hardware store', in M. Brady and R. Paul (eds.), *Robotics Research: The First International Symposium*, Chapter 8, The MIT Press, Cambridge, Massachusetts, 1984, pp. 679–693.
24. Gallardo, J. and Rico, J. M., 'Screw theory and helicoidal fields', in *ASME 1998 Design Engineering Technical Conferences*, CD-ROM, Paper DETC98/MECH-5893, Atlanta, 1998.

25. Innocenti, C., 'Forward kinematics in polynomial form of the general Stewart platform', in *ASME 1998 Design Engineering Technical Conferences*, CD-ROM, Paper DETC98/MECH-5894, Atlanta, 1998.
26. Husty, M. L., 'An algorithm for solving the direct kinematics of general Stewart-Gough platforms', *Mechanism and Machine Theory* **31**, 1996, 365–380.
27. Zlatanov, D., Fenton, R. G. and Benhabib, B., 'Analysis of the instantaneous kinematic and singular configurations of hybrid-chain manipulators', *Proceedings of the ASME 23rd Biennial Mechanisms Conference*, DE-Vol. 72, Minneapolis, USA, Sep. 1994, pp. 467–476.