The Unit Circle from Scratch

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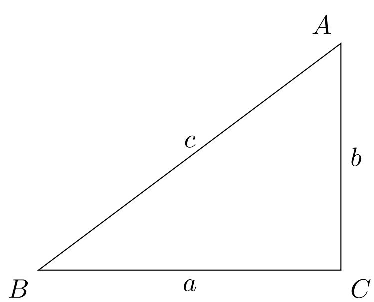
We will construct the unit circle and label it with values associated with $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ “from scratch”, *i.e.* beginning only with some basic geometric principles.

1 Basic results from trigonometry

First, we state some basic geometric theorems.

Theorem 1 (Pythagoras' theorem)**.** *Given a right triangle with hypotenuse of length c and sides respectively of length a and b, we have $a^2 + b^2 = c^2$.*



1.1 Determining the lengths of the sides of a 45-45-90 triangle when we know the length of the hypotenuse

The following can be derived from Pythagoras' theorem, basic arithmetic, and the fact that the adjacent and opposite sides of a 45-45-90 triangle have equal length:

Theorem 2 (45-45-90 triangle)**.** *In a* 45-45-90 *triangle with hypotenuse of length c, and opposite and adjacent sides have equal length $a = \frac{c}{\sqrt{2}}$.*

In particular, if we have a hypotenuse of unit length, *i.e.* $c = 1$, then $a = b = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

1.2 Determining the lengths of the sides of a 30-60-90 triangle when we know the length of the hypotenuse

Observe that if we set two 30-60-90 triangles back-to-back (see image) we obtain an equilateral triangle. image

Theorem 3 (30-60-90 triangle)**.** *In a* 30-60-90 *triangle with hypotenuse of length* c*, the shortest side has length* $a = \frac{c}{2}$*, and the other side has length* $b = \frac{c\sqrt{3}}{2}$*.*

Proof. Suppose c is the length of the hypotenuse, a is the length of the shortest side, and b is the length of the remaining side. By Pythagoras' theorem we have $a^2 + b^2 = c^2$, but since our triangle is 30-60-90 we know that $a = \frac{c}{2}$. So, we have $(\frac{c}{2})^2 + b^2 = c^2$, and hence $b^2 = c^2 - \frac{c^2}{4} = \frac{4c^2 - c^2}{4} = \frac{3c^2}{4}$. Thus, $b = \sqrt{\frac{3c^2}{4}} = \frac{c\sqrt{3}}{2}$. □

In particular, if we have a hypotenuse of unit length, *i.e.* $c = 1$, then $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$.

Indeed, the pair $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ should look familiar!

2 Constructing the unit circle

These are all the building blocks we need to construct the unit circle "from scratch", without any memorization.

We begin in the Cartesian plane.

2.1 Finding the coordinates of $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$ from what we know about 45-45-90 triangles

First, note that a rotation by $\frac{\pi}{4}$ radians is the same as a rotation by 45 degress. Thus, if we draw the line perpendicular to the x-axis from the point $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$, we obtain a 45-45-90 triangle with hypotenuse of length 1. We want to find the lengths of the other sides, *i.e.* the adjacent side A and the opposite side O. But we know that $A = O$, since this triangle is 45-45-90. Recalling that the Pythagorean theorem tells us that $A^2 + O^2 = 1^2 = 1$, since $A = 0$ we thus have $2A^2 = 1$, hence $A^2 = \frac{1}{2}$ and thus $A = \frac{1}{\sqrt{2}}$. But $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, since $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Thus we have found that $A = O = \frac{\sqrt{2}}{2}$, yielding $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

2.2 Finding the coordinates of $(\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$ and $(\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$ from what we know about 30-60-90 triangles