Chapter 8

Graphs of Trigonometric Functions

One of the most important uses of trigonometry is in describing periodic processes. We find many such processes in nature: the swing of a pendulum, the tidal movement of the ocean, the variation in the length of the day throughout the year, and many others.

All of these periodic motions can be described by one important family of functions, which all physicists use. These are the functions of the form

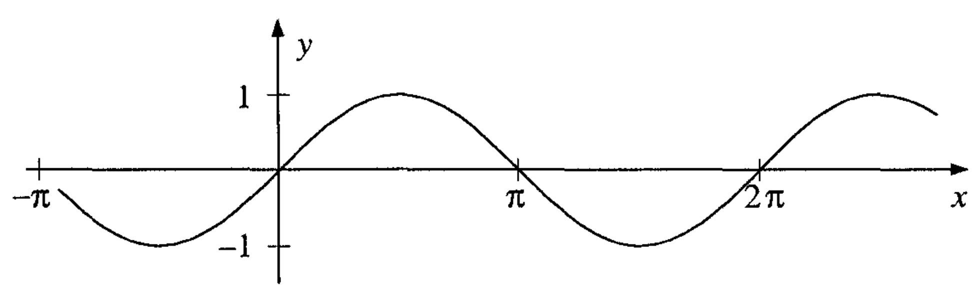
$$y = a \sin k(x - \beta) \, ,$$

where the constants a and k are positive, and β is arbitrary. In this chapter, we will describe their graphs, which we will call *sinusoidal* curves. Since they are so important, we will discuss them step-by-step, analyzing in turn each of the parameters a, k, and β.

1 Graphing the basic sine curve

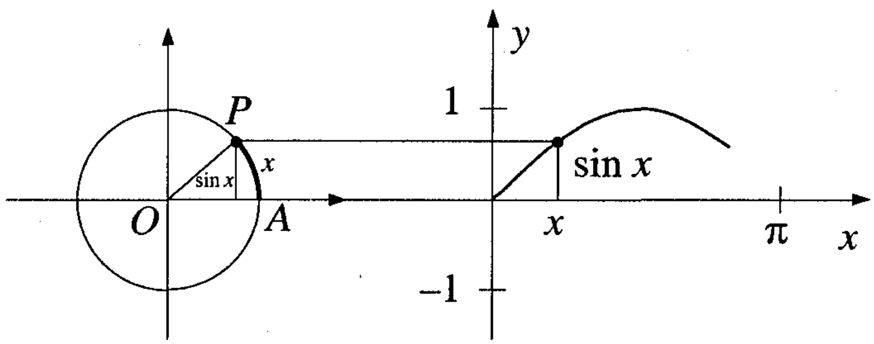
$$y = a \sin k(x - \beta) \quad \text{for } a = 1, k = 1, \beta = 0$$

In Chapter 5 we drew the graph of $y = \sin x$:



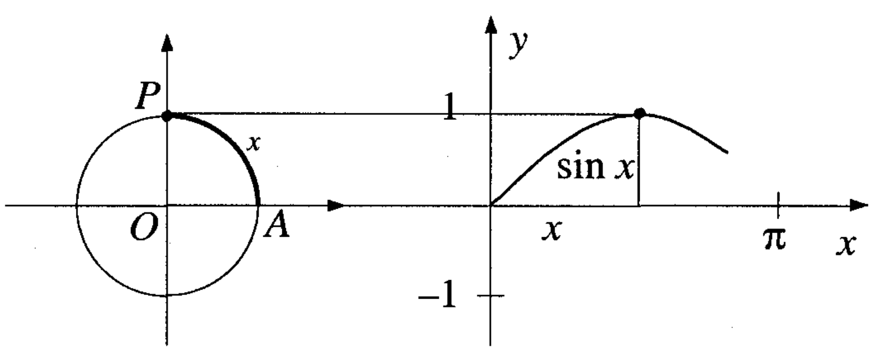
That is, we start with the case $a = 1$, $k = 1$, $\beta = 0$. Recall that we can take the sine of any real number (the *domain* of the function $y = \sin x$ is all real numbers), but that the values we get are all between -1 and 1 (the *range* of the function is the interval $-1 \leq y \leq 1$).

Let us review how we obtained this graph. On the left below is a circle with unit radius. Point P is rotating around it in a counterclockwise direction, starting at the point labeled A. If x is the length of the arc $\overset{\frown}{AP}$, then

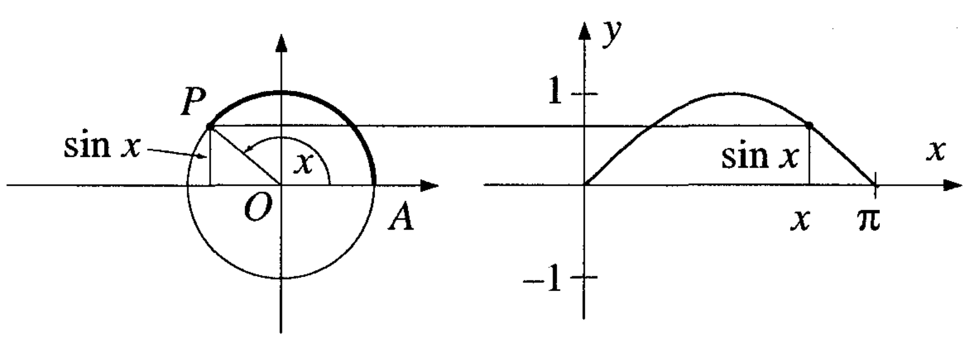


$\sin x$ is the vertical displacement of P. On the right, we have marked off the length x of arc $\overset{\frown}{AP}$. The height of the curve above the x-axis is $\sin x$.

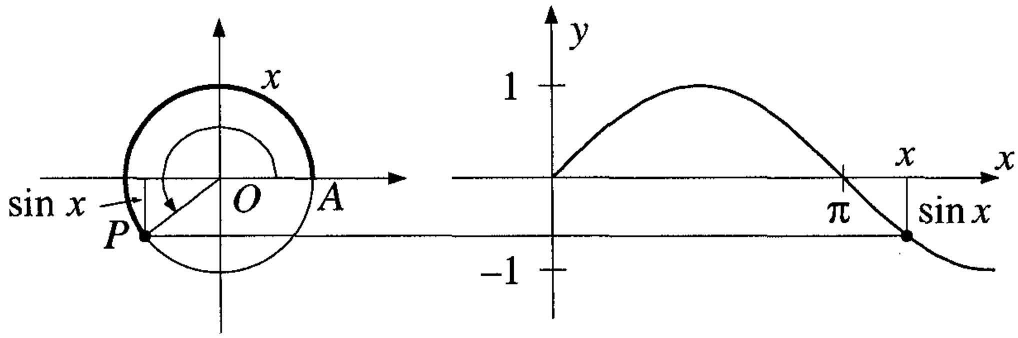
As the angle x goes from 0 to $\pi/2$, $\sin x$ grows from 0 to 1 (the picture for $x = \pi/2$ is shown below).

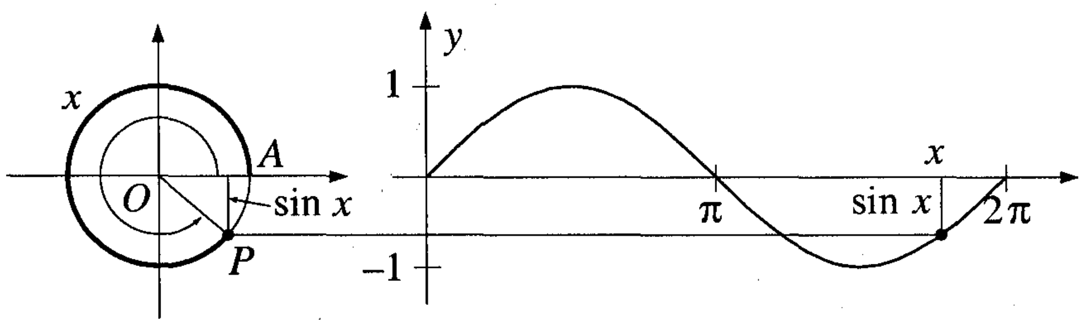


In fact, this is all we need to graph $y = \sin x$. As x goes from $\pi/2$ to π, the values of $\sin x$ repeat themselves "backwards":



And as x goes from π to 2π, the values are the negatives of the values in the first two quadrants:





2 The period of the function $y = \sin x$

As x grows larger than 2π, the values of $\sin x$ repeat on intervals of length 2π. For this reason, we say that the function $y = \sin x$ is *periodic*, with period 2π. Geometrically, this means that if we shift the whole graph 2π units to the right or to the left, we will still have the same graph. Algebraically, this means that

$$\sin (x + 2\pi) = \sin x$$

for any number x.

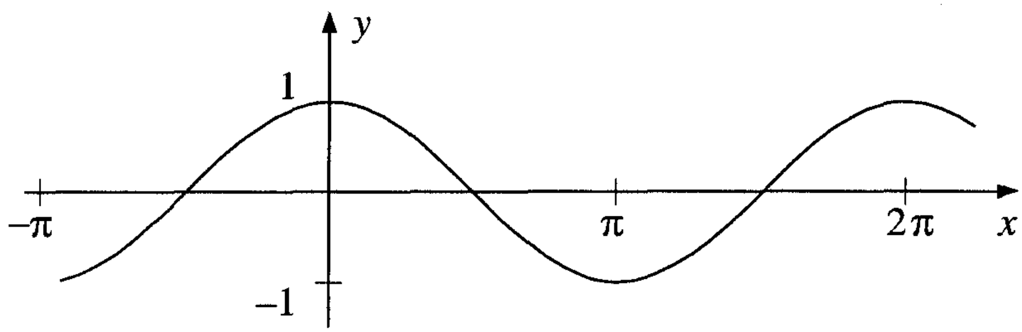
Definition: A function f has a period p if $f(x) = f(x + p)$ for all values of x for which $f(x)$ and $f(x + p)$ are defined.

The function $y = \sin x$ has a period of 2π. You can check that it also has periods of $4\pi, 6\pi, -2\pi$, and in general, $2\pi n$ for any integer n. This is no accident: if $f(x)$ is a periodic function with period p, then $f(x)$ is periodic with period np for any integer n. This is why we make the following definition:

Definition: The period of a periodic function $f(x)$ is the *smallest positive* real number p such that $f(x + p) = f(x)$ for all values of x for which $f(x)$ and $f(x + p)$ are defined.

Using this definition, we say that the period of $y = \sin x$ is 2π.

Let us also draw the graph of the function $y = \cos x$. Following the same methods, we find that the graph is as shown below:



The period of the function $y = \cos x$ is also 2π. We will see later that this curve can be described by an equation of the form $y = a \sin k(x - \beta)$.

3 Periods of other sinusoidal curves

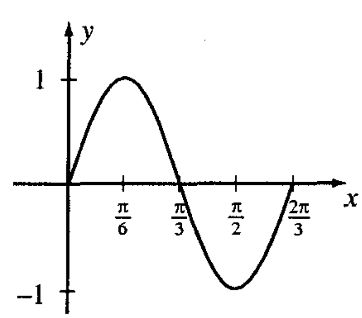
$$y = a \sin k(x - \beta) \quad \text{for } a = 1,\ \beta = 0,\ k > 0$$

Example 59 Find the period of the function $y = \sin 3x$.

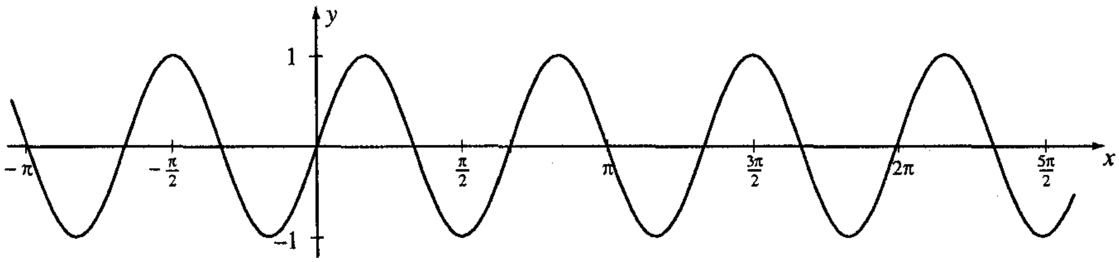
Solution: One period of this function is $2\pi/3$, since $\sin 3(x + 2\pi/3) = \sin(3x + 2\pi) = \sin 3x$. It is not difficult to see that this is the smallest positive period (for example, by looking at the values of x for which $\sin 3x = 0$).

Example 60 Draw the graph of the function $y = \sin 3x$.

Solution: The function $y = \sin x$ takes on certain values as x goes from 0 to 2π. The function $y = \sin 3x$ takes on these same values, but as x goes from 0 to $2\pi/3$. Hence one period of the graph looks like this:



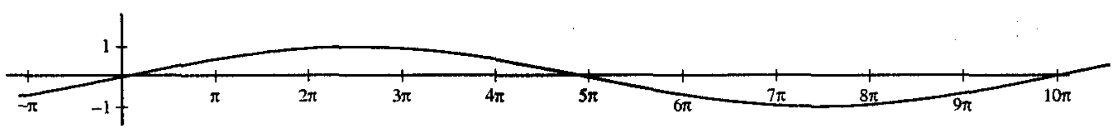
Having drawn one period, of course, it is easy to draw as much of the whole graph as we like (or have room for):



The graph is the same as that of y = sin x, but compressed by a factor of 3 in the x- direction. In general, we have the following result:

> For $k > 1$, the graph of $y = \sin kx$ is obtained from the graph $y = \sin x$ by compressing it in the x-direction by a factor of k.

What if $0 < k < 1$? Let us draw the graph of $y = \sin x/5$. Since the period of $y = \sin x/5$ is 10π, our function takes on the same values as the function $y = \sin x$, but stretched out over a longer period.



Again, we have a general result:

> For $0 < k < 1$, the graph of $y = \sin kx$ is obtained from the graph $y = \sin x$ by stretching it in the x-direction by a factor of k.

Analogous results hold for graphs of the functions $y = \cos kx$, $k > 0$.

Our basic family of functions is $y = a \sin k(x - \pi)$. What is the significance of the constant k here? We have seen that $2\pi/k$ is the period of the function. So in an interval of 2π, the function repeats its period k times. For this reason, the constant k is called the *frequency* of the function.

Exercises

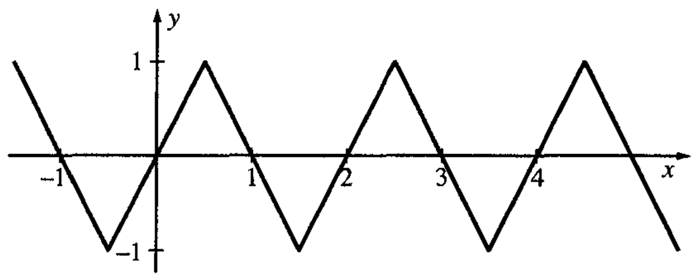
Find the period and frequency of the following functions:

1. $y = \sin 5x$ 2. $y = \sin x/4$ 3. $y = \cos 4x/5$ 4. $y = \cos 5x/4$.

Graph each of the following curves. Indicate the period of each. Check your work with a graphing calculator, if you wish.

5. $y = \sin 3x$ 6. $y = \sin x/3$ 7. $y = \sin 3x/2$ 8. $y = \sin 2x/3$
9. $y = \cos 2x/3$ 10. $y = \cos 3x/2$

11. The graph shown below has some equation $y = f(x)$.



(a) Draw the graph of the function $y = f(3x)$.

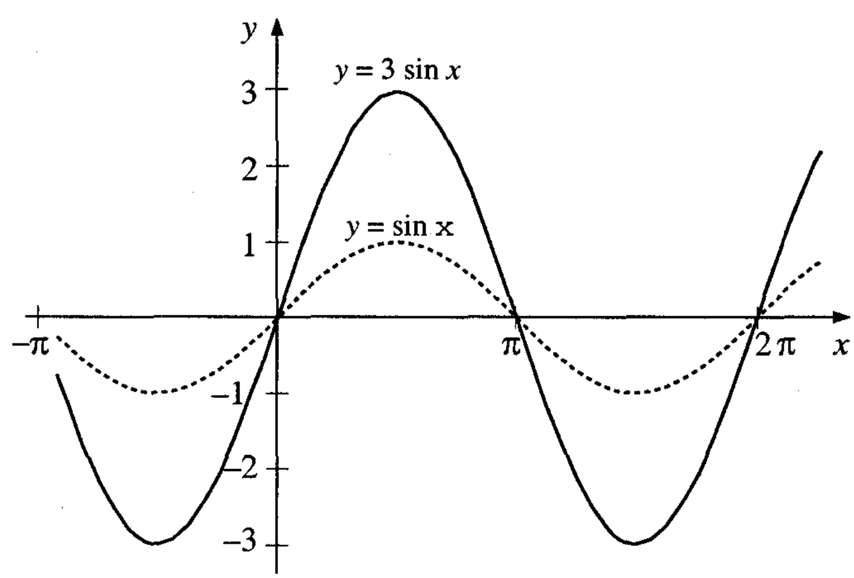
(b) Draw the graph of the function $y = f(x/3)$.

4 The amplitude of a sinusoidal curve

$$y = a \sin k(x - \beta); \quad a > 0, \beta = 0, k > 0$$

Example 61 Draw the graph of the function $y = 3 \sin x$.

Solution: The values of this function are three times the corresponding values of the function $y = \sin x$. Hence the graph will have the same period, but each y-value will be multiplied by 3:



We see that the graph of $y = 3\sin x$ is obtained from the graph of $y = \sin x$ by stretching in the y-direction. Similarly, it is not hard to see that the graph of $y = (1/2)\sin x$ is obtained from the graph of $y = \sin x$ by a compression in the y-direction.

We have the following general result:

> *For $a > 1$, the graph of $y = a\sin x$ is obtained from the graph $y = \sin x$ by stretching in the y-direction. For $0 < a < 1$, the graph of $y = a\sin x$ is obtained from the graph $y = \sin x$ by compressing in the y-direction.*

Analogous results hold for graphs of functions in which the period is not 1, and for equations of the form $y = a\cos x$. The constant a is called the *amplitude* of the function $y = a\sin k(x - \beta)$.

Exercises

Graph the following functions. Give the period and amplitude of each. As usual, you are invited to check your work, after doing it manually, with a graphing calculator.

1. $y = 2\sin x$	2. $y = (1/2)\sin x$	3. $y = 3\sin 2x$
4. $y = (1/2)\sin 3x$	5. $y = 4\cos x$	6. $y = (1/3)\cos 2x$

7. Suppose $y = f(x)$ is the function whose graph is given in Exercise 11 on page 178.

   (a) Draw the graph of the function $y = 3f(x)$.

   (b) Draw the graph of the function $y = (1/3)f(x)$.

5 Shifting the sine

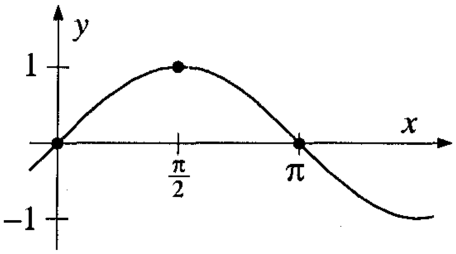
$$y = a\sin k(x - \beta); \quad a = 1, k = 1, \beta \text{ arbitrary}$$

We start with two examples, one in which β is positive and another in which β is negative.

Example 62 Draw the graph of the function $y = \sin(x - \pi/5)$.

Solution: We will graph this function by relating the new graph to the graph of $y = \sin x$. The positions of three particular points[1] on the original graph will help us understand how to do this:

x	$\sin x$
0	0
$\frac{\pi}{2}$	1
π	0

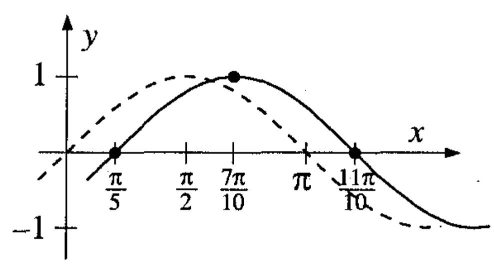


What are the analogous points on the graph of $y = \sin(x - \frac{\pi}{5})$? It is not convenient to use $x = 0$, because then $y = \sin(-\frac{\pi}{5})$, whose value is difficult to work with. Similarly, if we use $x = \frac{\pi}{2}$, we will need the value $y = \sin(\frac{\pi}{2} - \frac{\pi}{5}) = \sin\frac{3\pi}{10}$, which is still less convenient.

But if we let $x = \frac{\pi}{5}, \frac{\pi}{2} + \frac{\pi}{5}, \pi + \frac{\pi}{5}$, things will work out better:

x	$x - \frac{\pi}{5}$	$\sin(x - \frac{\pi}{5})$
$\frac{\pi}{5}$	0	0
$\frac{\pi}{2} + \frac{\pi}{5}$	$\frac{\pi}{2}$	1
$\pi + \frac{\pi}{5}$	π	0

That is, our choice of "analogous" points in our new function are those where the y-values are the same as those of the original function, not where the x-values are the same. The graph of $y = \sin(x - \frac{\pi}{5})$ looks just like the graph of $y = \sin x$, but shifted to the right by $\frac{\pi}{5}$ units:



But we must check this graph for more than three points. Are the other points on the graph shifted the same way? Let us take any point $(x_0, \sin x_0)$ on the graph $y = \sin x$. If we shift it to the right by $\frac{\pi}{5}$, we are merely adding this number to the point's x-coordinate, while leaving its y-coordinate the same. The new point we obtain is $(x_0 + \frac{\pi}{5}, \sin x_0)$, and this is in fact on the graph of the function $y = \sin(x - \frac{\pi}{5})$.

[1] Of course, with a calculator or a table of sines, you can get many more values. Or, if you have a good memory, you can remember the values of the sines of other particular angles. But these three points will serve us well for quite a while.

There is nothing special about the number $\frac{\pi}{5}$, except that it is positive. In general, the following statement is useful:

> *If $\beta > 0$, the graph of $y = \sin(x - \beta)$ is obtained from the graph of $y = \sin x$ by a shift of β units to the right.*

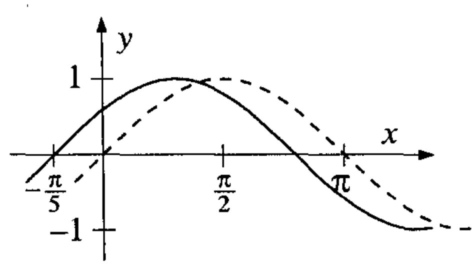
What if β is negative?

Example 63 Draw the graph of the function $y = \sin(x + \frac{\pi}{5})$.

Solution: In this example, $\beta = -\frac{\pi}{5}$. Again, we will relate this graph to the graph of $y = \sin x$. Using the method of the previous example, we seek values of x such that

$$\sin(x + \frac{\pi}{5}) = 0\,, \;\; \sin(x + \frac{\pi}{5}) = 1\,, \;\; \sin(x + \frac{\pi}{5}) = 0 \text{ (for a second time).}$$

It is not difficult to see that these values are $x = -\frac{\pi}{5}, \frac{\pi}{2} - \frac{\pi}{5}, \pi - \frac{\pi}{5}$, respectively. Using these values, we find that the graph of $y = \sin(x + \frac{\pi}{5})$ is obtained by shifting the graph of $y = \sin x$ by $\frac{\pi}{5}$ units to the *left*:



In general:

> *The graph of the function $y = \sin(x - \beta)$ is obtained from the graph of $y = \sin x$ by a shift of β units. The shift is towards the left if β is negative, and towards the right if β is positive.*

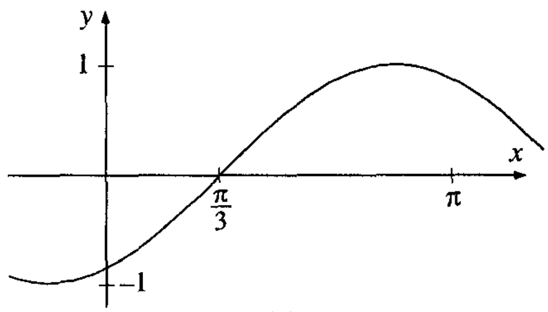
The number β is called the *phase angle* or *phase shift* of the curve. Analogous results hold for the graph of $y = \cos(x - \beta)$.

Exercises

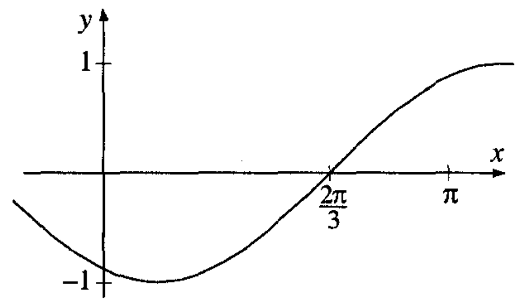
Sketch the graphs of the following functions:

1. $y = \sin(x - \frac{\pi}{6})$ 2. $y = \sin(x + \frac{\pi}{6})$ 3. $y = 2\sin(x - \frac{\pi}{2})$
4. $y = \frac{1}{2}\sin(x + \frac{\pi}{2})$ 5. $y = \cos(x - \frac{\pi}{4})$ 6. $y = 3\cos(x + \frac{\pi}{3})$
7. $y = \sin(x - 2\pi)$

8–11: Write equations of the form $y = \sin(x - \alpha)$ for each of the curves shown below:



(a)



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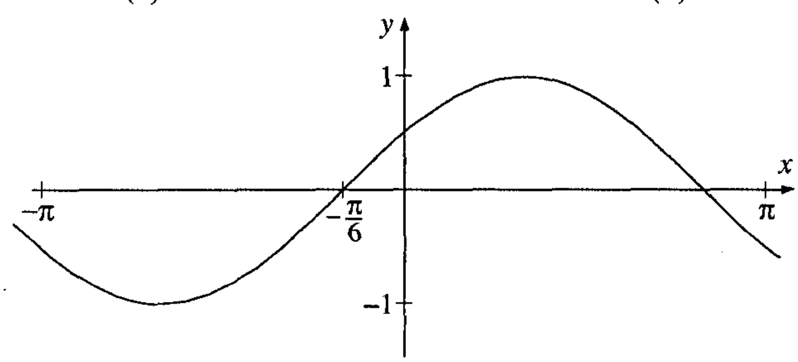

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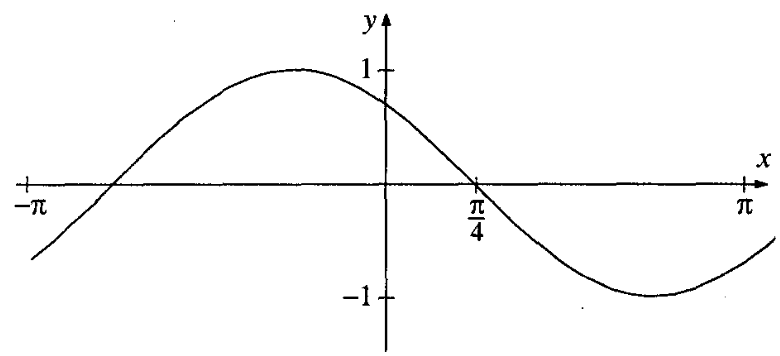


(d)

6 Shifting and stretching

Graphing $y = a \sin k(x - \beta)$

We run into a small difficulty if we combine a shift of the curve with a change in period.

Example 64 Graph the function $y = \sin(2x + \pi/3)$.

Solution: Let us write this equation in our standard form:

$$\sin(2x + \pi/3) = \sin 2(x + \pi/6)$$

We see that the graph is that of $y = \sin 2x$, shifted $\pi/6$ units to the left.

At first glance, one might have thought that the shift is $\pi/3$ units to the left. But this is incorrect. In the original equation, $\pi/3$ is added to $2x$, not to x. The error is avoided if we rewrite the equation in standard form.

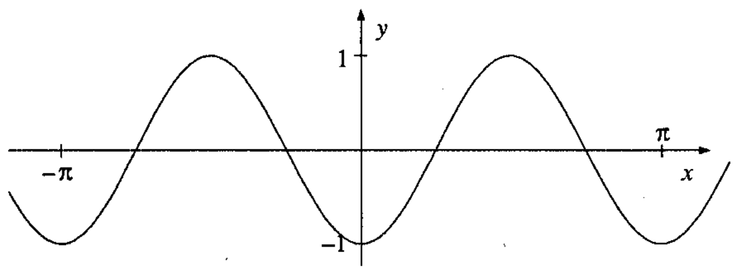
Exercises

Graph the following functions:

1. $y = \sin \frac{1}{2}(x - \frac{\pi}{6})$ 2. $y = \sin(\frac{1}{2}x - \frac{\pi}{6})$ 3. $y = \cos 2(x + \frac{\pi}{3})$

4–5: Write equations of the form $y = \sin k(x - \beta)$ for the following graphs:

4.



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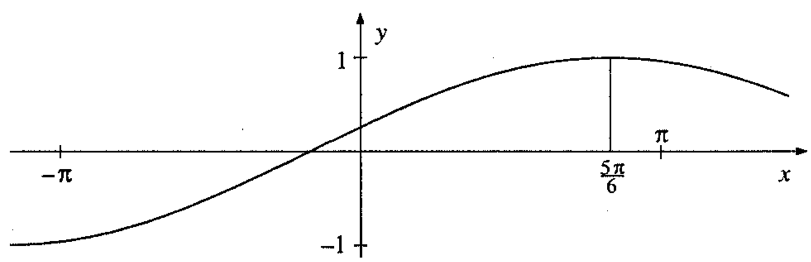

7 Some special shifts: Half-periods

We will see, in this section, that we have not lost generality by restricting a and k to be positive, or by neglecting the cosine function.

It is useful to write our general equation as $y = a \sin k(x + \gamma)$, where $\gamma = -\beta$. Then, for positive values of γ, we are shifting to the left. For the special value $c = 2\pi$, we already know what happens to the graph $y = \sin x$. Since 2π is a period of the function, the graph will coincide with itself after such a shift.

In fact, we can state the following alternative definition of a period of a function:

> *A function $y = f(x)$ has period p if the graph of the function coincides with itself after a shift to the left of p units.*