Physics 916: Homework #4

Due on April 13, 2020 at 5pm

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Problem 1

Consider a physical system whose three-dimensional state space is spanned by an orthonormal basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. In that state space, consider two operators L_z and S defined by:

$$L_z |u_1\rangle = |u_1\rangle, L_z |u_2\rangle = |0\rangle, L_z |u_3\rangle = -|u_3\rangle$$
$$S |u_1\rangle = |u_3\rangle, S |u_2\rangle = |u_2\rangle, S |u_3\rangle = |u_1\rangle$$

(a) Write the matrices which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operators L_z, L_z^2, S, and S^2. Are these operators observables?

(b) Give the form of the most general matrix, which represents an operator which commutes with L_z. Same for L_z^2 and S^2.

(c) Do L_z^2 and S^2 form a CSCO? Give a basis of common eigenvectors.

Solution

Part a

The matrix representation of these two operators is found by applying to each $|u_i\rangle$ and simply seeing what each row and column must be to bring about the given transformations. They are:

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then we square them to get the other two matrices:

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These are all observables because they are Hermitian:

$$L_z^\dagger = L_z, (L_z^2)^\dagger = L_z^2$$
$$S^\dagger = S, (S^2)^\dagger = S^2$$

Part b

To find the most general matrix, A, which commutes with L_z, we need to solve:

$$[L_z, A] = L_z A - A L_z = 0$$

Let's define A in a general way as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

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Then we have:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ 0 & 0 & 0\\ -a_{31} & -a_{32} & -a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & 0 & -a_{13}\\ a_{21} & 0 & -a_{23}\\ a_{31} & 0 & -a_{33}\end{pmatrix}$$

$$\Rightarrow a_{11} = a_{11},\ a_{12} = 0,\ a_{13} = -a_{13}$$

$$a_{21} = 0,\ a_{22} = a_{22},\ a_{23} = 0$$

$$a_{31} = 0,\ a_{23} = 0,\ a_{33} = a_{33}$$

$$\Rightarrow A = \begin{pmatrix}a_{11} & 0 & 0\\ 0 & a_{22} & 0\\ 0 & 0 & a_{33}\end{pmatrix}$$

Let's use the same general definition of A to find the general matrix which commutes with L_z^2:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ 0 & 0 & 0\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & 0 & a_{13}\\ a_{21} & 0 & a_{23}\\ a_{31} & 0 & a_{33}\end{pmatrix}$$

$$\Rightarrow a_{11} = a_{11},\ a_{12} = 0,\ a_{13} = a_{13}$$

$$a_{21} = 0,\ a_{22} = a_{22},\ a_{23} = 0$$

$$a_{31} = a_{31},\ a_{32} = 0,\ a_{33} = a_{33}$$

$$\Rightarrow A = \begin{pmatrix}a_{11} & 0 & a_{13}\\ 0 & a_{22} & 0\\ a_{31} & 0 & a_{33}\end{pmatrix}$$

And now for S^2:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

$$A = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

The commuting matrix for L_z must be diagonal, for L_z^2 it must have the structure shown above, and for S^2 everything commutes because $S^2 = I$, the identity.

Part c

The first thing to note is that both L_Z^2 and S^2 are degenerate. This means that the eigenvalues of either alone cannot fully specify the state of a vector.

Next, I want to see if L_z^2 commutes with S^2. If you refer back to the matrix representation of S^2 from part a, you'll see that $S^2 = I$, where I is the identity matrix. $\Rightarrow [S^2, L_z^2] = [I, L_Z^2] = 0$.

But do they form a CSCO? We can quickly find the eigenvectors for L_z^2. Since the matrix is diagonal, we can simply read off the eigenvalues, $\lambda = 1, 0$. Plugging these into the characteristic equation, we get:
For $\lambda = 0$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_1 = 0, c_3 = 0$$

$$|v_{\lambda=0}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_2 = 0$$

$$|v_{\lambda=1}\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

But we can split $|v_{\lambda=1}\rangle$ into two orthogonal vectors:

$$|v_{\lambda=1}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Renaming the vectors as v_1, v_2, v_3:

$$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now we have three mutually orthogonal eigenvectors fully specifying the eigenspace. These are easy to check:

$$L_z^2 |v_1\rangle = |v_1\rangle, \ S^2 |v_1\rangle = |v_1\rangle$$
$$L_z^2 |v_2\rangle = |0\rangle, \ S^2 |v_2\rangle = |v_2\rangle$$
$$L_z^2 |v_3\rangle = |v_3\rangle, \ S^2 |v_3\rangle = |v_3\rangle$$

This has not gotten rid of our degeneracy. We still have $\lambda = 1$ for v_1 and v_3. Thus, L_z^2 and S^2 do not form a CSCO.

Problem 2

Problem 3

Problem 4