Physics 926: Homework #12

Due on April 30, 2020 at 5pm

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Problem 1

Suppose that instead of introducing an $SU(2)$ doublet of complex fields to do the symmetry breaking that generates the boson masses, we used an $SU(2)$ triplet instead. In that representation, the generators of the group are

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \; T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} \; T^3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

(a) Take

$$\phi_0 = \begin{pmatrix}0\\ 0\\ v\end{pmatrix}$$

assign the hypercharge such that the field is electrically neutral. Calculate M_W/M_Z in this model.

(b) Now take

$$\phi_0 = \begin{pmatrix}0\\ v\\ 0\end{pmatrix}$$

and show that only the charged weak bosons acquire mass in this case.

Solution

(a) We want the field be be electrically neutral, this means that $Q\phi_0 = 0$

$$\begin{aligned}
Q\phi_0 =& e\left(T_3 + \frac{Y}{2}\right)\phi_0\\
=& eT_3\phi_0 + \frac{e}{2}Y\phi_0\\
=& -e\phi_0 + \frac{e}{2}Y\phi_0\\
=& 0
\end{aligned}$$

This means that a hypercharge of 2 gives us a field which is electrically neutral. In matrix form, we could write Y like this:

$$Y = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$$

Following from the lecture notes, we have the Lagrangian:

$$\mathcal{L} = \left|\left(i\partial_\mu - \frac{g}{2}\tau_a W_\mu^a - \frac{g'}{2}YB_\mu\right)\phi_0\right|^2 - V(\phi_0)$$

To find the masses, we only need the terms which are quadratic in the fields, and they come from the

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following piece, given by equation (24) from the lecture notes from 4/21:

$$
\begin{aligned}
&\left|\left(\frac{g}{2}\tau_a W_\mu^a + \frac{g'}{2} Y B_\mu\right)\phi_0\right|^2 \\
=&\frac{1}{4}\left| g \begin{pmatrix} W_\mu^3 & \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) & 0 \\ \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & 0 & \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ 0 & \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + g' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \right|^2 \\
=&\frac{1}{4}\left| \begin{pmatrix} gW_\mu^3 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) & 0 \\ \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & 0 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ 0 & \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + 2g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \right|^2 \\
=&\frac{1}{4}\begin{pmatrix} 0 & 0 & v \end{pmatrix} \begin{pmatrix} gW_\mu^3 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) & 0 \\ \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & 0 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ 0 & \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + 2g'B_\mu \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} gW_\mu^3 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) & 0 \\ \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & 0 & \frac{g}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ 0 & \frac{g}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + 2g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \\
=&\frac{1}{4}\begin{pmatrix} 0 & \frac{vg}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) & +v(-gW_\mu^3 + 2g'B_\mu) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{vg}{2}(W_\mu^1 - iW_\mu^2) \\ v(-gW_\mu^3 + 2g'B_\mu) \end{pmatrix} \\
=&\frac{v^2 g^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] + \frac{v^2}{4}\left[-gW_\mu^3 + 2g'B_\mu\right]^2
\end{aligned}
$$

What we see here is that we have two gauge bosons with the same mass such that $m_W = \frac{vg}{2}$. These are the two w-bosons. Then we have another mass term which is a superposition of the W and B fields.

Since we identify the mass terms in the Lagrangian in the form $\frac{1}{2}m_G^2 G^2$, where G is some field and m_G is its mass, we can associate this term with the second part of the above equation (since the photon mass is zero we know the only term we have must be the Z). We can now identify the Z-boson field (normalized):

$$Z_\mu = \frac{2g'B_\mu - gW_\mu}{\sqrt{4g'^2 + g^2}}$$

$$Z_\mu\sqrt{4g'^2 + g^2} = 2g'B_\mu - gW_\mu$$

Now compare the mass term with what was obtained above

$$
\begin{aligned}
&\frac{v^2}{4}(2g'B_\mu - gW_\mu)^2 = \frac{1}{2}\left[\frac{v^2}{2}(4g'^2 + g^2)\right] Z_\mu Z^\mu \\
&\Rightarrow m_Z = v\sqrt{\frac{4g'^2 + g^2}{2}}
\end{aligned}
$$

From the text, we see that we can express these results in terms of θ_W (equations 15.22 and 15.23)

$$
\begin{aligned}
Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \\
\cos\theta_W &= -g \\
\sin\theta_W &= -2g' \\
\tan\theta_W &= 2g'/g \\
g' &= \frac{g\tan\theta_W}{2}
\end{aligned}
$$

Let's use these identities to rewrite m_Z in terms of just g and then take the ratio of m_W/m_Z.

$$\begin{aligned}
m_Z =& v\sqrt{\frac{g^2\tan^2\theta_W + g^2}{2}} \\
=& vg\sqrt{\frac{\tan^2\theta_W + 1}{2}} \\
=& vg\frac{\sec\theta_W}{\sqrt{2}} \\
m_W/m_Z =& \frac{vg}{2}\frac{\sqrt{2}}{vg\sec\theta_W} \\
=& \frac{\cos\theta_W}{\sqrt{2}}
\end{aligned}$$

This clearly does not correspond to our physical reality. This would mean that if the w-boson had a mass of 80 GeV, then the z would have a mass of 129 GeV. And of course the z-mass is roughly 91 GeV.

(b) This time I'll carry out the same calculation but with the new given value of ϕ_0.

$$\begin{aligned}
&\left|\left(\frac{g}{2}\tau_a W^a_\mu + \frac{g'}{2}YB_\mu\right)\phi_0\right|^2 \\
=&\frac{1}{4}\left| g\begin{pmatrix} W^3_\mu & \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) & 0 \\ \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & 0 & \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \\ 0 & \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & -W^3_\mu \end{pmatrix}\begin{pmatrix}0\\v\\0\end{pmatrix} + g'\begin{pmatrix}0&0&0\\0&0&0\\0&0&2B_\mu\end{pmatrix}\begin{pmatrix}0\\v\\0\end{pmatrix}\right|^2 \\
=&\frac{1}{4}\left|\begin{pmatrix} gW^3_\mu & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) & 0 \\ \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & 0 & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \\ 0 & \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & -gW^3_\mu + 2g'B_\mu \end{pmatrix}\begin{pmatrix}0\\v\\0\end{pmatrix}\right|^2 \\
=&\frac{1}{4}\begin{pmatrix}0 & v & 0\end{pmatrix}\begin{pmatrix} gW^3_\mu & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) & 0 \\ \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & 0 & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \\ 0 & \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & -gW^3_\mu + 2g'B_\mu \end{pmatrix}\cdot \\
&\cdot\begin{pmatrix} gW^3_\mu & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) & 0 \\ \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & 0 & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \\ 0 & \frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & -gW^3_\mu + 2g'B_\mu \end{pmatrix}\begin{pmatrix}0\\v\\0\end{pmatrix} \\
=&v^2\begin{pmatrix}\frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu) & 0 & \frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu)\end{pmatrix}\begin{pmatrix}\frac{g}{\sqrt{2}}(W^1_\mu - iW^2_\mu)\\0\\\frac{g}{\sqrt{2}}(W^1_\mu + iW^2_\mu)\end{pmatrix} \\
=&\frac{v^2g^2}{2}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] + 0 + \frac{v^2g^2}{2}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right] \\
=&v^2g^2\left[(W^1_\mu)^2 + (W^2_\mu)^2\right]
\end{aligned}$$

Here we see that we only have two bosons acquiring mass such that $m = vg\sqrt{2}$ for both of them.

Problem 2

Solution

Problem 3

Consider the spontaneous symmetry breaking of an $SU(2)$ local gauge symmetry, as we did in class. We made a particular choice of vacuum. Show that for any choice of vacuum, all three gauge bosons still acquire the same mass.

Solution

The kinetic energy term will wind up giving us the mass of the boson(s) as seen in equations (17) AND (18) from the lecture notes.

$$\frac{g^2}{4}\left(\frac{1}{\sqrt{2}}\right)^2\left|\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2$$

We can let ϕ be any generic choice and let the vector in the above equation be $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where v_1 and v_2 are both complex numbers

$$\begin{aligned}
&\frac{g^2}{4}\left(\frac{1}{\sqrt{2}}\right)^2\left|\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|^2 = \\
=&\frac{g^2}{8}\left[\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right]^\dagger\left[\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] \\
=&\frac{g^2}{8}\left[\begin{pmatrix} v_1^* & v_2^* \end{pmatrix}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\right]\left[\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] \\
=&\frac{g^2}{8}\begin{pmatrix} W_\mu^3 v_1^* + (W_\mu^1 + iW_\mu^2)v_2^* & (W_\mu^1 - iW_\mu^2)v_1^* - W_\mu^3 v_2^* \end{pmatrix}\begin{pmatrix} W_\mu^3 v_1 + (W_\mu^1 - iW_\mu^2)v_2 \\ (W_\mu^1 + iW_\mu^2)v_1 - W_\mu^3 v_2 \end{pmatrix} \\
=&\frac{g^2}{8}(W_\mu^3)^2 v_1^2 + (W_\mu^1 + iW_\mu^2)(W_\mu^1 - iW_\mu^2)v_2^2 + W_\mu^3(W_\mu^1 - iW_\mu^2)v_1^* v_2 + W_\mu^3(W_\mu^1 + iW_\mu^2)v_1 v_2^* \\
+&\frac{g^2}{8}(W_\mu^1 - iW_\mu^2)(W_\mu^1 + iW_\mu^2)v_1^2 + (W_\mu^3)^2 v_2^2 - W_\mu^3(W_\mu^1 - iW_\mu^2)v_1^* v_2 - W_\mu^3(W_\mu^1 + iW_\mu^2)v_1 v_2^* \\
=&\frac{g^2}{8}(W_\mu^3)^2 v_1^2 + (W_\mu^1)^2 v_2^2 + (W_\mu^2)^2 v_2^2 + (W_\mu^3)^2 v_2^2 + (W_\mu^1)^2 v_1^2 + (W_\mu^2)^2 v_1^2 \\
=&\frac{g^2}{8}(v_1^2 + v_2^2)(W_\mu^1)^2 + (v_1^2 + v_2^2)(W_\mu^2)^2 + (v_1^2 + v_2^2)(W_\mu^3)^2 \\
=&\frac{g^2(v_1^2 + v_2^2)}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2\right]
\end{aligned}$$

And now we have three gauge bosons with the same mass such that

$$m = \frac{g\sqrt{v_1^2 + v_2^2}}{2}$$

Problem 4

The Lagrangian for the scalar field

$$\mathcal{L} = \left| \left(i\partial_\mu - \frac{g}{2}\tau_a W_\mu^a - \frac{g'}{2} Y B_\mu \right) \phi \right|^2 - V(\phi)$$

contains trilinear hW^+W^- and quadrilinear hhW^+W^- Higgs boson couplings. Use

$$\phi = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

to show that in the standard model the vertex factors are igM_W and $ig^2/4$, respectively.

Solution

We can follow the same procedure as before, but replacing ϕ with the one given in this problem. If we follow this procedure, the relevant parts of the Lagrangian become:

$$\begin{aligned} &\frac{1}{8} \left(g(v+h)^2(W_\mu^1 + iW\mu^2) \quad (v+h)(g'B_\mu - gW_\mu^3) \right) \begin{pmatrix} g(v+h)^2(W_\mu^1 - iW\mu^2) \\ (v+h)(g'B_\mu - gW_\mu^3) \end{pmatrix} \\ =&\frac{1}{8} \left[g^2(v+h)^2[(W_\mu^1)^2 + (W_\mu^2)^2] + (v+h)^2(g'B_\mu - gW_\mu^3)^2 \right] \end{aligned}$$

But as we've already learned, the second term corresponds to the z-boson and we want to understand the coupling of the w-bosons. So let's look at:

$$\begin{aligned} =&\frac{1}{8} \left(g^2(v+h)^2[(W_\mu^1)^2 + (W_\mu^2)^2] \right) \\ =&\frac{g^2}{8} \left(v^2[(W_\mu^1)^2 + (W_\mu^2)^2] + h^2[(W_\mu^1)^2 + (W_\mu^2)^2] + 2vh[(W_\mu^1)^2 + (W_\mu^2)^2] \right) \end{aligned}$$

Note the interaction terms between the h-field and the w-bosons. They are:

$$\frac{g^2}{8} h^2[(W_\mu^1)^2 + (W_\mu^2)^2], \ \frac{g^2}{8} 2vh[(W_\mu^1)^2 + (W_\mu^2)^2]$$

The first of these terms is the coupling of hhW^+W^- and the second term is the coupling of hW^+W^-.

So the strength of the hhW^+W^- coupling is:

$$\frac{ig^2}{4}$$

And the strength of the hW^+W^- coupling is:

$$\frac{ig^2 v}{2} = ig\frac{gv}{2} = igM_W$$

Because the w-boson mass is given by: $M_W = gv/2$ (equation 15.8 from H&M)