Physics 926: Homework #11

Due on April 14, 2020 at 5pm

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Problem 1

Show that

$$P(\nu_1 \to \nu_2) = \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E}$$

for a two neutrino system in which the mixing matrix is

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $\Delta m_{12}^2 = m_1^2 - m_2^2$

Solution

Equation (7) from the lecture gives us a good starting point. This equation is an approximation which is valid when mass is small, which in this case it is

$$P(\nu_1 \to \nu_2) = \left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2$$

Using this equation as a starting point, we can plug in the given values of U and explicitly do the sum.

$$\begin{aligned}
\left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2 &= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{1j}^* U_{2j} e^{-im_j^2 L/2E} \right]^* \\
&= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] \\
\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} &= \cos\theta(-\sin\theta) e^{-im_1^2 L/2E} + \sin\theta\cos\theta e^{-im_2^2 L/2E} \\
&= -\cos\theta\sin\theta e^{-im_1^2 L/2E} + \cos\theta\sin\theta e^{-im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \\
\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} &= -\sin\theta\cos\theta e^{im_1^2 L/2E} + \cos\theta\sin\theta e^{im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right)
\end{aligned}$$

$$\begin{aligned}
\left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] &= \cos^2\theta\sin^2\theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left(2 - e^{i(m_1^2 - m_2^2)L/2E} - e^{-i(m_1^2 - m_2^2)L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left(2 - 2Re\left[e^{i(m_1^2 - m_2^2)L/2E} \right] \right) \\
&= 2\cos^2\theta\sin^2\theta \left(1 - \cos\frac{\Delta m_{12}^2 L}{2E} \right)
\end{aligned}$$

Problem 1 continued on next page. . .

From here, use the trig identity: $1 - cos\theta = 2\sin^2\frac{\theta}{2}$ and then $2\cos\theta\sin\theta = \sin 2\theta$:

$$\begin{aligned}\left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E}\right]\left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E}\right] =& 2\cos^2\theta\sin^2\theta\left(2\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\ =& 4\cos^2\theta\sin^2\theta\left(\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\ =& \sin^2 2\theta\sin^2\frac{\Delta m_{12}^2 L}{4E}\end{aligned}$$

Problem 2

As an exercise in natural units, show that the quantity $\Delta m_{12}^2 L/4E$ that appears in the theory of neutrino oscillations is in fact equal to $1.27\Delta m_{12}^2(eV^2)L(km)/E(GeV)$.

Solution

In homework #1, we had to derive conversion factors between natural units and SI units. In this problem we will use this to convert between length in GeV^{-1} and km. The various ratios for conversions are:

$$L(GeV^{-1}) = \frac{5.067 \times 10^{15}\ GeV^{-1}}{1\ m}\frac{10^3\ m}{1\ km}L(km)$$
$$\Delta m^2(GeV^2) = \left(\frac{1\ GeV}{10^9\ eV}\right)^2 \Delta m^2(eV^2)$$

Now plug these into the argument of the sine function from the previous problem to see what we get:

$$\begin{aligned}\frac{\Delta m^2(GeV^2)L(GeV^{-1})}{4E(GeV)} &= \frac{\left(\frac{1\ GeV}{10^9\ eV}\right)^2 \Delta m^2(eV^2)\frac{5.067\times 10^{15}\ GeV^{-1}}{1\ m}\frac{10^3\ m}{1\ km}L(km)}{4E(GeV)} \\ &= \frac{10^3}{10^{18}}\frac{5.067 \times 10^{15}}{4}\frac{\Delta m^2(eV^2)L(km)}{E(GeV)} \\ &= 1.27\frac{\Delta m^2(eV^2)L(km)}{E(GeV)}\end{aligned}$$

Problem 3

As mentioned in class, experiments such as NOνA are taking advantage of the fact that neutrinos that are traveling off-axis of a neutrino beam have a narrower energy spread. Let's take a look.

(a) We want to make a neutrino beam from a beam of π^+ with $E_\pi = 20\ GeV$. How long should the decay pipe be to ensure the the great bulk of pions have decayed before they reach the absorber?

(b) Consider a pion with energy E_π in the laboratory frame. Find the energy of the neutrino E_ν in the decay $\pi^+ \to \mu^+\nu_\mu$ as a function of the laboratory angle θ that the emitted neutrino makes with the original flight direction of the π^+.

(c) Plot E_ν for E_π between 2 and 20 GeV in the case $\theta = 0$ and $\theta = 15\ mrad$.

Solution

(a) Starting with the energy given, $E = 20\ GeV$, we can calculate velocity and time and use basic special relativity to get a good estimate. I am going to assume that the bulk of pions decaying means betwen 5 and 10 lifetimes (This may be a bit of an overestimate, but might as well err on the side of caution).

Here are some relevant values:

$$\begin{aligned}
&\bar{\tau} = (2.603 \pm 0.005) \times 10^{-8} s\\
&m = 0.14\ GeV = 1.78 \times 10^{-27}\ kg\\
&E = 20\ GeV\\
&p = \sqrt{E^2 - m^2} \approx 20\ GeV = 1.07 \times 10^{-17}\ kg \cdot m/s
\end{aligned}$$

Where $\bar{\tau}$ is the mean lifetime, m is the mass, E is the energy, and p is the momentum.
Now I'll use these values to calculate velocity, lifetime in the rest frame of the particle, and thus the distance traveled.

$$\begin{aligned}
&p = \gamma m v = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}\\
&v = \frac{cp}{\sqrt{m^2c^2 + \frac{p^2}{c^2}}} = \frac{(2.998 \times 10^8)(1.07 \times 10^{-17})}{\sqrt{(1.78 \times 10^{-27} \cdot 2.998 \times 10^8)^2 + \left(\frac{1.07 \times 10^{-17}}{2.998 \times 10^8}\right)^2}}\\
&v = 2.9979 \times 10^8 m/s \approx 0.999976\ c\\
&\gamma = \frac{1}{\sqrt{1 - 0.999976^2}} = 143.3\\
&\tau_{0,lower} = 5\gamma\bar{\tau} = 1.87 \times 10^{-5} s\\
&\tau_{0,upper} = 10\gamma\bar{\tau} = 3.73 \times 10^{-5} s\\
&d_{lower} = (2.9979 \times 10^8)(1.87 \times 10^{-5}) = 5.6\ km\\
&d_{upper} = (2.9979 \times 10^8)(3.73 \times 10^{-5}) = 11.1\ km
\end{aligned}$$

Where $\tau_{0,lower}$ is the time estimate for five lifetimes, $\tau_{0,upper}$ is the time estimate for ten lifetimes, $d_{0,lower}$ is the distance using five lifetimes, and $d_{0,upper}$ is the distance using ten lifetimes.

So based on my estimates the bulk of pions will be gone between 5.6 and 11.1 kilometers.

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(b) In the lab frame, the four-momenta of each particle can be defined. Let the initial direction of the pion be in the x-direction. (σ is the index of the four-vectors while μ and ν are reserved for the muon and neutrino)

Before the decay:

$$P_\pi^\sigma = (E_\pi, p_\pi, 0, 0)$$

After the decay:

$$\begin{aligned} P_\mu^\sigma &= (E_\mu, \vec{p}_\mu) \\ P_\nu^\sigma &= (E_\nu, p_\nu \cos\theta, p_\nu \sin\theta, 0) \end{aligned}$$

Note: I left the muon momentum completely general because it won't matter what value it has in the end.

Due to conservation of four-momentum, we can write:

$$\begin{aligned} P_\pi^\sigma &= P_\mu^\sigma + P_\nu^\sigma \\ P_\pi^\sigma - P_\nu^\sigma &= P_\mu^\sigma \end{aligned}$$

We can contract each side with itself since this operation is Lorentz invariant:

$$\begin{aligned} (P_\pi - P_\nu)^\sigma (P_\pi - P_\nu)_\sigma &= P_\mu^\sigma P_{\mu,\sigma} \\ P_\pi^2 + P_\nu^2 - 2P_\nu^\sigma P_{\pi,\sigma} &= P_\mu^2 \\ m_\pi^2 + m_\nu^2 - 2(E_\pi E_\nu - p_\pi p_\nu \cos\theta) &= m_\mu^2 \end{aligned}$$

$p_\nu \approx E_\nu$ because $m_\nu \approx 0$ compared with the other masses in the problem.

$$\begin{aligned} m_\pi^2 - 2(E_\pi E_\nu - m_\pi E_\nu \cos\theta) &= m_\mu^2 \\ m_\pi^2 - m_\mu^2 &= 2E_\nu(E_\pi - p_\pi \cos\theta) \\ E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - p_\pi \cos\theta)} \end{aligned}$$

But don't forget the dispersion relation:$E^2 = p^2 + m^2$

$$\Rightarrow p_\pi = \sqrt{E_\pi^2 - m_\pi^2}$$

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2} \cos\theta)}$$

(c) Below is the plot of $E_\nu = \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2} \cos\theta)}$.

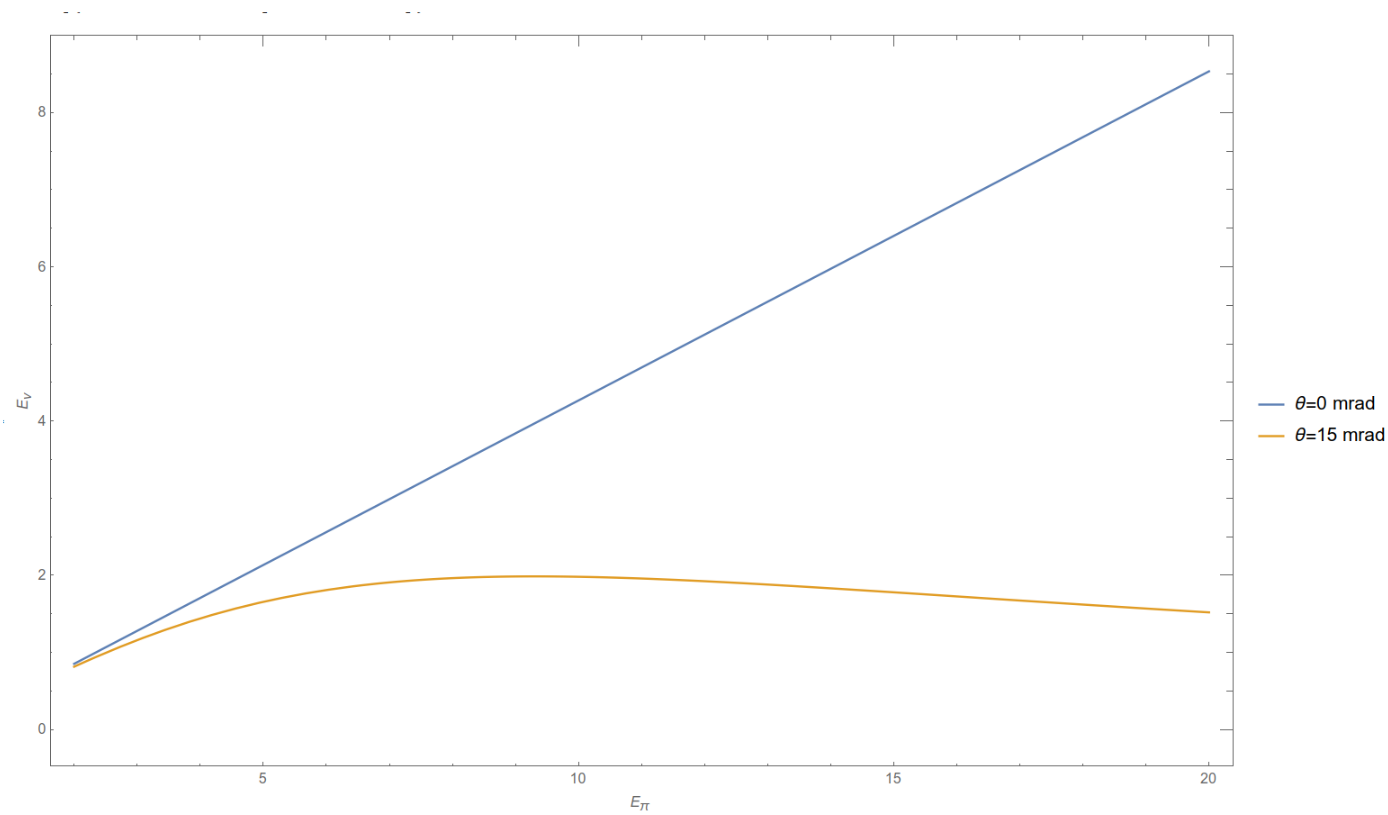

Figure 1: Neutrino energy as a function of pion energy for the decay, $\pi^+ \to \nu_\mu \mu^+$