Phys 916: Quantum Mechanics I UNL Spring 2020
Homework Assignment #5
Due date: Monday May 4, 2020

Problem 1:

Show that in general, any 2×2 matrix M can be represented in terms of the unit matrix, I, and the Pauli matrices, $\vec{\sigma}$, .i.e.,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = a_0 I + \vec{a}.\vec{\sigma}$$

where the expansion coefficients $a_0 = Tr\{M\}$, $a_x = Tr\{M\sigma_x\}$, $a_y = Tr\{M\sigma_y\}$, and $a_z = Tr\{M\sigma_z\}$.

Problem 2:

Consider the quantum operator H whose matrix representation in the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ writes:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where $H_{11}$and H_{22} are real numbers and $H_{12} = H_{21}^*$. It is thus obvious that H is Hermitian.

1. Show that:
$$H = \frac{1}{2}(H_{11} + H_{22})I + \widetilde{K} \equiv \frac{1}{2}(H_{11} + H_{22})I + \frac{1}{2}(H_{11} - H_{22})K \qquad (1)$$
where I is the identity operator, and the operators $\widetilde{K}$ and K must be determined in terms of the matrix elements of H. Are the operators $\widetilde{K}$ and K Hermitian?
2. A key result from the decomposition in Eq. (1) is that the operators $\widetilde{K}$, K and H all have the same eigenvectors $|\psi_\pm\rangle$. Let $\tilde{\kappa}_\pm$, $\kappa_\pm$, $E_\pm$ be the eigenvalues of operators $\widetilde{K}$, K, and H. Use Eq. (1) to establish the relation between $E_\pm$ and $\kappa_\pm$, and the relation between $E_\pm$ and $\tilde{\kappa}_\pm$. Show that these relations allow for a change of the eigenvalue origin.
3. Directly solve the secular equations for the operators K and H, and determine the corresponding eigenvalues. Check that the relation between $E_\pm$ and $\kappa_\pm$ established in query 2) is correct.

4. Let us define the angles $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$ be the angles defined as: $\tan\theta = \frac{2|H_{21}|}{H_{11}-H_{22}}$ and $H_{21} = |H_{21}|e^{i\varphi}$, when $H_{11} - H_{22} \neq 0$. Express the matrix K as well as the eigenvalues $\kappa_{\pm}$ in terms of these angles.
5. Show that $E_+ + E_- = Tr\{H\}$, and that $E_+E_- = Det\{H\}$.
6. Show that if H has a degenerate spectrum, then it is necessarily proportional to the identity operator.
7. Use the operator $K(\theta, \varphi)$ to calculate the normalized eigenvectors $|\psi_{\pm}\rangle$ in terms of these angles in the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$. You must find that the normalized eigenvectors $|\psi_{\pm}\rangle$ are collinear to the normalized eigenvectors $|\pm\rangle_u$ of the ½ spin operator $\hat{S}_u$, where $\hat{u}$ is an arbitrary unit vector defined by these polar and azimuthal angles.
8. Show that $K(\theta = 0, \varphi = 0)$ is proportional to the z-component of the Pauli operator, σ_z. What are the corresponding eigenvalues and eigenvectors?
9. When $\theta = \pi/2$, the operator K is not finite and we must now use $\widetilde{K}$. Show that $\widetilde{K}_x \equiv \widetilde{K}(\theta = \pi/2, \varphi = 0)$ is proportional to the x-component of the Pauli operator, σ_x. What are the corresponding eigenvalues and eigenvectors?
10. Show that $\widetilde{K}_y \equiv \widetilde{K}(\theta = \pi/2, \varphi = \pi/2)$ is proportional to the y-component of the Pauli operator, σ_y. What are the corresponding eigenvalues and eigenvectors?
11. Calculate the commutator, $[\widetilde{K}_x, \widetilde{K}_y]$, and show that it is proportional to the z-component of the Pauli operator, σ_z.

Problem 3:

The spin operator $\vec{S}$ of an electron is pointing in any direction and is related to the Pauli matrices as $\vec{S} \equiv \boldsymbol{S} = \frac{\hbar}{2}\vec{\sigma}$. In the orthonormal basis $\{|+\rangle, |-\rangle\}$ for S_z:

1. Write down the matrix for S_z, S_x, S_y, and S_u. Are they Hermitian?
2. Determine the eigenvalues of each component for the spin operator.
3. Determine the eigenvectors of each component for the spin operator.
4. Show that $[S_x, S_y] = i\hbar S_z$, $[S_y, S_z] = i\hbar S_x$, $[S_z, S_x] = i\hbar S_y$.
5. Show that $[\boldsymbol{S}^2, \boldsymbol{S}] = 0$. In query 4), since S_x and S_y do not commute with their commutator S_z, one must never use the commutator formula: $[A, F(B)] = [A, B]F'(B)$.

Problem 4:

Let us consider two Stern-Gerlach experiments where the first one prepares the atoms in a state and the second one measures a spin physical quantity. Here, the state of the ½ spin system is prepared such that the spin is pointing down along $\hat{u}$ in the x-z plane.

1. Immediately before measurement, what is the state $|\psi\rangle$ of the system in the basis of eigenvectors of $\hat{S}_z$.
2. What is the probability that a measurement of spin along the z-axis will find $-\hbar/2$? What is the state of the system immediately after the measurement?
3. What is the probability that a measurement of spin along the z-axis will find $+\hbar/2$? What is the state of the system immediately after the measurement?
4. What is the probability for spin down along the y-axis? What is the state of the system immediately after the measurement?
5. What is the probability for spin up along the x-axis? What is the state of the system immediately after the measurement?
6. What is the probability that a measurement of spin along the u-axis will find $-\hbar/2$?
7. What is the probability that a measurement of spin along the u-axis will find $+\hbar/2$?
8. Calculate the mean value $\langle\psi|S_z|\psi\rangle$ by two methods. First by using statistical analysis where the incident beam contains N silver atoms, and the direct calculation of the matrix element. How does this result compare with classical prediction?

Problem 5:

Consider a spin ½ particle placed in a magnetic field $\vec{B} = (B_x, 0, B_z) = \frac{1}{\sqrt{2}}(B_0, 0, B_0)$. In the orthonormal basis $\{|+\rangle, |-\rangle\}$ for S_z:

1. Calculate the matrix representing the Hamiltonian H of the system.
2. Calculate the eigenvalues and eigenvectors of H.
3. The system at time $t = 0$ is in the state $|-\rangle$. What values can be found if the energy is measured, and with what probabilities?
4. Calculate the state vector $|\psi(t)\rangle$ at time t. At this instant, S_x is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation of your results.