# Physics 926: Homework #11

Due on April 14, 2020 at 5pm

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*In addition to the lecture notes, the following resources were used to better understand the material:
`https://arxiv.org/ftp/arxiv/papers/1511/1511.06752.pdf`

# Problem 1

Show that

$$P(\nu_1 \to \nu_2) = \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E}$$

for a two neutrino system in which the mixing matrix is

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $\Delta m_{12}^2 = m_1^2 - m_2^2$

**Solution**

Equation (7) from the lecture gives us a good starting point. This equation is an approximation which is valid when mass is small, which in this case it is

$$P(\nu_1 \to \nu_2) = \left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2$$

Using this equation as a starting point, we can plug in the given values of $U$ and explicitly do the sum.

$$\begin{aligned}
\left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2 &= \left[ \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[ \sum_j U_{1j}^* U_{2j} e^{-im_j^2 L/2E} \right]^* \\
&= \left[ \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[ \sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] \\
\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} &= \cos\theta(-\sin\theta) e^{-im_1^2 L/2E} + \sin\theta\cos\theta e^{-im_2^2 L/2E} \\
&= -\cos\theta\sin\theta e^{-im_1^2 L/2E} + \cos\theta\sin\theta e^{-im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left( e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \\
\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} &= -\sin\theta\cos\theta e^{im_1^2 L/2E} + \cos\theta\sin\theta e^{im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left( e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right)
\end{aligned}$$

$$\begin{aligned}
\left[ \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[ \sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] &= \cos^2\theta\sin^2\theta \left( e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \left( e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left( 2 - e^{i(m_1^2 - m_2^2)L/2E} - e^{-i(m_1^2 - m_2^2)L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left( 2 - 2Re\left[ e^{i(m_1^2 - m_2^2)L/2E} \right] \right) \\
&= 2\cos^2\theta\sin^2\theta \left( 1 - \cos\frac{\Delta m_{12}^2 L}{2E} \right)
\end{aligned}$$

From here, use the trig identity: $1 - cos\theta = 2\sin^2\frac{\theta}{2}$ and then $2\cos\theta\sin\theta = \sin 2\theta$:

$$\begin{aligned}\left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E}\right]\left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E}\right] =& 2\cos^2\theta\sin^2\theta\left(2\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\ =& 4\cos^2\theta\sin^2\theta\left(\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\ =& \sin^2 2\theta\sin^2\frac{\Delta m_{12}^2 L}{4E}\end{aligned}$$

# Problem 2

As an exercise in natural units, show that the quantity $\Delta m_{12}^2 L/4E$ that appears in the theory of neutrino oscillations is in fact equal to $1.27\Delta m_{12}^2(eV^2)L(km)/E(GeV)$.

**Solution**

First we want to get the expression in terms of S.I. units as a starting point for the conversion. The factor in question, $\Delta m_{12}^2 L/4E$, has to be dimensionless since it is the argument of a sine function. So let's look at the dimensions:

$$\begin{aligned}\frac{[\Delta m_{12}^2][L]}{[E]} =& \frac{M^2 L}{ML^2/T^2}\\ =& \frac{MT^2}{L}\end{aligned}$$

To get this to be unitless, we need a factor with units of $L/MT^2$. Since we have been working under the paradigm that $c = \hbar = 1$ we need to plug in factors of these to give the needed units.

$$\begin{aligned}[c] =& \frac{L}{T}\\ [\hbar] =& \frac{ML^2}{T}\end{aligned}$$

To get the needed units of $L/MT^2$, we can see right away that $\hbar$ must be in the denominator with a power of one since it's the only unit with mass in it. The $T$ from $\hbar$ is going to cancel the $T$ from c, and we need a $T^2$ in the final result. This leads to the conclusion that $c$ must be to the third power.

$$\begin{aligned}\frac{[c]^3}{[\hbar]} =& \left(\frac{L^3}{T^3}\right)\left(\frac{T}{ML^2}\right)\\ =& \frac{L}{MT^2}\end{aligned}$$

These are the dimensions we needed to make the argument of the sine function dimensionless. Therefore we can rewrite the argument this way:

$$\frac{\Delta m_{12}^2 L}{4E}\frac{c^3}{\hbar}$$

Let's start with $kg, m, J$ and use the conversions we used earlier in the semester to convert to $eV, km, GeV$. Using the values I calculated back in Homework #1:

$$\begin{aligned}1kg =& 5.608 \times 10^{26} GeV\\ =& 5.608 \times 10^{35} eV\\ \Delta m_{12}^2(kg^2) =& \frac{1}{(5.608 \times 10^{35})^2}\Delta m_{12}^2(eV)\\ L(m) =& 10^3 L(km)\\ E(J) =& 1.602 \times 10^{-19} J \times 10^9 eV = 1.602 \times 10^{-10} E(GeV)\\ c =& 2.998 \times 10^8 m/s\\ \hbar =& 1.055 J \cdot s\end{aligned}$$

Now putting this all together, we get:

$$\left[\frac{1}{(5.608 \times 10^{35})^2}\Delta m_{12}^2(eV^2)\right]\left[\frac{10^3 L(km)}{4 \times 1.602 \times 10^{-10} E(GeV)}\right]\left[\frac{(2.998 \times 10^8 m/s)^3}{1.055 \times 10^{-34} J \cdot s}\right] = 1.27\frac{\Delta m_{12}^2(eV^2)L(km)}{E(GeV)}$$

# Problem 3

As mentioned in class, experiments such as NO$\nu$A are taking advantage of the fact that neutrinos that are traveling off-axis of a neutrino beam have a narrower energy spread. Let's take a look.

(a) We want to make a neutrino beam from a beam of $\pi^+$ with $E_\pi = 20\ GeV$. How long should the decay pipe be to ensure the the great bulk of pions have decayed before they reach the absorber?

(b) Consider a pion with energy $E_\pi$ in the laboratory frame. Find the energy of the neutrino $E_\nu$ in the decay $\pi^+ \to \mu^+\nu_\mu$ as a function of the laboratory angle $\theta$ that the emitted neutrino makes with the original flight direction of the $\pi^+$.

(c) Plot $E_\nu$ for $E_\pi$ between 2 and 20 $GeV$ in the case $\theta = 0$ and $\theta = 15\ mrad$.

**Solution**

(a)

(b) In the lab frame, the four-momenta of each particle can be defined. Let the initial direction of the pion be in the x-direction. ($\sigma$ is the index of the four-vectors while $\mu$ and $\nu$ are reserved for the muon and neutrino)

Before the decay:

$$P_\pi^\sigma = (E_\pi, p_\pi, 0, 0)$$

After the decay:

$$\begin{aligned} P_\mu^\sigma =& (E_\mu, \vec{p}_\mu) \\ P_\nu^\sigma =& (E_\nu, p_\nu \cos\theta, p_\nu \sin\theta, 0) \end{aligned}$$

Note: I left the muon momentum completely general because it won't matter what value it has in the end.

Due to conservation of four-momentum, we can write:

$$\begin{aligned} P_\pi^\sigma &= P_\mu^\sigma + P_\nu^\sigma \\ P_\pi^\sigma - P_\nu^\sigma &= P_\mu^\sigma \end{aligned}$$

We can contract each side with itself since this operation is Lorentz invariant:

$$\begin{aligned} (P_\pi - P_\nu)^\sigma (P_\pi - P_\nu)_\sigma =& P_\mu^\sigma P_{\mu,\sigma} \\ P_\pi^2 + P_\nu^2 - 2P_\nu^\sigma P_{\pi,\sigma} =& P_\mu^2 \\ m_\pi^2 + m_\nu^2 - 2(E_\pi E_\nu - p_\pi p_\nu \cos\theta) =& m_\mu^2 \end{aligned}$$

$p_\nu \approx E_\nu$ because $m_\nu \approx 0$ compared with the other masses in the problem.

$$\begin{aligned} m_\pi^2 - 2(E_\pi E_\nu - m_\pi E_\nu \cos\theta) &= m_\mu^2 \\ m_\pi^2 - m_\mu^2 &= 2E_\nu (E_\pi - p_\pi \cos\theta) \\ E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - p_\pi \cos\theta)} \end{aligned}$$

But don't forget the dispersion relation:$E^2 = p^2 + m^2$

$$\Rightarrow p_\pi = \sqrt{E_\pi^2 - m_\pi^2}$$

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2} \cos\theta)}$$

(c) Below is the plot of $E_\nu = \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2}\cos\theta)}$.

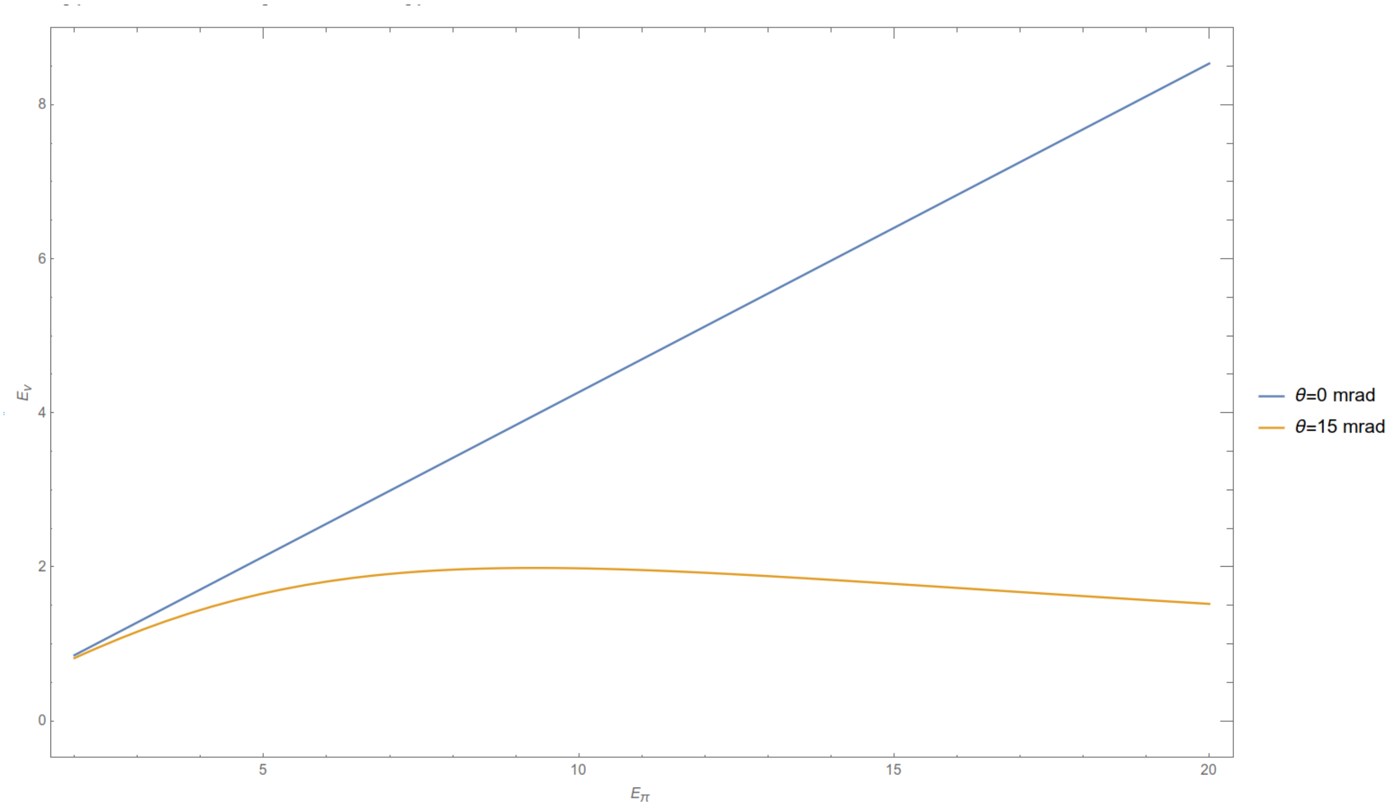


Figure 1: Neutrino energy as a function of pion energy for the decay, $\pi^+ \to \nu_\mu \mu^+$