Physics 926: Homework #11

Due on April 14, 2020 at 5pm

Professor Ken Bloom

Robert Tabb

*In addition to the lecture notes, the following resources were used to better understand the material:
`https://arxiv.org/ftp/arxiv/papers/1511/1511.06752.pdf`

Problem 1

Show that

$$P(\nu_1 \to \nu_2) = \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E}$$

for a two neutrino system in which the mixing matrix is

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $\Delta m_{12}^2 = m_1^2 - m_2^2$

Solution

Equation (7) from the lecture gives us a good starting point. This equation is an approximation which is valid when mass is small, which in this case it is

$$P(\nu_1 \to \nu_2) = \left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2$$

Using this equation as a starting point, we can plug in the given values of U and explicitly do the sum.

$$\begin{aligned}
\left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2 &= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{1j}^* U_{2j} e^{-im_j^2 L/2E} \right]^* \\
&= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] \\
\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} &= \cos\theta(-\sin\theta) e^{-im_1^2 L/2E} + \sin\theta\cos\theta e^{-im_2^2 L/2E} \\
&= -\cos\theta\sin\theta e^{-im_1^2 L/2E} + \cos\theta\sin\theta e^{-im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \\
\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} &= -\sin\theta\cos\theta e^{im_1^2 L/2E} + \cos\theta\sin\theta e^{im_2^2 L/2E} \\
&= \cos\theta\sin\theta \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right)
\end{aligned}$$

$$\begin{aligned}
\left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] &= \cos^2\theta \sin^2\theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left(2 - e^{i(m_1^2 - m_2^2)L/2E} - e^{-i(m_1^2 - m_2^2)L/2E} \right) \\
&= \cos^2\theta\sin^2\theta \left(2 - 2Re\left[e^{i(m_1^2 - m_2^2)L/2E} \right] \right) \\
&= 2\cos^2\theta\sin^2\theta \left(1 - \cos\frac{\Delta m_{12}^2 L}{2E} \right)
\end{aligned}$$

From here, use the trig identity: $1 - cos\theta = 2\sin^2\frac{\theta}{2}$ and then $2\cos\theta\sin\theta = \sin 2\theta$:

$$\begin{aligned}
\left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E}\right]\left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E}\right] =& 2\cos^2\theta\sin^2\theta\left(2\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\
=& 4\cos^2\theta\sin^2\theta\left(\sin^2\frac{\Delta m_{12}^2 L}{4E}\right)\\
=& \sin^2 2\theta\sin^2\frac{\Delta m_{12}^2 L}{4E}
\end{aligned}$$

Problem 2

As an exercise in natural units, show that the quantity $\Delta m_{12}^2 L/4E$ that appears in the theory of neutrino oscillations is in fact equal to $1.27\Delta m_{12}^2(eV^2)L(km)/E(GeV)$.

Solution

First we want to get the expression in terms of S.I. units as a starting point for the conversion. The factor in question, $\Delta m_{12}^2 L/4E$, has to be dimensionless since it is the argument of a sine function. So let's look at the dimensions:

$$\begin{aligned}\frac{[\Delta m_{12}^2][L]}{[E]} =& \frac{M^2 L}{ML^2/T^2}\\ =& \frac{MT^2}{L}\end{aligned}$$

To get this to be unitless, we need a factor with units of L/MT^2. Since we have been working under the paradigm that $c = \hbar = 1$ we need to plug in factors of these to give the needed units.

$$\begin{aligned}[c] =& \frac{L}{T}\\ [\hbar] =& \frac{ML^2}{T}\end{aligned}$$

To get the needed units of L/MT^2, we can see right away that $\hbar$ must be in the denominator with a power of one since it's the only unit with mass in it. The T from $\hbar$ is going to cancel the T from c, and we need a T^2 in the final result. This leads to the conclusion that c must be to the third power.

$$\begin{aligned}\frac{[c]^3}{[\hbar]} =& \left(\frac{L^3}{T^3}\right)\left(\frac{T}{ML^2}\right)\\ =& \frac{L}{MT^2}\end{aligned}$$

These are the dimensions we needed to make the argument of the sine function dimensionless. Therefore we can rewrite the argument this way:

$$\frac{\Delta m_{12}^2 L}{4E}\frac{c^3}{\hbar}$$

Let's start with kg, m, J and use the conversions we used earlier in the semester to convert to eV, km, GeV. Using the values I calculated back in Homework #1:

$$\begin{aligned}1kg =& 5.608\times 10^{26} GeV\\ =& 5.608\times 10^{35} eV\\ \Delta m_{12}^2(kg^2) =& \frac{1}{5.608\times 10^{35}}\Delta m_{12}^2(eV)\\ L(m) =& 10^3 L(km)\\ E(J) =& 1.602\times 10^{-19} J\times 10^9 eV = 1.602\times 10^{-10} E(GeV)\\ c =& 2.998\times 10^8 m/s\\ \hbar =& 1.055 J\cdot s\end{aligned}$$

Now putting this all together, we get:

$$\left[\frac{1}{(5.61\times 10^{35})^2}\Delta m_{12}^2(eV^2)\right]\left[\frac{10^3 L(km)}{4\times 1.61\times 10^{-10} E(GeV)}\right]\left[\frac{(2.998\times 10^8 m/s)^3}{1.055\times 10^{-34} J\cdot s}\right] = 1.27\frac{\Delta m_{12}^2(eV^2)L(km)}{E(GeV)}$$

Problem 3