Phys 916: Quantum Mechanics I UNL Spring 2020

Homework Assignment #4

Due date: Friday April 13, 2020

Problem 1:

Consider a physical system whose three-dimensional state space is spanned by an orthonormal basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. In that state space, consider two operators L_z and S defined by:

$L_z|u_1\rangle = |u_1\rangle$, $L_z|u_2\rangle = 0$, $L_z|u_3\rangle = -|u_3\rangle$

$S|u_1\rangle = |u_3\rangle$, $S|u_2\rangle = |u_2\rangle$, $S|u_3\rangle = |u_1\rangle$

a) Write the matrices, which represent, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the operators L_z, L_z^2, S, and S^2? Are these operators observables?
b) Give the form of the most general matrix, which represents an operator which commutes with L_z. Same question for L_z^2, then for S^2.
c) Do L_z^2 and S^2 form a C.S.C.O.? Give a basis of common eigenvectors.

Problem 2:

a) Consider a linear operator A in the state space. Show that the trace of this operator is invariant under the change of basis. The trace of a matrix A, noted as $Tr\{A\}$, is defined as the sum of its diagonal elements.
b) Assume that the eigenvalues, a_n, of the observable A are degenerate with a degree g_n. Calculate $Tr\{A\}$ in terms of the eigenvalues and the degree of degeneracy.

Problem 3:

Let $|\varphi_n\rangle$ be the eigenstates of a Hermitian operator H (H is for example the Hamiltonian of an arbitrary physical system). Assume that the states $|\varphi_n\rangle$ form a discrete orthonormal basis. The operator $U(m,n)$ is defined by:

$$U(m,n) = |\varphi_m\rangle\langle\varphi_n|$$

a) Calculate $U^\dagger(m,n)$ of $U(m,n)$.
b) Calculate the commutator, $[H, U(m,n)]$.

c) Prove the relation: $U(m,n)U^{\dagger}(m,n) = \delta_{nq}U(m,p)$.
d) Calculate $Tr\{U(m,n)\}$, the trace of the operator $U(m,n)$.
e) Let A be an operator, with matrix elements, $A_{mn} = \langle\varphi_m|A|\varphi_n\rangle$. Prove the relation: $A = \sum_{m,n} A_{mn}U(m,n)$.
f) Show that $A_{pq} = Tr\{AU^{\dagger}(m,n)\}$.

Problem 4:

Consider the even operator A and the odd operator B. Let $F(A)$ and $F(B)$ be the functions of these operators. It is always possible to expand the function $F(A)$ in a power series in A; same for $F(B)$.

a) Show that $F(A) = e^{A}$ does has a definite parity, whereas $F(B) = e^{B}$ does not.
b) Show that when $|\varphi_c\rangle$ is an eigenvector of an Hermitian operator C with the eigenvalue value c, $|\varphi_c\rangle$ is also an eigenvector of $F(C)$, with the eigenvalue $F(c)$.
c) If $|\varphi_a\rangle$ is an eigenvector of $F(A)$, calculate the matrix element $\langle\varphi_a|B|\varphi_a\rangle$.
d) As an application of result in b), calculate the function e^{C} if the matrix C is given by:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

e) Compare the three following functions: $F(A)F(B)$, $F(B)F(A)$, and $F(A+B)$. Are they equal? If not, why?
f) Repeat query e) if the operator B is replaced by the parity operator Π.

Problem 5:

Consider a three-dimensional state space and the following set of operators:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find all possible complete sets of commuting observables (C.S.C.O.). That is, determine whether or not each of the sets: $\{A\}$, $\{B\}$, $\{C\}$, $\{A,B\}$, $\{A,C\}$, $\{B,C\}$, $\{A,B,C\}$ constitutes a valid C.S.C.O..