Physics 926
Homework 12
Due Tuesday, April 14

1. Show that
$$P(\nu_1 \to \nu_2) = \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E} \tag{1}$$
for a two-neutrino system in which the mixing matrix is
$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2}$$
and $\Delta m_{12}^2 = m_1^2 - m_2^2$.

2. As an exercise in natural units, show that the quantity $\Delta m^2 L/4E$ that appears in the theory of neutrino oscillations is in fact equal to $1.27\Delta m^2(\mathrm{eV}^2)L(\mathrm{km})/E(\mathrm{GeV})$.

3. As mentioned in class, experiments such as NOνA are taking advantage of the fact that neutrinos that are traveling off-axis of a neutrino beam have a narrower energy spread. Let's take a look.

   (a) We want to make a neutrino beam from a beam of π^+ with $E_\pi = 20$ GeV. How long should the decay pipe be to ensure that the great bulk of the pions have decayed before they reach the absorber?

   (b) Consider a pion with energy E_π in the laboratory frame. Find the energy of the neutrino E_ν in the decay $\pi^+ \to \mu^+\nu_\nu$ as a function of the laboratory angle θ that the emitted neutrino makes with the original flight direction of the π^+. (This is probably most easily done in the lab frame.)

   (c) Plot E_ν for E_π between 2 and 20 GeV in the case $\theta = 0$ and $\theta = 15$ mrad.

4. We only briefly mentioned the possibility that neutrinos are their own antiparticles, *i.e.* are Majorana particles, and only briefly discussed the issue of CP violation. Let's explore these a little further. In class we said that for three-generation mixing,
$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin^2 \frac{\Delta m_{ij}^2 L}{4E} + 2\sum_{i>j} \mathrm{Im}(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*) \sin \frac{\Delta m_{ij}^2 L}{4E}. \tag{3}$$
Now, if neutrinos are Majorana particles, then the neutrino mixing matrix includes two extra Majorana phases, like so:
$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\eta_1/2} & 0 & 0 \\ 0 & e^{i\eta_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$
(Feel free to use Equation 14.34 in the Review of Particle Properties to save yourself some algebra.)

(a) Verify that these new phases have no effect on the oscillation probability.

(b) As discussed in class, there is CP violation in neutrino oscillation if the imaginary term of the oscillation probability is non-zero for $\alpha \neq \beta$. Use the unitarity of the mixing matrix ($\sum U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}$) to show that for a given pair of flavors α and β (with $\alpha \neq \beta$), the quantity $\mathrm{Im}(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j})$ is the same (up to a sign) for any i and j. This argument, based simply on unitarity, can be expanded by interchanging the roles of the rows and columns of U to conclude that for a given pair of i and j ($i \neq j$) the same quantity is independent of α and β up to a sign. Thus, that quantity is universal, independent of i, j, α and β.

(c) Use the expression for U above, without the Majorana phases, to work out the value of the same imaginary quantity, with your favorite choice of i, j, α and β (since it doesn't matter what you choose). Verify that this quantity is proportional to $s_{12}s_{13}s_{23}\sin\delta$. This shows that CP violation needs not only a complex phase δ but also non-trivial mixing between all three neutrino states. If any of the rotation angles θ_{ij} is equal to 0 or π, the CP violating effects disappear.