

Physics 926: Homework #12

Due on April 21, 2020 at 5pm

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Problem 1

If the vertex factor for the decay of a vector boson X into two spin-1/2 fermions f_1 and f_2 is

$$-igx\gamma^\mu \frac{1}{2}(c_v - c_A\gamma^5)$$

then show that

$$\Gamma(X \rightarrow f_1 \bar{f}_2) = \frac{g_X^2}{48\pi}(c_v^2 + c_A^2)M_X$$

where M_X is the mass of the boson and where we have neglected the mass of the fermions. Hints: use

$$\sum_\lambda \epsilon_\mu^{(\lambda)*} \epsilon_\nu^\lambda = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$$

to show that after summing over the fermions and averaging over the boson spins ,

$$\overline{|\mathcal{M}|^2} = \frac{1}{12}g_X^2(c_v^2 + c_A^2)(-g_{\mu\nu})Tr(\gamma^\mu \not{k} \gamma^\nu \not{k}')$$

where k and k' are the four-momenta of the fermions. Work in the boson rest frame, and use

$$\Gamma(X \rightarrow f_1 \bar{f}_2) \frac{p_f}{32\pi^2 m_X^2} \int \overline{|\mathcal{M}|^2} d\Omega$$

Solution

Using the vertex factor given in the problem, begin by writing the matrix element

$$\begin{aligned} \mathcal{M} &= \bar{u}(k) \left[-ig_X \gamma^\mu \frac{1}{2}(c_v - c_A\gamma^5) \right] v(k') \epsilon_\mu \\ &= -\frac{ig_X}{2} [\bar{u}(k) \gamma^\mu (c_v - c_A\gamma^5) v(k') \epsilon_\mu] \\ |\mathcal{M}|^2 &= \frac{g_X^2}{4} [\bar{u}(k) \gamma^\mu (c_v - c_A\gamma^5) v(k') \epsilon_\mu] [\bar{u}(k) \gamma^\nu (c_v - c_A\gamma^5) v(k') \epsilon_\nu]^* \end{aligned}$$

We now need to write the average by summing over the spins and polarizations. In this step, assume the fermion masses can be neglected and use the formula: $\sum_s [\bar{u}(a) \Gamma_1 v(b)] [\bar{u}(a) \Gamma_2 v(b)]^* = Tr[\Gamma_1 \not{a} \Gamma_2 \not{b}]$, recalling that $\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{3} \sum_{s,\lambda} |\mathcal{M}|^2 \\ &= \frac{g_X^2}{12} \sum_\lambda (\epsilon_\mu \epsilon_\nu^*) \sum_s [\bar{u}(k) \gamma^\mu (c_v - c_A\gamma^5) v(k')] [\bar{u}(k) \gamma^\nu (c_v - c_A\gamma^5) v(k')]^* \\ &= \frac{g_X^2}{12} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2} \right) Tr \left[\gamma^\mu (c_v - c_A\gamma^5) \not{k}' \gamma^0 (\gamma^\nu (c_v - c_A\gamma^5))^\dagger \gamma^0 \not{k} \right] \\ &= \frac{g_X^2}{12} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2} \right) Tr \left[(c_v \gamma^\mu \not{k}' - c_A \gamma^\mu \gamma^5 \not{k}') (c_v \gamma^0 \gamma^{\nu\dagger} \gamma^0 \not{k} - c_A \gamma^0 \gamma^{5\dagger} \gamma^0 \gamma^{\nu\dagger} \gamma^0 \not{k}) \right] \\ &= \frac{g_X^2}{12} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2} \right) Tr \left[(c_v \gamma^\mu \not{k}' - c_A \gamma^\mu \gamma^5 \not{k}') (c_v \gamma^\nu \not{k} + c_A \gamma^5 \gamma^\nu \not{k}) \right] \end{aligned}$$

Where the plus sign in the last line comes from letting $\gamma^0\gamma^5 \rightarrow \gamma^5\gamma^0$ Now let's evaluate that trace

$$\begin{aligned}
Tr \left[(c_v \gamma^\mu \not{k}' - c_A \gamma^\mu \gamma^5 \not{k}') (c_v \gamma^\nu \not{k} + c_A \gamma^5 \gamma^\nu \not{k}) \right] &= Tr \left[c_v^2 \gamma^\mu \not{k}' \gamma^\nu \not{k} - c_A^2 \gamma^\mu \gamma^5 \not{k}' \gamma^5 \gamma^\nu \not{k} \right] \\
&= c_v^2 Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] - c_A^2 Tr \left[\gamma^\mu \gamma^5 \not{k}' \gamma^5 \gamma^\nu \not{k} \right] \\
&= c_v^2 Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] - c_A^2 Tr \left[-\gamma^\mu \not{k}' \gamma^5 \gamma^5 \gamma^\nu \not{k} \right] \\
&= c_v^2 Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] + c_A^2 Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] \\
&= Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] (c_v^2 + c_A^2)
\end{aligned}$$

Putting all this together:

$$|\overline{\mathcal{M}}|^2 = \frac{g_X^2}{12} (-g_{\mu\nu}) Tr \left[\gamma^\mu \not{k}' \gamma^\nu \not{k} \right] (c_v^2 + c_A^2)$$

Now we calculate the decay rate, Γ , but first let's go ahead and evaluate that trace

$$\Gamma(X \rightarrow f_1 \bar{f}_2) = \frac{p_f}{32\pi^2 M_X^2} \int |\overline{\mathcal{M}}|^2 d\Omega$$

Problem 2

Using the result of the previous problem, compute the total widths and branching ratios for the Z and W decays into all possible final-state fermions. Use $\sin^2 \theta_W = 0.23$, $M_Z = 91 \text{ GeV}$, and $G_F = 1.17 \times 10^{-5} \text{ GeV}^{-2}$.

Solution

First, the Z decays. The Z boson does not change flavor, so all decays must be of the same flavor, have zero net charge, and conserve lepton number, meaning for example that you can't get a final product containing two neutrinos that are not antiparticle versions of each other. Note that the top quark is too heavy to be a decay product from the Z boson.

quarks	$Z \rightarrow u\bar{u}$	$Z \rightarrow d\bar{d}$	$Z \rightarrow c\bar{c}$	$Z \rightarrow s\bar{s}$	$Z \rightarrow b\bar{b}$
leptons	$Z \rightarrow e^+e^-$	$Z \rightarrow \mu^+\mu^-$	$Z \rightarrow \tau^+\tau^-$	X	X
neutrinos	$Z \rightarrow \nu_e\bar{\nu}_e$	$Z \rightarrow \nu_\mu\bar{\nu}_\mu$	$Z \rightarrow \nu_\tau\bar{\nu}_\tau$	X	X

The relation between G_F and g is given by $G_F = \frac{\sqrt{2}}{8} \frac{g^2}{M_X^2}$. So in the decay equation, we write:

$$\Gamma(X \rightarrow f_1\bar{f}_2) = \frac{8G_F M_X^2}{\sqrt{2}48\pi} (c_v^2 + c_A^2) M_X = \frac{G_F M_X^3}{6\pi\sqrt{2}} (c_v^2 + c_A^2)$$

The decay widths: (all of these have the form $Z \rightarrow ff$, so in the equations I just put which type of fermion it decays to). The values of the constant, c_v and c_A come from the textbook, p. 301 using $\sin^2 \theta_W = 0.23$.

$$\begin{aligned} \Gamma(q^+) &= \frac{G_F M_Z^3}{6\pi\sqrt{2}} (c_{v,q^+}^2 + c_{A,q^+}^2) \\ &= 0.095 \end{aligned}$$

$$\begin{aligned} \Gamma(q^-) &= \frac{G_F M_Z^3}{6\pi\sqrt{2}} (c_{v,q^-}^2 + c_{A,q^-}^2) \\ &= 0.12 \end{aligned}$$

$$\begin{aligned} \Gamma(l) &= \frac{G_F M_Z^3}{6\pi\sqrt{2}} (c_{v,l}^2 + c_{A,l}^2) \\ &= 0.083 \end{aligned}$$

$$\begin{aligned} \Gamma(\nu) &= \frac{G_F M_Z^3}{6\pi\sqrt{2}} (c_{v,\nu}^2 + c_{A,\nu}^2) \\ &= 0.165 \end{aligned}$$

$$\begin{aligned} \Gamma_Z &= \\ &= 0.47 \end{aligned}$$

Then the branching ratios are:

$$\Gamma(q^+)/\Gamma_Z = 0.20$$

$$\Gamma(q^-)/\Gamma_Z = 0.26$$

$$\Gamma(l)/\Gamma_Z = 0.18$$

$$\Gamma(\nu)/\Gamma_Z = 0.35$$

Now for the W decays. We have to be careful to separate the positive from the negative W boson. We also have to consider that the W boson allows for flavor changing interactions.

quarks	$W^+ \rightarrow u\bar{d}$	$W^+ \rightarrow u\bar{s}$	$W^+ \rightarrow u\bar{b}$	$W^+ \rightarrow c\bar{d}$	$W^+ \rightarrow c\bar{s}$	$W^+ \rightarrow c\bar{b}$
quarks	$W^- \rightarrow \bar{u}d$	$W^- \rightarrow \bar{u}s$	$W^- \rightarrow \bar{u}b$	$W^- \rightarrow \bar{c}d$	$W^- \rightarrow \bar{c}s$	$W^- \rightarrow \bar{c}b$
lepton ν	$W^+ \rightarrow e^+\nu_e$	$W^+ \rightarrow \mu^+\nu_\mu$	$W^+ \rightarrow \tau^+\nu_\tau$	X	X	X
lepton ν	$W^- \rightarrow e^-\bar{\nu}_e$	$W^- \rightarrow \mu^-\bar{\nu}_\mu$	$W^- \rightarrow \tau^-\bar{\nu}_\tau$	X	X	X

Since W goes to particles and antiparticles the same way, just with everything conjugated, the decay rates will be the same. So only one from each pair needs to be calculated. In the case of the quarks, we have to include the strength of the flavor-changing interactions from the CKM matrix. Because these are different for different quark pairs, unlike with the Z decay, we have to calculate different decay widths for each quark pair. Also note, $c_A = c_v = 1/\sqrt{2}$ for W decays.

$$\begin{aligned}\Gamma(l^+\nu) &= \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.114\end{aligned}$$

$$\begin{aligned}\Gamma(u\bar{d}) &= V_{ud} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.111\end{aligned}$$

$$\begin{aligned}\Gamma(u\bar{s}) &= V_{us} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.0257\end{aligned}$$

$$\begin{aligned}\Gamma(u\bar{b}) &= V_{ub} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.000400\end{aligned}$$

$$\begin{aligned}\Gamma(c\bar{d}) &= V_{cd} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.0257\end{aligned}$$

$$\begin{aligned}\Gamma(c\bar{s}) &= V_{cs} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.111\end{aligned}$$

$$\begin{aligned}\Gamma(c\bar{b}) &= V_{cb} \frac{G_F M_W^3}{12\pi\sqrt{2}} \\ &= 0.00470\end{aligned}$$

$$\Gamma_W = 0.785$$

The branching ratios are then:

$$\Gamma(l^+\nu)/\Gamma_W = 0.145$$

$$\Gamma(u\bar{d})/\Gamma_W = 0.141$$

$$\Gamma(u\bar{b})/\Gamma_W = 0.0327$$

$$\Gamma(u\bar{s})/\Gamma_W = 0.000510$$

$$\Gamma(c\bar{d})/\Gamma_W = 0.0327$$

$$\Gamma(c\bar{s})/\Gamma_W = 0.141$$

$$\Gamma(c\bar{b})/\Gamma_W = 0.00510$$

Problem 3

The Lagrangian for three interacting real fields ϕ_1, ϕ_2, ϕ_3 is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$, and where summation of ϕ_i^2 over i is implied. Show that it describes a massive field of mass $\sqrt{-2\mu^2}$ and two massless Goldstone bosons.

Solution