

Physics 926: Homework #11

Due on April 14, 2020 at 5pm

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*In addition to the lecture notes, the following resources were used to better understand the material:
<https://arxiv.org/ftp/arxiv/papers/1511/1511.06752.pdf>

Problem 1

Show that

$$P(\nu_1 \rightarrow \nu_2) = \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E}$$

for a two neutrino system in which the mixing matrix is

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and $\Delta m_{12}^2 = m_1^2 - m_2^2$

Solution

Equation (7) from the lecture gives us a good starting point. This equation is an approximation which is valid when mass is small, which in this case it is

$$P(\nu_1 \rightarrow \nu_2) = \left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2$$

Using this equation as a starting point, we can plug in the given values of U and explicitly do the sum.

$$\begin{aligned} \left| \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right|^2 &= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{1j}^* U_{2j} e^{-im_j^2 L/2E} \right]^* \\ &= \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] \\ \sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} &= \cos \theta (-\sin \theta) e^{-im_1^2 L/2E} + \sin \theta \cos \theta e^{-im_2^2 L/2E} \\ &= -\cos \theta \sin \theta e^{-im_1^2 L/2E} + \cos \theta \sin \theta e^{-im_2^2 L/2E} \\ &= \cos \theta \sin \theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \\ \sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} &= -\sin \theta \cos \theta e^{im_1^2 L/2E} + \cos \theta \sin \theta e^{im_2^2 L/2E} \\ &= \cos \theta \sin \theta \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right) \\ \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] &= \cos^2 \theta \sin^2 \theta \left(e^{-im_2^2 L/2E} - e^{-im_1^2 L/2E} \right) \left(e^{im_2^2 L/2E} - e^{im_1^2 L/2E} \right) \\ &= \cos^2 \theta \sin^2 \theta \left(2 - e^{i(m_1^2 - m_2^2)L/2E} - e^{-i(m_1^2 - m_2^2)L/2E} \right) \\ &= \cos^2 \theta \sin^2 \theta \left(2 - 2 \operatorname{Re} \left[e^{i(m_1^2 - m_2^2)L/2E} \right] \right) \\ &= 2 \cos^2 \theta \sin^2 \theta \left(1 - \cos \frac{\Delta m_{12}^2 L}{2E} \right) \end{aligned}$$

From here, use the trig identity: $1 - \cos\theta = 2 \sin^2 \frac{\theta}{2}$ and then $2 \cos\theta \sin\theta = \sin 2\theta$:

$$\begin{aligned}
 \left[\sum_i U_{1i}^* U_{2i} e^{-im_i^2 L/2E} \right] \left[\sum_j U_{2j}^* U_{1j} e^{im_j^2 L/2E} \right] &= 2 \cos^2 \theta \sin^2 \theta \left(2 \sin^2 \frac{\Delta m_{12}^2 L}{4E} \right) \\
 &= 4 \cos^2 \theta \sin^2 \theta \left(\sin^2 \frac{\Delta m_{12}^2 L}{4E} \right) \\
 &= \sin^2 2\theta \sin^2 \frac{\Delta m_{12}^2 L}{4E}
 \end{aligned}$$

Problem 2

As an exercise in natural units, show that the quantity $\Delta m_{12}^2 L/4E$ that appears in the theory of neutrino oscillations is in fact equal to $1.27 \Delta m_{12}^2 (eV^2) L(km)/E(GeV)$.

Solution

First we want to get the expression in terms of S.I. units as a starting point for the conversion. The factor in question, $\Delta m_{12}^2 L/4E$, has to be dimensionless since it is the argument of a sine function. So let's look at the dimensions:

$$\begin{aligned} \frac{[\Delta m_{12}^2][L]}{[E]} &= \frac{M^2 L}{ML^2/T^2} \\ &= \frac{MT^2}{L} \end{aligned}$$

To get this to be unitless, we need a factor with units of L/MT^2 . Since we have been working under the paradigm that $c = \hbar = 1$ we need to plug in factors of these to give the needed units.

$$\begin{aligned} [c] &= \frac{L}{T} \\ [\hbar] &= \frac{ML^2}{T} \end{aligned}$$

To get the needed units of L/MT^2 , we can see right away that \hbar must be in the denominator with a power of one since it's the only unit with mass in it. The T from \hbar is going to cancel the T from c , and we need a T^2 in the final result. This leads to the conclusion that c must be to the third power.

$$\begin{aligned} \frac{[c]^3}{[\hbar]} &= \left(\frac{L^3}{T^3} \right) \left(\frac{T}{ML^2} \right) \\ &= \frac{L}{MT^2} \end{aligned}$$

These are the dimensions we needed to make the argument of the sine function dimensionless. Therefore we can rewrite the argument this way:

$$\frac{\Delta m_{12}^2 L}{4E} \frac{c^3}{\hbar}$$

Let's start with kg, m, J and use the conversions we used earlier in the semester to convert to eV, km, GeV . Using the values I calculated back in Homework #1:

$$\begin{aligned} 1kg &= 5.608 \times 10^{26} GeV \\ &= 5.608 \times 10^{35} eV \\ \Delta m_{12}^2 (kg^2) &= \frac{1}{5.608 \times 10^{35}} \Delta m_{12}^2 (eV) \\ L(m) &= 10^3 L(km) \\ E(J) &= 1.602 \times 10^{-19} J \times 10^9 eV = 1.602 \times 10^{-10} E(GeV) \\ c &= 2.998 \times 10^8 m/s \\ \hbar &= 1.055 J \cdot s \end{aligned}$$

Now putting this all together, we get:

$$\left[\frac{1}{(5.61 \times 10^{35})^2} \Delta m_{12}^2 (eV^2) \right] \left[\frac{10^3 L(km)}{4 \times 1.61 \times 10^{-10} E(GeV)} \right] \left[\frac{(2.998 \times 10^8 m/s)^3}{1.055 \times 10^{-34} J \cdot s} \right] = 1.27 \frac{\Delta m_{12}^2 (eV^2) L(km)}{E(GeV)}$$

Problem 3

As mentioned in class, experiments such as NO ν A are taking advantage of the fact that neutrinos that are traveling off-axis of a neutrino beam have a narrower energy spread. Let's take a look.

- (a) We want to make a neutrino beam from a beam of π^+ with $E_\pi = 20 \text{ GeV}$. How long should the decay pipe be to ensure the the great bulk of pions have decayed before they reach the absorber?
- (b) Consider a pion with energy E_π in the laboratory frame. Find the energy of the neutrino E_ν in the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$ as a function of the laboratory angle θ that the emitted neutrino makes with the original flight direction of the π^+ .
- (c) Plot E_ν for E_π between 2 and 20 GeV in the case $\theta = 0$ and $\theta = 15 \text{ mrad}$.

Solution

(a)

(b) In the lab frame, the four-momenta of each particle can be defined. Let the initial direction of the pion be in the x-direction. (σ is the index of the four-vectors while μ and ν are reserved for the muon and neutrino)

Before the decay:

$$P_\pi^\sigma = (E_\pi, p_\pi, 0, 0)$$

After the decay:

$$\begin{aligned} P_\mu^\sigma &= (E_\mu, \vec{p}_\mu) \\ P_\nu^\sigma &= (E_\nu, p_\nu \cos \theta, p_\nu \sin \theta, 0) \end{aligned}$$

Note: I left the muon momentum completely general because it won't matter what value it has in the end.

Due to conservation of four-momentum, we can write:

$$\begin{aligned} P_\pi^\sigma &= P_\mu^\sigma + P_\nu^\sigma \\ P_\pi^\sigma - P_\nu^\sigma &= P_\mu^\sigma \end{aligned}$$

We can contract each side with itself since this operation is Lorentz invariant:

$$\begin{aligned} (P_\pi - P_\nu)^\sigma (P_\pi - P_\nu)_\sigma &= P_\mu^\sigma P_{\mu,\sigma} \\ P_\pi^2 + P_\nu^2 - 2P_\nu^\sigma P_{\pi,\sigma} &= P_\mu^2 \\ m_\pi^2 + m_\nu^2 - 2(E_\pi E_\nu - p_\pi p_\nu \cos \theta) &= m_\mu^2 \end{aligned}$$

$p_\nu \approx E_\nu$ because $m_\nu \approx 0$ compared with the other masses in the problem.

$$\begin{aligned} m_\pi^2 - 2(E_\pi E_\nu - m_\pi E_\nu \cos \theta) &= m_\mu^2 \\ m_\pi^2 - m_\mu^2 &= 2E_\nu(E_\pi - p_\pi \cos \theta) \\ E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - p_\pi \cos \theta)} \end{aligned}$$

But don't forget the dispersion relation: $E^2 = p^2 + m^2$

$$\begin{aligned} \Rightarrow p_\pi &= \sqrt{E_\pi^2 - m_\pi^2} \\ E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2} \cos \theta)} \end{aligned}$$

(c) Below is the plot of $E_\nu = \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \sqrt{E_\pi^2 - m_\pi^2} \cos \theta)}$

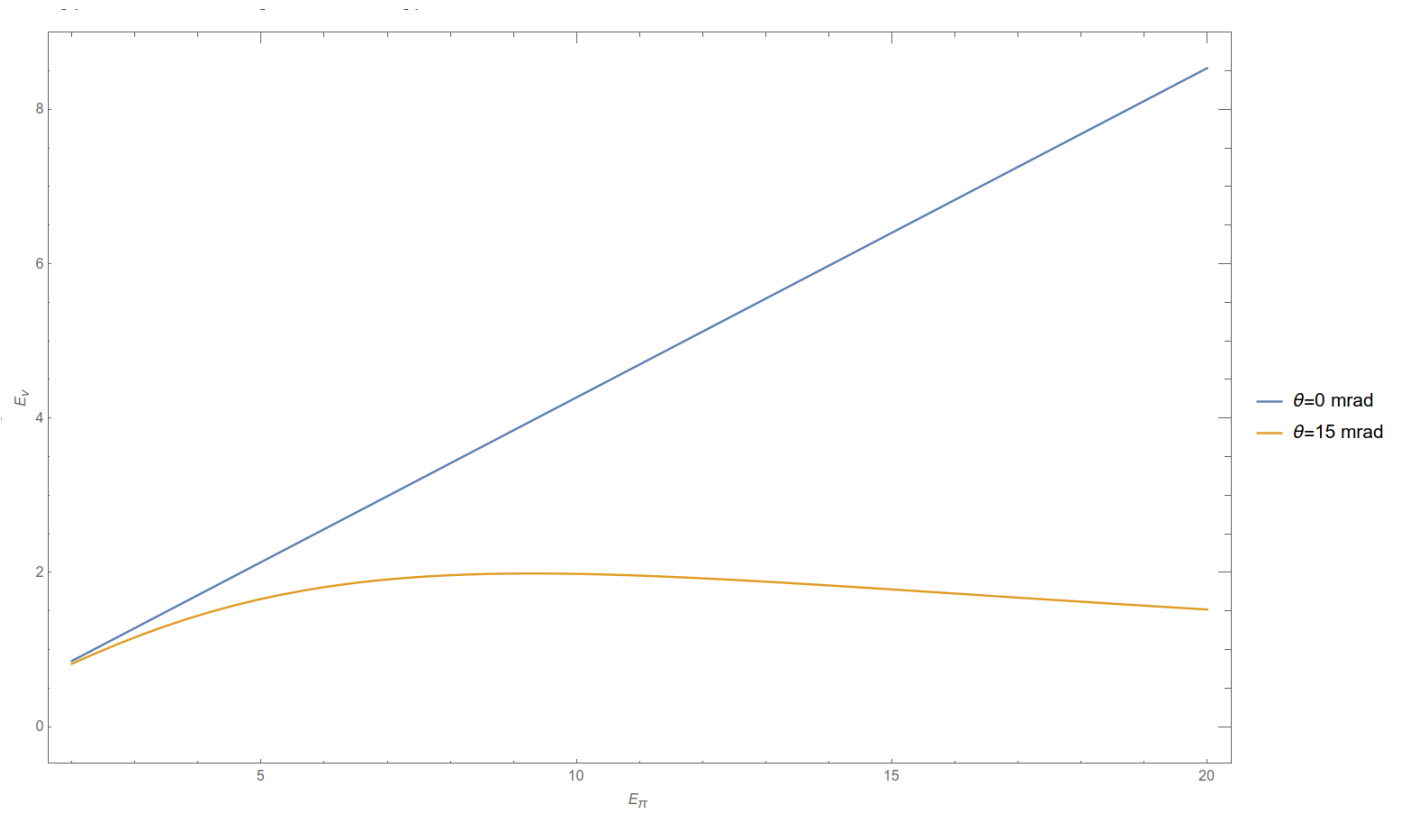


Figure 1: Neutrino energy as a function of pion energy for the decay, $\pi^+ \rightarrow \nu_\mu \mu^+$