Physics 926: Homework #10

Due on April 7, 2020 at 5pm

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*In addition to the lecture notes, the following resources were used to better understand the material:
`https://arxiv.org/pdf/hep-ph/0401236.pdf`
`https://www.hep.phy.cam.ac.uk/~thomson/lectures/partIIIparticles/Handout12_2009.pdf`

Problem 1

Show that:

$$\frac{\Gamma(K_L \to \pi^- e^+ \nu_e) - \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e))}{\Gamma(K_L \to \pi^- e^+ \nu_e) + \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e))} = 2Re(\epsilon)$$

to first order in ϵ. This asymmetry is evidence for indirect CP violation, and also allows us to unambiguously define electric charge - positive charge is assigned to the lepton that dominates in the K_L decay.

Solution

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|}}\left[\frac{1+\epsilon}{\sqrt{2}}|K^0\rangle - \frac{1-\epsilon}{\sqrt{2}}|\bar{K^0}\rangle\right]$$

$\bar{K^0} \to \pi^+ e^- \bar{\nu}_e$ and $K^0 \to \pi^- e^+ \nu_e$ (see Figure 1), therefore to get $K_L \to \pi^+ e^- \bar{\nu}_e$, take the inner product of $\bar{K^0}$ with K_L and to get $K_L \to \pi^- e^+ \nu_e$, take the inner product of K^0 with K_L.

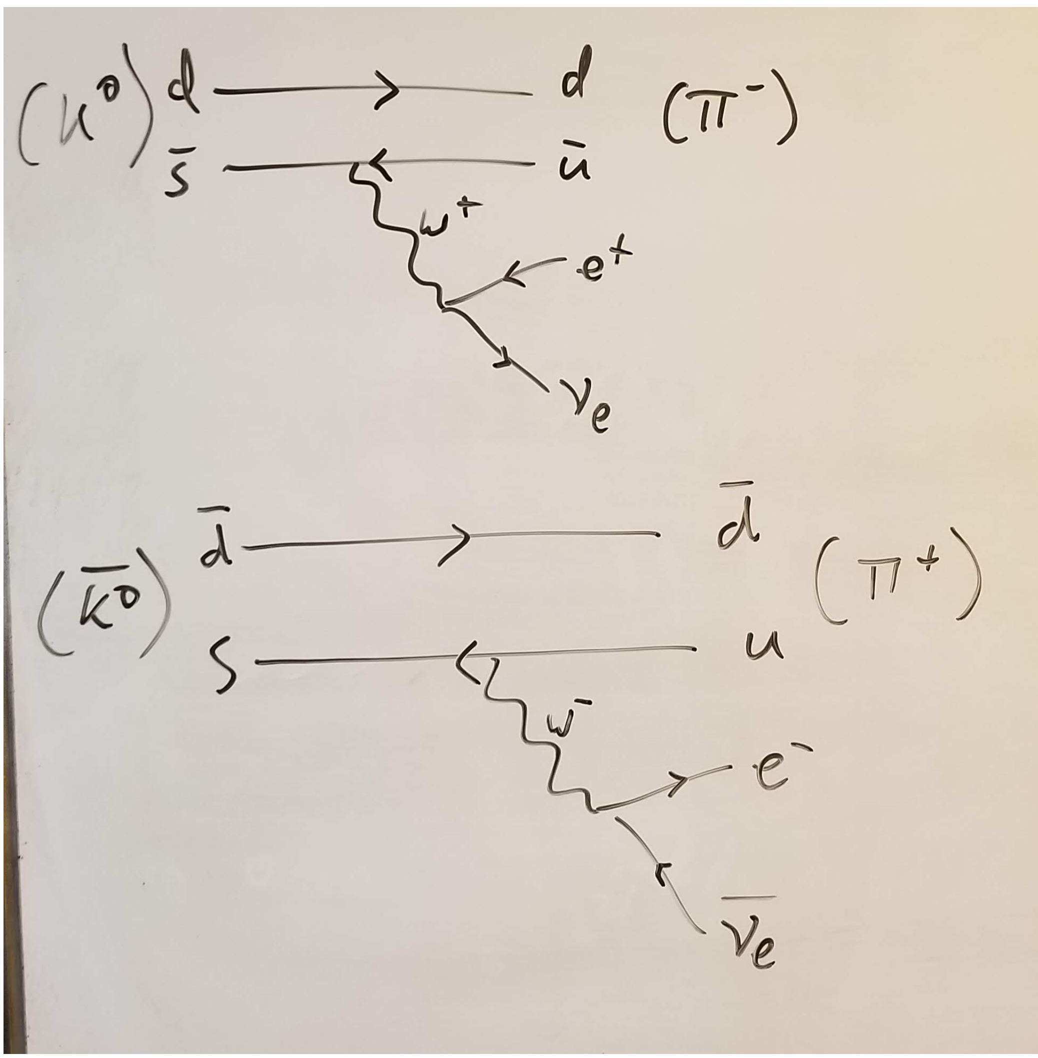

Figure 1: The two neutral kaon decays

$$\Gamma(K_L \to \pi^+ e^- \bar{\nu}_e) \propto \left|\langle \bar{K}^0 | K_L \rangle\right|^2 \propto |1-\epsilon|^2 = (1-\epsilon)(1-\epsilon^*) = 1 - \epsilon^* - \epsilon + |\epsilon|^2 \approx 1 - 2Re(\epsilon)$$
$$\Gamma(K_L \to \pi^- e^+ \nu_e) \propto \left|\langle K^0 | K_L \rangle\right|^2 \propto |1+\epsilon|^2 = (1+\epsilon)(1+\epsilon^*) = 1 + \epsilon^* + \epsilon + |\epsilon|^2 \approx 1 + 2Re(\epsilon)$$

Here I dropped the ϵ^2 term since we are only looking to first order in ϵ. I also dropped any common constants since they will be the same for each term and will divide out in the end.

$$\begin{aligned}\frac{\Gamma(K_L \to \pi^- e^+ \nu_e) - \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e))}{\Gamma(K_L \to \pi^- e^+ \nu_e) + \Gamma(K_L \to \pi^+ e^- \bar{\nu}_e))} =& \frac{1 + 2Re(\epsilon) - (1 - 2Re(\epsilon))}{1 + 2Re(\epsilon) + (1 - 2Re(\epsilon))} \\ =& \frac{4Re(\epsilon)}{2} = 2Re(\epsilon)\end{aligned}$$

Problem 2

Defining

$$\eta_{\pm} = |\eta_{\pm}|e^{i\phi_{\pm}} = \frac{\langle \pi^+\pi^- | K_L\rangle}{\langle \pi^+\pi^- | K_S\rangle}$$

calculate the probabilities of $\pi^+\pi^-$ decay as a function of proper time for an initial K^0 or $\bar{K^0}$ produced at $t = 0$. Express your answer, up to common proportionality constants, in terms of $\epsilon, |\eta_{\pm}|, \phi_{\pm}, \Delta m, \Gamma_S$, and Γ_L, where $\Delta m = K_S - K_L$. Keep only leading terms in ϵ. Using the experimental values for these quantities, plot the two probabilities as a function of time in units of the K_S lifetime, going out to 30 K_S lifetimes.

Solution

Here are the definitions of K_L and K_S in terms of K^0 and $\bar{K^0}$:

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|}}\left[\frac{1+\epsilon}{\sqrt{2}}|K^0\rangle - \frac{1-\epsilon}{\sqrt{2}}|\bar{K^0}\rangle\right]$$
$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|}}\left[\frac{1+\epsilon}{\sqrt{2}}|K^0\rangle + \frac{1-\epsilon}{\sqrt{2}}|\bar{K^0}\rangle\right]$$

And here they are in terms of the CP eigenstates:

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|}}\left[|K_2\rangle + \epsilon|K_1\rangle\right]$$
$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|}}\left[|K_1\rangle + \epsilon|K_2\rangle\right]$$

First, define the time-evolution of the wave function in the same way it was done in the lecture notes:

$$|K_S(t)\rangle = |K_S\rangle e^{im_S t - \Gamma_S t/2}$$
$$|K_L(t)\rangle = |K_L\rangle e^{im_L t - \Gamma_L t/2}$$

Here I will write the total wave function as a function of time but in the CP basis since we know that the $\pi^+\pi^-$ system is a CP eigenstate with an eigenvalue of +1. This will be helpful because we know the eigenvalues of $|K_1\rangle$ and $|K_2\rangle$

$$|\psi_{K^0}(t)\rangle = \frac{1}{\sqrt{2}}\left[|K_S(t)\rangle + |K_L(t)\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[|K_S\rangle e^{im_S t - \Gamma_S t/2} + |K_L\rangle e^{im_L t - \Gamma_L t/2}\right]$$
$$|\psi_{\bar{K^0}}(t)\rangle = \frac{1}{\sqrt{2}}\left[|K_S\rangle e^{im_S t - \Gamma_S t/2} - |K_L\rangle e^{im_L t - \Gamma_L t/2}\right]$$
$$|\psi_{K^0}(t)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\epsilon|}}\left[(|K_1\rangle + \epsilon|K_2\rangle)e^{im_S t - \Gamma_S t/2} + (|K_2\rangle + \epsilon|K_1\rangle)e^{im_L t - \Gamma_L t/2}\right]$$
$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\epsilon|}}\left[(e^{im_S t - \Gamma_S t/2} + \epsilon e^{im_L t - \Gamma_L t/2})|K_1\rangle + (e^{im_L t - \Gamma_L t/2} + \epsilon e^{im_S t - \Gamma_S t/2})|K_2\rangle\right]$$
$$|\psi_{\bar{K^0}}(t)\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+|\epsilon|}}\left[(e^{im_S t - \Gamma_S t/2} - \epsilon e^{im_L t - \Gamma_L t/2})|K_1\rangle + (-e^{im_L t - \Gamma_L t/2} + \epsilon e^{im_S t - \Gamma_S t/2})|K_2\rangle\right]$$

The subscript on the wave function state refers to the initial beam being either purely K^0 or purely $\bar{K^0}$.

Problem 2 continued on next page. . .

The probability to find the system in the state $|\pi^+\pi^-\rangle$ can be found like this:

$$
\begin{aligned}
|\langle K_1|\psi_{K^0}(t)\rangle|^2 =& \frac{1}{2+2|\epsilon|}\left[(e^{im_S t-\Gamma_S t/2}+\epsilon e^{im_L t-\Gamma_L t/2})(e^{im_S t-\Gamma_S t/2}+\epsilon e^{im_L t-\Gamma_L t/2})^*\right] \\
=& \frac{1}{2+2|\epsilon|}\left[e^{-\Gamma_S t}+|\epsilon|^2 e^{-\Gamma_L t}+\epsilon^* e^{i(m_S-m_L)t}e^{-(\Gamma_S+\Gamma_L)t/2}+\epsilon e^{-i(m_S-m_L)t}e^{-(\Gamma_S+\Gamma_L)t/2}\right] \\
\approx& \frac{1}{2+2|\epsilon|}\left[e^{-\Gamma_S t}+2Re(\epsilon e^{i\Delta m t}e^{-(\Gamma_S+\Gamma_L)t/2})\right] \\
|\langle K_1|\psi_{\bar{K^0}}(t)\rangle|^2 \approx& \frac{1}{2+2|\epsilon|}\left[e^{-\Gamma_S t}-2Re(\epsilon e^{i\Delta m t}e^{-(\Gamma_S+\Gamma_L)t/2})\right]
\end{aligned}
$$

The plot of the probabilities can be found in Figure 2.

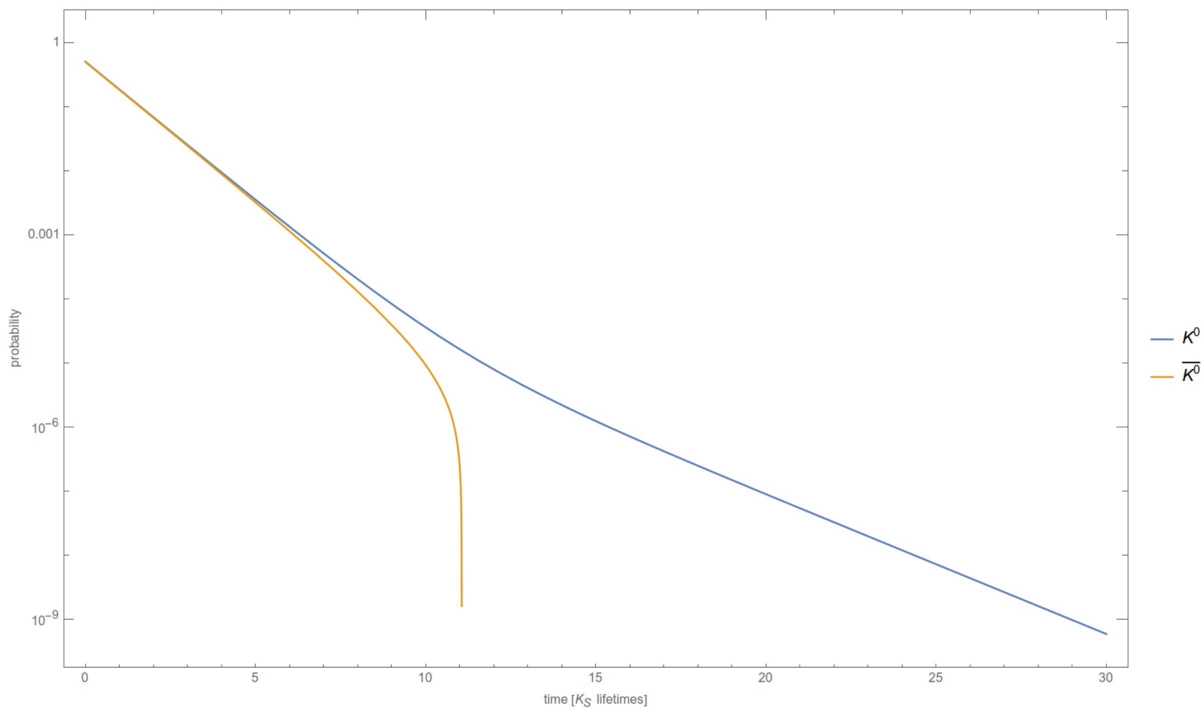

Figure 2: Probability for K^0 and $\bar{K^0}$ to decay to $\pi^+\pi^-$ as a function of K_S lifetime.

Problem 3

Calculate $P(A, t; B)$, where A and B are either K^0 or $\bar{K^0}$. This is defined to be the probability that a neutral kaon in state B at $t = 0$ has oscillated into a state A after a time t. Express your answer in terms of $\epsilon, \Delta m, \Gamma_S$, and Γ_L. Keep only leading terms in ϵ. Plot the probabilities as before, going out to 20 K_S lifetimes, and using a factor of ϵ which is a factor of ten larger than the experimental value.

Solution

Once again, here are K_L and K_S:

$$
\begin{aligned}
|K_L\rangle &= \frac{1}{\sqrt{1+|\epsilon|}}\left[\frac{1+\epsilon}{\sqrt{2}}|K^0\rangle - \frac{1-\epsilon}{\sqrt{2}}|\bar{K^0}\rangle\right] \\
|K_S\rangle &= \frac{1}{\sqrt{1+|\epsilon|}}\left[\frac{1+\epsilon}{\sqrt{2}}|K^0\rangle + \frac{1-\epsilon}{\sqrt{2}}|\bar{K^0}\rangle\right]
\end{aligned}
$$

We can use these equations to write K^0 and $\bar{K^0}$ in terms of K_L and K_S.

$$
\begin{aligned}
|K^0\rangle =& \sqrt{\frac{1+|\epsilon^2|}{2}}\frac{1}{1+\epsilon}\left[|K_L\rangle + |K_S\rangle\right] \\
|\bar{K^0}\rangle =& \sqrt{\frac{1+|\epsilon^2|}{2}}\frac{1}{1-\epsilon}\left[|K_L\rangle - |K_S\rangle\right]
\end{aligned}
$$

Now write the wave functions with time included.

$$
\begin{aligned}
|K^0(t)\rangle =& \sqrt{\frac{1+|\epsilon^2|}{2}}\frac{1}{1+\epsilon}\left[e^{im_L t - \Gamma_L t/2}|K_L\rangle + e^{im_S t - \Gamma_S t/2}|K_S\rangle\right] \\
|\bar{K^0}(t)\rangle =& \sqrt{\frac{1+|\epsilon^2|}{2}}\frac{1}{1-\epsilon}\left[e^{im_L t - \Gamma_L t/2}|K_L\rangle - e^{im_S t - \Gamma_S t/2}|K_S\rangle\right]
\end{aligned}
$$

Let's find the probability that a neutral kaon, K^0, at $t = 0$ decays into its anti-particle, $\bar{K^0}$ at time t.

$$
\begin{aligned}
\left|\langle K^0(t=0)|\bar{K^0}(t)\rangle\right|^2 =& \left|\frac{1+\left|\epsilon^2\right|}{2}\frac{1}{|1+\epsilon|^2}\left[\langle K_L| + \langle K_S|\right]\left[e^{im_L t - \Gamma_L t/2}|K_L\rangle - e^{im_S t - \Gamma_S t/2}|K_S\rangle\right]\right|^2 \\
=& \left(\frac{1+\left|\epsilon^2\right|}{2}\frac{1}{|1+\epsilon|^2}\right)^2\left|\left[e^{im_L t - \Gamma_L t/2} - e^{im_S t - \Gamma_S t/2}\right]\right|^2 \\
\approx& \frac{1}{4(1+4Re[\epsilon])}\left[e^{-\Gamma_L t} + e^{-\Gamma_S t} + 2Re\left(e^{-i\Delta m t}e^{-(\Gamma_S+\Gamma_L)t/2}\right)\right]
\end{aligned}
$$

Between the second and third lines, I dropped all factors of ϵ which were of order greater than 1. You will find the plot of this probability in Figure 3.

If you start with a particle in state $\bar{K^0}$ going to K^0, you will get the same plot. The only difference in the equation will be in the prefactor you will have a $1 - 4Re(\epsilon)$ instead of $1 + 4Re(\epsilon)$ in the denominator, but because ϵ is so small, the difference will be negligible.

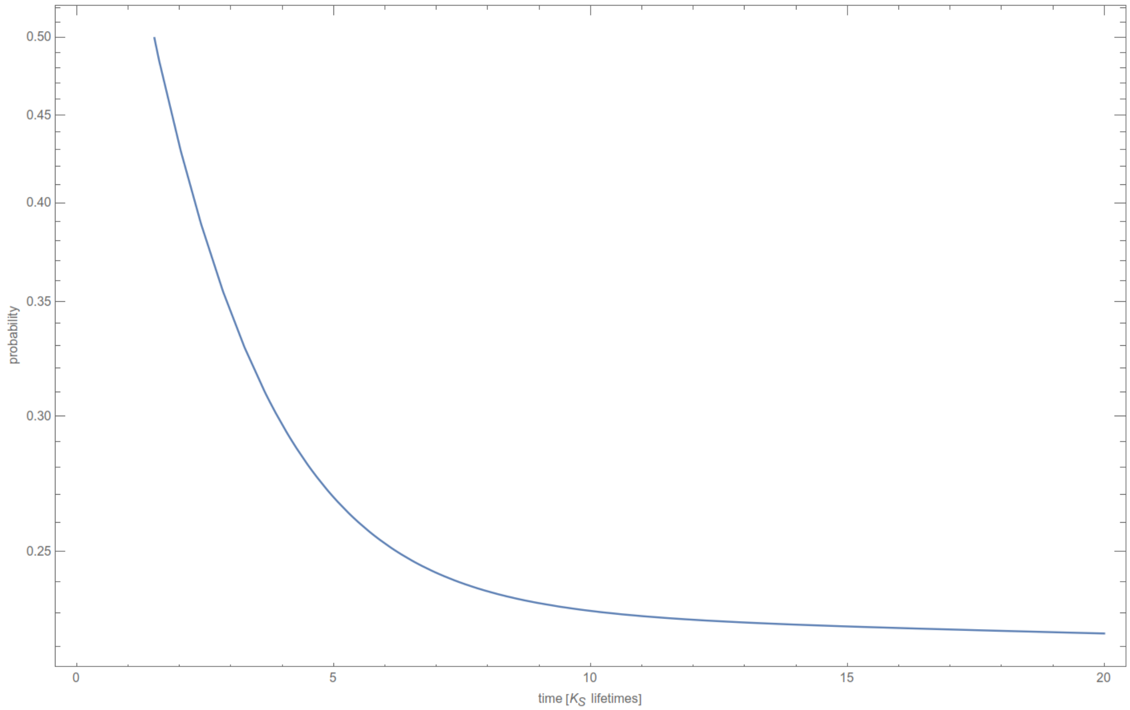

Figure 3: Probability for K^0 to oscillate to $\bar{K^0}$ as a function of time

Problem 4

In class we briefly mentioned the (approximate) $\Delta I = 1/2$ rule, which says that in strange particle decays, transitions with $\Delta I = 1/2$ are enhanced over those with $\Delta I = 3/2$. Use this rule to derive:

$$\Gamma(K_L \to 3\pi^0) = \frac{3}{2}\Gamma(K_L \to \pi^+\pi^-\pi^0)$$

and compare this result with experimental data. Make the reasonable assumption that all pairs of pions are in an $L = 0$ state. Don't forget that pions are bosons and you'll need to write totally symmetric wave functions.

Solution