Physics 916: Homework #5

Due on April , 2020

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Problem 1

Show that any 2×2 matrix can be represented in terms of the identity matrix and the pauli matrices.

Solution

First I will use a common convention and define σ_0 as the identity operator. So we have:

$$\begin{aligned} M = \vec{a} \cdot \vec{\sigma} =& a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ =& \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} \end{aligned}$$

Assuming that the elements of $\vec{a}$ are complex, they can be used to define any number in the matrix M which I defined above.

Problem 2

Show that

$$[S_x, S_y] = i\hbar S_z$$

Solution

$$
\begin{aligned}
S_x = \frac{\hbar}{2}\sigma_x,\ S_y &= \frac{\hbar}{2}\sigma_y,\ S_z = \frac{\hbar}{2}\sigma_z \\
S_x S_y =& \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \\
S_y S_x =& \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} \\
S_x S_y - S_y S_x =& \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} \\
=& \frac{\hbar^2}{4}\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
=& i\hbar\left(\frac{\hbar}{2}\sigma_z\right) \\
\Rightarrow [S_x, S_y] =& i\hbar S_z
\end{aligned}
$$