Physics 926: Homework #12

Due on April 21, 2020 at 5pm

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Problem 1

If the vertex factor for the decay of a vector boson X into two spin-1/2 fermions f_1 and f_2 is

$$-igx\gamma^\mu \frac{1}{2}(c_v - c_A\gamma^5)$$

then show that

$$\Gamma(X \to f_1\bar{f}_2) = \frac{g_X^2}{48\pi}(c_v^2 + c_A^2)M_X$$

where M_X is the mass of the boson and where we have neglected the mass of the fermions. Hints: use

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}\epsilon_\nu^\lambda = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}$$

to show that after summing over the fermions and averaging over the boson spins ,

$$\overline{|\mathcal{M}|^2} = \frac{1}{12}g_X^2(c_v^2 + c_A^2)(-g_{\mu\nu})Tr(\gamma^\mu \not{k}\gamma^\nu \not{k}')$$

where k and k' are the four-momenta of the fermions. Work in the boson rest frame, and use

$$\Gamma(X \to f_1\bar{f}_2)\frac{p_f}{32\pi^2 m_X^2}\int \overline{|\mathcal{M}|^2} d\Omega$$

Solution

Using the vertex factor given in the problem, begin by writing the matrix element

$$\begin{aligned}
\mathcal{M} =& \bar{u}(k)\left[-ig_X\gamma^\mu \frac{1}{2}(c_v - c_A\gamma^5)\right]v(k')\epsilon_\mu \\
=& -\frac{ig_X}{2}\left[\bar{u}(k)\gamma^\mu(c_v - c_A\gamma^5)v(k')\epsilon_\mu\right] \\
|\mathcal{M}|^2 =& \frac{g_X^2}{4}\left[\bar{u}(k)\gamma^\mu(c_v - c_A\gamma^5)v(k')\epsilon_\mu\right]\left[\bar{u}(k)\gamma^\nu(c_v - c_A\gamma^5)v(k')\epsilon_\nu\right]^*
\end{aligned}$$

We now need to write the average by summing over the spins and polarizations. In this step, assume the fermion masses can be neglected and use the formula: $\sum_s[\bar{u}(a)\Gamma_1 v(b)][\bar{u}(a)\Gamma_1 v(b)]^* = Tr[\Gamma_1 \not{b}\bar{\Gamma}_2 \not{a}]$, recalling that $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{1}{3}\sum_{s,\lambda}|\mathcal{M}|^2 \\
=& \frac{g_X^2}{12}\sum_\lambda(\epsilon_\mu\epsilon_\nu^*)\sum_s\left[\bar{u}(k)\gamma^\mu(c_v - c_A\gamma^5)v(k')\right]\left[\bar{u}(k)\gamma^\nu(c_v - c_A\gamma^5)v(k')\right]^* \\
=& \frac{g_X^2}{12}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2}\right)Tr\left[\gamma^\mu(c_v - c_A\gamma^5)\not{k}'\gamma^0(\gamma^\nu(c_v - c_A\gamma^5))^\dagger\gamma^0\not{k}\right] \\
=& \frac{g_X^2}{12}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2}\right)Tr\left[(c_v\gamma^\mu\not{k}' - c_A\gamma^\mu\gamma^5\not{k}')(c_v\gamma^0\gamma^{\nu\dagger}\gamma^0\not{k} - c_A\gamma^0\gamma^{5\dagger}\gamma^{\nu\dagger}\gamma^0\not{k})\right] \\
=& \frac{g_X^2}{12}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2}\right)Tr\left[(c_v\gamma^\mu\not{k}' - c_A\gamma^\mu\gamma^5\not{k}')(c_v\gamma^\nu\not{k} + c_A\gamma^5\gamma^\nu\not{k})\right]
\end{aligned}$$

Where the plus sign in the last line comes from letting $\gamma^0\gamma^5 \to \gamma^5\gamma^0$ Now let's evaluate that trace

$$\begin{aligned}
Tr\left[(c_v\gamma^\mu \not{k}' - c_A\gamma^\mu\gamma^5\not{k}')(c_v\gamma^\nu\not{k} + c_A\gamma^5\gamma^\nu\not{k})\right] =& Tr\left[c_v^2\gamma^\mu\not{k}'\gamma^\nu\not{k} - c_A^2\gamma^\mu\gamma^5\not{k}'\gamma^5\gamma^\nu\not{k} - 2c_vc_A\gamma^\mu\gamma^5\not{k}'\gamma^\nu\not{k}\right] \\
=& c_v^2 Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right] - c_A^2 Tr\left[\gamma^\mu\gamma^5\not{k}'\gamma^5\gamma^\nu\not{k}\right] - 2c_vc_A Tr\left[\gamma^\mu\gamma^5\not{k}'\gamma^\nu\not{k}\right] \\
=& c_v^2 Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right] - c_A^2 Tr\left[-\gamma^\mu\not{k}'\gamma^5\gamma^5\gamma^\nu\not{k}\right] - 2c_vc_A Tr\left[\gamma^\mu\gamma^5\not{k}'\gamma^\nu\not{k}\right] \\
=& c_v^2 Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right] + c_A^2 Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right] - 2c_vc_A Tr\left[\gamma^\mu\gamma^5\not{k}'\gamma^\nu\not{k}\right] \\
=& Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right](c_v^2 + c_A^2) - 2c_vc_A Tr\left[\gamma^\mu\gamma^5\not{k}'\gamma^\nu\not{k}\right]
\end{aligned}$$

From the trace theorems in Chapter 6 of the text, we see that our last trace is proportional to the levi-cevita tensor which is antisymmetric. Both $g_{\mu\nu}$ and $q_\mu q_\nu$ are symmetric. The inner product between an antisymmetric and a symmetric tensor is zero, therefore this term disappears.

Since we are working in the rest frame of the boson, we have the following four-momenta:

$$P_X^\sigma = (M_X, 0)\,,\; P_1^\sigma = \left(\frac{M_X}{2}, \vec{p}\right),\; P_2^\sigma = \left(\frac{M_X}{2}, -\vec{p}\right)$$

Putting all this together:

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{g_X^2}{12}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_X^2}\right) Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right](c_v^2 + c_A^2) \\
=& \frac{g_X^2}{12}\left(-g_{\mu\nu} Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right] + \frac{q_\mu q_\nu}{M_X^2} Tr\left[\gamma^\mu\not{k}'\gamma^\nu\not{k}\right]\right)(c_v^2 + c_A^2) \\
=& \frac{g_X^2}{12}\left(-g_{\mu\nu}4(k'^\mu k^\nu - g^{\mu\nu}(k'\cdot k) + k'^\mu k^\nu) + \frac{q_\mu q_\nu}{M_X^2}4(k'^\mu k^\nu - g^{\mu\nu}(k'\cdot k) + k'^\mu k^\nu)\right)(c_v^2 + c_A^2) \\
=& -\frac{g_X^2}{12}4\left(k'_\nu k^\nu - 4(k'\cdot k) + k'_\nu k^\nu + \left(\frac{1}{M_X^2}\right)\left[\left(\frac{M_X^2}{2}\right)^2 + \left(\frac{M_X^2}{2}\right)^2 - 2\frac{M_X^2}{4}M_X^2\right]\right)(c_v^2 + c_A^2) \\
=& -\frac{g_X^2}{3}\left(2(k'\cdot k) - 4(k'\cdot k)\right)(c_v^2 + c_A^2) \\
=& \frac{g_X^2}{3}\left(2(k'\cdot k)\right)(c_v^2 + c_A^2)
\end{aligned}$$

Now we calculate the decay rate, Γ, but first let's go ahead and evaluate that trace

$$\begin{aligned}
\Gamma(X \to f_1\bar{f}_2) =& \frac{p_f}{32\pi^2 M_X^2}\int \overline{|\mathcal{M}|^2} d\Omega \\
=& \frac{M_X}{2}\frac{2g_X^2}{96\pi^2 M_X^2}(c_v^2 + c_A^2)\int k'\cdot k\, d\Omega \\
=& \frac{g_X^2}{96\pi^2 M_X}(c_v^2 + c_A^2)4\pi\frac{M_X^2}{2} \\
=& \frac{g_X^2}{48\pi}(c_v^2 + c_A^2)M_X
\end{aligned}$$

Problem 2

Using the result of the previous problem, compute the total widths and branching ratios for the Z and W decays into all possible final-state fermions. Use $\sin^2\theta_W = 0.23$, $M_Z = 91\ GeV$, and $G_F = 1.17\times10^{-5}\ GeV^{-2}$.

Solution

First, the Z decays. The Z boson does not change flavor, so all decays must be of the same flavor, have zero net charge, and conserve lepton number, meaning for example that you can't get a final product containing two neutrinos that are not antiparticle versions of each other. Note that the top quark is too heavy to be a decay product from the Z boson.

quarks	$Z\to u\bar{u}$	$Z\to d\bar{d}$	$Z\to c\bar{c}$	$Z\to s\bar{s}$	$Z\to b\bar{b}$
leptons	$Z\to e^+e^-$	$Z\to\mu^+\mu^-$	$Z\to\tau^+\tau^-$	X	X
neutrinos	$Z\to\nu_e\bar{\nu}_e$	$Z\to\nu_\mu\bar{\nu}_\mu$	$Z\to\nu_\tau\bar{\nu}_\tau$	X	X

The relation between G_F and g is given by $G_F = \frac{\sqrt{2}}{8}\frac{g^2}{M_X^2}$. So in the decay equation, we write:

$$\Gamma(X\to f_1\bar{f}_2) = \frac{8G_F M_X^2}{\sqrt{2}48\pi}(c_v^2+c_A^2)M_X = \frac{G_F M_X^3}{6\pi\sqrt{2}}(c_v^2+c_A^2)$$

The decay widths: (all of these have the form $Z\to ff$, so in the equations I just put which type of fermion it decays to). The values of the constant, c_v and c_A come from the textbook, p. 301 using $\sin^2\theta_W = 0.23$.

$$\begin{aligned}
\Gamma(q^+) =& \frac{G_F M_Z^3}{6\pi\sqrt{2}}(c_{v,q^+}^2 + c_{A,q^+}^2) = 0.0950 \\
\Gamma(q^-) =& \frac{G_F M_Z^3}{6\pi\sqrt{2}}(c_{v,q^-}^2 + c_{A,q^-}^2) = 0.123 \\
\Gamma(l) =& \frac{G_F M_Z^3}{6\pi\sqrt{2}}(c_{v,l}^2 + c_{A,l}^2) = 0.0832 \\
\Gamma(\nu) =& \frac{G_F M_Z^3}{6\pi\sqrt{2}}(c_{v,\nu}^2 + c_{A,\nu}^2) = 0.165 \\
\Gamma_Z =& 0.467
\end{aligned}$$

Then the branching ratios are:

$$\begin{aligned}
\Gamma(q^+)/\Gamma_Z =& 0.20 \\
\Gamma(q^-)/\Gamma_Z =& 0.26 \\
\Gamma(l)/\Gamma_Z =& 0.18 \\
\Gamma(\nu)/\Gamma_Z =& 0.35
\end{aligned}$$

Now for the W decays. We have to be careful to separate the positive from the negative W boson. We also have to consider that the W boson allows for flavor changing interactions.

quarks	$W^+\to u\bar{d}$	$W^+\to u\bar{s}$	$W^+\to u\bar{b}$	$W^+\to c\bar{d}$	$W^+\to c\bar{s}$	$W^+\to c\bar{b}$
quarks	$W^-\to\bar{u}d$	$W^-\to\bar{u}s$	$W^-\to\bar{u}b$	$W^-\to\bar{c}d$	$W^+\to\bar{c}s$	$W^+\to\bar{c}b$
lepton ν	$W^+\to e^+\nu_e$	$W^+\to\mu^+\nu_\mu$	$W^+\to\tau^+\nu_\tau$	X	X	X
lepton ν	$W^-\to e^-\bar{\nu}_e$	$W^-\to\mu^-\bar{\nu}_\mu$	$W^-\to\tau^-\bar{\nu}_\tau$	X	X	X

Since W goes to particles and antiparticles the same way, just with everything conjugated, the decay rates will be the same. So only one from each pair needs to be calculated. In the case of the quarks, we have to include the strength of the flavor-changing interactions from the CKM matrix. Because these are different

Problem 2 continued on next page. . .

for different quark pairs, unlike with the Z decay, we have to calculate different decay widths for each quark pair. Also note, $c_A = c_v = 1/\sqrt{2}$ for W decays.

$$\begin{aligned}
\Gamma(l^+\nu) =& \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.114 \\
\Gamma(u\bar{d}) =& V_{ud}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.108 \\
\Gamma(u\bar{s}) =& V_{us}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.00577 \\
\Gamma(u\bar{b}) =& V_{ub}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 1.41 \times 10^{-6} \\
\Gamma(c\bar{d}) =& V_{cd}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.00577 \\
\Gamma(c\bar{s}) =& V_{cs}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.108 \\
\Gamma(c\bar{b}) =& V_{cb}^2 \frac{G_F M_W^3}{12\pi\sqrt{2}} = 0.000194 \\
\Gamma_W =& 0.684
\end{aligned}$$

The branching ratios are then:

$$\begin{aligned}
\Gamma(l^+\nu)/\Gamma_W =& 0.167 \\
\Gamma(u\bar{d})/\Gamma_W =& 0.158 \\
\Gamma(u\bar{b})/\Gamma_W =& 0.00844 \\
\Gamma(u\bar{s}/\Gamma_W =& 2.05 \times 10^{-6} \\
\Gamma(c\bar{d})/\Gamma_W =& 0.00844 \\
\Gamma(c\bar{s})/\Gamma_W =& 0.158 \\
\Gamma(c\bar{b})/\Gamma_W =& 0.000283
\end{aligned}$$

Problem 3

The Lagrangian for three interacting real fields ϕ_1, ϕ_2, ϕ_3 is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \phi_i^2 - \frac{1}{4}\lambda(\phi_i^2)^2$$

with $\mu^2 < 0$ and $\lambda > 0$, and where summation of ϕ_i^2 over i is implied. Show that it describes a massive field of mass $\sqrt{-2\mu^2}$ and two massless Goldstone bosons.

Solution

We can rewrite this in a more explicit way as:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2}(\partial_\mu \phi_3)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2 + \phi_3^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2 + \phi_3^2)^2$$

The last two terms describe the potential. For the case where $\mu^2 < 0$ we have a a breaking of the symmetry of the potential. We have a set of minima that satisfy the equation:

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = v^2 = -\frac{\mu^2}{\lambda}$$

This is the equation of a sphere and any point on this sphere represents a potential minimum. In order to expand the Lagrangian about the minimum, we have to choose one, thus breaking the symmetry. Let's choose $\phi_1 = v,\ \phi_2 = 0,\ \phi_3 = 0$. Let's do an expansion like:

$$\phi \propto [v + \eta(x) + i\xi(x) + j\rho(x)]$$

Then if we plug this into the Lagrangian, we get something like what was given in the lecture notes:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}(\partial_\mu \eta)^2 + \mu^2\eta^2 + \cdots$$

Just as in the notes, we have a massive field, *eta*, and two massless fields which are called Golstone bosons.

Using the rule for determining the mass of the field, let's look at the mass term:

$$\mu^2\eta^2 = \frac{1}{2}(-2\mu^2)\eta^2$$
$$\Rightarrow mass = \sqrt{-2\mu^2}$$