

# Physics 916: Homework #5

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## Problem 1

Show that in general, any  $2 \times 2$  matrix  $M$  can be represented in terms of the unit matrix,  $I$ , and the Pauli matrices. i.e.

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = a_0 I + \vec{a} \cdot \vec{\sigma}$$

where the expansion coefficients  $a_i = \frac{1}{2} \text{Tr}\{M\sigma_i\}$

### Solution

First I will use a common convention and define  $\sigma_0$  as the identity operator. So we have:

$$\begin{aligned} M = \vec{a} \cdot \vec{\sigma} &= a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} \end{aligned}$$

$$M_{11} = a_0 + a_3$$

$$M_{12} = a_1 - ia_2$$

$$M_{21} = a_1 + ia_2$$

$$M_{22} = a_0 - a_3$$

$$\begin{aligned} a_0 &= \frac{1}{2} \text{Tr}\{M\sigma_0\} = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \text{Tr} \left[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \\ &= \frac{1}{2} (M_{11} + M_{22}) = \frac{1}{2} (a_0 + a_3 + a_0 - a_3) \\ &= a_0 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{2} \text{Tr}\{M\sigma_1\} = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \text{Tr} \left[ \begin{pmatrix} M_{12} & M_{11} \\ M_{22} & M_{21} \end{pmatrix} \right] \\ &= \frac{1}{2} (M_{12} + M_{21}) = \frac{1}{2} (a_1 - ia_2 + a_1 + ia_2) \\ &= a_1 \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{1}{2} \text{Tr}\{M\sigma_2\} = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \text{Tr} \left[ \begin{pmatrix} iM_{12} & -iM_{11} \\ iM_{22} & -iM_{21} \end{pmatrix} \right] \\ &= \frac{1}{2} (iM_{12} - iM_{21}) = \frac{1}{2} (ia_1 - i^2a_2 - ia_1 - i^2a_2) \\ &= a_2 \end{aligned}$$

$$\begin{aligned} a_3 &= \frac{1}{2} \text{Tr}\{M\sigma_3\} = \frac{1}{2} \text{Tr} \left[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \text{Tr} \left[ \begin{pmatrix} M_{11} & -M_{12} \\ M_{21} & -M_{22} \end{pmatrix} \right] \\ &= \frac{1}{2} (M_{11} - M_{22}) = \frac{1}{2} (a_0 + a_3 - a_0 + a_3) \\ &= a_3 \end{aligned}$$

## Problem 2

Consider the quantum operator,  $H$ , whose matrix representation in the orthonormal basis  $\{|u_1\rangle, |u_2\rangle\}$  writes:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where  $H_{11}$  and  $H_{22}$  are real numbers and  $H_{12} = H_{21}^*$ . It is thus obvious that  $H$  is Hermitian.

1. Show that:

$H = \frac{1}{2}(H_{11} + H_{22})I + \tilde{K} \equiv \frac{1}{2}(H_{11} + H_{22})I + \frac{1}{2}(H_{11} - H_{22})K$  where  $I$  is the identity operator, and the operators  $\tilde{K}, K$  must be determined in terms of the matrix elements of  $H$ . Are  $\tilde{K}$  and  $K$  Hermitian?

2. A key result from the decomposition in part 1 is that the operators  $\tilde{K}, K$ , and  $H$  all have the same eigenvectors  $|\psi_{\pm}\rangle$ . Let  $\tilde{\kappa}_{\pm}, \kappa_{\pm}, E_{\pm}$  be the eigenvalues of  $\tilde{K}, K$ , and  $H$ . Use the result of part 1 to establish the relation between  $E_{\pm}$  and  $\kappa_{\pm}$ , and the relation between  $E_{\pm}$  and  $\tilde{\kappa}_{\pm}$ . Show that these relations allow for a change of the eigenvalue origin.

3. Directly solve the secular equations for  $K$  and  $H$  and determine the corresponding eigenvalues. Check that the relation between  $E_{\pm}$  and  $\kappa_{\pm}$  established in part 2 is correct.

4. Let us define angles  $0 \leq \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$  defined as:

$$\tan \theta = \frac{2|H_{21}|}{H_{11} - H_{22}} \quad \text{and} \quad H_{21} = |H_{21}|e^{i\phi}$$

5. Show that  $E_+ + E_- = \text{Tr}\{H\}$ , and that  $E_+ E_- = \det H$

6. Show that if  $H$  has a degenerate spectrum, then it is necessarily proportional to the identity operator.

7. Use the operator,  $K(\theta, \phi)$  to calculate normalized eigenvectors  $|\psi_{\pm}\rangle$  in terms of these angles in the orthonormal basis  $\{|u_1\rangle, |u_2\rangle\}$ . You must find that the eigenvectors  $|\psi_{\pm}\rangle$  are collinear to the eigenvectors  $|\pm\rangle$  of the  $1/2$  spin operator  $S_u$ , where  $u$  is an arbitrary unit vector defined by these angles.

8. Show that  $K(\theta = 0, \phi = 0)$  is proportional to the z-component of the Pauli operator,  $\sigma_z$ . What are the corresponding eigenvalues and eigenvectors?

9. When  $\theta = \pi/2$ , the operator  $K$  is not finite and we must use  $\tilde{K}$ . Show that  $\tilde{K}_x \equiv \tilde{K}(\theta = \pi/2, \phi = 0)$  is proportional to the x-component of the Pauli operator,  $\sigma_x$ . What are the corresponding eigenvalues and eigenvectors?

10. Show that  $\tilde{K}_y \equiv \tilde{K}(\theta = \pi/2, \phi = \pi/2)$  is proportional to the y-component of the Pauli operator,  $\sigma_y$ . What are the corresponding eigenvalues and eigenvectors?

11. Calculate the commutator,  $[\tilde{K}_x, \tilde{K}_y]$ , and show that it is proportional to the z-component of the Pauli operator,  $\sigma_z$ .

### Solution

Since they will be used several times later on in this problem, here are the definitions of the  $2 \times 2$  Pauli

operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. If we let the following be true we will have the relation suggested:

$$\begin{aligned} H_{11} &= \frac{1}{2}[H_{11}(1 + K_{11}) + H_{22}(1 - K_{11})] \\ H_{22} &= \frac{1}{2}[H_{11}(1 + K_{22}) + H_{22}(1 - K_{22})] \\ H_{12} &= \frac{1}{2}K_{12}(H_{11} - H_{22}) \\ H_{21} &= \frac{1}{2}K_{12}(H_{11} - H_{22}) \end{aligned}$$

From here we can determine a couple of things about  $K$  and  $\tilde{K}$

$$\begin{aligned} K_{11} &= 1, K_{22} = -1 \\ K_{12} &= \frac{2}{H_{11} - H_{22}} H_{12} = \frac{2}{H_{11} - H_{22}} H_{21}^* = K_{21}^* \end{aligned}$$

So we know the value of the diagonal of  $K$  and  $\tilde{K}$  and we also know that they are Hermitian.

2. Starting with the relation between  $E_{\pm}$  and  $\tilde{\kappa}_{\pm}$

$$\begin{aligned} H|\psi_{\pm}\rangle &= \frac{1}{2}(H_{11} + H_{22})I|\psi_{\pm}\rangle + \tilde{K}|\psi_{\pm}\rangle \\ &= \frac{1}{2}(H_{11} + H_{22})|\psi_{\pm}\rangle + \tilde{\kappa}_{\pm}|\psi_{\pm}\rangle \\ &= E_{\pm}|\psi_{\pm}\rangle \\ \Rightarrow E_{\pm} &= \frac{1}{2}(H_{11} + H_{22}) + \tilde{\kappa}_{\pm} \end{aligned}$$

Now for the relation between  $E_{\pm}$  and  $\kappa_{\pm}$

$$\begin{aligned} H|\psi_{\pm}\rangle &= \frac{1}{2}(H_{11} + H_{22})I|\psi_{\pm}\rangle + \frac{1}{2}(H_{11} - H_{22})K|\psi_{\pm}\rangle \\ &= \frac{1}{2}(H_{11} + H_{22})|\psi_{\pm}\rangle + \frac{1}{2}(H_{11} - H_{22})\kappa_{\pm}|\psi_{\pm}\rangle \\ &= E_{\pm}|\psi_{\pm}\rangle \\ \Rightarrow E_{\pm} &= \frac{1}{2}(H_{11} + H_{22}) + \frac{1}{2}(H_{11} - H_{22})\kappa_{\pm} \end{aligned}$$

In both of these cases, the eigenvalues are related by a shift by a constant,  $\frac{1}{2}(H_{11} + H_{22})$ , thus "shifting the origin" of the eigenvalues.

3. Start by solving the secular equation to find the eigenvalues of  $H$

$$\begin{aligned}
 \begin{vmatrix} H_{11} - \lambda & H_{12} \\ H_{21} & H_{22} - \lambda \end{vmatrix} &= (H_{11} - \lambda)(H_{22} - \lambda) - |H_{12}|^2 \\
 &= H_{11}H_{22} + \lambda^2 - \lambda H_{22} - \lambda H_{11} - |H_{12}|^2 \\
 &= \lambda^2 - \lambda(H_{11} + H_{22}) + H_{11}H_{22} - |H_{12}|^2 \\
 &= 0 \\
 \Rightarrow \lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - |H_{12}|^2)} \\
 &= \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{H_{11}^2 + H_{22}^2 + 2H_{11}H_{22} - 4H_{11}H_{22} + 4|H_{12}|^2} \\
 &= \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{H_{11}^2 + H_{22}^2 - 2H_{11}H_{22} + 4|H_{12}|^2} \\
 &= \frac{1}{2}(H_{11} + H_{22}) \pm \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}
 \end{aligned}$$

Now I'll do the same for the eigenvalues of  $K$ . Recall from part 1 that  $K_{11} = 1$ ,  $K_{22} = -1$

$$\begin{aligned}
 \begin{vmatrix} 1 - \lambda & K_{12} \\ K_{21} & -(1 + \lambda) \end{vmatrix} &= -(1 + \lambda)(1 - \lambda) - |K_{12}|^2 \\
 &= \lambda^2 - 1 - |K_{12}|^2 \\
 &= 0 \\
 \Rightarrow \lambda &= \pm \sqrt{1 + |K_{12}|^2}
 \end{aligned}$$

Recall from part 1:  $K_{12} = \frac{2}{H_{11} - H_{22}} H_{12}$

$$\begin{aligned}
 \lambda &= \pm \sqrt{1 + \left| \frac{2}{H_{11} - H_{22}} H_{12} \right|^2} \\
 &= \pm \frac{1}{H_{11} - H_{22}} \sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}
 \end{aligned}$$

4. We have:

$$\begin{aligned}
 \tan \theta &= \frac{2|H_{21}|}{H_{11} - H_{22}}, \quad H_{21} = |H_{21}|e^{i\phi} \\
 \Rightarrow e^{i\phi} \tan \theta &= \frac{2H_{21}}{H_{11} - H_{22}} \\
 K_{21} &= e^{i\phi} \tan \theta \\
 K_{12} &= e^{-i\phi} \tan \theta
 \end{aligned}$$

We already established that  $K_{11} = 1$  and  $K_{22} = -1$ , so we have:

$$K = \begin{pmatrix} 1 & e^{-i\phi} \tan \theta \\ e^{i\phi} \tan \theta & -1 \end{pmatrix}$$

Then the eigenvalue can be expressed using the equation in the last part of part 3:

$$\begin{aligned}
 \kappa_{\pm} &= \pm \sqrt{1 + \left| \frac{2}{H_{11} - H_{22}} H_{12} \right|^2} \\
 &= \pm \sqrt{1 + |e^{i\phi} \tan \theta|^2} \\
 &= \pm \sqrt{1 + \tan^2 \theta} \\
 &= \pm \sec \theta
 \end{aligned}$$

5. Taking the eigenvalues of  $H$  from part 3:

$$\begin{aligned}
 E_+ + E_- &= \frac{1}{2}(H_{11} + H_{22}) + \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} + \frac{1}{2}(H_{11} + H_{22}) - \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} \\
 &= \frac{1}{2}(H_{11} + H_{22}) + \frac{1}{2}(H_{11} + H_{22}) \\
 &= H_{11} + H_{22} \\
 &= \text{Tr}\{H\}
 \end{aligned}$$

Now to calculate  $E_+ E_-$ . The form of this will be  $(a + b)(a - b) = a^2 - b^2$ :

$$\begin{aligned}
 E_+ E_- &= \left( \frac{1}{2}(H_{11} + H_{22}) \right)^2 - \left( \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} \right)^2 \\
 &= \frac{1}{4}(H_{11}^2 + H_{22}^2 + 2H_{11}H_{22}) - \frac{1}{4}(H_{11} - H_{22})^2 - |H_{12}|^2 \\
 &= \frac{1}{4}(H_{11}^2 + H_{22}^2 + 2H_{11}H_{22}) - \frac{1}{4}(H_{11}^2 + H_{22}^2 - 2H_{11}H_{22}) - |H_{12}|^2 \\
 &= \frac{1}{4}2H_{11}H_{22} - \frac{1}{4}(-2H_{11}H_{22}) - |H_{12}|^2 \\
 &= H_{11}H_{22} - |H_{12}|^2 \\
 &= \det H
 \end{aligned}$$

6. Since  $H$  is only  $2 \times 2$ , there are only two eigenvalues, so for it to be degenerate means that the eigenvalues are the same.  $\Rightarrow E_+ = E_-$

$$\begin{aligned}
 \Rightarrow \frac{1}{2}(H_{11} + H_{22}) + \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} &= \frac{1}{2}(H_{11} + H_{22}) - \frac{1}{2}\sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2} \\
 (H_{11} - H_{22})^2 + 4|H_{12}|^2 &= 0
 \end{aligned}$$

For this to be true,  $|H_{12}| = 0$ . This is because otherwise  $H_{11}$  and/or  $H_{22}$  would have to be complex, and it was specified in the problem that they are real. Then if  $|H_{12}| = 0$ ,  $H_{11} = H_{22}$ . This means that  $H$  must be proportional to the identity matrix.

7. The eigenvalues of  $K$  are  $\kappa_{\pm} = \pm \sec \theta$ , and  $K$  is defined as:

$$K = \begin{pmatrix} 1 & e^{-i\phi} \tan \theta \\ e^{i\phi} \tan \theta & -1 \end{pmatrix}$$

To find the eigenvectors, starting with  $\kappa_+$

$$\begin{aligned}
 \begin{pmatrix} 1 - \sec \theta & e^{-i\phi} \tan \theta \\ e^{i\phi} \tan \theta & -(1 + \sec \theta) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 \Rightarrow c_2 &= c_1 \frac{\sec \theta - 1}{\tan \theta} e^{i\phi} = c_1 \tan \frac{\theta}{2} e^{i\phi} \\
 v_+ &= \begin{pmatrix} 1 \\ \tan \frac{\theta}{2} e^{i\phi} \end{pmatrix}
 \end{aligned}$$

Now to normalize it:

$$A_+^2(1 + \tan^2 \frac{\theta}{2}) = 1$$

$$A_+ = \sqrt{\frac{1}{1 + \tan^2 \frac{\theta}{2}}} = \sqrt{\frac{1}{\sec^2 \frac{\theta}{2}}} = \cos \frac{\theta}{2}$$

For the eigenvalue  $\kappa_+$ , we have:

$$|\psi_+\rangle = \cos \frac{\theta}{2} \begin{pmatrix} 1 \\ \tan \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

Now for  $\kappa_-$

$$\begin{pmatrix} \sec \theta + 1 & e^{-i\phi} \tan \theta \\ e^{i\phi} \tan \theta & \sec \theta - 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow c_2 = c_1 \frac{\tan \theta}{1 - \sec \theta} e^{i\phi} = -c_1 \cot \frac{\theta}{2} e^{i\phi}$$

$$v_- = \begin{pmatrix} 1 \\ -\cot \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

Now to normalize it:

$$A_-^2(1 + \cot^2 \frac{\theta}{2}) = 1$$

$$A_- = \sqrt{\frac{1}{1 + \cot^2 \frac{\theta}{2}}} = \sqrt{\frac{1}{\csc^2 \frac{\theta}{2}}} = \sin \frac{\theta}{2}$$

For the eigenvalue  $\kappa_-$ , we have:

$$|\psi_-\rangle = \sin \frac{\theta}{2} \begin{pmatrix} 1 \\ -\cot \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

8. The operator  $K(\theta, \phi)$  is defined in the matrix representation as:

$$K = \begin{pmatrix} 1 & e^{-i\phi} \tan \theta \\ e^{i\phi} \tan \theta & -1 \end{pmatrix}$$

If we let  $\theta = \phi = 0$ , we have:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is equal to the Pauli matrix,  $\sigma_z$ . The eigenvalues and eigenvectors can be found by solving the charac-

teristic equation:

$$\begin{vmatrix} 1 - \lambda & 0 \\ 0 & -(1 + \lambda) \end{vmatrix} = 0$$

$$\lambda = \pm 1$$

$$\lambda = 1$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9. Using the definitions from part 4, we can write:

$$\begin{aligned} H_{11} - H_{22} &= \frac{2|H_{12}|}{\tan \theta} \\ \tilde{K} &= \frac{1}{2}(H_{11} - H_{22})K = \begin{pmatrix} \frac{|H_{12}|}{\tan \theta} & e^{-i\phi} \tan \theta \frac{|H_{12}|}{\tan \theta} \\ e^{i\phi} \tan \theta \frac{|H_{12}|}{\tan \theta} & -\frac{|H_{12}|}{\tan \theta} \end{pmatrix} \\ &= \begin{pmatrix} \frac{|H_{12}|}{\tan \theta} & |H_{12}|e^{-i\phi} \\ |H_{12}|e^{i\phi} & -\frac{|H_{12}|}{\tan \theta} \end{pmatrix} \end{aligned}$$

For  $\theta = \pi/2$  and  $\phi = 0$ , we have:

$$\tilde{K}_x = |H_{12}| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \propto \sigma_x$$



Now we use the usual method to find eigenvalues and eigenvectors:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\lambda = 1$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = c_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -c_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

10.

$$\tilde{K} = \begin{pmatrix} \frac{|H_{12}|}{\tan \theta} & |H_{12}|e^{-i\phi} \\ |H_{12}|e^{i\phi} & -\frac{|H_{12}|}{\tan \theta} \end{pmatrix}$$

$$\tilde{K}(\pi/2, \pi/2) = \begin{pmatrix} 0 & |H_{12}|e^{-i\pi/2} \\ |H_{12}|e^{i\pi/2} & 0 \end{pmatrix}$$

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$e^{-i\pi/2} = \cos(\pi/2) - i\sin(\pi/2) = -i$$

$$\Rightarrow \tilde{K}(\pi/2, \pi/2) = |H_{12}| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \propto \sigma_y$$

Now we use the usual method to find eigenvalues and eigenvectors:

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 + i^2 = 0$$

$$\lambda = \pm 1$$

$$\lambda = 1$$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = ic_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda = -1$$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -ic_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

11.

$$[\tilde{K}_x, \tilde{K}_y] = \tilde{K}_x \tilde{K}_y - \tilde{K}_y \tilde{K}_x$$

$$\tilde{K}_x \tilde{K}_y = |H_{12}|^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = |H_{12}|^2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\tilde{K}_y \tilde{K}_x = |H_{12}|^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |H_{12}|^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\tilde{K}_x, \tilde{K}_y] = |H_{12}|^2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - |H_{12}|^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= |H_{12}|^2 \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$= 2i |H_{12}|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \propto \sigma_z$$

### Problem 3

The spin operator,  $\mathbf{S}$  of an electron is pointing in any direction and is related to the Pauli matrices as  $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$  in the orthonormal basis  $\{|+\rangle, |-\rangle\}$  for  $S_z$ .

1. Write down the matrix for  $S_x, S_y, S_z, S_u$ . Are they Hermitian?
2. Determine the eigenvalues of each component for the spin operator.
3. Determine the eigenvectors of each component for the spin operator.
4. Show that  $[S_x, S_y] = i\hbar S_z, [S_y, S_z] = i\hbar S_x, [S_z, S_x] = i\hbar S_y$
5. Show that  $[\mathbf{S}^2, \mathbf{S}] = 0$

### Solution

1. For  $S_x, S_y, S_z$ , we simply plug in the Pauli matrices.

$$\begin{aligned} S_x &= \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ S_y &= \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ S_z &= \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

To find  $S_u$  we use the definition:  $S_u = \mathbf{S} \cdot \mathbf{u}$ , where  $\mathbf{u}$  is the unit vector in 3-dimensional space.

$$\begin{aligned} S_u &= \mathbf{S} \cdot \mathbf{u} = S_x u_x + S_y u_y + S_z u_z \\ &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin \theta \cos \phi + \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin \theta \sin \phi + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos \theta \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta \cos \phi - i \sin \theta \sin \phi \\ \sin \theta \cos \phi + i \sin \theta \sin \phi & \cos \theta \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta (\cos \phi - i \sin \phi) \\ \sin \theta (\cos \phi + i \sin \phi) & \cos \theta \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & \cos \theta \end{pmatrix} \end{aligned}$$

Each of these matrices is symmetric except for the two cases with complex parts,  $S_y, S_u$ , but in both of those matrices,  $S_{21} = S_{12}^*$ , so these are all Hermitian.

2. We find the eigenvalues in the usual way by solving the characteristic equation.

For  $S_x$ :

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm \hbar/2$$

For  $S_y$ :

$$\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = \lambda^2 + i^2 \frac{\hbar^2}{4} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm \hbar/2$$

For  $S_z$ :

$$\begin{vmatrix} \hbar/2 - \lambda & 0 \\ 0 & \hbar/2 - \lambda \end{vmatrix} = -\left(\frac{\hbar}{2} + \lambda\right)\left(\frac{\hbar}{2} - \lambda\right) = 0 \Rightarrow \lambda = \pm \hbar/2$$

So each of the spin operators has the same eigenvalues.

3. For each of the spin operators, plug the eigenvalues calculated in the last part into the eigenvalue equation and calculate the eigenvectors.

For  $S_x, \lambda = +1$  (leaving off the factors of  $\hbar/2$  since they don't matter here):

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = c_1$$

$$\vec{v}_{\lambda=+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For  $S_x, \lambda = -1$ :

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -c_1$$

$$\vec{v}_{\lambda=-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For  $S_y, \lambda = +1$ :

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = ic_1$$

$$\vec{v}_{\lambda=+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For  $S_y, \lambda = -1$ :

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -ic_1$$

$$\vec{v}_{\lambda=-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For  $S_z, \lambda = +1$ :

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = 0$$

$$\vec{v}_{\lambda=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For  $S_z, \lambda = -1$ :

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 = 0$$

$$\vec{v}_{\lambda=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. Use the matrix definitions given in part 1 to explicitly calculate the commutators

$$S_x S_y = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$S_y S_x = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\begin{aligned} S_x S_y - S_y S_x &= \frac{\hbar^2}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{\hbar^2}{4} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \frac{\hbar^2}{4} \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = \frac{i\hbar^2}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar \left( \frac{\hbar}{2} \sigma_z \right) \end{aligned}$$

$$\Rightarrow [S_x, S_y] = i\hbar S_z$$

$$S_y S_z = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$S_z S_y = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\begin{aligned} S_y S_z - S_z S_y &= \frac{\hbar^2}{4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\ &= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \frac{i\hbar^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar \left( \frac{\hbar}{2} \sigma_x \right) \end{aligned}$$

$$\Rightarrow [S_y, S_z] = i\hbar S_x$$

$$S_z S_x = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S_x S_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned} S_z S_x - S_x S_z &= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{\hbar^2}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{\hbar^2}{4} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \frac{i\hbar^2}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= i\hbar \left( \frac{\hbar}{2} \sigma_y \right) \end{aligned}$$

$$\Rightarrow [S_z, S_x] = i\hbar S_y$$

5. To start with, calculate  $\mathbf{S}^2$

$$\begin{aligned}\mathbf{S} &= \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \\ \mathbf{S}^2 &= \frac{\hbar^2}{4} \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \\ &= \frac{\hbar^2}{4} [\sigma_x^2 + \sigma_y^2 + \sigma_z^2] \\ &= \frac{\hbar^2}{4} [I + I + I] \\ &= \frac{3\hbar^2}{4} I\end{aligned}$$

For each of the Pauli matrices,  $\sigma_i^2 = I$ , the identity matrix. So now we know:

$$[\mathbf{S}^2, \mathbf{S}] \propto [I, \mathbf{S}] = 0$$

Because the identity matrix commutes with everything.

## Problem 4

Let us consider two Stern-Gerlach experiments where the first one prepares the atoms in a state and the second one measures a spin physical quantity. Here, the state of the  $1/2$  spin system is prepared such that the spin is pointing down along  $\hat{u}$  in the  $x$ - $z$  plane.

1. Immediately before measurement, what is the state  $|\psi\rangle$  of the system in the basis of eigenvectors of  $S_z$ ?

### Solution

1. In the  $S_z$  basis, with the spin known to be in the  $x$ - $z$  plane ( $\phi = 0$ ), we can write:

$$|+\rangle_u = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad |-\rangle_u = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

In general, we can write the state of the system as:

$$|\psi\rangle = \alpha |+\rangle + \beta |-\rangle$$

2. Since we are starting in the state,  $|-\rangle_u$ , the probability of finding it in the  $-\hbar/2$  state is:

$$\mathcal{P} = |\langle - | + \rangle_u|^2 = \cos^2 \frac{\theta}{2}$$

Immediately following this measurement, the state of the system is spin down in  $S_z$ .

3. Since we are starting in the state,  $|-\rangle_u$ , the probability of finding it in the  $-\hbar/2$  state is:

$$\mathcal{P} = |\langle + | + \rangle_u|^2 = \sin^2 \frac{\theta}{2}$$

Immediately following this measurement, the state of the system is spin up in  $S_z$ .

## Problem 5

Consider a spin  $1/2$  particle placed in a magnetic field  $\vec{B} = (B_x \ 0 \ B_z) = \frac{B_0}{\sqrt{2}} (1 \ 0 \ 1)$  in the orthonormal basis  $\{|+\rangle, |-\rangle\}$  for  $S_z$ .

1. Calculate the matrix representing the Hamiltonian  $H$  of the system.
2. Calculate the eigenvalues and eigenvectors of  $H$ .
3. The system at time  $t = 0$  is in the state  $|-\rangle$ . What values can be found if the energy is measured, and with what probabilities?
4. Calculate the state vector  $|\psi(t)\rangle$  at time  $t$ . At this instant,  $S_x$  is measured; what is the mean value of the results that can be obtained? Give a geometrical interpretation of your results.

### Solution

1. The Hamiltonian of a particle of spin  $1/2$  in a magnetic field is:

$$\begin{aligned}
 H &= -\vec{\mu} \cdot \vec{B} \\
 \vec{\mu} &= \gamma \vec{S} \\
 \Rightarrow H &= -\gamma \vec{S} \cdot \vec{B} \\
 &= -\gamma(S_x B_x + S_y B_y + S_z B_z) \\
 &= -\frac{\gamma B_0 \hbar}{2\sqrt{2}} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
 &= -\frac{\gamma B_0 \hbar}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
 \end{aligned}$$

2. By the usual means, calculate eigenvalues then eigenvectors:

$$\begin{aligned}
 \begin{vmatrix} 1-\lambda & 1 \\ 1 & -(1+\lambda) \end{vmatrix} &= -(1+\lambda)(1-\lambda) - 1 = 0 \\
 &= -(1-\lambda^2) - 1 = \lambda^2 - 2 = 0 \\
 \Rightarrow \lambda &= \pm\sqrt{2}
 \end{aligned}$$

So the eigenvalues of  $H$  are:

$$E_{\pm} = \mp \frac{\gamma B_0 \hbar}{2}$$