Physics 916: Homework #5

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Problem 1

Show that in general, any 2×2 matrix M can be represented in terms of the unit matrix, I, and the Pauli matrices. i.e.

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = a_0 I + \vec{a} \cdot \vec{\sigma}$$

where the expansion coefficients $a_i = \frac{1}{2} Tr\{M\sigma_i\}$

Solution

First I will use a common convention and define σ_0 as the identity operator. So we have:

$$\begin{aligned}
M = \vec{a} \cdot \vec{\sigma} =& a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
=& \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} \\
M_{11} =& a_0 + a_3 \\
M_{12} =& a_1 - ia_2 \\
M_{21} =& a_1 + ia_2 \\
M_{22} =& a_0 - a_3
\end{aligned}$$

$$\begin{aligned}
a_0 =& \frac{1}{2} Tr\{M\sigma_0\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{11} + M_{22} \right) = \frac{1}{2} \left(a_0 + a_3 + a_0 - a_3 \right) \\
=& a_0 \\
a_1 =& \frac{1}{2} Tr\{M\sigma_1\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{12} & M_{11} \\ M_{22} & M_{21} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{12} + M_{21} \right) = \frac{1}{2} \left(a_1 - ia_2 + a_1 + ia_2 \right) \\
=& a_1 \\
a_2 =& \frac{1}{2} Tr\{M\sigma_2\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} iM_{12} & -iM_{11} \\ iM_{22} & -iM_{21} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(iM_{12} - iM_{21} \right) = \frac{1}{2} \left(ia_1 - i^2 a_2 - ia_1 - i^2 a_2 \right) \\
=& a_2 \\
a_3 =& \frac{1}{2} Tr\{M\sigma_3\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{11} & -M_{12} \\ M_{21} & -M_{22} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{11} - M_{22} \right) = \frac{1}{2} \left(a_0 + a_3 - a_0 + a_3 \right) \\
=& a_3
\end{aligned}$$

Problem 2

Show that

$$[S_x, S_y] = i\hbar S_z$$

Solution

$$S_x = \frac{\hbar}{2}\sigma_x,\ S_y = \frac{\hbar}{2}\sigma_y,\ S_z = \frac{\hbar}{2}\sigma_z$$

$$\begin{aligned}
S_x S_y =& \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\\
S_y S_x =& \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\
S_x S_y - S_y S_x =& \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\
=& \frac{\hbar^2}{4}\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
=& i\hbar\left(\frac{\hbar}{2}\sigma_z\right)\\
\Rightarrow [S_x, S_y] =& i\hbar S_z
\end{aligned}$$