Physics 916: Homework #5

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Problem 1

Show that in general, any 2×2 matrix M can be represented in terms of the unit matrix, I, and the Pauli matrices. i.e.

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = a_0 I + \vec{a} \cdot \vec{\sigma}$$

where the expansion coefficients $a_i = \frac{1}{2} Tr\{M\sigma_i\}$

Solution

First I will use a common convention and define σ_0 as the identity operator. So we have:

$$\begin{aligned}
M = \vec{a} \cdot \vec{\sigma} =& a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
=& \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} \\
M_{11} =& a_0 + a_3 \\
M_{12} =& a_1 - ia_2 \\
M_{21} =& a_1 + ia_2 \\
M_{22} =& a_0 - a_3
\end{aligned}$$

$$\begin{aligned}
a_0 =& \frac{1}{2} Tr\{M\sigma_0\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{11} + M_{22} \right) = \frac{1}{2} \left(a_0 + a_3 + a_0 - a_3 \right) \\
=& a_0 \\
a_1 =& \frac{1}{2} Tr\{M\sigma_1\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{12} & M_{11} \\ M_{22} & M_{21} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{12} + M_{21} \right) = \frac{1}{2} \left(a_1 - ia_2 + a_1 + ia_2 \right) \\
=& a_1 \\
a_2 =& \frac{1}{2} Tr\{M\sigma_2\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} iM_{12} & -iM_{11} \\ iM_{22} & -iM_{21} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(iM_{12} - iM_{21} \right) = \frac{1}{2} \left(ia_1 - i^2 a_2 - ia_1 - i^2 a_2 \right) \\
=& a_2 \\
a_3 =& \frac{1}{2} Tr\{M\sigma_3\} = \frac{1}{2} Tr \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = Tr \left[\begin{pmatrix} M_{11} & -M_{12} \\ M_{21} & -M_{22} \end{pmatrix} \right] \\
=& \frac{1}{2} \left(M_{11} - M_{22} \right) = \frac{1}{2} \left(a_0 + a_3 - a_0 + a_3 \right) \\
=& a_3
\end{aligned}$$

Problem 2

Consider the quantum operator, H, whose matrix representation in the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ writes:

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where H_{11} and H_{22} are real numbers and $H_{12} = H_{21}^*$. It is thus obvious that H is Hermitian.

1. Show that:
$H = \frac{1}{2}(H_{11} + H_{22})I + \tilde{K} \equiv \frac{1}{2}(H_{11} + H_{22})I + \frac{1}{2}(H_{11} - H_{22})K$ where I is the identity operator, and the operators $\tilde{K}, K$ must be determined in terms of the matrix elements of H. Are $\tilde{K}$ and K Hermitian?

2. A key result from the decomposition in part 1 is that the operators $\tilde{K}, K$, and H all have the same eigenvectors $|\psi_\pm\rangle$. Let $\tilde{\kappa}_\pm, \kappa_\pm, E_\pm$ be the eigenvalues of $\tilde{K}, K$, and H. Use the result of part 1 to establish the relation between $E_\pm$ and $\kappa_\pm$, and the relation between $E_\pm$ and $\tilde{\kappa}_\pm$. Show that these relations allow for a change of the eigenvalue origin.

3. Directly solve the secular equations for K and H and determine the corresponding eigenvalues. Check that the relation between $E_\pm$ and $\kappa_\pm$ established in part 2 is correct.

4. Let us define angles $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ defined as:
$\tan\theta = \frac{2|H_{21}|}{H_{11} - H_{22}}$ and $H_{21} = |H_{21}|e^{i\phi}$

5. Show that $E_+ + E_- = Tr\{H\}$, and that $E_+ E_- = \det H$

6. Show that if H has a degenerate spectrum, then it is necessarily proportional to the identity operator.

7. Use the operator, $K(\theta, \phi)$ to calculate normalized eigenvectors $|\psi_\pm\rangle$ in terms of these angles in the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$. You must find that the eigenvectors $|\psi_\pm\rangle$ are collinear to the eigenvectors $|\pm\rangle$ of the 1/2 spin operator S_u, where u is an arbitrary unit vector defined by these angles.

8. Show that $K(\theta = 0, \phi = 0)$ is proportional to the z-component of the Pauli operator, σ_z. What are the corresponding eigenvalues and eigenvectors?

9. When $\theta = \pi/2$, the operator K is not finite and we must use $\tilde{K}$. Show that $\tilde{K}_x \equiv \tilde{K}(\theta = \pi/2, \phi = 0)$ is proportional to the x-component of the Pauli operator, σ_x. What are the corresponding eigenvalues and eigenvectors?

10. Show that $\tilde{K}_y \equiv \tilde{K}(\theta = \pi/2, \phi = \pi/2)$ is proportional to the y-component of the Pauli operator, σ_y. What are the corresponding eigenvalues and eigenvectors?

11. Calculate the commutator, $[\tilde{K}_x, \tilde{K}_y]$, and show that it is proportional to the z-component of the pauli operator, σ_z.

Solution

Since they will be used several times later on in this problem, here are the definitions of the 2×2 Pauli

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operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. If we let the following be true we will have the relation suggested:

$$\begin{aligned} H_{11} =& \frac{1}{2}[H_{11}(1+K_{11}) + H_{22}(1-K_{11})] \\ H_{22} =& \frac{1}{2}[H_{11}(1+K_{22}) + H_{22}(1-K_{22})] \\ H_{12} =& \frac{1}{2}K_{12}(H_{11} - H_{22}) \\ H_{21} =& \frac{1}{2}K_{12}(H_{11} - H_{22}) \end{aligned}$$

From here we can determine a couple of things about K and $\tilde{K}$

$$\begin{aligned} & K_{11} = 1,\ K_{22} = -1 \\ K_{12} =& \frac{2}{H_{11} - H_{22}} H_{12} = \frac{2}{H_{11} - H_{22}} H_{21}^{*} = K_{21}^{*} \end{aligned}$$

So we know the value of the diagonal of K and $\tilde{K}$ and we also know that they are Hermitian.

2. Starting with the relation between $E_\pm$ and $\tilde{\kappa}_\pm$

$$\begin{aligned} H\left|\psi_\pm\right\rangle =& \frac{1}{2}(H_{11} + H_{22}) I \left|\psi_\pm\right\rangle + \tilde{K}\left|\psi_\pm\right\rangle \\ =& \frac{1}{2}(H_{11} + H_{22})\left|\psi_\pm\right\rangle + \tilde{\kappa}_\pm\left|\psi_\pm\right\rangle \\ =& E_\pm\left|\psi_\pm\right\rangle \\ \Rightarrow E_\pm =& \frac{1}{2}(H_{11} + H_{22}) + \tilde{\kappa}_\pm \end{aligned}$$

Now for the relation between $E_\pm$ and $\kappa_\pm$

$$\begin{aligned} H\left|\psi_\pm\right\rangle =& \frac{1}{2}(H_{11} + H_{22}) I \left|\psi_\pm\right\rangle + \frac{1}{2}(H_{11} - H_{22}) K \left|\psi_\pm\right\rangle \\ =& \frac{1}{2}(H_{11} + H_{22})\left|\psi_\pm\right\rangle + \frac{1}{2}(H_{11} - H_{22})\kappa_\pm\left|\psi_\pm\right\rangle \\ =& E_\pm\left|\psi_\pm\right\rangle \\ \Rightarrow E_\pm =& \frac{1}{2}(H_{11} + H_{22}) + \frac{1}{2}(H_{11} - H_{22})\kappa_\pm \end{aligned}$$

In both of these cases, the eigenvalues are related by a shift by a constant, $\frac{1}{2}(H_{11} + H_{22})$, thus "shifting the origin" of the eigenvalues.

3. Start by solving the secular equation to find the eigenvalues of H

$$\begin{aligned}
\begin{vmatrix} H_{11}-\lambda & H_{12} \\ H_{21} & H_{22}-\lambda \end{vmatrix} =&(H_{11}-\lambda)(H_{22}-\lambda)-|H_{12}|^2 \\
=&H_{11}H_{22}+\lambda^2-\lambda H_{22}-\lambda H_{11}-|H_{12}|^2 \\
=&\lambda^2-\lambda(H_{11}+H_{22})+H_{11}H_{22}-|H_{12}|^2 \\
=&0 \\
\Rightarrow \lambda =&\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
=&\frac{1}{2}(H_{11}+H_{22})\pm\frac{1}{2}\sqrt{(H_{11}+H_{22})^2-4(H_{11}H_{22}-|H_{12}|^2)} \\
=&\frac{1}{2}(H_{11}+H_{22})\pm\frac{1}{2}\sqrt{H_{11}^2+H_{22}^2+2H_{11}H_{22}-4H_{11}H_{22}+4|H_{12}|^2} \\
=&\frac{1}{2}(H_{11}+H_{22})\pm\frac{1}{2}\sqrt{H_{11}^2+H_{22}^2-2H_{11}H_{22}+4|H_{12}|^2} \\
=&\frac{1}{2}(H_{11}+H_{22})\pm\frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2}
\end{aligned}$$

Now I'll do the same for the eigenvalues of K. Recall from part 1 that $K_{11}=1,\ K_{22}=-1$

$$\begin{aligned}
\begin{vmatrix} 1-\lambda & K_{12} \\ K_{21} & -(1+\lambda) \end{vmatrix} =&-(1+\lambda)(1-\lambda)-|K_{12}|^2 \\
=&\lambda^2-1-|K_{12}|^2 \\
=&0 \\
\Rightarrow \lambda =&\pm\sqrt{1+|K_{12}|^2}
\end{aligned}$$

Recall from part 1: $K_{12}=\frac{2}{H_{11}-H_{22}}H_{12}$

$$\begin{aligned}
\lambda =&\pm\sqrt{1+\left|\frac{2}{H_{11}-H_{22}}H_{12}\right|^2} \\
=&\pm\frac{1}{H_{11}-H_{22}}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2}
\end{aligned}$$

4. We have:

$$\begin{aligned}
\tan\theta =&\frac{2|H_{21}|}{H_{11}-H_{22}},\ H_{21}=|H_{21}|e^{i\phi} \\
\Rightarrow e^{i\phi}\tan\theta =&\frac{2H_{21}}{H_{11}-H_{22}} \\
K_{21} =&e^{i\phi}\tan\theta \\
K_{12} =&e^{-i\phi}\tan\theta
\end{aligned}$$

We already established that $K_{11}=1$ and $K_{22}=-1$, so we have:

$$K=\begin{pmatrix} 1 & e^{-i\phi}\tan\theta \\ e^{i\phi}\tan\theta & -1 \end{pmatrix}$$

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Then the eigenvalue can be expressed using the equation in the last part of part 3:

$$\begin{aligned}
\kappa_\pm &= \pm\sqrt{1+\left|\frac{2}{H_{11}-H_{22}}H_{12}\right|^2} \\
&= \pm\sqrt{1+\left|e^{i\phi}\tan\theta\right|^2} \\
&= \pm\sqrt{1+\tan^2\theta} \\
&= \pm\sec\theta
\end{aligned}$$

5. Taking the eigenvalues of H from part 3:

$$\begin{aligned}
E_+ + E_- =& \frac{1}{2}(H_{11}+H_{22}) + \frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2} + \frac{1}{2}(H_{11}+H_{22}) - \frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2} \\
=& \frac{1}{2}(H_{11}+H_{22}) + \frac{1}{2}(H_{11}+H_{22}) \\
=& H_{11}+H_{22} \\
=& Tr\{H\}
\end{aligned}$$

Now to calculate E_+E_-. The form of this will be $(a+b)(a-b)=a^2-b^2$:

$$\begin{aligned}
E_+E_- =& \left(\frac{1}{2}(H_{11}+H_{22})\right)^2 - \left(\frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2}\right)^2 \\
=& \frac{1}{4}\left(H_{11}^2+H_{22}^2+2H_{11}H_{22}\right) - \frac{1}{4}(H_{11}-H_{22})^2 - |H_{12}|^2 \\
=& \frac{1}{4}\left(H_{11}^2+H_{22}^2+2H_{11}H_{22}\right) - \frac{1}{4}(H_{11}^2+H_{22}^2-2H_{11}H_{22}) - |H_{12}|^2 \\
=& \frac{1}{4}2H_{11}H_{22} - \frac{1}{4}(-2H_{11}H_{22}) - |H_{12}|^2 \\
=& H_{11}H_{22} - |H_{12}|^2 \\
=& \det H
\end{aligned}$$

6. Since H is only 2×2, there are only two eigenvalues, so for it to be degenerate means that the eigenvalues are the same. $\Rightarrow E_+ = E_-$

$$\begin{aligned}
&\Rightarrow \frac{1}{2}(H_{11}+H_{22}) + \frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2} = \frac{1}{2}(H_{11}+H_{22}) - \frac{1}{2}\sqrt{(H_{11}-H_{22})^2+4|H_{12}|^2} \\
&(H_{11}-H_{22})^2+4|H_{12}|^2 = 0
\end{aligned}$$

For this to be true, $|H_{12}|=0$. This is because otherwise H_{11} and/or H_{22} would have to be complex, and it was specified in the problem that they are real. Then if $|H_{12}|=0$, $H_{11}=H_{22}$. This means that H must be proportional to the identity matrix.

7. The eigenvalues of K are $\kappa_\pm = \pm\sec\theta$, and K is defined as:

$$K = \begin{pmatrix} 1 & e^{-i\phi}\tan\theta \\ e^{i\phi}\tan\theta & -1 \end{pmatrix}$$

To find the eigenvectors, starting with κ_+

$$\begin{aligned}
&\begin{pmatrix} 1-\sec\theta & e^{-i\phi}\tan\theta \\ e^{i\phi}\tan\theta & -(1+\sec\theta) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Rightarrow c_2 = c_1\frac{\sec\theta-1}{\tan\theta}e^{i\phi} = c_1\tan\frac{\theta}{2}e^{i\phi} \\
&v_+ = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2}e^{i\phi} \end{pmatrix}
\end{aligned}$$

Now to normalize it:

$$A_+^2(1+\tan^2\frac{\theta}{2}) = 1$$
$$A_+ = \sqrt{\frac{1}{1+\tan^2\frac{\theta}{2}}} = \sqrt{\frac{1}{\sec^2\frac{\theta}{2}}} = \cos\frac{\theta}{2}$$

For the eigenvalue κ_+, we have:

$$|\psi_+\rangle = \cos\frac{\theta}{2}\begin{pmatrix}1\\ \tan\frac{\theta}{2}e^{i\phi}\end{pmatrix}$$

Now for κ_-

$$\begin{pmatrix}\sec\theta+1 & e^{-i\phi}\tan\theta\\ e^{i\phi}\tan\theta & \sec\theta-1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$
$$\Rightarrow c_2 = c_1\frac{\tan\theta}{1-\sec\theta}e^{i\phi} = -c_1\cot\frac{\theta}{2}e^{i\phi}$$
$$v_- = \begin{pmatrix}1\\ -\cot\frac{\theta}{2}e^{i\phi}\end{pmatrix}$$

Now to normalize it:

$$A_-^2(1+\cot^2\frac{\theta}{2}) = 1$$
$$A_- = \sqrt{\frac{1}{1+\cot^2\frac{\theta}{2}}} = \sqrt{\frac{1}{\csc^2\frac{\theta}{2}}} = \sin\frac{\theta}{2}$$

For the eigenvalue κ_+, we have:

$$|\psi_-\rangle = \sin\frac{\theta}{2}\begin{pmatrix}1\\ -\cot\frac{\theta}{2}e^{i\phi}\end{pmatrix}$$

8. The operator $K(\theta,\phi)$ is defined in the matrix representation as:

$$K = \begin{pmatrix}1 & e^{-i\phi}\tan\theta\\ e^{i\phi}\tan\theta & -1\end{pmatrix}$$

If we let $\theta=\phi=0$, we have:

$$K = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

This is equal to the Pauli matrix, σ_z. The eigenvalues and eigenvectors can be found by solving the characteristic equation:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -(1+\lambda) \end{vmatrix} = 0$$

$$\lambda = \pm 1$$

$$\lambda = 1$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -1$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9. Using the definitions from part 4, we can write:

$$H_{11} - H_{22} = \frac{2|H_{12}|}{\tan\theta}$$

$$\tilde{K} = \frac{1}{2}(H_{11} - H_{22})K = \begin{pmatrix} \frac{|H_{12}|}{\tan\theta} & e^{-i\phi}\tan\theta\frac{|H_{12}|}{\tan\theta} \\ e^{i\phi}\tan\theta\frac{|H_{12}|}{\tan\theta} & -\frac{|H_{12}|}{\tan\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{|H_{12}|}{\tan\theta} & |H_{12}|e^{-i\phi} \\ |H_{12}|e^{i\phi} & -\frac{|H_{12}|}{\tan\theta} \end{pmatrix}$$

For $\theta = \pi/2$ and $\phi = 0$, we have:

$$\tilde{K}_x = |H_{12}| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \propto \sigma_x$$

Now we use the usual method to find eigenvalues and eigenvectors:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\lambda = 1$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = c_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -c_1$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

10.

$$\tilde{K} = \begin{pmatrix} \frac{|H_{12}|}{\tan\theta} & |H_{12}|e^{-i\phi} \\ |H_{12}|e^{i\phi} & -\frac{|H_{12}|}{\tan\theta} \end{pmatrix}$$

$$\tilde{K}(\pi/2, \pi/2) = \begin{pmatrix} 0 & |H_{12}|e^{-i\pi/2} \\ |H_{12}|e^{i\pi/2} & 0 \end{pmatrix}$$

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

$$e^{-i\pi/2} = \cos(\pi/2) - i\sin(\pi/2) = -i$$

$$\Rightarrow \tilde{K}(\pi/2, \pi/2) = |H_{12}| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \propto \sigma_y$$

Problem 2 continued on next page. . .

Now we use the usual method to find eigenvalues and eigenvectors:

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 + i^2 = 0$$
$$\lambda = \pm 1$$

$$\lambda = 1$$
$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$c_2 = ic_1$$
$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda = -1$$
$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$c_2 = -ic_1$$
$$\Rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

11.

$$\begin{aligned}
[\tilde{K}_x, \tilde{K}_y] =& \tilde{K}_x \tilde{K}_y - \tilde{K}_y \tilde{K}_x \\
\tilde{K}_x \tilde{K}_y =& |H_{12}|^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = |H_{12}|^2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\tilde{K}_y \tilde{K}_x =& |H_{12}|^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |H_{12}|^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
[\tilde{K}_x, \tilde{K}_y] =& |H_{12}|^2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - |H_{12}|^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
=& |H_{12}|^2 \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \\
=& 2i|H_{12}|^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \propto \sigma_z
\end{aligned}$$

Problem 3

The spin operator, $\boldsymbol{S}$ of an electron is pointing in any direction and is related to the Pauli matrices as $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ in the orthonormal basis $\{|+\rangle, |-\rangle\}$ for S_z.

1. Write down the matrix for $S_x,\ S_y,\ S_z,\ S_u$. Are they Hermitian?
2. Determine the eigenvalues of each component for the spin operator.
3. Determine the eigenvectors of each component for the spin operator.
4. Show that $[S_x, S_y] = i\hbar S_z$, $[S_y, S_z] = i\hbar S_x$, $[S_z, S_x] = i\hbar S_y$
5. Show that $[\boldsymbol{S}^2, \boldsymbol{S}] = 0$

Solution

1. For S_x, S_y, S_z, we simply plug in the Pauli matrices.

$$\begin{aligned}
S_x =& \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
S_y =& \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
S_z =& \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

To find S_u we use the definition: $S_u = \boldsymbol{S}\cdot\boldsymbol{u}$, where $\boldsymbol{u}$ is the unit vector in 3-dimensional space.

$$\begin{aligned}
S_u =& \boldsymbol{S}\cdot\boldsymbol{u} = S_x u_x + S_y u_y + S_z u_z \\
=& \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta \\
=& \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & \cos\theta \end{pmatrix} \\
=& \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & \cos\theta \end{pmatrix} \\
=& \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{pmatrix}
\end{aligned}$$

Each of these matrices is symmetric except for the two cases with complex parts, S_y, S_u, but in both of those matrices, $S_{21} = S_{12}^*$, so these are all Hermitian.

2. We find the eigenvalues in the usual way by solving the characteristic equation.
For S_x:

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm\hbar/2$$

For S_y:

$$\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = \lambda^2 + i^2\frac{\hbar^2}{4} = \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \lambda = \pm\hbar/2$$

For S_x:

$$\begin{vmatrix} \hbar/2 - \lambda & 0 \\ 0 & \hbar/2 - \lambda \end{vmatrix} = -\left(\frac{\hbar}{2} + \lambda\right)\left(\frac{\hbar}{2} - \lambda\right) = 0 \Rightarrow \lambda = \pm\hbar/2$$

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So each of the spin operators has the same eigenvalues.

3. For each of the spin operators, plug the eigenvalues calculated in the last part into the eigenvalue equation and calculate the eigenvectors.
For S_x,$\lambda = +1$ (leaving off the factors of $\hbar/2$ since they don't matter here):

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = c_1$$

$$\vec{v}_{\lambda=+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For S_x,$\lambda = -1$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -c_1$$

$$\vec{v}_{\lambda=-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For S_y,$\lambda = +1$:

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = ic_1$$

$$\vec{v}_{\lambda=+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For S_y,$\lambda = -1$:

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = -ic_1$$

$$\vec{v}_{\lambda=-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For S_z,$\lambda = +1$:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_2 = 0$$

$$\vec{v}_{\lambda=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For S_z,$\lambda = -1$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 = 0$$

$$\vec{v}_{\lambda=+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Problem 3 continued on next page...

4. Use the matrix definitions given in part 1 to explicitly calculate the commutators

$$S_xS_y = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$
$$S_yS_x = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$$

$$\begin{aligned}
S_xS_y - S_yS_x =& \frac{\hbar^2}{4}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\
=& \frac{\hbar^2}{4}\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
=& i\hbar\left(\frac{\hbar}{2}\sigma_z\right)\\
\Rightarrow [S_x, S_y] =& i\hbar S_z
\end{aligned}$$

$$S_yS_z = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$
$$S_zS_y = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$$

$$\begin{aligned}
S_yS_z - S_zS_y =& \frac{\hbar^2}{4}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\\
=& \frac{\hbar^2}{4}\begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
=& i\hbar\left(\frac{\hbar}{2}\sigma_x\right)\\
\Rightarrow [S_y, S_z] =& i\hbar S_x
\end{aligned}$$

$$S_zS_x = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$
$$S_xS_z = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned}
S_zS_x - S_xS_z =& \frac{\hbar^2}{4}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\
=& \frac{\hbar^2}{4}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\\
=& i\hbar\left(\frac{\hbar}{2}\sigma_y\right)\\
\Rightarrow [S_z, S_x] =& i\hbar S_y
\end{aligned}$$

Problem 3 continued on next page. . .

5. To start with, calculate $\boldsymbol{S}^2$

$$\boldsymbol{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{S}^2 =& \frac{\hbar^2}{4} \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \\ =& \frac{\hbar^2}{4} [\sigma_x^2 + \sigma_y^2 + \sigma_z^2] \\ =& \frac{\hbar^2}{4} [I + I + I] \\ =& \frac{3\hbar^2}{4} I \end{aligned}$$

For each of the Pauli matrices, $\sigma_i^2 = I$, the identity matrix. So now we know:

$$\left[\boldsymbol{S}^2, \boldsymbol{S}\right] \propto [I, \boldsymbol{S}] = 0$$

Because the identity matrix commutes with everything.