Physics 926
Homework 9
Due Tuesday, March 31

1. Show that in $\pi \to \mu\nu$ decay, $|\vec{p}_\mu| = |\vec{p}_\nu| = (m_\pi^2 - m_\mu^2)/2m_\pi$. Assume the neutrino is massless.

2. (H&M Exercise 12.13) Predict the ratio of the $K^- \to e^-\bar{\nu}_e$ and $K^- \to \mu^-\bar{\nu}_\mu$ rates. Given that the lifetime of the K^- is $\tau = 1.2 \times 10^{-8}$ s and the $K \to \mu\nu$ branching ratio is 64%, estimate the decay constant f_K. Comment on your assumptions and your result.

3. (Halzen & Martin, Exercise 12.16) If the weak charged current had had a structure $\gamma^\mu(a + b\gamma^5)$, show that for neutrino-electron scattering
$$\frac{d\sigma}{d\Omega} = \frac{G^2 s}{32\pi^2}(A^+ + A^- \cos^4(\theta/2)) \tag{1}$$
for both $\nu_e e$ and $\bar{\nu}_e e$ scattering. If this were the case, then, in contrast to Equation 12.64, we would have $\sigma(\nu_e e) = \sigma(\bar{\nu}_e e)$.

4. (Halzen & Martin, Exercise 12.21 plus a little) Estimate the relative rates for the following decay modes of the $D^0 = c\bar{u}$ meson: $D^0 \to K^-\pi^+$, $\pi^-\pi^+$ and $K^+\pi^-$. Compare your estimates to the values given in the PDG. Note that the last of these decays was first observed in the mid-1990's, well after the textbook was written.

5. (Halzen & Martin, Exercise 12.22) Given that the partial rate $\Gamma(K^+ \to \pi^0 e^+ \nu) = 4 \times 10^6\ \mathrm{s}^{-1}$, calculate the rate for $D^0 \to K^- e^+ \nu$. Hence, estimate the lifetime of the D^0 meson.

6. Because the CKM matrix is unitary, there are constraints among its elements. In particular, the six "dot products" $(\mathrm{row})_i(\mathrm{row})_j^*$ and $(\mathrm{column})_i(\mathrm{column})_j^*$, with $i \neq j$, are all equal to zero. Thus, the six quantities can be expressed graphically in terms of six triangles in the complex plane. These are called unitarity triangles.

   (a) Using the Wolfenstein approximation for the CKM matrix that we used in class,
   $$V_{CKM} = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{2}$$
   which is good up to factors of λ^4, to show that four of the six unitarity triangles are squashed, *i.e.* the length of one of the three sides is much smaller than that of the other two. The other two triangles all have sides of the same order of magnitude. Remember that A, ρ and η are all of order unity and $\lambda \simeq 0.2$.

   (b) Show that to leading order in λ the two remaining triangles are really the same triangle. The sides of this triangle involve V_{td}, V_{cb} and V_{ub}, which all control aspects of decays of particles containing the b quark.

(c) A general representation of the CKM matrix is

$$
\begin{aligned}
V_{CKM} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3) \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (4)
\end{aligned}
$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ and the θ_{ij} are three rotation angles while δ is a phase. (Let's hope I typed that correctly.) Show that in this representation the area of all six triangles is given by

$$\frac{1}{2}|s_{12}s_{13}s_{23}c_{12}c_{13}^2c_{23}\sin\delta|. \quad (5)$$

(Hint – start by finding a general expression for the area of a triangle in the complex plane, which is simplified by dividing the unitarity constraint $A+B+C = 0$ by a phase factor so that one side of the triangle (say A) lies along the x axis.)
Note: This turns out to be one half of the Jarlskog invariant

$$J = |\mathrm{Im}(V_{ij}V_{il}^*V_{kj}^*V_{kl})|,\ i \neq k,\ j \neq l. \quad (6)$$

It can be shown that in the standard model, all CP-violating effects are proportional to J, and thus that the area of the of this triangle must be non-zero to have CP violation. This implies not only the need for a non-zero phase, but the mixing among all the generations must be non-trivial, so that none of the sin and cos factors are zero. We'll address this issue in the neutrino sector in an upcoming homework problem.