Calculus II

Cesar O. Aguilar

4. Parametric Equations and Polar Coordinates

In this chapter, we introduce parametric equations on the plane and polar coordinates.

4.1 Parametric Equations

Consider the following curve C in the plane:

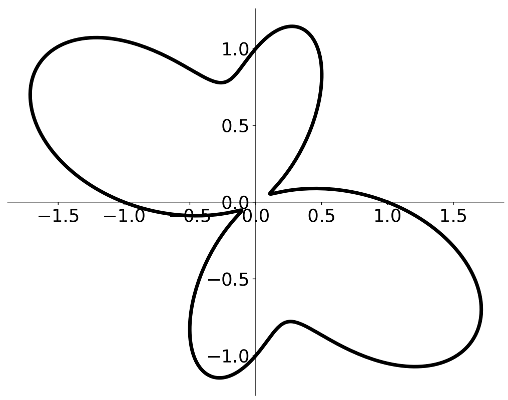

Fig. 4.1: A curve that is not the graph of a function $y = f(x)$

The curve cannot be expressed as the graph of a function $y = f(x)$ because there are points x associated to multiple values of y, that is, the curve does not pass the vertical line test. We may still be interested in describing the points (x, y) on the curve. For example, if the curve is the trajectory of a particle moving on a plane then the position (x, y) of the particle is a function of time t:

$$x = x(t)$$
$$y = y(t)$$

This is an example of a set of **parametric equations** and the variable t is called the **parameter** of the parametrization. In some examples, the parameter could instead be an angle variable θ:

$$x = x(\theta)$$
$$y = y(\theta)$$

The main point is that the points (x, y) can be expressed or depend on a third parameter. Parametric equations also come with a domain for the parameter, usually we denote the domain with $I = [a, b]$, and it could be infinite $I = [a, \infty)$, or $I = (-\infty, \infty)$, etc.

Example 4.1.1
Make a sketch of the curve C parametrized by

$$x = x(t) = t^2$$
$$y = y(t) = t + 1.$$

The domain of the parameter is $-3 \leq t \leq 3$. Eliminate the parameter t to find a Cartesian equation of the curve.

Solution: Partition the interval $I = [-3, 3]$ into $t_0 = -3, t_1 = -2, t_2 = -1, \ldots, t_7 = 3$ and evaluate $(x(t_i), y(t_i))$ and plot points. The resulting curve is given below. The orientation is clockwise.

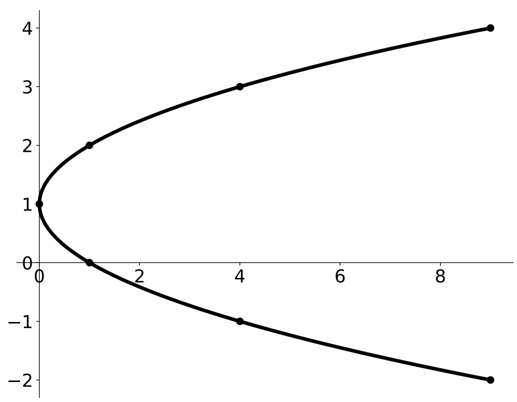

Fig. 4.2: Curve $x(t) = t^2$, $y(t) = t + 1$ for $-3 \leq t \leq 3$

The initial point is $(9, -2)$ and final point is $(9, 4)$. It seems as though the curve is a parabola. To find a Cartesian equation, start with

$$x = t^2$$
$$y = t + 1$$

and from the y-equation we get $t = y - 1$ and thus $x = (y - 1)^2$. ■

Example 4.1.2
Sketch the curve C parametrized by the equations below on the interval $I = [0, \frac{3\pi}{2}]$. Indicate the orientation of the parametrization with arrows. Eliminate the parameter to find a Cartesian equation of the curve.

$$x = x(\theta) = 2\cos(\theta)$$
$$y = y(\theta) = 3\sin(\theta)$$

Solution: Use the estimate $\sqrt{2} \approx 1.4$. Evaluate the parametric equations along convenient θ values:

θ	**0**	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$
$x(\theta)$	2	$2\frac{\sqrt{2}}{2}$	0	$-2\frac{\sqrt{2}}{2}$	-2	$-2\frac{\sqrt{2}}{2}$	0
$y(\theta)$	0	$3\frac{\sqrt{2}}{2}$	3	$3\frac{\sqrt{2}}{2}$	0	$-3\frac{\sqrt{2}}{2}$	-3

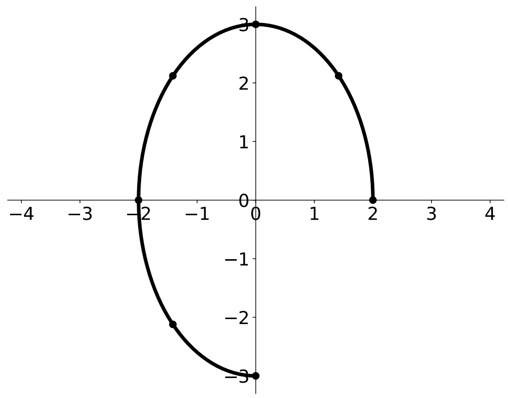

Fig. 4.3: Curve $x(\theta) = 2\cos(\theta)$, $y(\theta) = 3\sin(\theta)$ for $0 \leq \theta \leq 3\pi/2$

The curve seems to be a portion of an ellipse. Recall that an ellipse centered at (x_0, y_0) with horizontal radius a and vertical radius b has Cartesian equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

Thus, we suspect that the Cartesian equation of the curve is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

To eliminate the parameter start with

$$\begin{aligned} x &= 2\cos(\theta) \\ y &= 3\sin(\theta) \end{aligned}$$

and then

$$\begin{aligned} \frac{x}{2} &= \cos(\theta) \\ \frac{y}{3} &= \sin(\theta) \end{aligned}$$

Now $\cos^2(\theta) + \sin^2(\theta) = 1$ and thus

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2(\theta) + \sin^2(\theta)$$

and thus

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

■

In general, a parametrization of a general ellipse

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

is given by

$$\begin{aligned} x = x(t) &= x_0 \pm a\cos(t) \\ y = y(t) &= y_0 \pm b\sin(t) \end{aligned}$$

with interval I depending on how much of the ellipse we want to parametrize. To get a full rotation of the ellipse, we need an interval of length 2π, and if we take $I = [0, 2\pi]$ we start at $(a, 0)$ and get a counter-clockwise (CCW) orientation with a full rotation.

Example 4.1.3
Draw the ellipse and find a parametrization starting at the point $(3, 1)$ with a full rotation with CCW orientation.

$$\frac{(x - 3)^2}{7} + \frac{(y + 2)^2}{9} = 1$$

Example 4.1.4
Sketch the curve parametrized by the equations

$$\begin{aligned} x(t) &= -\cos(t) \\ y(t) &= 2 + \sin(t) \end{aligned}$$

where $-\frac{\pi}{2} \leq t \leq 2\pi$. Indicate the terminal and final point of the parametrization.

Example 4.1.5
Find a parametrization of the ellipse centered at $(-1,3)$, with **clockwise** orientation, starting at $(-1,2)$ and passing through the point $(-7,3)$, and going around one and a half times (end point is $(-1,4)$).

Solution: First determine a CCW orientation and then change the signs accordingly. The ellipse is:

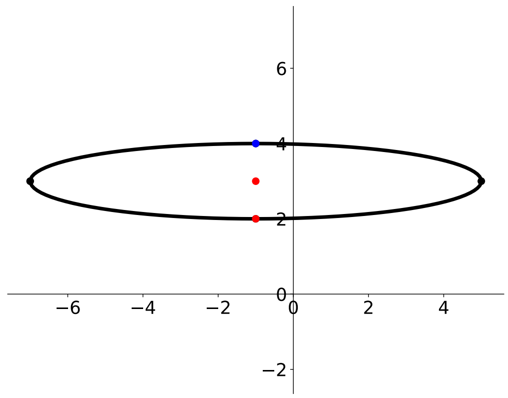

Fig. 4.4: Ellipse centered at $(-1,3)$ of radii $a=6$ and $b=1$

The general equations of the parametrization of an ellipse in this case is

$$x(t) = -1 + 6\cos(t)$$
$$y(t) = 3 + 1\sin(t)$$

We need the interval to be $I = [-\frac{\pi}{2}, \frac{5\pi}{2}]$. This gives a CCW orientation. To change the orientation, we can change the sign in front of the $\cos(t)$ term:

$$x(t) = -1 - 6\cos(t)$$
$$y(t) = 3 + 1\sin(t)$$

■

A familiar type of curve is the graph of a function:

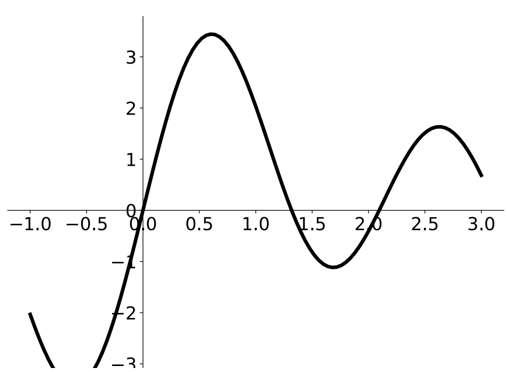

Fig. 4.5: A curve C is the graph of a function f if whenever (x, y) is a point on C then $y = f(x)$

Every point (x, y) on the curve has $y = f(x)$. Therefore, a parametrization is

$$\begin{aligned} x &= x(t) = t \\ y &= y(t) = f(t) \end{aligned}$$

with $a \leq t \leq b$.

Example 4.1.6
Parameterize the graph of the function $y = 2x^3$ where $-1 \leq x \leq 2$ with left-to-right orientation. Then find a right-to-left orientation.

Solution: A parametrization is

$$\begin{aligned} x &= x(t) = t \\ y &= y(t) = 2t^3 \end{aligned}$$

with $-1 \leq t \leq 2$. We can find a right-to-left parametrization by changing all t's to $-t$ and changing the interval to $-b \leq t \leq -a$. In this case, a right-to-left parametrization is then

$$\begin{aligned} x &= x(t) = -t \\ y &= y(t) = 2(-t)^3 = -2t^3 \end{aligned}$$

with interval $-2 \leq t \leq 1$. ■

Example 4.1.7
Parameterize the line segment through the points $(-1, 2)$ and $(\frac{2}{7}, 11)$.

Solution: The slope of the line is

$$m = \frac{11 - 2}{\frac{2}{7} + 1} = \frac{9}{9/7} = 7$$

The equation of the line is then $y = 7x + 9$. A parametrization of the entire line is

$$x = t$$
$$y = 7t + 9$$

and since we only want the line segment where $-1 \leq x \leq \frac{2}{7}$ then the interval is $I = [-1, \frac{2}{7}]$. ■

Example 4.1.8
For each set of parametric equations, eliminate the parameter to find a Cartesian equation for the curve.

(a) $x(t) = \sqrt{t+1}$, $y(t) = \sqrt{t}$

(b) $x(t) = -4 + 2\cos(t)$, $y(t) = 3 - 3\sin(t)$

(c) $x(t) = t^2$, $y(t) = t^6 - 2t^4$

(d) $x(t) = \sec^2(t) - 1$, $y(t) = \tan(t)$

Recall that the equation of the line tangent to the graph of $y = f(x)$ through the point $(x_0, f(x_0))$ is

$$y = y_0 + m(x - x_0)$$

where $m = f'(x_0) = \frac{dy}{dx}(x_0)$.

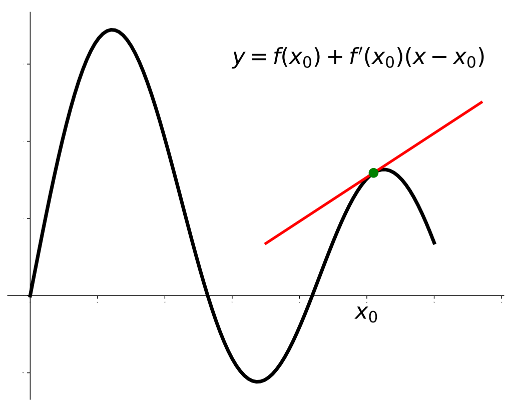

Fig. 4.6: Equation of tangent line at x_0 is $y = f(x_0) + f'(x_0)(x - x_0)$

Example 4.1.9

Find equation of line tangent to the graph of $f(x) = -\sqrt{9\left(1 - \frac{x^2}{4}\right)}$ through the point $(1, -\frac{3\sqrt{3}}{2})$.

Example 4.1.10
Find the equation of the line tangent to the given ellipse and passing through the point $P = (1, -\frac{3\sqrt{3}}{2})$.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Solution: We could solve for y in terms of x:

$$y = -\sqrt{9\left(1 - \frac{x^2}{4}\right)}$$

However, in some cases we may only have a parametric equation for a curve and even if we had a Cartesian equation we may not be able to solve for y (could use implicit differentiation). Use instead a parametrization

$$\begin{aligned} x(t) &= 2\cos(t) \\ y(t) &= 3\sin(t) \\ 0 &\leq t \leq 2\pi \end{aligned}$$

We now need a way to find the slope of the tangent line in terms of the parametric equations. We do know that $y = f(x)$ near P and thus $y(t) = f(x(t))$. Therefore, by the chain rule

$$y'(t) = f'(x(t)) \cdot x'(t)$$

and therefore

$$f'(x(t)) = \frac{y'(t)}{x'(t)}$$

Because $f'(x) = \frac{dy}{dx}$ this is sometimes written as

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

Hence, in this case

$$\frac{dy}{dx} = \frac{3\cos(t)}{-2\sin(t)}$$

At what t value do we evaluate? It has to correspond to the t value where the parametrization passes through the point P. The value t^* where the parametrization passes through the point P occurs when

$$2\cos(t^*) = 1, \qquad 3\sin(t^*) = -3\frac{\sqrt{3}}{2}.$$

From $\cos(t^*) = 1/2$ either $t^* = \pi/3$ or $t^* = 5\pi/3$. In this case, need to take $t^* = 5\pi/3$. Hence, we obtain

$$m = \frac{3\cos(t)}{-2\sin(t)}\Big|_{t=t^*} = \frac{3(1/2)}{-2(-\sqrt{3}/2)} = \sqrt{3}/2$$

Therefore, equation of line is

$$y = y_0 + m(x - x_0) = -\frac{3\sqrt{3}}{2} + \sqrt{3}/2(x-1)$$

which simplifies to

$$y = \sqrt{3}/2x - 2\sqrt{3}.$$

■

Recall that a line is horizontal when its slope is zero and a vertical line could be thought of as a line with infinite slope. Since

$$m = \frac{y'(t)}{x'(t)}$$

the tangent line is horizontal at a t value when $y'(t) = 0$ and is vertical when $x'(t) = 0$.

Example 4.1.11
Consider the parametrized curve

$$\begin{aligned} x(t) &= t^2 \\ y(t) &= t^3 - 3t + 3 \\ t &\in (-\infty, \infty) \end{aligned}$$

(a) Find the equation of the line tangent to the curve at the point $(4, 1)$.

(b) Find the points on the curve where the tangent line is horizontal.

(c) Find the points on the curve where the tangent line is vertical.

(d) Find a Cartesian equation for the curve.

Solution: The curve is shown in Figure 4.7.

1. We compute

$$m = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 3}{2t}$$

The curve passes through the point $P = (4, 1)$ when $t^2 = 4$, so $t = \pm 2$ but need $y(t) = t^3 - 3t = 1$. At $t = 2$ get $y(2) = 8 - 6 + 3 = 5$ but at $y(-2) = -2 + 3 = 1$. Hence, t value is $t^* = -2$. Hence,

$$m = \frac{3(-2)^2 - 3}{2(-2)} = -\frac{9}{4}.$$

Equation of line is

$$y = -\frac{9}{4}(x - 4) + 1 = -\frac{9}{4}x + 10$$

2. Tangent line is horizontal when $y'(t) = 0$ which occurs at t values $t = \pm 1$. The points are therefore $(x(1), y(1)) = (1, 1)$ and $(x(-1), y(-1)) = (1, 5)$.
3. Tangent line is vertical when $x'(t) = 0$ which occurs at $t = 0$. The point is $(x(0), y(0)) = (0, 0)$.
4. To find a Cartesian equation notice that

$$(y - 3)^2 = t^6 - 6t^4 + 9t^2 = (t^2)^3 - 6(t^2)^2 + 9t^2$$

and thus $(y - 3)^2 = x^3 - 6x^2 + x$ is a Cartesian equation.

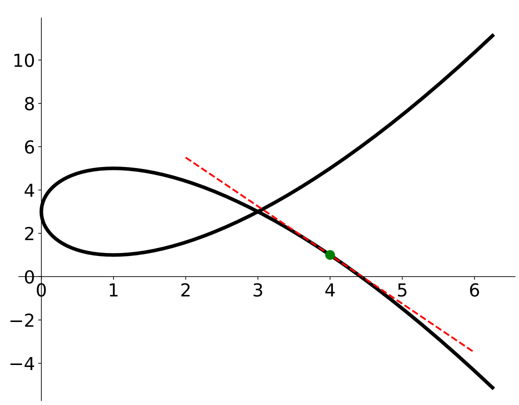

Fig. 4.7: Curve $x(t) = t^2$, $y(t) = t^3 - 3t + 3$

■

The **arc length** of a curve C is the length of the curve. For example, the arc length of a circle of radius r is $L = 2\pi r$. In general, given a parametrization $x = x(t)$, $y = y(t)$, on the interval $I = [a, b]$ of a curve C, the arc length can be computed as

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 4.1.12

Find the arc length of the given parametric curve.

(a) $x(t) = t^3$, $y(t) = \frac{3}{2}t^2$, $0 \le t \le \sqrt{3}$

$$\begin{aligned} L &= \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}\, dt \\ &= \int_0^{\sqrt{3}} \sqrt{9t^4 + 9t^2}\, dt \\ &= \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1}\, dt \\ &= 7 \text{ (by substitution)} \end{aligned}$$

(b) Of the graph of $y = f(x) = x^{3/2}$ for $0 \le x \le 4$. A parametrization is $x(t) = t$ and $y(t) = t^{3/2}$. Then

$$\begin{aligned} L &= \int_0^4 \sqrt{(1)^2 + (3/2t^{1/2})^2}\, dt \\ &= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dt \\ &= \frac{8}{27}\left[(1 + 27/4)^{3/2} - 1\right] \text{ (by substitution)} \end{aligned}$$

(c) Show that the arc length of a circle of radius r is $2\pi r$.

4.2 Polar Coordinates

Polar coordinates is a coordinate system to represent points in 2D space; it is an alternative to the Cartesian coordinate system. In some problems, it is more natural to use polar coordinates than Cartesian coordinates.

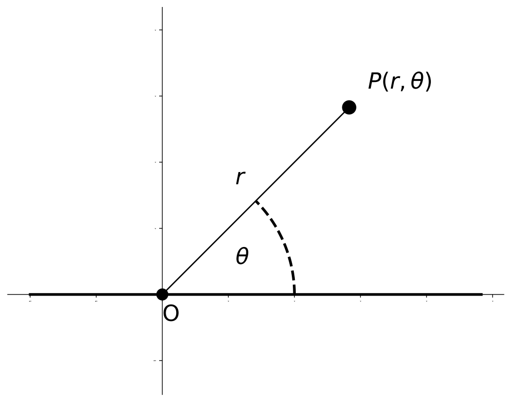

Fig. 4.8: A point P with polar coordinates (r, θ)

- r - distance from origin (directed) and can be negative
- θ - angle measured from a chosen horizontal axis and increasing CCW (can be negative)

Polar coordinates rely on the idea that once an origin is fixed, every point in the 2D plane lies on some circle. It is convention to list polar coordinates with first r and then θ like (r, θ), e.g., the polar coordinates $(4.3, 1)$ means $r = 4.3$ and $\theta = 1$.

Example 4.2.1
Draw the points with given polar coordinates.

(a) $(r, \theta) = (5, 4\pi/3)$

(b) $(r, \theta) = (-2, 5\pi/6)$

(c) $(r, \theta) = (2, -\pi/6)$

(d) $(r, \theta) = (-3, 3\pi/2)$

Given the polar coordinates (r, θ) of a point P, its Cartesian coordinates (x, y) are

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

On the other hand, given the Cartesian coordinates (x, y) of a point P then a set of polar coordinates (r, θ) of P are

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \pm \{\pi, 2\pi\}$$

May need to add π or 2π if want $0 \leq \theta < 2\pi$ since the range of $\arctan$ is $(-\pi/2, \pi/2)$. Notice we take $r > 0$ and may need $r < 0$ if we ask that $-\pi/2 \leq \theta < \pi/2$.

Example 4.2.2
Find the Cartesian coordinates of the points with given polar coordinates.

(a) $(r, \theta) = (5, 4\pi/3)$

(b) $(r, \theta) = (-2, 5\pi/6)$

Example 4.2.3
Find the polar coordinates of the points with given Cartesian coordinates. (Note: $\arctan(3) \approx 1.25$)

(a) $P_1(1, 3)$

(b) $P_2(1, -3)$

(c) $P_3(-1, 3)$

(d) $P_4(-1, -3)$

Unless specified otherwise, in this course, we will use the following convention:

$$r \geq 0 \text{ and } 0 \leq \theta < 2\pi$$

In some problems, it is easier to work with mathematical objects expressed in polar coordinates.

Example 4.2.4
Convert the given equations from Cartesian coordinates to polar coordinates.

(a) $y = x^2$

(b) $x^2 + y^2 = 121$

(c) $x^2 + (y-3)^2 = 9$

Solution:

1. $r\sin(\theta) = r^2\cos^2(\theta)$ which can be factored as

$$r(r\cos^2(\theta) - \sin(\theta)) = 0$$

   Now $r = 0$ represents only the origin. The other points on $y = x^2$ therefore satisfy $r\cos^2(\theta) - \sin(\theta) = 0$ which can be written as

$$r = \tan(\theta)\sec(\theta), \theta \neq \frac{\pi}{2}$$

2. Since $x^2 + y^2 = r^2$ then $r^2 = 7$ or $r = \sqrt{7}$.
3. Expanding gives $x^2 + y^2 - 6y + 9 = 9$ and then $r^2 - 6r\sin(\theta) = 0$ or $r(r - 6\sin(\theta)) = 0$. Use $r = 6\sin(\theta)$ because when $\theta = \pi$ get also $r = 0$.

■

Conversely, we may want to convert an equation from polar coordinates to Cartesian coordinates.

Example 4.2.5
Convert the given equations from polar coordinates to Cartesian coordinates and identify the curve.

(a) $r\cos(\theta) - 4 = 0$

(b) $r^2 = 4r\cos(\theta)$

(c) $r = \frac{4}{2\cos(\theta) - \sin(\theta)}$

(d) $r = 2\cos(\theta) - \sin(\theta)$

Solution:

1. $x - 4 = 0$ or $x = 4$ (vertical line)
2. $x^2 + y^2 = 4x$ and completing square gives $(x-2)^2 + y^2 = 4$ (circle at $(2, 0)$ of radius 4)
3. Write as $2r\cos(\theta) - r\sin(\theta) = 4$ and then $2x - y = 4$ or $y = 2x - 4$ (line)

4. Get $r = 2\frac{x}{r} - \frac{y}{r}$ and thus $r^2 = 2x - y$ or $x^2 - 2x + y^2 + y = 0$ and complete square to get $(x-1)^2 + (y + \frac{1}{2})^2 = \frac{5}{4}$.

■

Regions and curves in polar coordinates

Example 4.2.6
Sketch the region on the 2D plane with given polar coordinates description.

(a) $1 \leq r \leq 3$ and $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

(b) $-\infty < r < \infty$ and $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$

Example 4.2.7
Sketch the curve in the 2D plane with polar coordinates description $r = 1 + \sin(\theta)$.
(Note: $\sqrt{2}/2 \approx 0.7$)

Solution: Create a table of (r, θ) coordinates by varying θ at step-size $\frac{\pi}{4}$:

θ	**0**	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$r(\theta)$	1	$1 + \frac{\sqrt{2}}{2}$	2	$1 + \frac{\sqrt{2}}{2}$	1	$1 - \frac{\sqrt{2}}{2}$	0	$1 - \frac{\sqrt{2}}{2}$

Table 4.2: (r, θ) pairs for $r = 1 + \sin(\theta)$

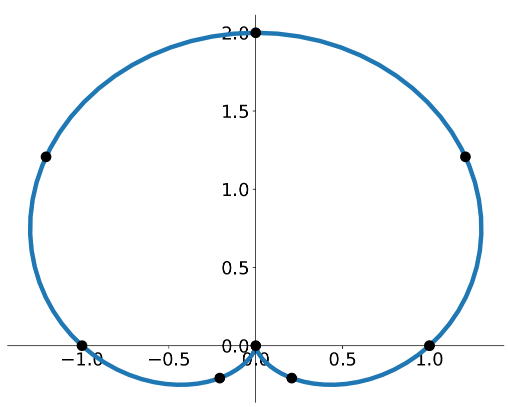

Fig. 4.9: Cardioid

This curve is called a **cardioid**. Since $r(\theta) = 1 + \sin(\theta)$, we can obtain a parametrization for this curve as follows:

$$\begin{aligned} x(\theta) &= r(\theta)\cos(\theta) = (1+\sin(\theta))\cos(\theta) \\ y(\theta) &= r(\theta)\sin(\theta) = (1+\sin(\theta))\sin(\theta) \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

In general, if a curve is expressed in polar coordinates as $r = r(\theta)$ for $a \leq \theta \leq b$ then a parametrization for the curve is:

$$\begin{aligned} x(\theta) &= r(\theta)\cos(\theta) \\ y(\theta) &= r(\theta)\sin(\theta) \\ a &\leq \theta \leq b \end{aligned}$$

■

Example 4.2.8
Find the equation of the line tangent to the cardioid through the point when $\theta = \frac{\pi}{4}$.

Solution: Recall that $m = \frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ if given a parametrization $x = x(t)$ and $y = y(t)$. Here we have

$$\begin{aligned} x(\theta) &= (1+\sin(\theta))\cos(\theta) \\ y(\theta) &= (1+\sin(\theta))\sin(\theta) \end{aligned}$$

We get

$$m = \frac{\cos(\theta)\sin(\theta) + (1+\sin(\theta))\cos(\theta)}{\cos^2(\theta) - (1+\sin(\theta))\sin(\theta)}$$

and evaluating at $\theta = \pi/4$ we obtain $m = -(1+\sqrt{2})$. The point on the curve at $\theta = \pi/4$ is $(x_0, y_0) = (1+\sqrt{2}/2)\sqrt{2}/2, (1+\sqrt{2}/2)\sqrt{2}/2)$ and thus the equation of the line is $y = y_0 + m(x - x_0)$. ■

Arc length in polar coordinates

Recall that the arc length of a curve parametrized by $x = x(t)$, $y = y(t)$, for $a \leq t \leq b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Given a curve in polar coordinates $r = r(\theta)$, for say $\theta_0 \leq \theta \leq \theta_1$, we have a parametrization

$$x(\theta) = r(\theta)\cos(\theta)$$
$$y(\theta) = r(\theta)\sin(\theta)$$

Then the arc length of the curve $r = r(\theta)$ is

$$L = \int_{\theta_0}^{\theta_1} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

This simplifies to

$$L = \int_{\theta_0}^{\theta_1} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 4.2.9
Setup, but do not evaluate, the integral that evaluates to the arc length of the cardioid $r = 1 + \sin(\theta)$, for $0 \leq \theta \leq 2\pi$.

Solution: A complete revolution of the cardioid requires $0 \leq \theta \leq 2\pi$. Now $r(\theta) = 1 + \sin(\theta)$ and then

$$x(\theta) = (1 + \sin(\theta))\cos(\theta)$$
$$y(\theta) = (1 + \sin(\theta))\sin(\theta)$$

Now $r'(\theta) = \cos(\theta)$ and thus

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin(\theta)}\, d\theta \end{aligned}$$

■

Example 4.2.10
Find the arc length of the curve $r = \theta^2$, where $0 \leq \theta \leq \sqrt{5}$.

Solution: Compute

$$
\begin{aligned}
L = \int_0^{\sqrt{5}} \sqrt{r^2 + (r')^2}\, d\theta &= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta \\
&= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta \\
&= \frac{19}{3}
\end{aligned}
$$

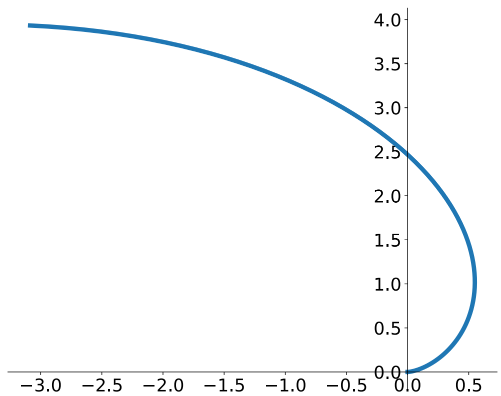

Fig. 4.10: The curve $r = \theta^2$

■

Areas in polar coordinates

The area of a wedge of radius r and sweeping an angle of θ is

$$A_{\text{wedge}} = \frac{1}{2} r^2 \theta.$$

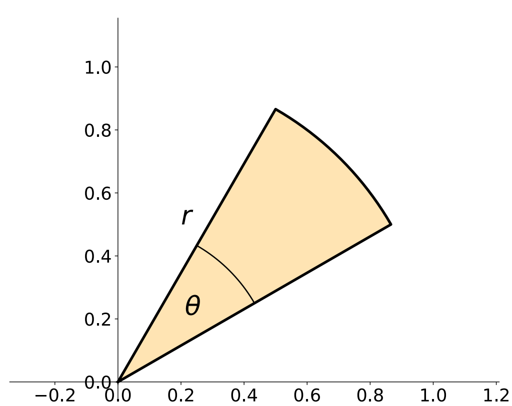

Fig. 4.11: Area of a wedge is $A = \frac{1}{2} r^2 \theta$

Given say the cardioid $r = 1 + \sin(\theta)$, we can divide the interval $0 \leq \theta \leq 2\pi$ so that we obtain a partition of wedges of the area enclosed by the cardioid:

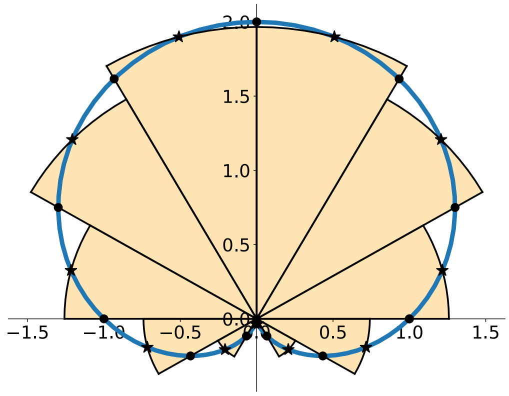

Fig. 4.12: Approximating the area enclosed by a cardioid with wedges

The sum of the area of the wedges approximates the true area A enclosed by the cardioid:

$$A \approx \sum_{i=1}^{n} \frac{1}{2} r(\theta_i^*)^2 \Delta\theta$$

As $n \to \infty$ we obtain the true area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r(\theta_i^*)^2 \Delta\theta = \int_0^{2\pi} \frac{1}{2} r(\theta)^2 \, d\theta$$

In general, for a curve given in polar coordinates $r = r(\theta)$, the area enclosed by the curve as θ ranges from $\theta_0 \leq \theta \leq \theta_1$ is

$$A = \int_{\theta_0}^{\theta_1} \frac{1}{2} r(\theta)^2 \, d\theta.$$

Example 4.2.11
Find the area enclosed by the cardioid $r = 1 + \sin(\theta)$ above the x-axis.

Solution: Apply the formula:

$$
\begin{aligned}
A &= \int_0^{\pi} \frac{1}{2}(1+\sin(\theta))^2\, d\theta \\
&= \int_0^{\pi} \frac{1}{2}(1+2\sin(\theta)+\sin^2(\theta))\, d\theta \\
&= \frac{\pi}{2} + 2 + \frac{1}{2}\int_0^{\pi} \frac{1-\cos(2\theta)}{2}\, d\theta \\
&= \frac{\pi}{2} + 2 + \frac{\pi}{4} = 2 + 3\frac{\pi}{4}
\end{aligned}
$$

■

Example 4.2.12
Consider the curve $r = 2 - \cos(\theta)$ for $0 \le \theta \le 2\pi$.

(a) Sketch the curve.

(b) Find the area enclosed by the curve.

(c) Setup the integral that evaluates to the arc length of the curve. Simplify the integrand but do not attempt to evaluate the integral.

(d) Use the Trapezoidal rule with $n = 4$ to estimate the arc length of the curve.

(e) Find the points on the curve where the tangent line is vertical.

Solution:

1. Evaluating $x(\theta) = r(\theta)\cos(\theta)$ and $y(\theta) = r(\theta)\sin(\theta)$ from $\theta = 0$ to $\theta = 2\pi$ at step-size of $\pi/4$, we obtain the following graph:

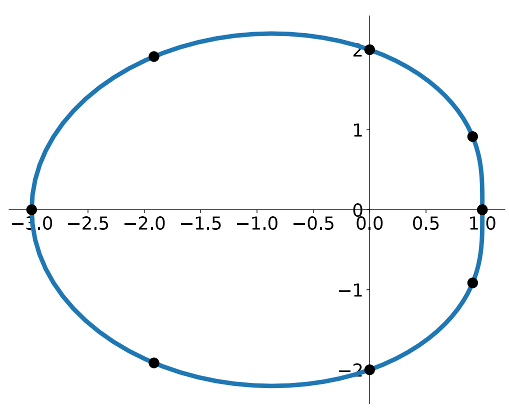

Fig. 4.13: Limacon

2.
$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r(\theta)^2\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(2-\cos(\theta))^2\, d\theta \\ &= \frac{1}{2}\left(\int_0^{2\pi} 4\, d\theta - 4\int_0^{2\pi} \cos(\theta)\, d\theta + \int_0^{2\pi} \cos^2(\theta)\, d\theta\right) \\ &= \frac{9\pi}{2} \end{aligned}$$

3.
$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(2-\cos(\theta))^2 + \sin^2(\theta)}\, d\theta \\ &= \int_0^{2\pi} \sqrt{5 - 4\cos(\theta)}\, d\theta \end{aligned}$$

4. Recall that the Trapezoidal rule is for estimating an integral

$$\int_a^b f(x)\, dx.$$

Partitioning $[a, b]$ into n equal subintervals, each subinterval has length $\Delta x = \frac{b-a}{n}$. The Trapezoidal rule is then

$$T = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $x_0, x_1, \ldots, x_n$ are the points obtained after subdividing the interval $[a, b]$. We wish to apply the Trapezoidal rule to

$$L = \int_0^{2\pi} \sqrt{5 - 4\cos(\theta)}\, d\theta$$

Hence, here $f(\theta) = \sqrt{5 - 4\cos(\theta)}$, $n = 4$, $\Delta\theta = \frac{2\pi}{n} = \frac{\pi}{2}$, and the points to evaluate f are $\theta_0 = 0$, $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \pi$, $\theta_3 = 3\pi/2$, and $\theta_4 = 2\pi$. Hence,

$$\begin{aligned}T &= \frac{\Delta\theta}{2}[f(\theta_0) + 2f(\theta_1) + 2f(\theta_2) + 2f(\theta_3) + f(\theta_4)] \\ &= \frac{\pi}{4}\left[\sqrt{1} + 2\sqrt{5} + 2\sqrt{9} + 2\sqrt{5} + 1\right] \\ &= \frac{\pi}{4}(8 + 4\sqrt{5}) \approx L\end{aligned}$$

5. Recall that given a parametrized curve $x = x(t)$ and $y = y(t)$, the slope of the line tangent to the curve at t is

$$m = \frac{y'(t)}{x'(t)}$$

The tangent line is vertical when $x'(t) = 0$. Here,

$$x(\theta) = (2 - \cos(\theta))\cos(\theta) = 2\cos(\theta) - \cos^2(\theta)$$

$$y(\theta) = (2 - \cos(\theta))\sin(\theta) = 2\sin(\theta) - \cos(\theta)\sin(\theta)$$

Then

$$\begin{aligned}x'(\theta) &= -2\sin(\theta) + 2\cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)(\cos(\theta) - 1)\end{aligned}$$

In the interval $0 \leq \theta \leq 2\pi$, $x'(\theta) = 0$ when $\theta = 0$ and $\theta = \pi$. Hence, the points are $(x(0), y(0)) = (1, 0)$ and $(x(\pi), y(\pi)) = (-3, 0)$. This agrees with the sketch of the curve.

■

Example 4.2.13
Consider the following polar curve $r(\theta) = \sin(2\theta)$, for $0 \leq \theta \leq 2\pi$.

(a) Sketch the curve.

(b) Find an expression for the arc length of the curve for $0 \leq \theta \leq \pi/2$.

(c) Use Simpson's rule with $n = 4$ subintervals to estimate the arc length of the curve on the interval $0 \leq \theta \leq \pi/2$. Label the grid points θ_0, θ_1, θ_2, θ_3, θ_4, and leave your answer in symbolic form.

(d) Find the area enclosed by the curve.

Solution:

1. Evaluating $x(\theta) = r(\theta)\cos(\theta)$ and $y(\theta) = r(\theta)\sin(\theta)$ from $\theta = 0$ to $\theta = 2\pi$ at step-size of $\pi/16$, we obtain the curve shown in Figure 4.14.

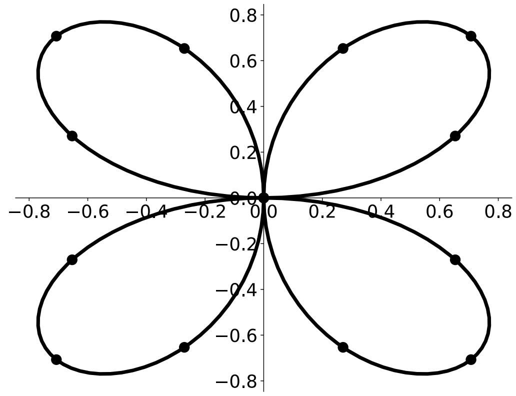

Fig. 4.14: The polar curve $r(\theta) = \sin(2\theta)$

2. Since $r'(\theta) = 2\cos(\theta)$, the arc length of the curve along for $0 \leq \theta \leq \pi/2$ is

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{r(\theta) + (r'(\theta))^2}\, d\theta \\ &= \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(\theta)}\, d\theta \end{aligned}$$

3. Let $f(\theta) = \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}$. Applying Simpson's rule, we obtain that $\Delta\theta = \frac{\pi/2}{4} = \pi/8$ and then the grid points are $\theta_0 = 0$, $\theta_1 = \pi/8$, $\theta_2 = \pi/4$, $\theta_3 = 3\pi/8$, and $\theta_4 = \pi/2$. The symbolic form of Simpson's rule is then

$$S = \frac{\Delta\theta}{3}\left[f(\theta_0) + 4f(\theta_1) + 2f(\theta_2) + 4f(\theta_3) + f(\theta_4)\right]$$

4. The area enclosed by the curve is

$$
\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2} r^2(\theta)\, d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \sin^2(2\theta)\, d\theta \\
&= \frac{1}{4} \int_0^{2\pi} (1 - \sin(4\theta))\, d\theta \\
&= \frac{1}{4} (\theta + \frac{1}{4} \cos(4\theta)) \Big|_0^{2\pi} \\
&= \frac{1}{4} (2\pi + \frac{1}{4} \cos(8\pi) - \frac{1}{4} \cos(0)) \\
&= \pi/2
\end{aligned}
$$

■