$$\lim_{w \to w_{max}} \left(\frac{1 - F(w)}{f(w)} \right)' = ?$$

P1

- First, try to simplify the function:

$$\left(\frac{1-F}{f} \right)' = \frac{(1-F)' f - f'(1-F)}{f^2} = \frac{-f^2 - f'(1-F)}{f^2}$$

$$= \frac{-f^2}{f^2} - \frac{f'(1-F)}{f^2} = -1 - \frac{f'}{f} \cdot \frac{1-F}{f}$$

assuming $f(w) \neq 0$

$$= -\left(1 + \frac{f'}{f} \cdot \frac{1-F}{f}\right) \quad (*)$$

- Couple of formulaes that we use while computing f':

$$\frac{\partial}{\partial w} \mathcal{W}_l\left(A w^{\frac{1}{\alpha-1}}\right) = \frac{1}{\alpha-1} \cdot \frac{\mathcal{W}_l\left(A w^{\frac{1}{\alpha-1}}\right)}{1 + \mathcal{W}_l\left(A w^{\frac{1}{\alpha-1}}\right)} \cdot \frac{1}{w}$$

$$\frac{\partial}{\partial w} \exp\left(\frac{-X_l(w)}{\gamma}\right) = \frac{\mathcal{W}_l\left(A w^{\frac{1}{\alpha-1}}\right)}{1 + \mathcal{W}_l\left(A w^{\frac{1}{\alpha-1}}\right)} \cdot \frac{1}{w} \exp\left(\frac{-X_l(w)}{\gamma}\right)$$

(we established these in Gamma LRT.pdf)

The density of weights is:

$$f(w) = \frac{\partial}{\partial w} \exp\left(\frac{-X_l(w)}{\gamma}\right) - \frac{\partial}{\partial w} \exp\left(\frac{-X_u(w)}{\gamma}\right)$$

In equation (54) in the p.d.f file, we showed that:

$$f(w) = \frac{W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}{1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)} \cdot \frac{1}{w} \exp\left(\frac{-X_l(w)}{\gamma}\right) - \frac{W_u\left(Aw^{\frac{1}{\alpha-1}}\right)}{1+W_u\left(Aw^{\frac{1}{\alpha-1}}\right)} \cdot \frac{1}{w} \exp\left(\frac{-X_u(w)}{\gamma}\right)$$

Start with this. My goal is to compute $f'(w)$; plug in the formula (*) on P1 and "hopefully" things cancel out. The derivative of first component in $f(w)$ is:

$$= \frac{\partial}{\partial w}\left(\frac{W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}{1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}\right) \cdot \frac{1}{w} \exp\left(\frac{-X_l(w)}{\gamma}\right) + \frac{W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}{1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)} \frac{\partial}{\partial w}\left(\frac{1}{w} \cdot \exp\left(\frac{-X_l(w)}{\gamma}\right)\right)$$

$$= \frac{\frac{\partial}{\partial w}\left(W_l\left(Aw^{\frac{1}{\alpha-1}}\right)\right)\left(1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)\right) - \frac{\partial}{\partial w}\left(1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)\right) W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}{\left(1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)\right)^2} \cdot \frac{1}{w} \exp\left(\frac{-X_l(w)}{\gamma}\right)$$

$$+ \frac{W_l\left(Aw^{\frac{1}{\alpha-1}}\right)}{1+W_l\left(Aw^{\frac{1}{\alpha-1}}\right)} \left\{ \frac{\partial}{\partial w}\left(\frac{1}{w}\right) \exp\left(\frac{-X_l(w)}{\gamma}\right) + \frac{1}{w} \frac{\partial}{\partial w}\left(\exp\left(\frac{-X_l(w)}{\gamma}\right)\right)\right\}$$

$$= \frac{\frac{1}{\alpha-1} \cdot \frac{1}{\omega} \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}}{\left(1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})\right)^2} \cdot \frac{1}{\omega} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) +$$

$$\frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \left(\frac{-1}{\omega^2} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) + \frac{1}{\omega} \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \cdot \frac{1}{\omega} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) \right)$$

$$= \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) \cdot \frac{1}{\omega^2} \cdot \frac{1}{\alpha-1} \frac{1}{\left(1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})\right)^2} +$$

$$\frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) \left\{ \frac{-1}{\omega^2} + \frac{1}{\omega^2} \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \right\}$$

$$= \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) \left\{ \frac{1}{\omega^2} \cdot \frac{1}{\alpha-1} \cdot \frac{1}{\left(1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})\right)^2} + \frac{1}{\omega^2} \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} - \frac{1}{\omega^2} \right\}$$

$$= \frac{\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-\chi_\ell(\omega)}{\gamma}\right) \left\{ \frac{1}{\omega^2(\alpha-1)} \frac{1}{\left(1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})\right)^2} - \frac{1}{\omega^2} \frac{1}{1+\mathcal{W}_\ell(A\omega^{\frac{1}{\alpha-1}})} \right\}$$

$$= \frac{1}{\omega^2} \frac{W_l(A\omega^{\frac{1}{\alpha-1}})}{1+W_l(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-X_l(\omega)}{\gamma}\right) \left\{ \frac{1}{\alpha-1} \frac{1}{\left(1+W_l(A\omega^{\frac{1}{\alpha-1}})\right)^2} - \frac{1}{1+W_l(A\omega^{\frac{1}{\alpha-1}})} \right\}$$

$$= \frac{1}{\omega^2} \frac{W_l(A\omega^{\frac{1}{\alpha-1}})}{1+W_l(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-X_l(\omega)}{\gamma}\right) \frac{1-(\alpha-1)\left(1+W_l(A\omega^{\frac{1}{\alpha-1}})\right)}{(\alpha-1)\left(1+W_l(A\omega^{\frac{1}{\alpha-1}})\right)^2}$$

with a similar work; the second component for $f'(\omega)$ is:

$$\frac{1}{\omega^2} \frac{W_u(A\omega^{\frac{1}{\alpha-1}})}{1+W_u(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-X_u(\omega)}{\gamma}\right) \frac{1-(\alpha-1)\left(1+W_u(A\omega^{\frac{1}{\alpha-1}})\right)}{(\alpha-1)\left(1+W_u(A\omega^{\frac{1}{\alpha-1}})\right)^2}$$

combining twos we can write:

$$f'(\omega) = \frac{1}{\omega^2} \frac{W_l(A\omega^{\frac{1}{\alpha-1}})}{1+W_l(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-X_l(\omega)}{\gamma}\right) \underbrace{\frac{1-(\alpha-1)\left(1+W_l(A\omega^{\frac{1}{\alpha-1}})\right)}{(\alpha-1)\left(1+W_l(A\omega^{\frac{1}{\alpha-1}})\right)^2}}_{=g_l(A,\omega)}$$

$$- \frac{1}{\omega^2} \frac{W_u(A\omega^{\frac{1}{\alpha-1}})}{1+W_u(A\omega^{\frac{1}{\alpha-1}})} \exp\left(\frac{-X_u(\omega)}{\gamma}\right) \underbrace{\frac{1-(\alpha-1)\left(1+W_u(A\omega^{\frac{1}{\alpha-1}})\right)}{(\alpha-1)\left(1+W_u(A\omega^{\frac{1}{\alpha-1}})\right)^2}}_{=g_u(A,\omega)}$$