Selected Topics

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Topics

- Adaptive Pareto Smoothed Importance Sampling
- Multilevel Causal Mediation Analysis
- Modelling Tuberculosis in Foreign-Born Canadians

Topics

- **Adaptive Pareto Smoothed Importance Sampling**
- Multilevel Causal Mediation Analysis
- Modelling Tuberculosis in Foreign-Born Canadians

Outline

- Importance sampling
- Measuring performance
- Improving performance
  - Modifications
  - Optimization

Importance Sampling

- Need to compute an expected value
  - $\mathbb{E}_F \varphi(X)$
- Can't do the sum/integral
- Monte Carlo approximation
  - Simulating from F might be hard

Importance Sampling

- Introduce "proposal distribution", G:

$$\begin{aligned}\mathbb{E}_F \varphi(X) &= \mathbb{E}_G \left[\varphi(X) \cdot \frac{f(X)}{g(X)}\right] \\ &= \mathbb{E}_G \left[\varphi(X) \cdot w(X)\right]\end{aligned}$$

- G can be nearly anything*
  - *Some choices will be better than others

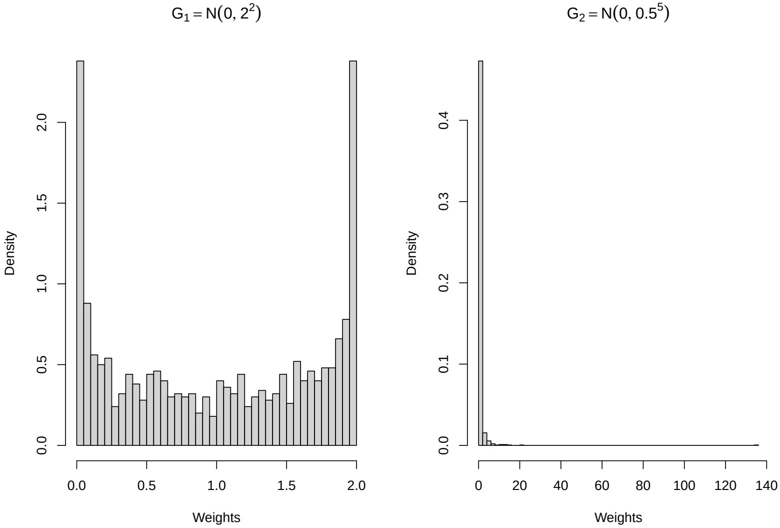
Example: Mystery Target

- f unknown, but can be evaluated
- $\varphi(X) = X^2$

- Try some proposals:
  - $G_1 \sim N(0, 2^2)$
  - $G_2 \sim N(0, 0.6^2)$

- Use $M = 1000$ samples from proposal
  - $\hat{\mathbb{E}}_1 = 0.99$, $\hat{\text{SD}} = 1.97$
  - $\hat{\mathbb{E}}_2 = 1.10$, $\hat{\text{SD}} = 2.32$

Example: Mystery Target



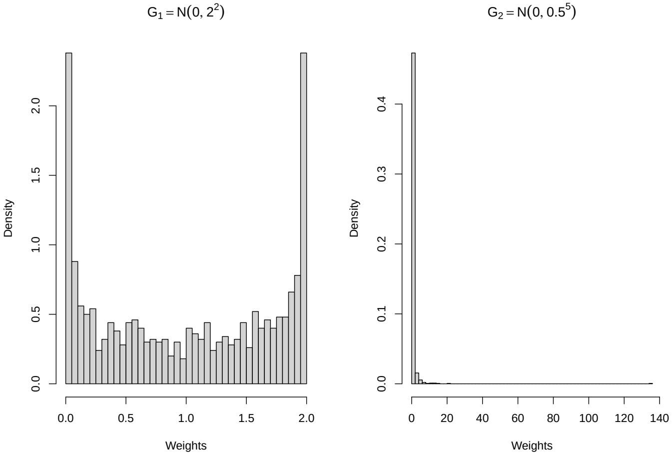
Importance Sampling

- G_1 weights look fine
- G_2 weights dominated by one large value

- We can make this difference precise
- "Effective Sample Size":

$$ESS = \frac{\left[\sum_i w(X_i)\right]^2}{\sum_i w(X_i)^2}$$

Example: Mystery Target



$ESS_1 \approx 662$ $\qquad$ $ESS_2 \approx 54$

Importance Sampling

- Problem: Low ESS $\rightarrow$ hard to estimate means
- But ESS is based on means
  - (Chatterjee and Diaconis, 2018)

Improving IS

- Choose a good proposal
- Modify large weights
    - Truncated IS
    - Pareto Smoothed IS

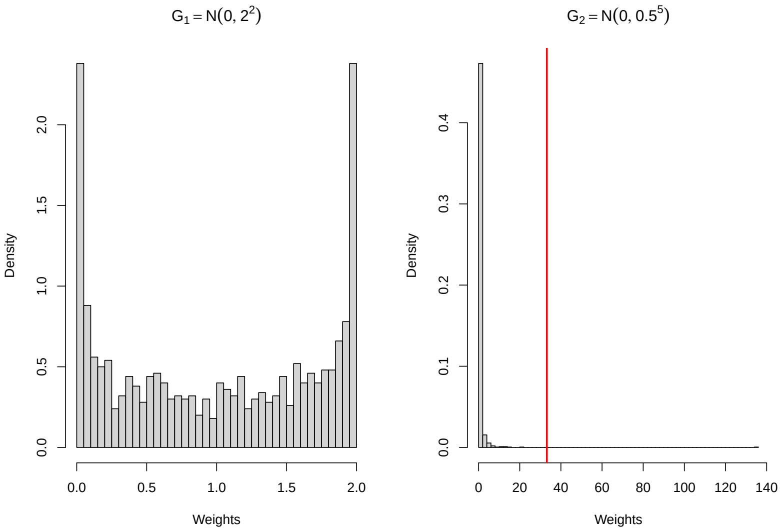
Improving IS

- Truncated Importance Sampling:
  - (Ionides, 2008)

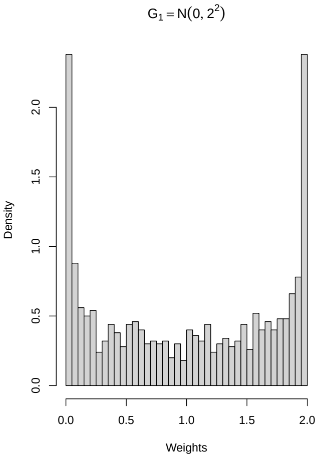
1. Choose a threshold
2. Apply hard thresholding to any large weights

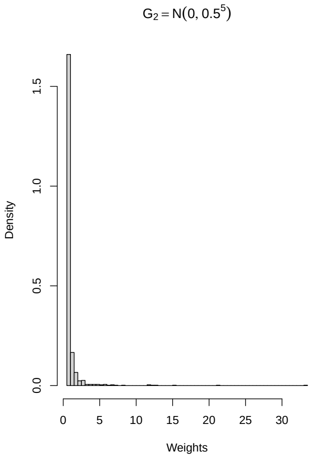
- Still consistent for the target

Example: Mystery Target



Example: Mystery Target





$$ESS_1 \approx 662$$
$$ESS_1^{(\text{trunc})} \approx 662$$

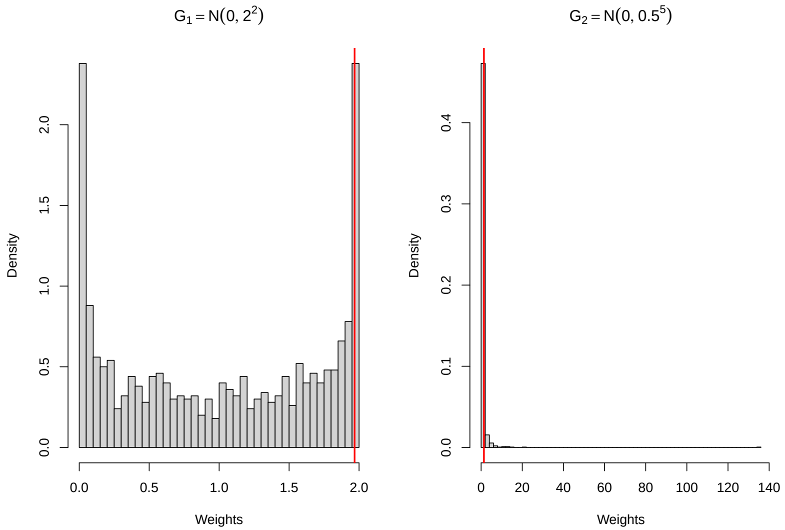
$$ESS_2 \approx 54$$
$$ESS_2^{(\text{trunc})} \approx 245$$

Improving IS

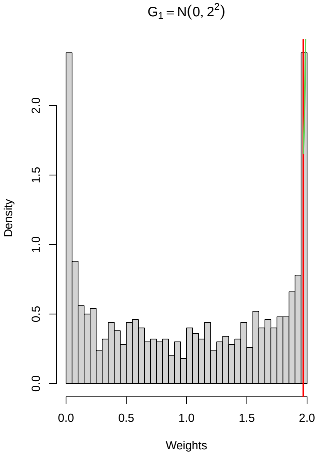
- Pareto Smoothed Importance Sampling:
    - (Vehtari et al., 2024)

1. Choose a threshold
    - Weights above threshold represent tail of their dist.
2. Approximate tail with Generalized Pareto Dist.
    - Fit GPD to weights above threshold
    - (Zhang and Stephens, 2009)
3. Replace large weights with quantiles of fitted GPD

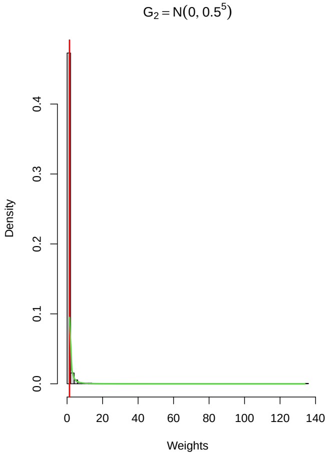
Example: Mystery Target



Example: Mystery Target

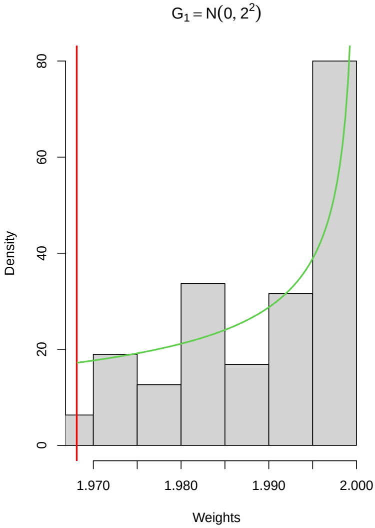


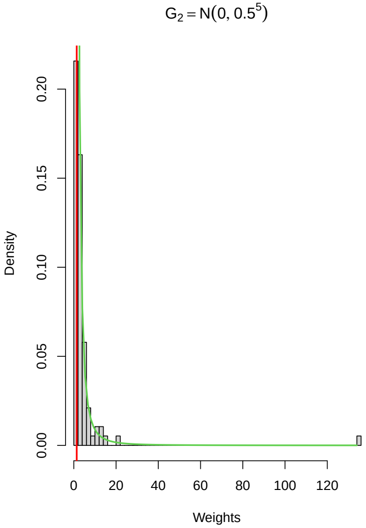
$\hat{k}_1 \approx -1.81$



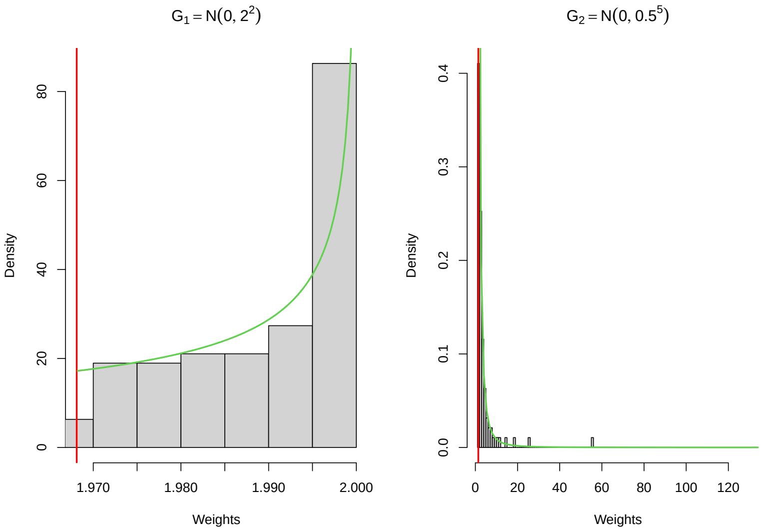
$\hat{k}_2 \approx 0.72$

Example: Mystery Target





Example: Mystery Target



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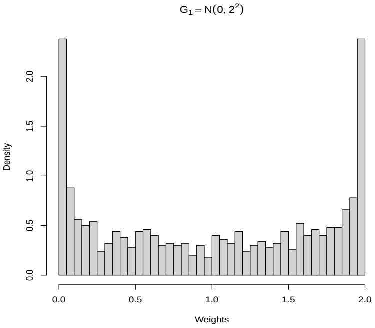
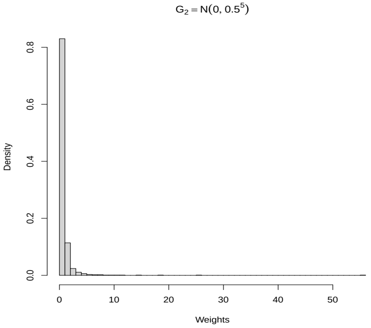




$$ESS_1 \approx 662$$
$$ESS_1^{(\text{trunc})} \approx 662$$
$$ESS_1^{(\text{PS})} \approx 662$$

$$ESS_2 \approx 54$$
$$ESS_2^{(\text{trunc})} \approx 245$$
$$ESS_2^{(\text{PS})} \approx 160$$

Adaptive IS

- Alternative approach: directly optimize ESS
- Adaptive Importance Sampling:
  - (Akyildiz and Míguez, 2021)

1. Choose a (parametric) family of proposals
2. Iteratively update the proposal to maximize ESS

Stochastic Approximation

- Actually, we want to maximize a population-level analog: $\rho \approx \frac{N}{ESS}$
- If we had ρ, we would do gradient descent
  - $\theta_{k+1} = \theta_k - \alpha \nabla \rho(\theta_k)$
- Instead, do gradient descent on $\hat{\rho}$
  - $\hat{\theta}_{k+1} = \hat{\theta}_k - \alpha_k \nabla \hat{\rho}(\hat{\theta}_k)$

Stochastic Approximation

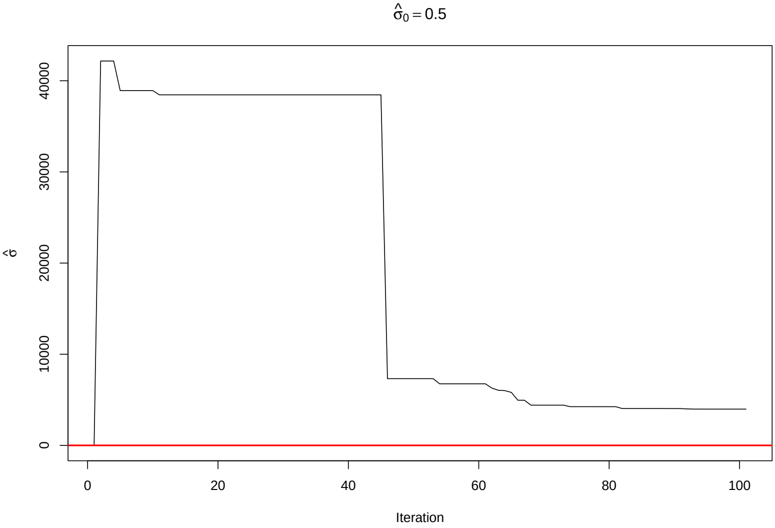
- Originally developed for root finding with noise
    - (Robbins and Monro, 1951)
- Quickly adapted for optimization
    - Use noisy evaluations for finite difference
    - (Kiefer and Wolfowitz, 1952)

Stochastic Approximation

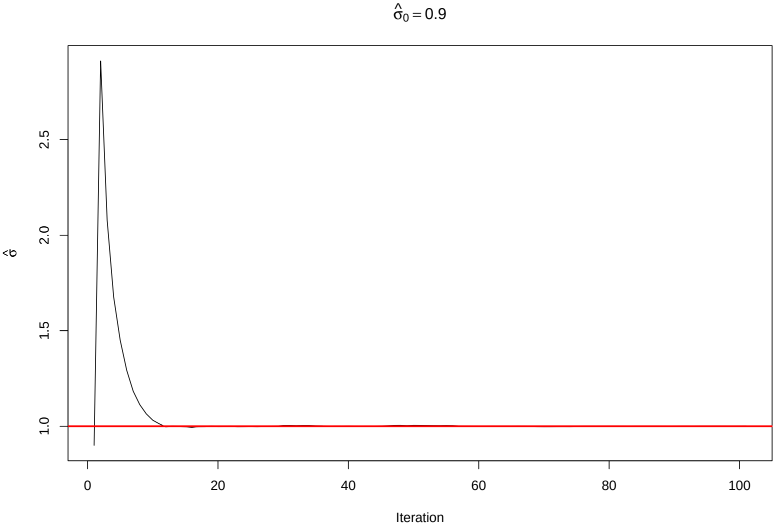
- Very well developed theory
- Step size $\to 0$

- Stochastic gradient descent
  - Resample a (very) large dataset

Example: Mystery Target



Example: Mystery Target



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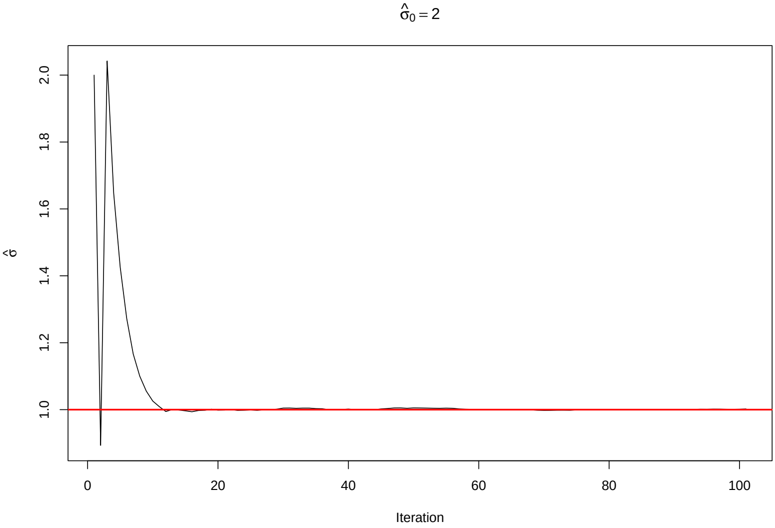
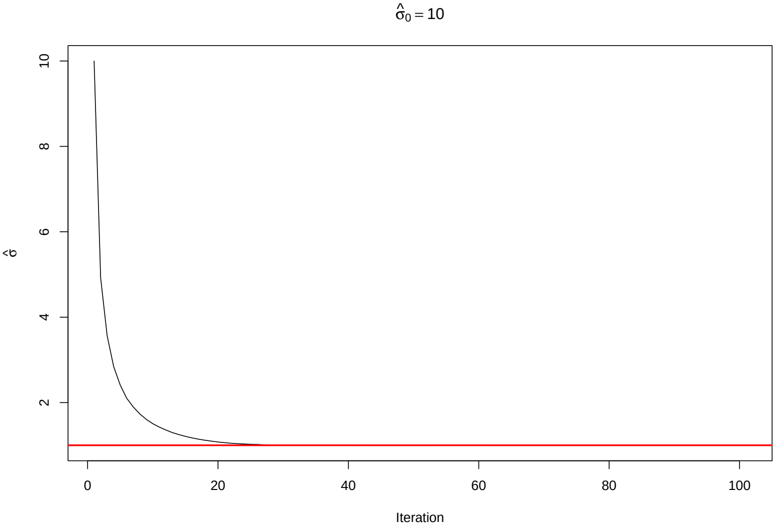


Example: Mystery Target



Our Method

- Recall: Be careful using IS means to diagnose IS
- Vehtari et al. give an alternative
  - Shape parameter of fitted tail distribution, $\hat{k}$

Our Method

- Use diagnostic as objective function
- Apply stochastic approximation to minimize $\hat{k}$
  - More precisely, its population analog: $k(\theta)$

Our Method

- Results: plot trajectories for the same values of sigma used above for ESS

Recap

- Importance sampling and extensions
    - Truncation
    - Pareto Smoothing
- Diagnostics for importance sampling
    - Effective sample size
    - Pareto tail index
- Adaptive importance sampling
    - Stochastic approximation

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- **Multilevel Causal Mediation Analysis**
- Modelling Tuberculosis in Foreign-Born Canadians

Topics

- Add slides from my SSC talk

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- Adaptive Pareto Smoothed Importance Sampling
- Multilevel Causal Mediation Analysis
- **Modelling Tuberculosis in Foreign-Born Canadians**

Topics

- Give brief overview and mention directions for future research
- One of the profs in the department, Cristina Anton, does numerical SDEs. Mention potential collaboration

Acknowledgements

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Thank You

Some References

Akyildiz, Ö. D. and Míguez, J. (2021). Convergence rates for optimized adaptive importance samplers. *Statistics and Computing*, 31(12).

Chatterjee, S. and Diaconis, P. (2018). The sample size required in importance sampling. *The Annals of Applied Probability*, 28(2).

Ionides, E. L. (2008). Truncated importance sampling. *Journal of Computational and Graphical Statistics*, 17(2).

Kiefer, J. and Wolfowitz, J. (1952). Stochastic estimation of the maximum of a regression function. *The Annals of Mathematical Statistics*, 23(3).

Robbins, H. and Monro, S. (1951). A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3).

Vehtari, A., Simpson, D., Gelman, A., Yao, Y., and Gabry, J. (2022). Pareto smoothed importance sampling. *ArXiv*.

Vehtari, A., Simpson, D., Gelman, A., Yao, Y., and Gabry, J. (2024). Pareto smoothed importance sampling. *Journal of Machine Learning Research*, 25(72).

Zhang, J. and Stephens, M. A. (2009). A new and efficient estimation method for the generalized Pareto distribution. *Technometrics*, 51(3).