Stat 330 - Tutorial 8

Today's tutorial will cover two topics: variance and moment generating functions. You are probably already familiar with variances, so we will go through this ~~m~~ fairly quickly to leave more time for the less familiar topic of MGFs. You also have a midterm next week, so I will ~~en~~ end a bit early to leave time for questions.

Variance

If X is a random variable, then its variance is defined as $V(X) = E[(X-E(X))^2]$, or $E[(X-\mu)^2]$ if $\mu = E(X)$. This definition is great for ~~making sense~~ interpreting variance, but there is an equivalent definition that is ~~m~~ usually easier to work with: $V(X) = E(X^2) - [E(X)]^2$. Let's do an example to see how this looks in practice.

e.g 1:

Let $X \sim \text{Gamma}(\alpha, \beta)$. That is, the density of X is $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ for $x > 0$, and 0 otherwise. Find $V(X)$.

Regardless of which formula we use, we will need to find $E(X)$. Let's start with that.

$$E(X) = \int_0^\infty \frac{x}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(\alpha+1)-1} e^{-x/\beta}\,dx$$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{(\alpha+1)-1} e^{-x/\beta}\,dx$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \beta \cdot 1$$

$$= \alpha \cdot \beta$$

The last step of this derivation uses the property of the gamma function that, for any positive real number, x, $\Gamma(x+1) = x \cdot \Gamma(x)$.

I much prefer working with our second definition of variance: $V(X) = E(X^2) - [E(X)]^2$. To use it, the only thing we still need is $E(X^2)$. Let's compute this.

$$E(X^2) = \int_0^\infty \frac{x^2}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(\alpha+2)-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha} \int \frac{1}{\Gamma(\alpha+2)\beta^{\alpha+2}} x^{(\alpha+2)-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \beta^2 \cdot 1$$

$$= (\alpha+1)\cdot\alpha\cdot\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

Putting everything together, we get that the variance of X is

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - [\alpha\beta]^2$$

$$= \alpha\beta^2$$

$\diamond$

~~The~~ Moment Generating Functions (MGFs)

The MGF of a random variable is a special function that is related to the random variable and tells us a lot of information about the random variable. If X is a R.V., we define its MGF ~~as~~ to be the function $m_X(t) = E(e^{tX})$. There are a few things to notice here:

i) The MGF of a R.V., evaluated at a point, say 5, is a number. E.g. $m_X(5) = 17$. This means that the MGF of a random variable is an entire function, unlike the mean which is a single number.

ii) We can use the MGF of a R.V. to calculate the expected value of positive integer powers of the random variable, e.g. $E(X), E(X^2), E(X^9)$. These expected powers of X are called the moments of X. We get the first moment~~s~~ of X from its MGF ~~in the following way~~: by differentiating $m_X(t)$ with respect to t, then evaluating at $t=0$. ~~The pro~~ More formally,

$$\frac{d}{dt} m_X(t) = \frac{d}{dt} E(e^{tX}) \overset{*}{=} E\left(\frac{d}{dt} e^{tX}\right) = E(Xe^{tX})$$

Evaluating at $t=0$ gives $E(Xe^{0 \cdot X}) = E(X)$. Higher-order derivatives lead to higher-order moments as well:

$$\left.\frac{d^n}{dt^n} m_X(t)\right|_{t=0} = \left.E\left(\frac{d^n}{dt^n} e^{tX}\right)\right|_{t=0} = \left.E(X^n e^{tX})\right|_{t=0} = E(X^n)$$

There is a hidden detail in this argument in the step marked with a $\clubsuit$. We have assumed that it is okay to move the derivative inside the expected value. It is usually okay to do this, but it is something that we need to either assume or prove. Sadly, ~~proving~~ the proof is beyond the scope of this course, so we will need to just assume that nothing goes wrong.

That's a bunch of theory, now let's see how these MGFs actually work.

e.g. 2

Let $X \sim Ber(p)$. Find the function $m_X(t)$.

This one is pretty straightforward. We can just directly compute the expected value of e^{tX}.

$$E(e^{tX}) = (e^{t1}) \cdot p + (e^{t0}) \cdot (1-p)$$

$$= pe^t + (1-p) \cdot 1$$

$$= pe^t + 1 - p$$

Let's also check that $m_X(t)$ correctly generates the moments of X. First, we compute $E(X^n)$ directly.

$$E(X^n) = 1^n \cdot p + 0^n (1-p) = p$$

Next, we take the n^{th}-order t-derivative of $m_X(t)$.

$$\frac{d^n}{dt^n}[pe^t + 1 - p] = pe^t$$

Evaluating this at $t=0$ gives $pe^0 = p = E(X^n)$.

$\diamond$

6

One application of the MGF of a R.V. is to compute the variance of that R.V. Remember that, ~~to get~~ we need $E(X)$ and $E(X^2)$ to get $V(X)$. ~~These are~~ But $E(X)$ and $E(X^2)$ are just the first two moments of X, so we can get them from $m_X(t)$. Let's use this strategy to verify that we got the right answer in e.g. 2.

e.g. 3

Let $X \sim \text{Gamma}(\alpha, \beta)$. (a) Find the MGF of X, $m_X(t)$. (b) Use $m_X(t)$ to find ~~the~~ $V(X)$.

(a) $m_X(t) = E(e^{tX})$

$$= \int_0^\infty \frac{e^{tX}}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} \exp\left(-\frac{x}{\beta} + tx\right) dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} \exp\left[-x\Big/\left(\frac{\beta}{1-t\beta}\right)\right] dx$$

$$= \frac{(\beta')^\alpha}{\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)(\beta')^\alpha} x^{\alpha-1} e^{-x/\beta'}\, dx$$

$$= (\beta')^\alpha / \beta^\alpha$$

$$= \frac{1}{(1-t\beta)^\alpha}$$

Aside:

$$\frac{-x}{\beta} + tx$$

$$= \frac{-x + \beta t x}{\beta}$$

$$= x\left(\frac{t\beta - 1}{\beta}\right)$$

$$= -x \Big/ \left(\frac{\beta}{1-t\beta}\right)$$

Let $\beta' = \dfrac{\beta}{1-t\beta}$

(b) Now we need to compute the first and second derivatives of $m_X(t)$.

$$\frac{d}{dt} m_X(t) = \frac{d}{dt} \frac{1}{(1-t\beta)^{\alpha}}$$

$$= \frac{-\alpha}{(1-t\beta)^{\alpha+1}} \cdot (-\beta)$$

$$= \frac{\alpha\beta}{(1-t\beta)^{\alpha+1}} \qquad (I)$$

$$\frac{d^2}{dt^2} m_X(t) = \frac{d}{dt} \frac{\alpha\beta}{(1-t\beta)^{\alpha+1}}$$

$$= \frac{-\alpha\beta(\alpha+1)}{(1-t\beta)^{\alpha+2}} \quad (-\beta)$$

$$= \frac{\alpha\beta^2(\alpha+1)}{(1-t\beta)^{\alpha+2}} \qquad (II)$$

Next, we get the first two moments of X by evaluating (I) and (II) at $t=0$.

$$E(X) = \frac{d}{dt} m_X(t)\Big|_{t=0} = \frac{\alpha\beta}{(1-0\beta)^{\alpha+1}} = \alpha\beta$$

$$E(X^2) = \frac{d^2}{dt^2} m_X(t)\Big|_{t=0} = \frac{\alpha\beta^2(\alpha+1)}{(1-0\beta)^{\alpha+2}} = \alpha\beta^2(\alpha+1) = \alpha^2\beta^2 + \alpha\beta^2$$

All that remains is to put these together and get $V(X)$

$$V(X) = E(X^2) - [E(X)]^2 = \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Fortunately, this matches the value we got in e.g. 1.