Sept. ~~24-29~~ 23-27

Stat 330 Tutorial - Week 3

The midterm is in 2 weeks (Oct. 7). Next week's tutorial will be a combination of practice questions and review. This week is going to follow the usual format of going through a few practice questions.

2.1.3 Let $S=\{1,2,3,4,5\}$ be our sample space. Define two different non-constant random variables on S, X and Y. Let $Z=X+Y^2$. Compute all possible values of Z.

To start, let's define X and Y. We can do anything we want for these (as long as they are non-constant and different). Here are two possibilities:

i) $X(w)=I(w=1)$,
$\quad Y(w)=w$

ii) $X(w)=-w$
$\quad Y(w)=\sqrt{w}$

where $w=1,2,\dots,5$

In case (i), $Z(w)=X(w)+Y(w)^2$, so the possible values of Z are

$Z(1)=X(1)+Y(1)^2=I(1=1)+1^2=1+1=2$

$Z(2)=I(2=1)+2^2=0+4=4$

$Z(3)=0+3^2=9$

$Z(4)=16$

$Z(5)=25$

In case 2, $Z(w)=X(w)+Y(w)^2=-w+(\sqrt{w})^2=0$, so Z is always equal to zero.

2.2.4) Let Z be the outcome on the roll of a fair 6-sided dice. Let $W = Z^3 + 4$ and $V = \sqrt{Z}$. Compute the following PMFs for every $x_1, \ldots, x_5 \in \mathbb{R}$

a) $p(W = x_1)$ b) $p(V = x_2)$ c) $p(ZW = x_3)$ d) $p(ZV = x_4)$ e) $p(W + V = x_5)$

We can get all of these probabilities from a table of values.

Z	prob.	x_1	x_2	x_3	x_4	x_5
1	1/6	5	1	~~4~~ 5	1	6
2	1/6	~~13~~ 12	$\sqrt{2}$	~~26~~ 24	$2\sqrt{2}$	$12+\sqrt{2}$
3	1/6	31	$\sqrt{3}$	93	$3\sqrt{3}$	$31+\sqrt{3}$
4	1/6	68	2	272	8	70
5	1/6	129	$\sqrt{5}$	645	$5\sqrt{5}$	$129+\sqrt{5}$
6	1/6	~~216~~ 220	$\sqrt{6}$	~~1272~~ 1320	$6\sqrt{6}$	~~216~~ $220+\sqrt{6}$

All other values of $x_1, \ldots, x_5$ have prob. zero.

2.3.16) An urn contains 4 black balls and 5 white balls. Consider drawing balls from this urn with replacement.

a) What is the probability that 5 black balls are observed over 15 draws?

b) What is the probability that 15 draws are required before the first black ball is drawn?

c) What is the probability that 15 draws are required before the fifth black ball is drawn? (This is slightly different from part c in the book)

a) The probability of drawing a black ball is $4/9$. Call this quantity p. The number of black balls out of 15 draws follows a binomial distribution with 15 trials and success ~~probabiti~~ probability p. That is, a $Bin(15, p)$ distribution. The probability of getting 5 successes under this distribution is

$$\binom{15}{5} p^5 (1-p)^{15-5} = \binom{15}{5} \left(\frac{4}{9}\right)^5 \left(\frac{5}{9}\right)^{10}$$

b) The number of draws required to get our first black ball follows a geometric distribution with success probability p. The probability that 15 draws are required is

This distribution is denoted $Geo(p)$.

$$(1-p)^{14} p^1 = \left(\frac{5}{9}\right)^{14} \left(\frac{4}{9}\right)$$

c) The number of draws required before our fifth black ball follows a negative binomial distribution with 5 required successes and success probability p. That is, a Neg-Bin$(5, p)$. The probability that we require 15 draws to get our 5^{th} black ball is

$$\binom{14}{4} p^5 (1-p)^{15-5} = \binom{14}{4}\left(\frac{4}{9}\right)^5\left(\frac{5}{9}\right)^{10}$$

2.4.14 Let $Y \sim \text{Exp}(\lambda)$. That is, the density of Y is $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$. [redacted]. Show that Y has the memoryless property. That is, show that for any two positive real numbers, x and y,

$$p(Y \geq x+y \mid Y \geq x) = p(Y \geq y)$$

Proof: To start, recall that the survival function of Y is $\bar{F}_Y(y) = p(Y > y) = e^{-\lambda y}$. We will use this later. Now, let's expand the left hand side of our equality.

$$
\begin{aligned}
&p(Y \geq x+y \mid Y \geq x) \\
&= \frac{p(Y \geq x+y \;\&\; Y \geq x)}{p(Y \geq x)} \\
&= \frac{p(Y \geq x+y)}{p(Y \geq x)} \\
&= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\
&= e^{-\lambda y} \\
&= p(Y \geq y)
\end{aligned}
$$

$\square$

Question: Why was it okay for me to be sloppy with $>$ and $\geq$ in this proof?

(W.1)

Let $F(x) = (1-e^{-x})I(x \geq 0)$ be the standard exponential CDF. Verify that F satisfies all the properties required to be a valid CDF.

From the discussion in the book following the proof of Theorem 2.5.2, there are five properties we must check:

i) $0 \leq F(x) \leq 1$ for all x

ii) $F(x) \leq F(y)$ whenever $x \leq y$

iii) $\lim_{x \to \infty} F(x) = 1$

iv) $\lim_{x \to -\infty} F(x) = 0$

v) $F(x^+) := \lim_{n \to \infty} F(x + \frac{1}{n}) = F(x)$

We are going to skip property (i) for now because it is a consequence of properties (ii)-(iv). Let's start with property (ii). We will do this by checking that the derivative of F is always non-negative when $x > 0$ ($x \leq 0$ is easy because F is constant)

$$\frac{d}{dx} F(x) = -\frac{d}{dx} e^{-x} = -e^{-x}(-1) = e^{-x} > 0$$

Next, properties (iii) and (iv) are easy to check because F is continuous (as a function; we have not proved that F corresponds to a continuous distribution).

$$\lim_{x \to \infty} F(x) = 1 - \lim_{x \to \infty} e^{-x} = 1$$

$\lim_{x \to -\infty} F(x) = 0$ because $F(x) = 0$ for all $x \leq 0$.

Now we get property (i) because we have shown that F increases monotonically from 0 to 1. All that is left is property (v). We will handle this by checking some cases.

Case 1 - $x < 0$: Here $F(x) = 0$ and $F(x + \frac{1}{n}) = 0$ for all sufficiently large n, so $\lim_{n \to \infty} F(x + \frac{1}{n}) = 0 = F(x)$.

Case 2 - $x \geq 0$: Here we can use the continuity of e^{-x}.

$$\lim_{n \to \infty} F(x + 1/n) = \lim_{n \to \infty} \left[1 - e^{-(x + 1/n)}\right] = 1 - e^{-\lim_{n \to \infty}(x + 1/n)} = 1 - e^{-x} = F(x)$$

This completes the proof. $\square$