1.3.2) Al watches the six-o'clock news 2/3 of the time, the eleven-o'clock news 1/2 of the time, and both 1/3 of the time. What is the prob. that Al watches only the six-o'clock news on a randomly selected day?

Here is a few different approaches. Let $p(6)$ & $p(11)$ be the prob. of Al watching the six-o'clock & 11-o'clock news, ~~respect~~

i) I need to get $p(6 \cap 11^c)$. I don't know how to get that, but I can get $p(6 \cup 11)$.

$$p(6 \cup 11) = p(6) + p(11) - p(6 \cap 11)$$
$$= 2/3 + 1/2 - 1/3$$
$$= 5/6$$

How does this help? If 6 were a superset of 11 (i.e. $6 \supseteq 11$), then we could use $p(6 \cap 11^c) = p(6) - p(11)$. This doesn't work because it is possible that Al watches the 6-o'clock news but not the 11-o'clock. However, $6 \cup 11$ is a superset of 11. Therefore,

$$p[(6 \cup 11) \cap 11^c] = p(6 \cup 11) - p(11) = 5/6 - 1/2 = 1/3$$

But $(6 \cup 11) \cap 11^c = (6 \cap 11^c) \cup (11 \cap 11^c) = 6 \cap 11^c$, so ~~thatpe~~ we actually just calculated $p(6 \cap 11^c)$, which is what we wanted.

ii) By the law of total probability,

$$p(6) = p(6 \cap 11) + p(6 \cap 11^c)$$
$$2/3 = 1/3 + p(6 \cap 11^c)$$
$$p(6 \cap 11^c) = 1/3$$

The moral of the story is that, when calculating probabilities, there are often many approaches. Furthermore, some of these approaches can be much easier than others. If you're stuck, see if there's another theorem that you can try.

1.4.4 Consider being dealt a hand of 5 cards from a well-shuffled deck. What is the probability of getting

a) all four aces and the king of spades?

b) all ~~cards type~~ 5 cards spades?

c) no pairs (i.e. all 5 cards different)?

d) a full house (i.e. 3 of one kind and 2 of another)?

a) This scenario describes a unique hand, so the numerator has 1 possibility. There are $\binom{52}{5}$ different possible 5-card hands, so the denominator will always be $\binom{52}{5}$. The probability here is therefore $1/\binom{52}{5}$.

b) There are 13 spades in the deck, so our numerator is $\binom{13}{5}$ and our probability is $\binom{13}{5}/\binom{52}{5}$

c) There are 13 types of card in the deck, so we have $\binom{13}{5}$ possibilities for card type. After selecting these types, there are 4 possible suits for each card. This gives an additional $\binom{4}{1}^5 = 4^5$ possibilities for a total of $\binom{13}{5}4^5$. Our probability is $\binom{13}{5}4^5/\binom{52}{5}$.

Question: Why can't we do $\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{9}{1}4^5$ for the numerator?

Answer: This approach assumes that order matters. You can check this mathematically by verifying that $\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{9}{1} = \binom{13}{5}\cdot 5!$.

That is, you get the number on the left by accounting for the 5! possible combinations of the 5 objects you selected with the $\binom{13}{5}$ term.

d) First, we must select the two types of card we will get, $\binom{13}{2}$ (not $\binom{13}{1}\binom{12}{1}$!!). Next, we need 3 suits for one and 2 suits for the other, which ~~[illegible]~~ is an additional $\binom{4}{3}\binom{4}{2}$ possibilities. This brings the total number of possibilities to $\binom{13}{2}\binom{4}{3}\binom{4}{2}$, for a probability of $\binom{13}{2}\binom{4}{3}\binom{4}{2}/\binom{52}{5}$.

1.5.4) ~~[illegible]~~ Imagine ~~[illegible]~~ being dealt 5 cards from a well-shuffled deck. What is the probability of getting 5 spades given that we got at least 4 spades?

From the definition of conditional probability, we know that

$$p(5 \text{ spades} \mid \text{at least 4 spades}) = \frac{p(5 \text{ spades } \& \text{ at least 4 spades})}{p(\text{at least 4 spades})}$$

$$= \frac{p(5 \text{ spades})}{p(\text{at least 4 spades})}$$

Let's calculate these two probabilities separately.

i) There are ~~[illegible]~~ 13 spades in the deck, so there are ~~[illegible]~~ $\binom{13}{5}$ ways to get exactly 5 of them. This gives a probability of $\binom{13}{5}/\binom{52}{5}$

ii) To get at least 4 spades, we can get either 4 spades or 5 spades. We know that there are $\binom{13}{5}$ ways to get 5 spades. To get 4 spades, we need 4 spades and 1 non-spade. This is $\binom{13}{4}\binom{39}{1}$, for a total probability of $\left[\binom{13}{5} + \binom{13}{4}\binom{39}{1}\right]/\binom{52}{5}$.

(4) The conditional probability we set out to calculate was

$$\frac{p(5 \text{ spades})}{p(\text{at least } 4 \text{ spades})}$$

$$= \frac{\binom{13}{5}/\binom{52}{5}}{\binom{13}{5}/\binom{52}{5} + \binom{13}{4}\binom{39}{1}/\binom{52}{5}}$$

$$= \cdots = \frac{3}{68} \quad \text{(check this)}$$

We can verify that we did everything correctly by writing a short simulation in R.