Oct. 21-25

Stat 330 - Tutorial 7

Today, we are talking about expected values. Specifically, how to calculate expected values. There is a simple trick that comes up embarrassingly often in these types of questions. This trick is actually just one of Kolmogorov's axioms: $p(S)=1$, where S is the sample space. This identity looks a bit different in the discrete and continuous cases:

$$\sum_{x} p(X=x) = 1 \qquad (I)$$

$$\int f_X(x)\,dx = 1 \qquad (II)$$

where f_X is the density of the random variable X. Let's do some examples to see these identities in action.

Discrete R.V.s

e.g. 1: Bernoulli

Let $X \sim \text{Bernoulli}(p)$. Finding the mean of X is pretty straightforward.

$$E(X) = \sum_{x=0}^{1} x \cdot p(X=x) = 0\cdot(1-p) + 1\cdot p = p$$

$\diamond$

We didn't even need to use (I) because it was so easy to do the sum directly. Now let's try a more involved example.

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e.g. 2: Binomial

Let $X \sim Bin(n,p)$. To find the mean of X this time, we will use identity (I).

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} \quad \text{Let } y = x-1$$

$$= \sum_{y=0}^{n-1} \frac{n!}{y!(n-y-1)!} p^{y+1} (1-p)^{n-y-1}$$

$$= np \underbrace{\sum_{y=0}^{n-1} \frac{(n-1)!}{y![(n-1)-y]!} p^y (1-p)^{(n-1)-y}}_{\text{This is the PMF of a } Bin(n-1,p) \text{ R.V.}}$$

$$= np(1) \quad \text{by (I)}$$

$$= np$$

$\diamond$

e.g. 3: The St. Petersberg Paradox

We like to think of the mean of a R.V. in terms of betting. Specifically, it is usually considered a 'fair bet' if you pay E(X) dollars to receive X dollars back, where X is the outcome of the R.V. ~~This example sh~~ The St. Petersberg example shows that this reasoning is not always valid. The intuition for this example is a game where the payout starts at \$1 and at every step you flip a coin. If the result is heads, you win the current payout and stop. If the result is tails, you double the payout and flip again. How much money should you be willing to pay to play this game?

Consider a R.V., X, which takes the value 2^x with probability 2^{-x}.
The mean of X is $E(X) = \sum_{x=0}^{\infty} 2^x p(X=2^x) = \sum_{x=0}^{\infty} 2^x \cdot 2^{-x} = \sum_{x=0}^{\infty} 1 = \infty$.

Therefore, you should be willing to pay any amount of money to play a game that pays $\$2^x$ when $X = 2^x$. This feels unreasonable however, because it is extremely unlikely that X will take a large value. For example, $p(X > 1000) \approx 0.002$, or $\frac{1}{5}$ of 1%. So should you really be willing to pay any amount of money to pay this game? I wouldn't.

Many people have offered solutions to this paradox. I like the one in the book. They point out that you probably couldn't win an arbitrarily large amount of money, so in reality your winnings would have some upper bound, say $\$2^{47}$ (which is more than 1.5 times the 2018 GDP of the entire world). If the payout would double more than 47 times, it stays at $\$2^{47}$. The expected payout now is

$$\sum_{x=0}^{47} 2^x \cdot p(X=x) + \sum_{x=48}^{\infty} 2^{47} p(X=x)$$

$$= \sum_{x=0}^{47} 1 + \sum_{x=48}^{\infty} 2^{47} \cdot 2^{-x}$$

$$= 48 + 2^{47} \cdot (2^{-47})$$

$$= 49$$

That is, the fair price to play this game is \$49. Pretty far from $\$\infty$.

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④ Continuous R.V.s

e.g. 4: Uniform

Let $X \sim Unif(a,b)$. We can find the mean of X without doing anything fancy.

$$E(X) = \int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{x^2}{2(b-a)}\Bigg|_{x=a}^{b}$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$\Diamond$

Another tool that is often important for calculating the mean of a continuous R.V. is integration by parts. The next example will use this, as well as ~~pro~~ identity (II).

e.g. 5: Exponential

Let $X \sim Exp(\lambda)$. There are different parameterizations of the exponential distribution. We will use the one with density $f(x) = \lambda e^{-\lambda x} I(x \geq 0)$.

$$E(X)=\int_0^\infty x\lambda e^{-\lambda x}\,dx \qquad \text{Let } u=x,\ dv=\lambda e^{-\lambda x}dx$$
$$v=-e^{-\lambda x},\ du=dx$$

$$=\int_0^\infty u\,dv$$

$$=uv\Big|_{x=0}^{\infty}-\int_0^\infty v\,du$$

$$=-xe^{-\lambda x}\Big|_{x=0}^{\infty}+\int_0^\infty e^{-\lambda x}dx$$

$$=(-0+0)+\frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda x}dx$$

$$=\frac{1}{\lambda}(1) \quad \text{by (II)}$$

$$=\frac{1}{\lambda}$$

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<u>e.g. 6: Normal</u>

Let $X\sim N(\mu,1)$. It turns out that showing $E(X)=\mu$ takes some work. Let's go through this so you can say you've seen it.

$$E(X)=\int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2}\right]dx$$

$$=\underbrace{\int_{-\infty}^{\infty}\frac{(x-\mu)}{\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2}\right]dx}_{A}+\underbrace{\int_{-\infty}^{\infty}\frac{\mu}{\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)}{2}\right]dx}_{B}$$

We will use different tactics to simplify A & B, so let's do them separately.

⑥

$$B = \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right] dx$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right] dx$$

$$= \mu \cdot 1 \quad \text{by (II)}$$

$$= \mu$$

A is ~~considerably~~ a bit longer and involves doing integration by substitution. I hate tracking the ~~possible values~~ range of integration when I use this tool, so we are going to do the relevant indefinite integral and evaluate at $\pm\infty$ as the last step.

$$\int \frac{x-\mu}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right] dx \quad \text{Let } u=(x-\mu)^2,\ du = 2(x-\mu)\,dx$$

$$= \int \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{-u}{2}\right) du$$

$$= \frac{1}{\sqrt{2\pi}} \int \frac{1}{2} e^{-u/2}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \left(-e^{-u/2}\right)$$

$$= \frac{-1}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right]$$

Now we can use this to get the value of A.

$$A = \int_{-\infty}^{\infty} \frac{(x-\mu)}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right] dx$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right]\Big|_{x=-\infty}^{\infty}$$

$$= 0 - 0 = 0$$

Putting this all together,

$$E(X) = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2}\right] dx$$

$$= A + B$$

$$= 0 + \mu$$

$$= \mu$$

$\diamond$

Next week, we will go over these ideas again with variances.