Stat 330 - Tutorial 6

This week's tutorial is optional, so I am going to cover some bonus material that isn't part of the course: multidimensional change of variables (chapter 2.9). Note that, even though this topic isn't part of the course, it is still important. Many interesting statistical problems rely on transforming a collection of random variables. The tactic I will describe for solving this problem is a bit longer than the univariate version, but most of the ideas are similar. Let's start with the general idea.

Overall strategy

Let's start with two variables, X_1 and X_2. Imagine that we want to find the distribution of some function, $g(X_1, X_2)$. Let $Y_1 = g(X_1, X_2)$. Outside of some simple scenarios, the math turns out to work better if our transformation ends up with the same number of variables it started with. This means we have to define another new variable, $Y_2 := h(X_1, X_2)$. Our next step is to compute the joint distribution of Y_1 and Y_2. Finally, we only really care about Y_1, so we sum or integrate ~~or~~ the joint distribution over all values of Y_2, leaving the marginal distribution of Y_1.

As with unidimensional change of variables, we handle the discrete and continuous cases separately.

② Discrete Random Variables

We will stick to the bivariate case. Handling more than 2 variables is essentially the same. Let X_1, X_2 be discrete r.v.s with PMF p_{X_1,X_2}. Let $Y_1 = g(X_1, X_2)$. We want to find the distribution of Y_1.

First, we must define a new random variable, $Y_2 = h(X_1, X_2)$. We have to be careful here though, because the bivariate transformation $(X_1, X_2) \to (Y_1, Y_2)$ must be invertible. In an abuse of notation that (I think) makes this whole business cleaner, let's define functions $Y_1(X_1, X_2) := Y_1$ and $Y_2(X_1, X_2) := Y_2$ (i.e. as functions, $Y_1 = g$ and $Y_2 = h$). Since the transformation $(X_1, X_2) \to (Y_1, Y_2)$ is invertible, it makes sense to talk about the inverse transformation, $(Y_1, Y_2) \to (X_1, X_2)$. In keeping with our abuse of notation, let's call the components of this inverse transformation $X_1(Y_1, Y_2)$ and $X_2(Y_1, Y_2)$.

We went to a lot of trouble to define the inverse transformation $(Y_1, Y_2) \mapsto (X_1(Y_1, Y_2), X_2(Y_1, Y_2))$, but in practice this step is usually pretty easy. In particular, as long as the forward transformations, $Y_1(X_1, X_2)$ and $Y_2(X_1, X_2)$, are defined with formulas, we can just solve the ~~of equation~~ following system of equations for X_1 and X_2:

$$Y_1 = Y_1(X_1, X_2)$$
$$Y_2 = Y_2(X_1, X_2)$$

This will probably make more sense when we do an example.

e.g. 1: Let $X_1 \sim \text{Poisson}(3)$, $X_2 \sim \text{Poisson}(5)$, $X_1 \perp\!\!\!\perp X_2$ and $Y_1 = X_1 + X_2$.

Show that $Y_1 \sim \text{Poisson}(8)$.

We don't yet have everything we need to do this question, but we can work out the transformations.

First, we need to define a new variable. It is often best to use something very simple. Let's go with $Y_2 = X_2$. Note: we did need to choose Y_2 so that the transformation is invertible. I used to struggle with this part, but as long as you are able to solve for X_1 and X_2 in the next step, then your transformation was invertible.

Next let's solve the following system of equations for X_1 and X_2:

$$\begin{matrix} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{matrix} \iff \begin{matrix} X_1 = Y_1 - Y_2 \\ X_2 = Y_2 \end{matrix}$$

You can use more complicated transformations if you want (e.g. $Y_2 = X_1 - X_2$ has a certain symmetry appeal), but you just end up making more work for yourself if you do. I always go with the simplest transformation I can think of that is still invertible. Unfortunately, if you set one of your variables to a constant, the transformation won't be invertible (this is too simple).

$\diamond$

Once we have solved for the functions $X_1(Y_1, Y_2)$ and $X_2(Y_1, Y_2)$, we need to get the joint distribution of Y_1 and Y_2. This works in basically the same way as the univariate case: plug in $X_1(Y_1, Y_2)$ and $X_2(Y_1, Y_2)$ to the joint distribution of X_1 and X_2.

e.g. 1 cont.: Now we want the joint distribution of Y_1 and Y_2. Remember that $X_1 \sim \text{Poisson}(3)$, $X_2 \sim \text{Poisson}(5)$ and X_1, X_2 are independent. Therefore, the joint distribution of X_1, X_2 is

$$P_{X_1 X_2}(x_1, x_2) = \left(\frac{e^{-3} 3^{x_1}}{x_1!}\right)\left(\frac{e^{-5} 5^{x_2}}{x_2!}\right) = e^{-8} \frac{3^{x_1} 5^{x_2}}{x_1! \, x_2!}$$

Above, we found that $X_1 = Y_1 - Y_2$ and $X_2 = Y_2$, so the joint distribution of Y_1 and Y_2 is

$$P_{Y_1 Y_2}(y_1, y_2) = e^{-8} \frac{3^{x_1(y_1, y_2)} 5^{x_2(y_1, y_2)}}{[x_1(y_1, y_2)]! \, [x_2(y_1, y_2)]!}$$

$$= e^{-8} \frac{3^{y_1 - y_2} 5^{y_2}}{(y_1 - y_2)! \, y_2!}$$

$$= e^{-8} \left(\frac{1}{(y_1 - y_2)! \, y_2!}\right) 3^{y_1 - y_2} 5^{y_2}$$

$$= \frac{e^{-8}}{y_1!} 8^{y_1} \left(\frac{y_1!}{(y_1 - y_2)! \, y_2!}\right) \left(\frac{3}{8}\right)^{y_1 - y_2} \left(\frac{5}{8}\right)^{y_2}$$

$$= \left[\frac{e^{-8} 8^{y_1}}{y_1!}\right] \left[\binom{y_1}{y_2} \left(\frac{5}{8}\right)^{y_2} \left(1 - \frac{5}{8}\right)^{y_1 - y_2}\right]$$

This looks like the product of a Poisson and a Binomial PMF. However, we're missing a part: the domain of y_1 and y_2. Remember that $X_1, X_2 \geq 0$. Therefore, $Y_1 - Y_2 \geq 0$ and $Y_2 \geq 0$ or, $Y_1 \geq Y_2 \geq 0$.

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All that's left is to find the marginal distribution of Y_1. We do this by summing the joint distribution of Y_1, Y_2 over all possible values of Y_2. In symbols,

$$P_{Y_1}(y_1) = \sum_{y_2} P_{Y_1 Y_2}(y_1, y_2)$$

Then we're done.

<u>e.g. 1 cont.</u> We have the joint distribution of Y_1, Y_2 and we know that $0 \le Y_2 \le Y_1$. Let's do the summation.

$$P_{Y_1}(y_1) = \sum_{y_2=0}^{y_1} P_{Y_1 Y_2}(y_1, y_2)$$

$$= \sum_{y_2=0}^{y_1} \left[\frac{e^{-8} 8^{y_1}}{y_1!}\right] \left[\binom{y_1}{y_2} \left(\frac{5}{8}\right)^{y_2} \left(1-\frac{5}{8}\right)^{y_1}\right]$$

$$= \frac{e^{-8} 8^{y_1}}{y_1!} \underbrace{\sum_{y_2=0}^{y_1} \binom{y_1}{y_2} \left(\frac{5}{8}\right)^{y_2} \left(1-\frac{5}{8}\right)^{y_1}}$$

This is just the sum of the PMF of a $Bin(y_1, 5/8)$ random variable over all of its possible values, so the sum is 1.

$$= \frac{e^{-8} 8^{y_1}}{y_1!}$$

This is the PMF of a Poisson(8) r.v., so $Y_1 = X_1 + X_2 \sim \text{Poisson}(8)$

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In fact, the relationship in example 1 holds in general for sums of independent Poisson r.v.s. That is, if $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$ and $X_1 \perp\!\!\!\perp X_2$, then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$. Verifying this is a good exercise.

I wanted to do the continuous case as well, but this tutorial is already long. I might cover continuous r.v.s if we have another bonus tutorial (maybe the week of Rememberance day).