WSABI Summer Research Lab | **Summer 2025**

Lecture 20: Intro to Game Theory

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20.1 Introduction to Strategic Interactions

20.1.1 Motivation

Game theory provides a mathematical framework for analyzing situations where multiple decision-makers interact strategically. In these scenarios, the outcome for each participant depends not only on their own choices but also on the choices of others. Strategic interactions are ubiquitous in economics, political science, biology, sports, and many other fields.

Key Questions: How do rational agents behave when their payoffs depend on others' actions? What outcomes can we expect when multiple decision-makers interact? How do we model and solve these strategic situations?

20.1.2 Examples of Strategic Interactions

Strategic interactions occur in many real-world contexts:

- **Economics:** Firms competing for market share, consumers choosing products
- **Politics:** Countries setting trade policies, candidates positioning in elections
- **Sports:** Teams choosing offensive and defensive strategies, players making tactical decisions
- **Biology:** Animals competing for resources, species evolving strategies
- **Everyday Life:** Roommates deciding who cleans, drivers choosing routes

Game theory provides three essential tools for analyzing these situations:

- **Models (Games):** Mathematical representations of strategic situations
- **Solution Concepts (Equilibria):** Predictions about how rational players will behave
- **Comparative Statics:** How outcomes change when the game parameters change

20.2 Normal-Form Games

20.2.1 Formal Definition

Definition 20.1 (Normal-Form Game). *A normal-form game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of:*

- *$N = \{1, 2, \ldots, n\}$: Set of players*
- *S_i: Strategy set for player i (all possible actions player i can take)*
- *$u_i : S \to \mathbb{R}$: Payoff function for player i, where $S = \prod_{i \in N} S_i$ is the set of all strategy profiles*

20.2.2 Types of Strategies

Definition 20.2 (Pure Strategy)**.** *A pure strategy for player i is a single element $s_i \in S_i$. This represents choosing one specific action with certainty.*

Definition 20.3 (Mixed Strategy)**.** *A mixed strategy for player i is a probability distribution σ_i over S_i. This represents randomizing between different actions according to specified probabilities.*

20.2.3 Best Response

Definition 20.4 (Best Response)**.** *A strategy s_i is a best response to s_{-i} if s_i maximizes player i's payoff given that other players choose s_{-i}:*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \textit{for all } s_i' \in S_i$$

The concept of best response is fundamental to game theory. If a player is rational, they will always choose a best response to what they believe others will do.

20.2.4 Example: Florida Gators Football

Consider a 3rd and 1 situation where the Florida Gators must decide whether to run or pass, and the opponent must decide whether to defend against the run or the pass. The payoff matrix (or strategy form) for this game is:

	Run Defense	Pass Defense
Run	$(40, 60)$	$(70, 30)$
Pass	$(60, 40)$	$(20, 80)$

In this game:

- The Gators choose rows (Run, Pass)
- The opponent chooses columns (Run, Pass)
- Payoffs are written as (Gators' probability of converting, Opponent's probability of stopping them). Note that these probabilities add up to 100%.

Question: What would we expect to happen if the Gators were to call a run play against a run defense?

Solution:

If the Gators call a run play against a run defense, they will have a 40% chance of converting.

20.3 Dominance and Iterated Elimination

20.3.1 Dominance Concepts

Definition 20.5 (Strict Dominance)**.** *A strategy s_i strictly dominates s_i' if:*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \textit{for all } s_{-i}$$

That is, s_i always gives a higher payoff than s_i' regardless of what other players do.

Definition 20.6 (Weak Dominance)**.** *A strategy s_i weakly dominates s_i' if:*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \textit{for all } s_{-i}$$

and the inequality is strict for at least one s_{-i}.

If a strategy is strictly dominated, a rational player will never choose it. This provides a powerful tool for simplifying games.

20.3.2 Iterated Elimination of Strictly Dominated Strategies

Algorithm 1 IESDS Algorithm

```
1: Start with the original game
2: repeat
3:     Identify any strictly dominated strategies
4:     Remove all strictly dominated strategies from the game
5:     Update the reduced game
6: until no strictly dominated strategies remain
```

20.3.3 Example: Simple Dominance

Consider the following game, where Player 1 chooses between strategies a and b, and Player 2 chooses between strategies x, y, and z. The payoffs matrix is:

	x	y	z
a	$(3, 2)$	$(1, 4)$	$(2, 1)$
b	$(4, 1)$	$(2, 3)$	$(5, 0)$

Problem: Apply IESDS to simplify this game.

Solution:

Strategy b strictly dominates strategy a for Player 1 ($4 > 3$, $2 > 1$, $5 > 2$). After eliminating strategy a, we have a 1×3 game:

	x	y	z
b	$(4, 1)$	$(2, 3)$	$(5, 0)$

No further elimination is possible since there's only one strategy left for Player 1.

20.4 Nash Equilibrium

20.4.1 Definition and Intuition

Definition 20.7 (Nash Equilibrium)**.** *A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a Nash equilibrium if for every player i:*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \textit{for all } s_i \in S_i$$

That is, no player can unilaterally improve their payoff by deviating from their equilibrium strategy.

The key insight of Nash equilibrium is that it represents a situation where each player is playing a best response to the strategies of all other players. No player has an incentive to change their strategy given what others are doing.

20.4.2 Finding Nash Equilibria

For games with small strategy sets, we can find Nash equilibria using the following method:

- For each player, identify their best response to each possible strategy profile of other players
- Mark the best response payoffs in the payoff matrix
- A strategy profile is a Nash equilibrium if every player's payoff is marked as a best response

For continuous strategy spaces, we solve the first-order conditions for each player's optimization problem.

Proposition 20.8 (IESDS and Nash Equilibrium)**.** *If IESDS leads to a unique strategy profile, then that profile is the unique Nash equilibrium of the game.*

20.4.3 Example: Prisoner's Dilemma

The Prisoner's Dilemma is one of the most famous examples in game theory. Two criminals are arrested and placed in separate cells. Each prisoner must choose whether to confess or remain silent. The payoff matrix for this game is:

	Confess	Silent
Confess	$(2, 2)$	$(3, 0)$
Silent	$(0, 3)$	$(1, 1)$

In this game:

- Each prisoner chooses rows/columns (Confess, Silent)
- Payoffs represent years in prison (lower is better)

Problem: Apply IESDS and/or best response checking and find the pure strategy Nash equilibrium.

Solution:

Method 1: IESDS Confess strictly dominates Silent for both players ($10 > 1$, $5 > 0$). Therefore, the unique Nash equilibrium is (Confess, Confess) with payoffs (5, 5).

Method 2: Best Response Checking

- When Player 2 chooses Confess: Player 1's payoffs are 5 (Confess) vs 0 (Silent). Best response: Confess
- When Player 2 chooses Silent: Player 1's payoffs are 10 (Confess) vs 1 (Silent). Best response: Confess
- When Player 1 chooses Confess: Player 2's payoffs are 5 (Confess) vs 0 (Silent). Best response: Confess
- When Player 1 chooses Silent: Player 2's payoffs are 10 (Confess) vs 1 (Silent). Best response: Confess

Both players always choose Confess as their best response, so (Confess, Confess) is the unique Nash equilibrium.

20.4.4 Example: 4×4 Game Analysis

Consider the following 4×4 game:

	w	x	y	z
a	$(15,15)$	$(20,6)$	$(15,8)$	$(12,9)$
b	$(22,6)$	$(18,10)$	$(16,12)$	$(17,12)$
c	$(8,3)$	$(19,25)$	$(6,3)$	$(15,11)$
d	$(10,14)$	$(21,15)$	$(11,18)$	$(20,5)$

Problem: Apply IESDS and find all Nash equilibria.

Solution:

Let's find all Nash equilibria using best response checking. We'll systematically identify each player's best responses to every possible strategy of the other player.

Player 1's best responses to each of Player 2's strategies

When Player 2 chooses w:

- Player 1's payoffs: a=15, b=22, c=8, d=10
- Best response: b (payoff 22)

When Player 2 chooses x:

- Player 1's payoffs: a=20, b=18, c=19, d=21
- Best response: d (payoff 21)

When Player 2 chooses y:

- Player 1's payoffs: a=15, b=16, c=6, d=11
- Best response: b (payoff 16)

When Player 2 chooses z:

- Player 1's payoffs: a=12, b=17, c=15, d=20
- Best response: d (payoff 20)

Player 2's best responses to each of Player 1's strategies

When Player 1 chooses a:

- Player 2's payoffs: w=15, x=6, y=8, z=9
- Best response: w (payoff 15)

When Player 1 chooses b:

- Player 2's payoffs: w=6, x=10, y=12, z=12
- Best responses: y and z (both payoff 12)

When Player 1 chooses c:

- Player 2's payoffs: w=3, x=25, y=3, z=11
- Best response: x (payoff 25)

When Player 1 chooses d:

- Player 2's payoffs: w=14, x=15, y=18, z=5
- Best response: y (payoff 18)

Identify Nash equilibria

Result: The unique pure strategy Nash equilibrium is (b,y) with payoffs (16,12).

20.4.5 Example: Corn Farmers

Consider two farmers simultaneously choosing to grow 0, 40, or 60 tons of corn. The demand for corn is given by $P = 120 - Q$, where $Q = q_1 + q_2$ is the total quantity of corn produced by the two farmers. Assume that the cost of running a farm is constant at \$1500. Then the payoff (profit) functions are:

$$\text{Farmer 1 profit} = \begin{cases} 0 & \text{if } q_1 = 0 \\ (120 - q_1 - q_2)q_1 - 1500 & \text{otherwise} \end{cases}$$
$$\text{Farmer 2 profit} = \begin{cases} 0 & \text{if } q_2 = 0 \\ (120 - q_1 - q_2)q_2 - 1500 & \text{otherwise} \end{cases}$$

Problem: Write the strategic form and find all Nash equilibria.

Solution:

Construct the payoff matrix First, let's construct the payoff matrix by calculating profits for each combination:

	0	40	60
0	$(0,0)$	$(0,1700)$	$(0,2100)$
40	$(1700,0)$	$(100,100)$	$(-700,-300)$
60	$(2100,0)$	$(-300,-700)$	$(-1500,-1500)$

Find best responses If the rival produces $0, 40, 60$ the profit-maximising quantities are respectively $60, 40, 0$. Hence:

Identify Nash equilibria The Nash equilibrium set is:

$$\text{NE set } = \{(60,0),(40,40),(0,60)\}.$$

No strategy is strictly dominated, so IESDS cannot reduce the game further.

Interpretation: There are three pure strategy Nash equilibria. The symmetric equilibrium (40,40) gives both farmers positive profits, while the asymmetric equilibria give one farmer zero profit and the other a high profit.

20.5 Mixed Strategy Nash Equilibrium

20.5.1 Motivation

Some games have no pure strategy Nash equilibrium. For example, in Matching Pennies:

- Player 1 wins if both choose the same side (both heads or both tails)
- Player 2 wins if they choose different sides

In such cases, players may randomize between strategies, leading to mixed strategy equilibria.

20.5.2 Properties of Mixed Strategy Equilibria

Lemma 20.9 (Best Response Support)**.** *In a mixed strategy Nash equilibrium, players only put positive probability on strategies that are best responses to the equilibrium strategies of other players.*

Lemma 20.10 (Indifference Principle)**.** *For a mixed strategy to be optimal, the expected payoff from each strategy in its support must be equal. This gives us the key condition for finding mixed strategy equilibria: players must be indifferent between all strategies they play with positive probability.*

20.5.3 Example: Florida Gators Football

Let's return to our Florida Gators example and find the mixed strategy equilibrium. The payoff matrix was:

	Run-D	Pass-D
Run	$(40, 60)$	$(70, 30)$
Pass	$(60, 40)$	$(20, 80)$

Problem: Find the mixed strategy Nash equilibrium.

Solution: Let the Gators *Run* with probability p and the defence choose *Run-D* with probability q.

Gators' expected gain from:

- Running: $40q + 70(1 - q) = 70 - 30q$
- Passing: $60q + 20(1 - q) = 20 + 40q$

Indifference $\Rightarrow 70 - 30q = 20 + 40q \Rightarrow q = \frac{5}{7}$.

Defence's expected stop-rate from:

- Run-D: $60p + 20(1 - p) = 20 + 40p$
- Pass-D: $30p + 80(1 - p) = 80 - 50p$

Indifference $\Rightarrow 20 + 40p = 80 - 50p \Rightarrow p = \frac{2}{3}$.

The mixed strategy Nash equilibrium is:

$$\boxed{\text{Gators } \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \quad \text{Defence } \left(\tfrac{5}{7}, \tfrac{2}{7}\right)}$$

The conversion probability is $\frac{1}{3}(70) + \frac{2}{3}(20) = \frac{110}{3} \approx 36.7\%$.

20.5.4 Example: Soccer Penalty Kick

A kicker chooses where to shoot (L, C, R); a goalkeeper chooses where to dive (L, M, R). The pair (score prob., save prob.) for each match-up is

	Dive L	Stay M	Dive R
Shoot L	$(0.4, 0.6)$	$(0.9, 0.1)$	$(0.9, 0.1)$
Shoot C	$(0.9, 0.1)$	$(0.2, 0.8)$	$(0.9, 0.1)$
Shoot R	$(0.9, 0.1)$	$(0.9, 0.1)$	$(0.4, 0.6)$

Because save prob. $= 1 -$ score prob. the game is *zero–sum*. Work with the kicker's payoff matrix

$$A = \begin{pmatrix} 0.40 & 0.90 & 0.90 \\ 0.90 & 0.20 & 0.90 \\ 0.90 & 0.90 & 0.40 \end{pmatrix}.$$

to find the mixed strategy Nash equilibrium for both players.

Solution:

1. Goalkeeper mix that keeps the kicker indifferent.

Let $q = (q_1, q_2, q_3)$ be the probabilities of Dive L, Stay M, Dive R with $q_3 = 1 - q_1 - q_2$. The kicker's expected score when shooting:

$$\pi_L = 0.40q_1 + 0.90q_2 + 0.90q_3,$$
$$\pi_C = 0.90q_1 + 0.20q_2 + 0.90q_3,$$
$$\pi_R = 0.90q_1 + 0.90q_2 + 0.40q_3.$$

Setting $\pi_L = \pi_C = \pi_R$ gives:

$$q_3 = q_1, \qquad q_2 = \frac{5}{7}q_1.$$

Normalising ($q_1 + q_2 + q_3 = 1$) yields:

$$\boxed{q_1 = q_3 = \tfrac{7}{19} \approx 0.368}, \qquad \boxed{q_2 = \tfrac{5}{19} \approx 0.263}.$$

Each shot now scores with probability:

$$v = \pi_L = \pi_C = \pi_R = \frac{13.6}{19} \approx 0.716.$$

2. Kicker mix that keeps the goalkeeper indifferent.

Let $p = (p_1, p_2, p_3)$ be the probabilities of shooting L, C, R ($p_3 = 1 - p_1 - p_2$). The kicker's expected score if the keeper:

$$s_L = 0.40p_1 + 0.90p_2 + 0.90p_3,$$
$$s_M = 0.90p_1 + 0.20p_2 + 0.90p_3,$$
$$s_R = 0.90p_1 + 0.90p_2 + 0.40p_3.$$

Solving $s_L = s_M = s_R$ gives:

$$\boxed{p_1 = p_3 = \tfrac{7}{19} \approx 0.368}, \qquad \boxed{p_2 = \tfrac{5}{19} \approx 0.263}.$$

Mixed-strategy Nash equilibrium:

$$\textbf{Kicker: } \left(\tfrac{7}{19}, \tfrac{5}{19}, \tfrac{7}{19}\right), \qquad \textbf{Goalkeeper: } \left(\tfrac{7}{19}, \tfrac{5}{19}, \tfrac{7}{19}\right), \qquad v = \tfrac{13.6}{19} \approx 0.716.$$

Interpretation: Both players use the same symmetric mix: about 37% left, 26% middle, 37% right. At these frequencies the kicker converts $\approx 71.6\%$ of penalties and the goalkeeper is exactly indifferent among all three dives, so neither has a profitable deviation.

20.6 Bayesian and Signalling Games

20.6.1 Motivation

Many strategic interactions involve *private information*: one player knows something that others do not. Bayesian games provide a framework for analyzing such situations where players have incomplete information about each other's characteristics or payoffs.

Key Questions: How do players behave when they have private information? How can players signal their type to others? What equilibria emerge when information is asymmetric?

20.6.2 Signalling Games

A signalling game is a dynamic game with incomplete information that proceeds in three stages:

1. Nature selects the Sender's *type* $t \in T$ according to a known prior distribution $p(t)$
2. The Sender observes their type t and chooses a *signal* $s \in S$
3. The Receiver observes the signal s (but not the type t), forms beliefs about the Sender's type, and chooses an *action* $a \in A$

Definition 20.11 (Perfect Bayesian Nash Equilibrium)**.** *A perfect Bayesian Nash equilibrium (PBNE) consists of:*

(i) A strategy profile where each player's strategy is optimal given their beliefs

(ii) A belief system where beliefs are updated using Bayes' rule whenever possible

(iii) Beliefs are consistent with the equilibrium strategies

20.6.3 Example: Bluffing in One-Card Poker

A \$2 pot (each player has contributed \$1) is on the table. Player 1's card is *High* (H) with probability $\frac{1}{2}$ or *Low* (L) with probability $\frac{1}{2}$. Move order:

1. Player 1 **Bet**s \$1 or **Check**s.

2. If a bet is made, Player 2 **Call**s or **Fold**s.

	Call	Fold
Bet, H	$+2$	$+1$
Bet, L	-2	$+1$
Check, H	$+1$	$-$
Check, L	-1	$-$

Notation.

$$\sigma = \Pr(\text{Low bets}), \qquad c = \Pr(\text{Call} \mid \text{Bet}).$$

High cards always bet (we verify that below).

Low card's incentive

$$\mathbb{E}[\text{Bluff} \mid L] = (1-c)(+1) \;+\; c(-2) = 1 - 3c,$$
$$\mathbb{E}[\text{Check} \mid L] = -1.$$

The low card prefers

$$\text{bluff} \iff 1 - 3c > -1 \quad \iff \quad c < \tfrac{2}{3}.$$

He is indifferent at $c = \frac{2}{3}$.

Caller's incentive

Posterior that the bet comes from a high card:

$$\pi \;=\; \Pr(H \mid \text{Bet}) \;=\; \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}\sigma} \;=\; \frac{1}{1+\sigma}.$$

Expected value of **Call** vs. **Fold** (-1):

$$\mathbb{E}[\text{Call}] = (-2)\pi + (+2)(1-\pi) = 2 - 4\pi = 2 - \frac{4}{1+\sigma}.$$

Call is preferred when $2-4\pi > -1$, i.e. $\sigma > \frac{1}{3}$; fold is preferred when $\sigma < \frac{1}{3}$; indifferent at $\sigma = \frac{1}{3}$.

Mutual best responses

Caller's c	Low card's best σ
$c < \frac{2}{3}$	$\sigma = 1$
$c = \frac{2}{3}$	σ any in $[0, 1]$
$c > \frac{2}{3}$	$\sigma = 0$

Low card's σ	Caller's best c
$\sigma < \frac{1}{3}$	$c = 0$
$\sigma = \frac{1}{3}$	c any in $[0, 1]$
$\sigma > \frac{1}{3}$	$c = 1$

The tables intersect uniquely at

$$\boxed{\sigma = \frac{1}{3}, \qquad c = \frac{2}{3}}.$$

Perfect Bayesian Nash Equilibrium

$$\boxed{\begin{cases} \textbf{High card}: & \text{Bet} \\ \textbf{Low card}: & \text{Bet with prob. } \frac{1}{3},\ \text{Check with } \frac{2}{3} \\ \textbf{Caller}: & \text{Call with prob. } \frac{2}{3} \text{ after any bet.} \end{cases}}$$

Verification.

- Given $c = \frac{2}{3}$, the low card's expected bluff payoff $1 - 3c = -1$ equals checking, so mixing is optimal.
- Given $\sigma = \frac{1}{3}$, the caller's expected value of calling is $2 - 4 \cdot \frac{3}{4} = -1$, identical to folding, so mixing is optimal.
- High cards earn $\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 1 = \frac{5}{3} > 1$ by betting, so betting is optimal.

Hence the strategy–belief pair above satisfies sequential rationality and Bayes' rule, completing the PBNE.

20.6.4 Example: Education as a Signal

Consider a labor market where workers have private information about their productivity and can signal their type through education.

Two types of workers exist: High productivity (H) and Low productivity (L). High-type productivity = \$120, Low-type productivity = \$60. Workers can choose to acquire a college degree or not. Degree costs: \$20 for high-type, \$40 for low-type. The firm pays \$120 if degree observed, \$60 otherwise.

The payoff matrix for workers is:

Type	Degree	No Degree
H	$120 - 20 = 100$	$60 - 0 = 60$
L	$120 - 40 = 80$	$60 - 0 = 60$

Problem: Does a separating perfect Bayesian Nash equilibrium exist where high-types get degrees and low-types do not?

Solution:

Check high-type incentives High-type payoff with degree: $120 - 20 = 100$ High-type payoff without degree: $60 - 0 = 60$ Since $100 > 60$, high-types prefer to get degrees.

Check low-type incentives Low-type payoff with degree: $120 - 40 = 80$ Low-type payoff without degree: $60 - 0 = 60$ Since $80 > 60$, low-types also prefer degrees.

Analyze equilibrium If both types get degrees, the signal becomes uninformative. The firm's beliefs would be:

$$P(H|\text{Degree}) = \frac{P(\text{Degree}|H)P(H)}{P(\text{Degree})} = \frac{1 \cdot \frac{1}{2}}{1} = \frac{1}{2}$$

Expected productivity $= \frac{1}{2} \cdot 120 + \frac{1}{2} \cdot 60 = 90$

If the firm pays \$90 to all degree-holders, then:

- High-type payoff: $90 - 20 = 70 < 60$ (prefers no degree)
- Low-type payoff: $90 - 40 = 50 < 60$ (prefers no degree)

This creates a contradiction. For a separating equilibrium, the degree cost for low-types must be sufficiently high.

Alternative: Higher degree cost for low-types If degree costs were \$20 for high-types and \$70 for low-types:

- High-type payoff with degree: $120 - 20 = 100 > 60$
- Low-type payoff with degree: $120 - 70 = 50 < 60$

Then the separating equilibrium exists:

$$\boxed{\text{High type: get degree;} \quad \text{Low type: skip degree.}}$$

$$\boxed{\text{Firm: pay \$120 if degree, \$60 if none.}}$$

Separating condition. A degree separates types if and only if

$$c_H \leq 60 < c_L.$$

For example $c_H = 20,\ c_L = 70$ satisfies the condition:

$$\text{High type: } 120 - 20 = 100 > 60 \qquad \text{(earns the degree)}$$

$$\text{Low type: } 120 - 70 = 50 < 60 \qquad \text{(skips the degree)}$$

The firm then rationally pays \$120 to degree-holders and \$60 otherwise.

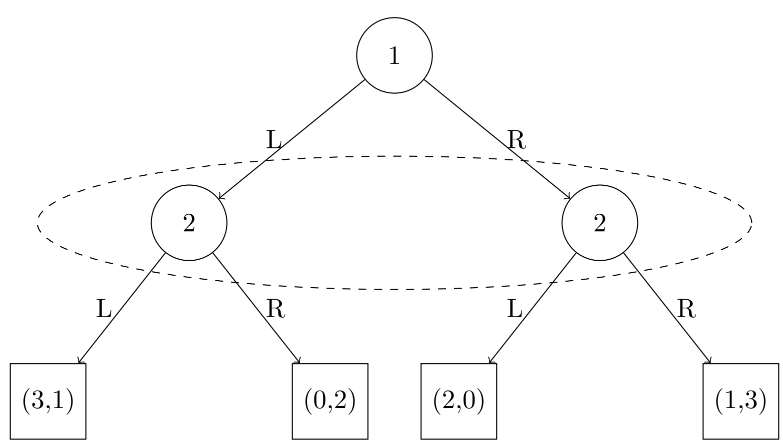
20.7 Extensive-Form Games

20.7.1 Game Trees and Information Sets

Extensive-form games model sequential interactions using game trees:

- Nodes represent decision points
- Edges represent possible actions
- Information sets group nodes where a player cannot distinguish between them
- Payoffs are assigned to terminal nodes

Example 20.12 (Simple Game Tree)**.** Consider a simple sequential game where Player 1 moves first, then Player 2 responds:



How to Read the Game Tree:

- **Decision Nodes** (circles): Points where players make choices. The number inside indicates which player moves.
- **Terminal Nodes** (rectangles): End points showing final payoffs for both players.
- **Edges** (arrows): Represent possible actions, labeled with the action name (L or R).

- **Information Sets** (dashed ellipses): Group nodes where a player cannot distinguish between them. Here, Player 2 doesn't know whether Player 1 chose L or R first.

Information Sets and Perfect vs. Imperfect Information:

- **Perfect Information**: Each player knows all previous moves when making their decision.
- **Imperfect Information**: Players may not know some previous moves, represented by information sets.
- In this example, Player 2 has imperfect information because they don't observe Player 1's choice.

Problem: Find all Nash equilibria in this game.

Solution:

Since Player 2 doesn't know Player 1's move, we analyze Player 2's best responses:

- If Player 1 chose L: Player 2 prefers R ($2 > 1$)
- If Player 1 chose R: Player 2 prefers R ($3 > 0$)

Player 1 anticipates this and chooses L ($3 > 0$). The Nash equilibrium is (L, R).

Note: This is also the subgame perfect Nash equilibrium, but we'll learn more about that concept later.

20.7.2 Backward Induction

Algorithm 2 Backward Induction

1. Start from the terminal nodes of the game tree
2. For each decision node, find the optimal action for the player at that node
3. Replace each decision node with the payoff from the optimal action
4. Move up the tree and repeat until reaching the root

20.7.3 Subgame Perfect Nash Equilibrium

Definition 20.13 (Subgame Perfect Nash Equilibrium)**.** *A strategy profile is subgame perfect if it induces a Nash equilibrium in every subgame of the original game.*

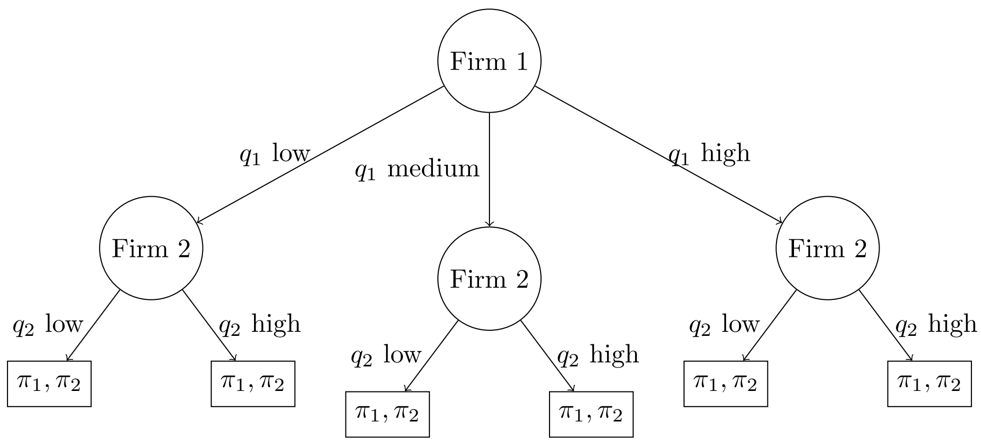
Subgame perfection eliminates Nash equilibria that rely on non-credible threats by requiring that strategies be optimal in every subgame.

20.7.4 Example: Stackelberg Duopoly

Consider a Stackelberg duopoly where Firm 1 (leader) chooses quantity first, then Firm 2 (follower) observes Firm 1's choice and responds. This is a sequential game where:

- **Firm 1 (Leader)** moves first and chooses quantity q_1
- **Firm 2 (Follower)** observes q_1 and then chooses quantity q_2
- **Market price** is determined by total supply: $P = 100 - (q_1 + q_2)$
- **Profits** are: $\pi_1 = (100 - q_1 - q_2)q_1$ and $\pi_2 = (100 - q_1 - q_2)q_2$

The game tree for this sequential interaction is:



Key Features of Stackelberg Competition:

- **Perfect Information**: Firm 2 observes Firm 1's choice before making its own
- **First-Mover Advantage**: Firm 1 can commit to a quantity, knowing Firm 2 will respond optimally
- **Strategic Interaction**: Firm 1's choice affects Firm 2's best response, which in turn affects Firm 1's optimal choice

Problem: Find the subgame perfect Nash equilibrium using backward induction.

Solution:

Using backward induction:

Firm 2's best response function For any q_1 chosen by Firm 1, Firm 2 maximizes:

$$\pi_2 = (100 - q_1 - q_2)q_2$$
$$\frac{\partial \pi_2}{\partial q_2} = 100 - q_1 - 2q_2 = 0$$
$$q_2 = \frac{100 - q_1}{2}$$

Firm 1 anticipates Firm 2's response Firm 1 maximizes its profit, knowing Firm 2 will choose $q_2 =$

$\frac{100-q_1}{2}$:

$$\pi_1 = \left(100 - q_1 - \frac{100-q_1}{2}\right) q_1$$
$$= \left(50 - \frac{q_1}{2}\right) q_1$$
$$= 50q_1 - \frac{q_1^2}{2}$$

Firm 1's optimal choice

$$\frac{\partial \pi_1}{\partial q_1} = 50 - q_1 = 0$$
$$q_1^* = 50$$

Firm 2's equilibrium choice

$$q_2^* = \frac{100-50}{2} = 25$$

The subgame perfect Nash equilibrium is $(q_1^*, q_2^*) = (50, 25)$.

Interpretation: The leader produces more than the follower, giving Firm 1 a first-mover advantage. Total output is 75, which is more than monopoly output (50) but less than Cournot duopoly output (66.7).

20.8 Repeated Games

20.8.1 Finitely Repeated Games

Proposition 20.14 (Folk Theorem for Finitely Repeated Games)**.** *If a stage game has a unique Nash equilibrium, then the unique subgame perfect equilibrium of the finitely repeated game is to play the stage game Nash equilibrium in every period.*

This result follows from backward induction: in the last period, players must play the stage game equilibrium. Given this, in the second-to-last period, they also play the stage game equilibrium, and so on.

20.8.2 Infinitely Repeated Games

In infinitely repeated games, cooperation can be sustained through trigger strategies.

Definition 20.15 (Grim Trigger Strategy)**.** *A grim trigger strategy specifies:*

- *Cooperate as long as all players have cooperated in the past*
- *Defect forever if any player has ever defected*

20.8.3 Condition for Cooperation

Grim trigger strategies can sustain cooperation if:

$$\delta \geq \frac{\pi_D - \pi_C}{\pi_D - \pi_N}$$

where π_C is the cooperative payoff, π_D is the deviation payoff, π_N is the punishment payoff, and δ is the discount factor.

20.8.4 Example: Finitely Repeated PD

Example 20.16 (Two periods)**.** The stage game is the familiar Prisoner's Dilemma with payoffs (years in prison; lower is better):

	Confess	Silent
Confess	$(2,2)$	$(3,0)$
Silent	$(0,3)$	$(1,1)$

The game is played exactly twice.

Problem: Find the subgame perfect Nash equilibrium using backward induction.

Solution:

Stage pay-offs $(C, D) = (2, 1, 3, 0)$ in the usual order. Backward induction $\Rightarrow$ (D, D) in both rounds.

Stage 2 The continuation game is the one–shot PD, whose unique Nash equilibrium is (D, D).

Stage 1 Anticipating (D, D) next period, each player again prefers D now (because $3 > 2$ if the opponent confesses and $1 > 0$ if the opponent stays silent).

Hence the unique subgame-perfect Nash equilibrium is (D, D) in both rounds.

20.8.5 Example: Infinite Collusion Game with Grim Trigger

Example 20.17 (Collusion)**.** Two firms repeatedly face

	Collude	Defect
Collude	$(2,1)$	$(1,4)$
Defect	$(3,3)$	$(0,0)$

They discount future payoffs by $\delta \in (0, 1)$. Both adopt the *grim-trigger* strategy: collude forever unless somebody defects, in which case defect forever.

Problem: Find the minimum discount factor required for grim trigger to be a subgame perfect Nash equilibrium.

Solution:

With pay-offs $\pi_C = 2, 1$, $\pi_D = 3, 4$, $\pi_N = 0, 0$, the deviation test gives $\delta_1 \geq \frac{1}{3}$, $\delta_2 \geq \frac{3}{4}$. Grim trigger is an SPNE iff $\delta \geq \frac{3}{4}$.

Firm 1's deviation check

$$\text{Cooperate path: } \frac{2}{1-\delta},$$
$$\text{One-shot deviation: } 3 + \frac{\delta \cdot 0}{1-\delta}.$$

Sustainability requires $2/(1-\delta) \geq 3 \Rightarrow \delta \geq \frac{1}{3}$.

Firm 2's deviation check

$$\frac{1}{1-\delta} \geq 4 \implies \delta \geq \frac{3}{4}.$$

Threshold Both constraints must hold, so grim trigger is an SPNE iff

$$\boxed{\delta \geq \frac{3}{4}.}$$

The follower (Firm 2) needs a higher discount factor to resist the temptation of an immediate payoff 4, hence the tougher bound.