WSABI Summer Research Lab **Summer 2025**

Lab 20: Intro to Game Theory

Instructor: Jonathan Pipping *Author: JP*

20.1 Penalty Kicks

20.1.1 Review from Lecture 20

In Lecture 20, we derived one mixed-strategy Nash equilibrium by hand. Our payoff matrix was

$$A = \begin{pmatrix} 0.40 & 0.90 & 0.90 \\ 0.90 & 0.20 & 0.90 \\ 0.90 & 0.90 & 0.40 \end{pmatrix}$$

where A_{ij} represents the probability that the kicker scores when shooting to position i (Left, Center, Right) and the goalkeeper dives to position j (Left, Center, Right). The diagonal elements represent the scoring probability when both players choose the same direction, while off-diagonal elements represent scoring probabilities when they choose different directions.

In this zero-sum game, the kicker (row player) wants to maximize the scoring probability, while the goalkeeper (column player) wants to minimize it. We found the mixed-strategy Nash equilibrium where both players randomize their strategies optimally for the given payoff matrix in lecture:

- Kicker's equilibrium strategy: $p^\star = (p_L^\star = \frac{7}{19}, p_C^\star = \frac{5}{19}, p_R^\star = \frac{7}{19})$
- Goalkeeper's equilibrium strategy: $q^\star = (q_L^\star = \frac{7}{19}, q_C^\star = \frac{5}{19}, q_R^\star = \frac{7}{19})$
- Equilibrium scoring probability: $v^\star \approx 0.716$

In this lab, we'll explore how these equilibrium strategies change as we vary the payoff matrix parameters.

20.1.2 Your Task

1. Write a function in `R` that takes a 3×3 payoff matrix A (kicker's scoring probabilities) that returns a row vector containing 7 values: the kicker's equilibrium strategy $p^\star = (p_L^\star, p_C^\star, p_R^\star)$, the goalkeeper's equilibrium strategy $q^\star = (q_L^\star, q_C^\star, q_R^\star)$, and the equilibrium scoring probability $v^\star$.
2. Assume that the player and goalkeeper are equally strong going to their left and right sides. Then the payoff matrix has the form:
$$A = \begin{pmatrix} a & b & c \\ d & e & d \\ c & b & a \end{pmatrix}$$
Consider a grid of values for (a, b, c, d, e) where $a, e \in [0.0, 0.5]$ and $b, c, d \in [0.5, 1.0]$. For each combination of these values, call the function you wrote in Part 1 to find the equilibrium strategies and scoring probabilities.

3. Plot a 2-dimensional heat map of the equilibrium probability that the kicker shoots to the center as a function of d and e.

4. Plot a 3-dimensional heat map of the equilibrium probability that the goalkeeper dives to the left as a function of a, d, and c.

5. Now we will break the symmetry of the game by assuming that the shooter is better at shooting to the left than to the right. Then the payoff matrix has the form:

$$A = \begin{pmatrix} a & b & c \\ d & e & d \\ f & g & h \end{pmatrix}$$

   Consider a constrained grid of values for (a, b, c, d, e, f, g, h) where $a, e, h \in [0.0, 0.5]$ and $b, c, d, f, g \in [0.5, 1.0]$, where $a > h, b > g, c > f$ (representing the shooter's better left-side shooting). For each combination of these values, call the function you wrote in Part 1 to find the equilibrium strategies and scoring probabilities.

6. Plot a 2-dimensional heat map of the equilibrium probability that the kicker shoots to the center as a function of d and e. Compare this heat map to the one in Part 3.

7. Plot a 3-dimensional heat map of the equilibrium probability that the goalkeeper dives to the left as a function of a, d, and f. Compare this heat map to the one in Part 4.