Introduction to Numerical Analysis and Applications

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Homework #2 (**DUE**: April 19th)

This homework covers the materials from the past weeks, including 1) *Linear Systems* and 2) *Regression & Curve Fitting*.

1. We've been shown that iterative methods for solving linear system of equations may not converge, and tricks that make them do. Now let's have a look the circuit problem in Figure 1.
   a. (10%) <u>Figure.</u> Develop your Jacobi method for matrix inverse, and find the currents flowing through each resistors in the circuit with the following specs: $[R_{12}, R_{23}, R_{34}, R_{45}, R_{56}, R_{25}] = [8,12,5,15,30,25]\ \Omega$, and $[V_1, V_6] = [100, 0]\ V$. Does your Jacobi method converge? If so, plot your solutions at each iteration. (Let's use the stopping criterion of $|\varepsilon_t| < 0.1\%$)

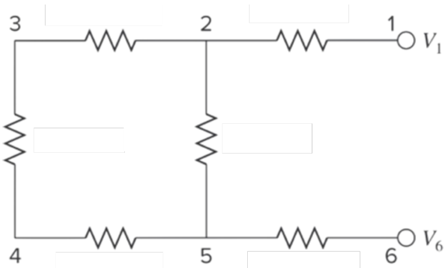

Figure 1 Circuit Layout.

   b. (10%) <u>Figure.</u> Redo the above with Gauss-Seidel method instead.
   c. (20%) <u>Figure.</u> Use relaxation with $\lambda = [0.3, 0.7, 1.2]$, and discuss how these weights affect the convergence of each iterative method.
2. (15%) <u>Figure</u>. Use the Monte Carlo method to determine the value of π with the area of a circle. Investigate how this method converges (i.e., the expected value of the absolute error vs. number of random numbers used to determine the area).
3. Let's have some fun investigating noise characteristics and the impact on the linear regression. *Been there, done that* so I know it's not difficult at all ☺. MRI signal is literally *complex*, not just how MR signal is formed, but also it consists of real and imaginary components, i.e. complex numbers. "Phase", defined as $\arctan(y/x)$, is one of the many essential information MRI provides to help differentiate diseased tissues from healthy surroundings, but for the time being let's simply focus on the numerical processing. Before we start, there are several tools we need:
   a. (5%) Random number generator (RNG). Use any package or write yourself one, to generate a group of random numbers (more than 20,000 points, for example) that has a distribution of mean = 0 and standard deviation = 1. Show the distribution with a histogram in a <u>figure</u> and verify your RNG with the analytical form of normal distribution $\mathcal{N}(\mu, \sigma)$: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$.
   Hint: 1. Notice that this is a normalized form, i.e. $\int f(x) = 1$, *so you should normalize your histogram accordingly to make them fit (i.e. use probability density function instead of counts). 2. Also pay attention to the width of each "bin" of your histogram.*

b. (5%) Similar to the slides I showed in the class, let's have a closer look at the noise distribution of the phase measurement. Let's start with a 2D vector of length (r) = 200, with a phase angle (θ) of 45° with respect to x-axis. As in a polar coordinate, the real and imaginary parts of this vector are $r\cos\theta$ and $r\sin\theta$, respectively. Now add your artificial noise ($\mu = 0, \sigma = 1$) to both real and imaginary parts, with 2 distinct groups of randomly generated number (i.e. the noise you add to the real part is NOT the same as that you add to the imaginary part). Find θ of this group of noisy vectors: you should get a group of *N* measurements of θ, where *N* is the amount of randomly generated number in your code (supposedly *N* > 20,000). Use a histogram again and observe the distribution of θ. Does it follow a normal distribution?

c. (10%) Do the above simulation but with the vector length (r) equal to 80, 40, 20, and 10. How does the distribution of θ change accordingly in terms of μ_θ and σ_θ? Show your results in separate <u>figures</u> for each parameter.

d. (5%) I believe that you see it now. The distribution of θ estimates also follows normal distribution, with σ_θ as a function of r. Specify their relationship, and discuss if this observation make sense to you? *Hint: a little trigonometry would be helpful to explain what you found.*

e. (10%) All the above should be just a piece of cake for you. Let's now get the real deal: the complex signal can be partially modeled as $S(t) = S_0 e^{-\frac{t}{T_2}+i(\omega t+\theta_0)}$, meaning that it not only decays in magnitude, but also accumulates phase angle along with time. The model parameters are S_0, T_2 and ω. Simulate the complex signal at t = 10, 50, and 90, with S_0 = 100, T_2 = 32 and $\omega = {}^{\pi}/_{200}$. Again use your RNG to emulate normal random noise ($\sigma = 1$) on both real and imaginary components, and then do *N* (>20,000) regressions on θ vs. t to estimate ω. How does your ω look in a histogram (<u>figure</u>)? Include μ_ω and σ_ω in your figure.

f. (10%) Lastly use your knowledge of the relationship between σ_θ and the signal magnitude, $|S(t)|$, design a weighting function to "equalize" the noise distribution before doing the regression. Compare the regression results with and without such weighting function.

More on this:

For Q3, the relationship between of σ_θ and r is in fact a little bit more complicated than what we found in section Q1-d. For more on this, an optional reading (1) is provided for those who are interested.

1. Gudbjartsson H, Patz S. The Rician distribution of noisy MRI data. Magn Reson Med 1995;34(6):910-914.