Problem 2.32

Prove that
$\sum_{k\geqslant 0} \min(k, x \dot{-} k) = \sum_{k\geqslant 0} (x \dot{-} (2k+1)) \quad \forall x \in \mathbb{R}^+$
where $a \dot{-} b = \max(0, a-b)$

$\sum_{k\geqslant 0} A_k = \sum_{k\geqslant 0} B_k$

Derivation using Iverson brackets:

$$\max(0, x \dot{-} k) = 0[0 > x-k] + (x-k)[x-k \geqslant 0] = (x-k)[x-k \geqslant 0]$$

$$\begin{aligned} A_k &= \min(k, x \dot{-} k) \\ &= \min(k, \max(0, x-k)) \\ &= k[k < \max(0, x-k)] + \max(0, x-k)[k \geqslant \max(0, x-k)] = \\ &= k[k < (x-k)[x-k \geqslant 0]] + (x-k)[x-k \geqslant 0][k \geqslant (x-k)[x-k \geqslant 0]] \end{aligned}$$

Case I $x-k \geqslant 0$

$$\begin{aligned} A_k &= k\,[k < x-k] + (x-k)[x-k \geqslant 0]\,[k \geqslant x-k] \\ &= k\left[k < \left\lfloor \frac{x}{2} \right\rfloor\right] + (x-k)[x-k \geqslant 0]\left[k \geqslant \left\lfloor \frac{x}{2} \right\rfloor\right] \\ &= k\left[k < \left\lfloor \frac{x}{2} \right\rfloor\right] + (x-k)\left[k \geqslant \left\lfloor \frac{x}{2} \right\rfloor\right] \end{aligned}$$

Case II $x-k < 0$

$$A_k = k[k<0] + (x-k)\cdot 0 \cdot [k \geqslant 0] = 0$$

$$A_k = \begin{cases} 0 & , k > x \\ k & , 0 \leqslant k < \left\lfloor \frac{x}{2} \right\rfloor \\ x-k & , \left\lfloor \frac{x}{2} \right\rfloor \leqslant k \leqslant 2\left\lfloor \frac{x}{2} \right\rfloor - 1 \end{cases}$$

The upper bound in the last branch above is the largest integer k that doesn't exceed x (if it did we'd be in the 1st branch instead, and all branches need to be disjoint).

$$\begin{aligned} B_k &= x \dot{-} (2k+1) = \max(0, x-2k-1) \\ &= (x-2k-1)[x-2k-1 \geqslant 0] \\ &= (x-2k-1)\left[k \leqslant \left\lfloor \frac{x-1}{2} \right\rfloor\right] \end{aligned}$$

$$B_k = \begin{cases} 0 & , k > \left\lfloor \frac{x-1}{2} \right\rfloor \\ x-2k-1 & , 0 \leqslant k \leqslant \left\lfloor \frac{x-1}{2} \right\rfloor \end{cases}$$

Suppose $2n \leqslant x < 2n+1 \implies \left\lfloor \frac{x}{2} \right\rfloor = n, \left\lfloor \frac{x-1}{2} \right\rfloor = n-1$

$$\begin{aligned} \sum_{k\geqslant 0} A_k &= \sum_{0\leqslant k < \lfloor \frac{x}{2} \rfloor} k + \sum_{\lfloor \frac{x}{2} \rfloor \leqslant k \leqslant 2\lfloor \frac{x}{2} \rfloor - 1} (x-k) \\ &= \sum_{0\leqslant k\leqslant n-1} k + \sum_{n\leqslant k\leqslant 2n-1} (x-k) \\ &= \sum_{0\leqslant k\leqslant n-1} k + \sum_{0\leqslant k\leqslant n-1} (x-k-n) \\ &= \sum_{0\leqslant k\leqslant n-1} (k+x-k-n) \\ &= \sum_{0\leqslant k\leqslant n-1} (x-n) = n(x-n) \end{aligned}$$

$$\sum_{k\geqslant 0} B_k = \sum_{0\leqslant k\leqslant \lfloor \frac{x-1}{2} \rfloor} (x-2k-1) = \sum_{0\leqslant k\leqslant n-1} x - 2\sum_{0\leqslant k\leqslant n-1} k - \sum_{0\leqslant k\leqslant n-1} 1 =$$
$$xn - n(n-1) - n = xn - n^2 + n - n = xn - n^2 = n(x-n)$$

TODO: Case $2n-1 \leqslant x < 2n$