CM summary - chapter 2

Terminology

H_n	harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $H_0 = 0$
Δ	difference operator $\Delta f(x) = f(x+1) - f(x)$
E	shift operator $Ef(x) = f(x+1)$
D	derivative operator
$\sum f(x)\delta x$	indefinite sum
$\sum_a^b f(x)\delta x$	definite sum
$\sum$	antidifference operator $\Delta f(n) = g(n) \iff \sum g(n)\delta n = f(n)$ $\Delta f(n) = g(n) \implies \sum_a^b g(n) = f(b) - f(a)$ $\Delta f(n) = g(n) \implies \sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x$

Finite calculus analogies with integral calculus

$$\begin{tikzcd}
\Delta f(x) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \sum f(x)\delta x \arrow[d, leftrightarrow] \\
Df(x) \arrow[r, leftrightarrow] & \int f(x)dx
\end{tikzcd}$$

In the above $\delta x = 1$ (reference)

Falling/Rising factorial identities

Assume $m > 0$

$$x^{\underline{-m}} = \frac{1}{(x+1)\cdots(x+m)}$$
$$x^{\underline{m}} = (x-m+1)\cdots(x-1)x$$
$$x^{\underline{+2}} = x(x-1)$$
$$x^{\underline{+1}} = x$$
$$x^{\underline{0}} = 1$$
$$x^{\underline{-1}} = \frac{1}{x+1}$$
$$x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$$

$x^{\underline{m+1}} = x^{\underline{m}}(x-m)^{\underline{n}}$ (law of exponents)

$$x^{\overline{-m}} = \frac{1}{(x-m)^{\overline{m}}} = \frac{1}{(x-1)^{\underline{m}}}$$

$(x+y)^{\underline{m}}$ analog of binomial theorem applies

$$\frac{x^{\underline{m}}}{(x-n)^{\underline{m}}} = \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}}$$
$$x^{\overline{m}} = (-1)^m(-x)^{\underline{m}} = (x+m-1)^{\underline{m}} = \frac{1}{(x-1)^{\underline{-m}}}$$
$$x^{\underline{m}} = (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{\overline{-m}}}$$

Summation properties

Double-counting: $[k \in K] + [k \in K'] = [k \in K \cap K'] + [k \in K \cup K']$

Interchanging order of summation: $\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]$ Interchanging summation order **vanilla version**: $\sum_{j \in J} \sum_{k \in K} a_{j,k} = \sum_{\substack{j \in J \\ k \in K}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$

Interchanging summation order **rocky road version**: $\sum_{j \in J} \sum_{k \in K(j)} a_{j,k} = \sum_{k \in K'} \sum_{j \in J'(k)} a_{j,k}$

where the following holds: $[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]$

Application of rocky road: $[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]$

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} = \sum_{1 \le j \le k \le n} a_{j,k} = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k}$$

Iverson bracket properties

◥ + ◣ = ■ + ╲ $\quad [1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le j,k \le n] + [1 \le j = k \le n]$

◹ + ◺ = ■ − ╲ $\quad [1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j,k \le n] - [1 \le j = k \le n]$

$\max(a,b) = a \cdot [a > b] + b \cdot [b \ge a]$

$\min(a,b) = a \cdot [a < b] + b \cdot [b \le a]$

Summation by parts

Difference of a product	$\Delta(u \cdot v) = u\Delta v + E\ v\Delta u$
Summation by parts	$\sum u \cdot \Delta v = uv - \sum E\ v\Delta u$ E is applied only to the term immediately after it

Finite difference table

$\mathbf{f(x)}$	$\mathbf{\Delta f(x)}$	note
$x^{\underline{m}}$	$m \cdot x^{\underline{m-1}}$	
H_x	$\frac{1}{x+1} = x^{\underline{-1}}$	
c^x	$(c-1)c^x$	
$\frac{x^{\underline{m+1}}}{m+1}$	$x^{\underline{m}}$	$m \neq -1$
$c \cdot u$	$c \cdot \Delta u$	
$u + v$	$\Delta u + \Delta v$	
$\frac{c^x}{c-1}$	c^x	
2^x	2^x	
$c^{\underline{x}}$	$\frac{c^{\underline{x+2}}}{c-x}$	

(see table 55 in the book)

Sum/Product properties

sum law	product law	name
$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$	$\prod_{k \in K} a_k^c = \left(\prod_{k \in K} a_k\right)^c$	distributive law (2.15)
$\sum_{k \in K} a_k + b_k = \sum_{k \in K} a_k + \sum_{k \in K} b_k$	$\prod_{k \in K} a_k b_k = \prod_{k \in K} a_k \cdot \prod_{k \in K} b_k$	associative law(2.16)
$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j\right) \cdot \left(\sum_{k \in K} b_k\right)$		general distributive law (2.28)
$\sum_{k \in K} a_k = \sum_{k \in K} a_{p(k)}$	$\prod_{k \in K} a_k = \prod_{k \in K} a_{p(k)}$	commutative law(2.17)
$\sum_{\substack{k \in K \\ j \in J}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$	$\prod_{\substack{k \in K \\ j \in J}} a_{j,k} = \prod_{k \in K} \prod_{j \in J} a_{j,k}$	
$\sum_{k \in K} a_k = \sum_k a_k \cdot [k \in K]$	$\prod_{k \in K} a_k = \prod_k a_k^{[k \in K]}$	
$\sum_{k \in K} 1 = \lvert K \rvert$	$\prod_{k \in K} c = c^{\lvert K \rvert}$	

General techniques for recurrences or sums

Summation factor

Recurrence type: $a_n T_n = b_n T_{n-1} + c_n$

Step 1. Multiply both sides by $s_n = \frac{a_{n-1}a_{n-2}\cdots a_1}{b_n b_{n-1}\cdots b_2}$

Note: $s_n b_n = s_{n-1}a_{n-1}$

Step 2. Build $S_n = S_{n-1} + s_n c_n$

Step 3. $S_n = s_0 a_0 T_0 + \sum k = 1^n s_k c_k = s_1 b_1 T_0 \sum_{k=1}^{n} s_k c_k$

Step 4. Closed form: $T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$

Repertoire method

We have a recurrence R_n with an initial condition and a recurrence relation.

Step 1. Write the general form $R_n = A(n)\alpha + B(n)\beta + C(n)\delta + D(n)\gamma$

Step 2. Set R_n to be $1, n, n^2, n^3, \ldots$ (the repertoire) successively and determine $\alpha, \beta, \delta, \gamma$ for each.

Step 3. Build a system of linear equations in $A(n), B(n), C(n), D(n)$ where the right-hand side will be the functions from the repertoire.

Step 4. Solve the system to determine the functions A, B, C, D and thereby finding a closed form for R_n

Perturbation method/scheme

Step 1. Rewrite the sum $S_{n+1} = \sum_{0 \le k \le n+1} a_k$ by splitting off the last and first term:

$S_{n+1} = S_n + a_{n+1}$

$S_{n+1} = a_0 + \sum_{1 \le k \le n+1} a_k = a_0 + \sum_{0 \le k \le n} a_{k+1}$

Step 2. Make the last sum look like S_n.

Step 3. Use these two expressions to find a closed form for S_n

Replace the sum by integrals

Step 1. Replace the sum $S_n = \sum_{k=1}^{n} f(k)$ by an integral $I_n = \int_0^n f(x)dx$, solve the integral.

Step 2. Look at the error in the approximation $E_n = S_n - I_n$, find a closed form for it which leads to a closed form for S_n.