

CM summary - chapter 2

Terminology

H_n	harmonic number $H_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ $H_0 = 0$
Δ	difference operator $\Delta f(x) = f(x+1) - f(x)$
E	shift operator $E f(x) = f(x+1)$
D	derivative operator
$\sum f(x)\delta x$	indefinite sum
$\sum_a^b f(x)\delta x$	definite sum
\sum	antidifference operator $\Delta f(n) = g(n) \iff \sum g(n)\delta n = f(n)$ $\Delta f(n) = g(n) \implies \sum_g^b g(n) = f(b) - f(a)$ $\Delta f(n) = g(n) \implies \sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x$

Finite calculus analogies with integral calculus

$$\Delta f(x) \longleftrightarrow \sum f(x)\delta x$$

\updownarrow
$$Df(x) \longleftrightarrow \int f(x)dx$$

In the above $\delta x = 1$ ([reference](#))
Falling/Rising factorial identities
Assume $m > 0$

$$\begin{aligned} x^{-m} &= \frac{1}{(x+1)\cdots(x+m)} \\ x^{\overline{m}} &= (x-m+1)\cdots(x-1)x \\ x^{\pm 2} &= x(x-1) \\ x^{\pm 1} &= x \\ x^0 &= 1 \\ x^{-1} &= \frac{1}{x+1} \\ x^{-2} &= \frac{1}{(x+1)(x+2)} \end{aligned}$$

$$\left\{ \begin{aligned} x^{\overline{m+1}} &= x^{\overline{m}}(x-m)^{\underline{n}} \text{ (law of exponents)} \\ x^{-\overline{m}} &= \frac{1}{(x-m)^{\overline{m}}} = \frac{1}{(x-1)^{\overline{m}}} \\ (x+y)^{\overline{m}} &\text{ analog of binomial theorem applies} \\ \frac{x^{\overline{m}}}{(x-n)^{\overline{m}}} &= \frac{x^{\underline{n}}}{(x-m)^{\underline{n}}} \\ x^{\overline{m}} &= (-1)^m(-x)^{\overline{m}} = (x+m-1)^{\overline{m}} = \frac{1}{(x-1)^{-\overline{m}}} \\ x^{\overline{m}} &= (-1)^m(-x)^{\overline{m}} = (x-m+1)^{\overline{m}} = \frac{1}{(x+1)^{-\overline{m}}} \end{aligned} \right.$$

Summation properties

Double-counting: $[k \in K] + [k \in K'] = [k \in K \cap K'] + [k \in K \cup K']$
Interchanging order of summation: $\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]$ Interchanging
summation order **vanilla version:** $\sum_{j \in J} \sum_{k \in K} a_{j,k} = \sum_{\substack{j \in J \\ k \in K}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$

Interchanging summation order **rocky road version:** $\sum_{j \in J} \sum_{k \in K(j)} a_{j,k} = \sum_{k \in K'} \sum_{j \in J'(k)} a_{j,k}$

where the following holds: $[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]$
Application of rocky road: $[1 \leq j \leq n][j \leq k \leq n] = [1 \leq j \leq k \leq n] = [1 \leq k \leq n][1 \leq j \leq k]$
 $\sum_{j=1}^n \sum_{k=j}^n a_{j,k} = \sum_{1 \leq j \leq k \leq n} a_{j,k} = \sum_{k=1}^n \sum_{j=1}^k a_{j,k}$

Iverson bracket properties

$$\blacktriangledown + \blacktriangle = \blacksquare + \blacktriangledown$$
$$[1 \leq j \leq k \leq n] + [1 \leq k \leq j \leq n] = [1 \leq j, k \leq n] + [1 \leq j = k \leq n]$$

$$\blacktriangledown + \blacktriangle = \blacksquare - \blacktriangledown$$
$$[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n]$$

Summation by parts

Difference of a product $\Delta(u \cdot v) = u\Delta v + E \ v\Delta u$
Summation by parts $\sum u \cdot \Delta v = uv - \sum E \ v\Delta u$
 E is applied only to the term immediately after it

Finite difference table

f(x)	Δf(x)	note
$x^{\overline{m}}$	$m \cdot x^{\overline{m-1}}$	
H_x	$\frac{1}{x+1} = x^{-1}$	
c^x	$(c-1)c^x$	
$\frac{x^{\overline{m+1}}}{m+1}$	$x^{\overline{m}}$	$m \neq -1$
$c \cdot u$	$c \cdot \Delta u$	
$u + v$	$\Delta u + \Delta v$	
$\frac{c^x}{c-1}$	c^x	
2^x	2^x	
$c^{\underline{x}}$	$\frac{c^{\overline{x+2}}}{c-x}$	

(see table 55 in the book)

Sum/Product properties

sum law	product law	name
$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$	$\prod_{k \in K} a_k^c = \left(\prod_{k \in K} a_k \right)^c$	distributive law (2.15)
$\sum_{k \in K} a_k + b_k = \sum_{k \in K} a_k + \sum_{k \in K} b_k$	$\prod_{k \in K} a_k b_k = \prod_{k \in K} a_k \cdot \prod_{k \in K} b_k$	associative law(2.16)
$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j \right) \cdot \left(\sum_{k \in K} b_k \right)$		general distributive law (2.28)
$\sum_{k \in K} a_k = \sum_{k \in K} a_{p(k)}$	$\prod_{k \in K} a_k = \prod_{k \in K} a_{p(k)}$	commutative law(2.17)
$\sum_{\substack{k \in K \\ j \in J}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$	$\prod_{\substack{k \in K \\ j \in J}} a_{j,k} = \prod_{k \in K} \prod_{j \in J} a_{j,k}$	
$\sum_{k \in K} a_k = \sum_k a_k \cdot [k \in K]$	$\prod_{k \in K} a_k = \prod_k a_k^{[k \in K]}$	
$\sum_{k \in K} 1 = K $	$\prod_{k \in K} c = c^{ K }$	

General techniques for recurrences or sums

Summation factor
Recurrence type: $a_n T_n = b_n T_{n-1} + c_n$
Step 1. Multiply both sides by $s_n = \frac{a_{n-1} a_{n-2} \cdots a_1}{b_n b_{n-1} \cdots b_2}$

Note: $s_n b_n = s_{n-1} a_{n-1}$
Step 2. Build $S_n = S_{n-1} + s_n c_n$

Step 3. $S_n = s_0 a_0 T_0 + \sum k = 1^n s_k c_k = s_1 b_1 T_0 \sum_{k=1}^n s_k c_k$

Step 4. Closed form: $T_n = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^n s_k c_k \right)$

Repertoire method

We have a recurrence R_n with an initial condition and a recurrence relation.
Step 1. Write the general form $R_n = A(n)\alpha + B(n)\beta + C(n)\delta + D(n)\gamma$
Step 2. Set R_n to be $1, n, n^2, n^3, \dots$ (the repertoire) successively and determine $\alpha, \beta, \delta, \gamma$ for each.
Step 3. Build a system of linear equations in $A(n), B(n), C(n), D(n)$ where the right-hand side will be the functions from the repertoire.
Step 4. Solve the system to determine the functions A, B, C, D and thereby finding a closed form for R_n

Perturbation method/scheme

Step 1. Rewrite the sum $S_{n+1} = \sum_{0 \leq k \leq n+1} a_k$ by splitting off the last and first term:

$$S_{n+1} = S_n + a_{n+1}$$
$$S_{n+1} = a_0 + \sum_{1 \leq k \leq n+1} a_k = a_0 + \sum_{0 \leq k \leq n} a_{k+1}$$

Step 2. Make the last sum look like S_n .
Step 3. Use these two expressions to find a closed form for S_n

Replace the sum by integrals

Step 1. Replace the sum $S_n = \sum_{k=1}^n f(k)$ by an integral $I_n = \int_0^n f(x)dx$, solve the integral.

Step 2. Look at the error in the approximation $E_n = S_n - I_n$, find a closed form for it which leads to a closed form for S_n .