CM summary - chapter 2

Terminology

H_n	harmonic number $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $H_0 = 0$
Δ	difference operator $\Delta f(x) = f(x+1) - f(x)$
E	shift operator $Ef(x) = f(x+1)$
D	derivative operator
$\sum f(x)\delta x$	indefinite sum
$\sum_a^b f(x)\delta x$	definite sum
$\sum$	antidifference operator $\Delta f(n) = g(n) \iff \sum g(n)\delta n = f(n)$ $\Delta f(n) = g(n) \implies \sum_a^b g(n) = f(b) - f(a)$ $\Delta f(n) = g(n) \implies \sum_a^b g(x)\delta x = -\sum_b^a g(x)\delta x$

Finite calculus analogies with integral calculus

$$\begin{tikzcd}
\Delta f(x) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \sum f(x)\delta x \arrow[d, leftrightarrow] \\
Df(x) \arrow[r, leftrightarrow] & \int f(x)dx
\end{tikzcd}$$

In the above $\delta x = 1$ (reference)

Falling/Rising factorial identities

$x^{\underline{-m}}$	$= \frac{1}{(x+1)\cdots(x+m)}$	for $m > 0$
$x^{\underline{m}}$	$= \frac{x!}{(x-n)!} = (x-n+1)(x-n+2)\cdots(x-1)x$	for $m > 0$
$x^{\underline{+2}}$	$= x(x-1)$	
$x^{\underline{+1}}$	$= x$	
$x^{\underline{0}}$	$= 1$	
$x^{\underline{-1}}$	$= \frac{1}{x+1}$	
$x^{\underline{-2}}$	$= \frac{1}{(x+1)(x+2)}$	

Summation properties

Double-counting: $[k \in K] + [k \in K'] = [k \in K \cap K'] + [k \in K \cup K']$

Interchanging order of summation: $\sum_j \sum_k a_{j,k}[P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k}[P(j,k)]$

Interchanging summation order **vanilla version**: $\sum_{j \in J} \sum_{k \in K} a_{j,k} = \sum_{\substack{j \in J \\ k \in K}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$

Interchanging summation order **rocky road version**: $\sum_{j \in J} \sum_{k \in K(j)} a_{j,k} = \sum_{k \in K'} \sum_{j \in J'(k)} a_{j,k}$

where the following holds: $[j \in J][k \in K(j)] = [k \in K'][j \in J'(k)]$

Application of rocky road: $[1 \le j \le n][j \le k \le n] = [1 \le j \le k \le n] = [1 \le k \le n][1 \le j \le k]$

$$\sum_{j=1}^{n} \sum_{k=j}^{n} a_{j,k} = \sum_{1 \le j \le k \le n} a_{j,k} = \sum_{k=1}^{n} \sum_{j=1}^{k} a_{j,k}$$

Iverson bracket properties

◸ + ◺ = ■ − ⁄

$[1 \le j \le k \le n] + [1 \le k \le j \le n] = [1 \le j, k \le n] + [1 \le j = k \le n]$

◤ + ◣ = ■ + ⁄

$[1 \le j < k \le n] + [1 \le k < j \le n] = [1 \le j, k \le n] - [1 \le j = k \le n]$

Summation by parts

Difference of a product	$\Delta(u \cdot v) = u\Delta v + E\ v\Delta u$
Summation by parts	$\sum u \cdot \Delta v = uv - \sum E\ v\Delta u$ E is applied only to the term immediately after it

Finite difference table

$\mathbf{f(x)}$	$\mathbf{\Delta f(x)}$	**note**
$x^{\underline{m}}$	$m \cdot x^{\underline{m-1}}$	
H_x	$\frac{1}{x+1} = x^{\underline{-1}}$	
c^x	$(c-1)c^x$	
$\frac{x^{\underline{m+1}}}{m+1}$	$x^{\underline{m}}$	$m \neq -1$
$c \cdot u$	$c \cdot \Delta u$	
$u + v$	$\Delta u + \Delta v$	
$\frac{c^x}{c-1}$	c^x	
2^x	2^x	

(see table 55 in the book)

Sum/Product properties

sum law	product law	name
$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$	$\prod_{k \in K} a_k^c = \left(\prod_{k \in K} a_k\right)^c$	distributive law (2.15)
$\sum_{k \in K} a_k + b_k = \sum_{k \in K} a_k + \sum_{k \in K} b_k$	$\prod_{k \in K} a_k b_k = \prod_{k \in K} a_k \cdot \prod_{k \in K} b_k$	associative law(2.16)
$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j\right) \cdot \left(\sum_{k \in K} a_k\right)$		general distributive law (2.28)
$\sum_{k \in K} a_k = \sum_{k \in K} a_{p(k)}$	$\prod_{k \in K} a_k = \prod_{k \in K} a_{p(k)}$	commutative law(2.17)
$\sum_{\substack{k \in K \\ j \in J}} a_{j,k} = \sum_{k \in K} \sum_{j \in J} a_{j,k}$	$\prod_{\substack{k \in K \\ j \in J}} a_{j,k} = \prod_{k \in K} \prod_{j \in J} a_{j,k}$	
$\sum_{k \in K} a_k = \sum_k a_k \cdot [k \in K]$	$\prod_{k \in K} a_k = \prod_k a_k^{[k \in K]}$	
$\sum_{k \in K} 1 = \|K\|$	$\prod_{k \in K} c = c^{\|K\|}$	