Prove that
$$\sum_{k\geqslant 0} \min(k, x \doteq k) = \sum_{k\geqslant 0} (x \doteq (2k+1)) \quad \forall x \in \mathbb{R}^+$$
 where $a \doteq b = \max(0, a-b)$

$$\sum_{k \ge 0} A_k = \sum_{k \ge 0} B_k$$

Derivation using Iverson brackets:

$$\max(0, x - k) = 0[0 > x - k] + (x - k)[x - k \ge 0] = (x - k)[x - k \ge 0]$$

$$\begin{array}{ll} A_k &= \min(k, x \doteq k) \\ &= \min(k, \max(0, x - k)) \\ &= k[k < \max(0, x - k)] + \max(0, x - k)[k \geqslant \max(0, x - k)] = \\ &= k[k < (x - k)[x - k \geqslant 0]] + (x - k)[x - k \geqslant 0][k \geqslant (x - k)[x - k \geqslant 0]] \end{array}$$

Case I $x - k \ge 0$

$$A_{k} = k \left[k < x - k \right] + (x - k) \left[x - k \geqslant 0 \right] \left[k \geqslant x - k \right]$$

$$= k \left[k < \left\lfloor \frac{x}{2} \right\rfloor \right] + (x - k) \left[x - k \geqslant 0 \right] \left[k \geqslant \left\lfloor \frac{x}{2} \right\rfloor \right]$$

$$= k \left[k < \left\lfloor \frac{x}{2} \right\rfloor \right] + (x - k) \left[k \geqslant \left\lfloor \frac{x}{2} \right\rfloor \right]$$
Cose If $x = k < 0$

$$A_k = k[k < 0] + (x - k) \cdot 0 \cdot [k \ge 0] = 0$$

$$A_k = \begin{cases} 0 & , k > x \\ k & , 0 \le k < \left\lfloor \frac{x}{2} \right\rfloor \\ x - k & , \left\lfloor \frac{x}{2} \right\rfloor \le k \le 2 \left\lfloor \frac{x}{2} \right\rfloor - 1$$

The upper bound in the last branch above is the largest integer k that doesn't exceed x (if it did we'd be in the 1st branch instead, and all branches need to be disjoint).

$$B_{k} = x \div (2k+1) = \max(0, x-2k-1)$$

$$= (x-2k-1)[x-2k-1 \ge 0]$$

$$= (x-2k-1) \left[k \le \left\lfloor \frac{x-1}{2} \right\rfloor \right]$$

$$B_{k} = \begin{cases} 0 & ,k > \left\lfloor \frac{x-1}{2} \right\rfloor \\ x-2k-1 & ,0 \le k \le \left\lfloor \frac{x-1}{2} \right\rfloor \end{cases}$$
Suppose $2n \le x < 2n+1 \implies \left\lfloor \frac{x}{2} \right\rfloor = n, \left\lfloor \frac{x-1}{2} \right\rfloor = n-1$

$$\sum_{k\geqslant 0} A_k = \sum_{0\leqslant k<\left\lfloor\frac{x}{2}\right\rfloor} k + \sum_{\left\lfloor\frac{x}{2}\right\rfloor\leqslant k\leqslant 2\left\lfloor\frac{x}{2}\right\rfloor-1} (x-k)$$

$$= \sum_{0\leqslant k\leqslant n-1} k + \sum_{n\leqslant k\leqslant 2n-1} (x-k)$$

$$= \sum_{0\leqslant k\leqslant n-1} k + \sum_{0\leqslant k\leqslant n-1} (x-k-n)$$

$$= \sum_{0\leqslant k\leqslant n-1} (k+x-k-n)$$

$$= \sum_{0\leqslant k\leqslant n-1} (x-n) = n(x-n)$$

$$\sum_{k\geqslant 0} B_k = \sum_{0\leqslant k\leqslant \left\lfloor\frac{x-1}{2}\right\rfloor} (x-2k-1) = \sum_{0\leqslant k\leqslant n-1} x-2 \sum_{0\leqslant k\leqslant n-1} k - \sum_{0\leqslant k\leqslant n-1} 1 = xn - n(n-1) - n = xn - n^2 + n - n = xn - n^2 = n(x-n)$$

TODO: Case $2n - 1 \le x < 2n$