

Effective and Efficient Attributed Community Search

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Abstract Given a graph G and a vertex $q \in G$, the *community search* query returns a subgraph of G that contains vertices related to q . Communities, which are prevalent in *attributed graphs* such as social networks and knowledge bases, can be used in emerging applications such as product advertisement and setting up of social events. In this paper, we investigate the *attributed community query* (or ACQ), which returns an *attributed community* (AC) for an *attributed graph*. The AC is a subgraph of G , which satisfies both *structure cohesiveness* (i.e., its vertices are tightly connected) and *keyword cohesiveness* (i.e., its vertices share common keywords). The AC enables a better understanding of how and why a community is formed (e.g., members of an AC have a common interest in music, because they all have the same keyword “music”). An AC can be “personalized”; for example, an ACQ user may specify that an AC returned should be related to some specific keywords like “research” and “sports”.

To enable efficient AC search, we develop the CL-tree index structure and three algorithms based on it. We evaluate our solutions on four large graphs, namely Flickr, DBLP, Tencent, and DBpedia. Our results show that ACs are more effective and efficient than existing community retrieval approaches. Moreover, an AC contains more precise and personalized information than that of existing community search and detection methods.

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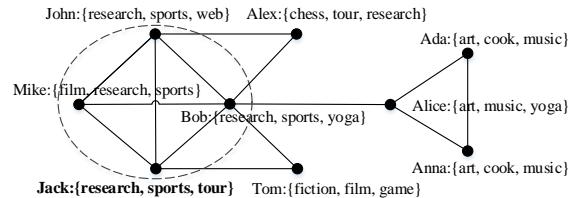


Fig. 1 Attributed graph and AC (circled).

1 Introduction

Due to the recent developments of gigantic social networks (e.g., Flickr, Facebook, and Twitter), the topic of *attributed graphs* has attracted attention from industry and research communities [31, 3, 6, 14, 16, 34, 17, 10]. An attributed graph is essentially a graph associated with text strings or keywords. Figure 1 illustrates an attributed graph, where each vertex represents a social network user, and its keywords describe the interest of that user.

In this paper, we investigate the *attributed community query* (or ACQ). Given an attributed graph G and a vertex $q \in G$, the ACQ returns one or more subgraphs of G known as *attributed communities* (or ACs). An AC is a kind of *community*, which consists of vertices that are closely related [28, 5, 4, 15, 24, 11]. Particularly, an AC satisfies *structure cohesiveness* (i.e., its vertices are closely linked to each other) and *keyword cohesiveness* (i.e., its vertices have keywords in common). Figure 1 illustrates an AC (circled), which is a connected subgraph with vertex degree 3; its vertices {Jack, Bob, John, Mike} have two keywords (i.e., “research” and “sports”) in common.

Prior works. The problems related to retrieving communities from a graph can generally be classified into *community detection* (CD) and *community search*

Table 1 Classification of works in community retrieval.

Graph Type	Community detection (CD)	Community search (CS)
Non-attributed	[24, 11]	[28, 5, 4, 15, 18]
Attributed	[35, 23, 20, 31, 25]	ACQ (This paper)

(CS). In general, CD algorithms aim to retrieve all communities for a graph [24, 11, 35, 23, 20, 31, 25]. These solutions are not “query-based”, i.e., they are not customized for a query request (e.g., a user-specified query vertex). Moreover, they can take a long time to find all the communities for a large graph, and so they are not suitable for quick or *online* retrieval of communities. To solve these problems, CS solutions have been recently developed [28, 5, 4, 15]. These approaches are query-based, and are able to derive communities in an “online” manner. However, existing CS algorithms assume *non-attributed* graphs, and only use the graph structure information to find communities. The ACQ is a class of CS problem for attributed graphs. As we will show, the use of keyword information can significantly improve the effectiveness of the communities retrieved. Table 1 summarizes some representative existing works in this area.

Features of ACs. We now present more details about ACs.

- **Ease of interpretation.** As demonstrated in Figure 1, an AC contains tightly-connected vertices with similar contexts or backgrounds. Thus, an ACQ user can focus on the common keywords or features of these vertices (e.g., the vertices of the AC in this example contain “research” and “sports”, reflecting that all members of this AC like research and sports). We call the set of common keywords among AC vertices the *AC-label*. In our experiments, the AC-labels facilitate understanding of the vertices that form the AC.

The design of ACs allows it to be used in setting up of social events. For example, if a Twitter member has many keywords about traveling (e.g., he posted a lot of photos about his trips, with keywords), issuing an ACQ with this member as the query vertex may return other members interested in traveling, because their vertices also have keywords related to traveling. A group tour can then be recommended to these members.

- **Personalization.** The user of an ACQ can control the semantics of the AC, by specifying a set of S of keywords. Intuitively, S decides the meaning of the AC based on the user’s need. If we let $q=Jack$, $k=2$ and $S=\{\text{“research”}\}$, the AC is formed by {Jack, Bob, John, Mike, Alex}, who are all interested in research. Let us consider another example in the DBLP bibliographical network, where each vertex’s attribute is repre-

sented by the top-20 frequent keywords in their publications. Let $q=Jim\ Gray$. If S is the set of keywords {transaction, data, management, system, research}, we obtain the AC in Figure 2(a), which contains six prominent database researchers closely related to Jim. On the other hand, when S is {sloan, digital, sky, survey, SDSS}, the ACQ yields another AC in Figure 2(b), which indicates the seven scientists involved in the SDSS project¹. Thus, with the use of different keyword sets S , different “personalized” communities can be obtained.

Existing CS algorithms, which do not handle attributed graphs, may not produce the two ACs above. For example, the CS algorithm in [28] returns the community with *all* the 14 vertices shown in Figures 2(a) and (b). The main reasons are: (1) these vertices are heavily linked with Jim; and (2) the keywords are not considered. In contrast, the use of set S in the ACQ places these vertices into two communities, containing vertices that are cohesive in terms of *structure* and *keyword*. This allows a user to focus on the important vertices that are related to S . For example, using the AC of Figure 2(a), a database conference organizer can invite speakers who have a close relationship with Jim.

The personalization feature is also useful in marketing. Suppose that Mary, a yoga lover, is a customer of a gym. An ACQ can be issued on a social network, with Mary as the query vertex and $S=\{\text{“yoga”}\}$. Since members of the AC contain the keyword “yoga”, they can be the gym’s advertising targets. On the other hand, current CS algorithms may return a community that contains one or more vertices without the keyword “yoga”. It is not clear whether the corresponding user of this vertex is interested in yoga.

- **Online evaluation.** Similar to other CS solutions, we have developed efficient ACQ algorithms for large graphs, allowing ACs to be generated quickly upon a query request. On the contrary, existing CD algorithms [35, 25, 23, 20] that generate all communities for a graph are often considered to be offline solutions, since they are often costly and time-consuming, especially on very large graphs.

Technical challenges and our contributions. We face two important questions: (1) What should be a sound definition of an AC? (2) How to evaluate ACQ efficiently? For the first question, we define an AC based on the *minimum degree*, which is one of the most common structure cohesiveness metrics [24, 11, 28, 5]. This measure requires that every vertex in the community has a degree of k or more. We formulate the keyword cohesiveness as maximizing the number of shared keywords in keyword set S . The shared keywords naturally reveal the common features among vertices (e.g., com-

¹ URL of the SDSS project: <http://www.sdss.org>.

mon interest of social network users). We can also use these shared keywords to explain how a community is formed.

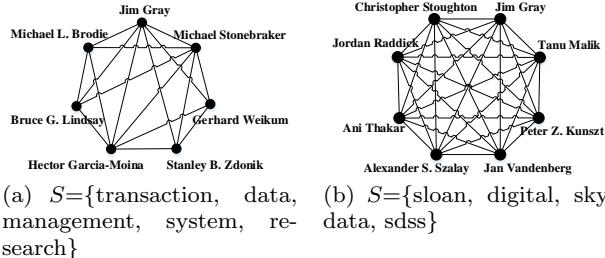


Fig. 2 Two ACs of Jim Gray.

The second question is not easy to answer, because the attributed graph G to be explored can be very large, and the (structure and keyword) cohesiveness criteria can be complex to handle. A simple way is first to consider all the possible keyword combinations, and then return the subgraphs, which satisfy the minimum degree constraint and have the most shared keywords. This solution, which requires the enumeration of all the subsets of q 's keyword set, has a complexity exponential to the size l of q 's keyword set. In our experiments, for some queries, l can be up to 30, resulting in the consideration of $2^{30} = 1,073,741,824$ subsets of q . The algorithm is impractical, especially when q 's keyword set is large.

We observe the *anti-monotonicity* property, which states that given a set S of keywords, if it appears in every vertex of an AC, then for every subset S' of S , there exists an AC in which every vertex contains S' . We use this intuition to propose better algorithms. We further develop the *CL-tree*, an index that organizes the vertex keyword data in a hierarchical structure. The CL-tree has a space and construction time complexity linear to the size of G . We have developed three different ACQ algorithms based on the CL-tree, and they are able to achieve a superior performance.

We have performed extensive experiments on four large real graph datasets (namely Flickr, DBLP, Tencent, and DBpedia). We found that a large number of common keywords appear across vertices in our graph datasets. In DBLP, for instance, an AC with one common keyword contains over 5,000 vertices on average; an AC with two common keywords contains over 700 vertices. Hence, using shared keywords among vertices as keyword cohesiveness makes sense. We have also studied how to quantify the quality of a community, based on occurrence frequencies of keywords and similarity between the keyword sets of two vertices. We conducted

a detailed case study on DBLP. These results confirm the superiority of the AC over the communities returned by existing community detection and community search algorithms, in terms of community quality. The performance of our best algorithm is 2 to 3 order-of-magnitude better than solutions that do not use the CL-tree. Another advantage of our approaches is that they organize and search vertex keywords for ACs effectively, achieving a higher efficiency than existing community search solutions (that do not use vertex keywords in the community search process).

Organization. We review the related work in Section 2, and define the ACQ problem formally in Section 3. Section 4 presents the basic solutions, and Section 5 discusses the CL-tree index. We present the query algorithms in Section 7. Our experimental results are reported in Section 9. We conclude in Section 10.

2 Related Work

Community detection (CD). A large class of studies aim to discover or *detect* all the communities from an entire graph. Table 1 summarises these works. Earlier solutions, such as [24, 11], employ link-based analysis to obtain these communities. However, they do not consider the textual information associated with graphs. Recent works focus on attributed graphs, and use clustering techniques to identify communities. For instance, Zhou et al. [35] considered both links and keywords of vertices to compute the vertices' pairwise similarities, and then clustered the graph. Ruan et al. [25] proposed a method called **CODICIL**. This solution augments the original graphs by creating new edges based on content similarity, and then uses an effective graph sampling to boost the efficiency of clustering. We will compare ACQ with this method experimentally.

Another common approach is based on topic models. In [23, 20], the **Link-PLSA-LDA** and **Topic-Link LDA** models jointly model vertices' content and links based on the **LDA** model. In [31], the attributed graph is clustered based on probabilistic inference. In [26], the topics, interaction types and the social connections are considered for discovering communities. **CESNA** [32] detects overlapping communities by assuming communities "generate" both the link and content. A discriminative approach [33] has also been considered for community detection. As discussed before, CD algorithms are generally slow, as they often consider the pairwise distance/similarity among vertices. Also, it is not clear how they can be adapted to perform online ACQ. In this paper, we propose online algorithms for finding communities on attributed graphs.

Community search (CS). Another class of solutions aims to obtain communities in an “online” manner, based on a query request. For example, given a vertex q , several existing works [28, 5, 18, 4, 15] have developed fast algorithms to obtain a community for q . To measure the structure cohesiveness of a community, the *minimum degree* is often used [28, 5, 18]. Sozio et al. [28] proposed the first algorithm **Global** to find the k -*core* containing q . Cui et al. [5] proposed **Local**, which uses local expansion techniques to enhance the performance of **Global**. We will compare these two solutions in our experiments. Other definitions, including k -clique [4] and k -truss [15], have also been considered for searching communities. A recent work [18] finds communities with high influence. These works assume non-attributed graphs, and overlook the rich information of vertices that come with attributed graphs. As we will see, performing CS on attributed graphs is better than on non-attributed graphs.

Graph keyword search. Given an attributed graph G and a set Q of keywords, graph keyword search solutions output a tree structure, whose nodes are vertices of G , and the union of these vertices’ keyword sets is a superset of Q [3, 6, 16]. Recent work studies the use of a subgraph of G as the query output [17]. These works are substantially different from the ACQ problem. First, they do not specify query vertices as required by the ACQ problem. Second, the tree or subgraph produced do not guarantee structure cohesiveness. Third, keyword cohesiveness is not ensured; there is no mechanism that enforces query keywords to be shared among the keyword sets of all query output’s vertices. Thus, graph keyword search solutions are not designed to find ACs.

3 The ACQ Problem

We now discuss the attributed graph model, the k -core, and the AC. In the CS and CD literature, most existing works assume that the underlying graph is undirected [28, 18, 31, 25]. We also suppose that an attributed graph $G(V, E)$ is undirected, with vertex set V and edge set E . Each vertex $v \in V$ is associated with a set of keywords, $W(v)$. Let n and m be the corresponding sizes of V and E . The degree of a vertex v of G is denoted by $\deg_G(v)$. Table 2 lists the symbols used in the paper.

A community is often a subgraph of G that satisfies *structure cohesiveness* (i.e., the vertices contained in the community are linked to each other in some way). A common notion of structure cohesiveness is that the *minimum degree* of all the vertices that appear in the community has to be k or more [28, 27, 2, 7, 5, 18]. This

Table 2 Symbols and meanings.

Symbol	Meaning
$G(V, E)$	An attributed graph with vertex set V and edge set E
$W(v)$	The keyword set of vertex v
$\deg_G(v)$	The degree of vertex v in G
$G[S']$	The largest connected subgraph of G s.t. $q \in G[S']$, and $\forall v \in G[S'], S' \subseteq W(v)$
$G_k[S']$	The largest connected subgraph of G s.t. $q \in G_k[S']$, and $\forall v \in G_k[S'], \deg_{G_k[S']}v \geq k$ and $S' \subseteq W(v)$

is used in the k -core and the AC. Let us discuss the k -core first.

Definition 1 (k -core [27, 2]) Given an integer k ($k \geq 0$), the k -core of G , denoted by H_k , is the largest subgraph of G , such that $\forall v \in H_k$, $\deg_{H_k}(v) \geq k$.

We say that H_k has an order of k . Notice that H_k may not be a connected graph [2], and its connected components, denoted by k -*cores*, are usually the “communities” returned by k -*core* search algorithms.

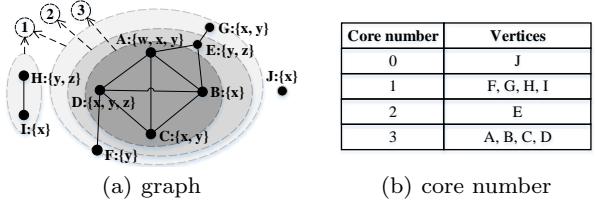


Fig. 3 Illustrating the k -core and the AC.

Example 1 In Figure 3(a), $\{A, B, C, D\}$ is both a 3-core and a 3-*core*. The 1-core has vertices $\{A, B, C, D, E, F, G, H, I\}$, and is composed of two 1-*core* components: $\{A, B, C, D, E, F, G\}$ and $\{H, I\}$. The number k in each circle represents the k -*core* contained in that ellipse.

Observe that k -cores are “nested” [2]: given two positive integers i and j , if $i < j$, then $H_j \subseteq H_i$. In Figure 3(a), H_3 is contained in H_2 , which is nested within H_1 .

Definition 2 (Core number) Given a vertex $v \in V$, its core number, denoted by $\text{core}_G[v]$, is the highest order of a k -core that contains v .

A list of core numbers and their respective vertices for Example 1 are shown in Figure 3(b). In [2], an $O(m)$ algorithm was proposed to compute the core number of every vertex.

We now formally define the ACQ problem as follows.

Problem 1 (ACQ) Given a graph $G(V, E)$, a positive integer k , a vertex $q \in V$ and a set of keywords $S \subseteq W(q)$, return a set \mathcal{G} of graphs, such that $\forall G_q \in \mathcal{G}$, the following properties hold:

- **Connectivity.** $G_q \subseteq G$ is connected and $q \in G_q$;
- **Structure cohesiveness.** $\forall v \in G_q, \deg_{G_q}(v) \geq k$;
- **Keyword cohesiveness.** The size of $L(G_q, S)$ is maximal, where $L(G_q, S) = \cap_{v \in G_q} (W(v) \cap S)$ is the set of keywords shared in S by all vertices of G_q .

We call G_q the *attributed community* (or AC) of q , and $L(G_q, S)$ the *AC-label* of G_q . In Problem 1, the first two properties are also specified by the k -*core* of a given vertex q [28]. The *keyword cohesiveness* (Property 3), which is unique to Problem 1, enables the retrieval of communities whose vertices have common keywords in S . We use S to impose semantics on the AC produced by Problem 1. By default, $S = W(q)$, which means that the AC generated should have keywords common to those associated with q . If $S \subset W(q)$, it means that the ACQ user is interested in forming communities that are related to some (but not all) of the keywords of q . A user interface could be developed to display $W(q)$ to the user, allowing her to include the desired keywords into S . For example, in Figure 3(a), if $q=A$, $k=2$ and $S=\{w, x, y\}$, the output of Problem 1 is $\{A, C, D\}$, with AC-label $\{x, y\}$, meaning that these vertices share the keywords x and y .

We require $L(G_q, S)$ to be maximal in Property 3, because we wish the AC(s) returned only contain(s) the most related vertices, in terms of the number of common keywords. Let us use Figure 3(a) to explain why this is important. Using the same query ($q=A, k=2, S=\{w, x, y\}$), without the “maximal” requirement, we can obtain communities such as $\{A, B, E\}$ (which do not share any keywords), $\{A, B, D\}$, or $\{A, B, C\}$ (which share 1 keyword). Note that there does not exist an AC with AC-label being exactly $\{w, x, y\}$. Our experiments (Section 9) show that imposing the “maximal” constraint yields the best result. Thus, we adopt Property 3 in Problem 1. If there is no AC whose vertices share one or more keywords (*i.e.*, $|L(G_q, S)|=0$), we return the subgraph of G that satisfies Properties 1 and 2 only.²

There are other candidates for structure cohesiveness (e.g., k -truss, k -clique) and *keyword cohesiveness* (e.g., Jaccard similarity and string edit distance). An AC can also be defined in different ways. For example, an ACQ user may specify that an AC returned must have vertices that contain a specific set of keywords.

² In practice, the query user can be alerted by the system when there is no sharing among the vertices.

An interesting direction is to extend ACQ to support for these criteria, and study their effectiveness.

4 Basic Solutions

For ease of presentation, we say that v contains a set S' of keywords, if $S' \subseteq W(v)$. We use $G[S']$ to denote the largest connected subgraph of G , where each vertex contains S' and $q \in G[S]$. We use $G_k[S']$ to denote the largest connected subgraph of $G[S']$, in which every vertex has degree being at least k in $G_k[S']$. We call S' a qualified keyword set for the query vertex q on the graph G , if $G_k[S']$ exists.

Given a query vertex q , a straightforward method to answer ACQ performs three steps. First, all non-empty subsets of S , $S_1, S_2, \dots, S_{2^l-1}$ ($l=|S|$), are enumerated. Then, for each subset S_i ($1 \leq i \leq 2^l-1$), we verify the existence of $G_k[S_i]$ and compute it when it exists (We postpone to discuss the details). Finally, we output the subgraphs having the most shared keywords among all $G_k[S_i]$.

One major drawback of the straightforward method is that we need to compute 2^l-1 subgraphs (*i.e.*, $G_k[S_i]$). For large values of l , the computation overhead renders the method impractical, and we do not further consider this method in the paper. To alleviate this issue, we propose the following two-step framework.

4.1 Two-Step Framework

The two-step framework is mainly based on the following *anti-monotonicity* property.

Lemma 1 (Anti-monotonicity) *Given a graph G , a vertex $q \in G$ and a set S of keywords, if there exists a subgraph $G_k[S]$, then there exists a subgraph $G_k[S']$ for any subset $S' \subseteq S$.*

All the proofs of lemmas studied in this paper can be found in Appendix A. The anti-monotonicity property allows us to stop examining all the super sets of S' ($S' \subseteq S$), once have verified that $G_k[S']$ does not exist. The basic solution begins with examining the set, Ψ_1 , of size-1 candidate keyword sets, *i.e.*, each candidate contains a single keyword of S . It then repeatedly executes the following two key steps, to retrieve the size-2 (size-3, ...) qualified keyword subsets until no qualified keyword sets are found.

• **Verification.** For each candidate S' in Ψ_c (initially $c=1$), mark S' as a qualified set if $G_k[S']$ exists.

• **Candidate generation.** For any two current size- c qualified keyword sets which only differ in one keyword, union them as a new expanded candidate with

size- $(c+1)$, and put it into set Ψ_{c+1} , if all its subsets are qualified, by Lemma 1.

Among the above steps, the key issue is how to compute $G_k[S']$. Since $G_k[S']$ should satisfy the *structure cohesiveness* (i.e., minimum degree at least k) and *keyword cohesiveness* (i.e., every vertex contains keyword set S'). Intuitively, we have two approaches to compute $G_k[S']$: either searching the subgraph satisfying degree constraint first, followed by further refining with keyword constraints (called **basic-g**); or vise versa (called **basic-w**). These two algorithms form our baseline solutions. Their pseudocodes are presented in Appendix B.

5 CL-tree Index

The major limitation of **basic-g** and **basic-w** is that they need to find the k -cores and do keyword filtering repeatedly. This makes the community search very inefficient. To achieve higher query efficiency, we propose a novel index, called **CL-tree** (Core Label tree), which organizes both the k -cores and keywords into a tree structure. Based on the index, the efficiency of answering ACQ and its variants can be improved significantly. We first introduce the index in Section 5.1, and then propose two index construction methods in Section 5.2.

5.1 Index Overview

The CL-tree index is built based on the key observation that cores are nested. Specifically, a $(k+1)$ -core must be contained in a k -core. The rationale behind is, a subgraph has a minimum degree at least $k+1$ implies that it has a minimum degree at least k . Thus, all k -cores can be organized into a tree structure³. We illustrate this in Example 2.

Example 2 Consider the graph in Figure 3(a). All the k -cores can be organized into a tree as shown in Figure 4(a). The height of the tree is 4. For each tree node, we attach the core number and vertex set of its corresponding k -core.

From the tree structure in Figure 4(a), we conclude that, if a $(k+1)$ -core (denoted as \mathcal{C}_{k+1}) is contained in a k -core (denoted as \mathcal{C}_k), then there is a tree node corresponding to \mathcal{C}_{k+1} and its parent node corresponds to \mathcal{C}_k . Besides, the height of the tree is at most $k_{max}+1$, where k_{max} is the maximum core number.

The tree structure in Figure 4(a) can be stored compactly, as shown in Figure 4(b). The key observation

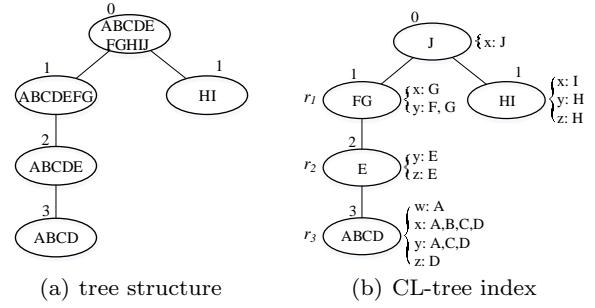


Fig. 4 An example CL-tree index.

is that, for any internal node p in the tree, the vertex sets of its child nodes are the subsets of p 's vertex set, because of the inclusion relationship. To save space cost, we can remove the redundant vertices that are shared by p 's child nodes from p 's vertex set. After such removal, we obtain a compressed tree, where each graph vertex appears only once. This structure constitutes the CL-tree index, the nodes of which are further augmented by inverted lists (Figure 4(b)). For each keyword e that appears in a CL-tree node, a list of IDs of vertices whose keyword sets contain e is stored. For example, in node r_3 , the inverted list of keyword y contains $\{A, C, D\}$. As discussed later, given a keyword set T , these inverted lists allow efficient retrieval of vertices whose keyword sets contain T . To summarize, each CL-tree node contains five elements:

- *coreNum*: the core number of the k -core;
- *vertexSet*: a set of graph vertices;
- *invertedList*: a list of $\langle key, value \rangle$ pairs, where the *key* is a keyword contained by vertices in *vertexSet* and the *value* is the list of vertices in *vertexSet* containing *key*;
- *childList*: a list of child nodes.

[**Yankai**: • *fatherNode*: the father node of the current node.]

Figure 4(b) depicts the CL-tree index for the example graph in Figure 3(a), the elements of each tree node are labeled explicitly. Using the CL-tree, the following two key operations used by our query algorithms (Section 7), can be performed efficiently.

• **Core-locating**. Given a vertex q and a core number c , find the k -core with core number c containing q , by traversing the CL-tree.

• **Keyword-checking**. Given a k -core, find vertices which contain a given keyword set, by intersecting the inverted lists of keywords contained in the keyword set.

Remarks. The CL-tree can also support k -core queries on general graphs without keywords. For example, it can be applied to finding k -core in previous community search methods [28].

³ We use “node” to mean “CL-tree node” in this paper.

Space cost. Since each graph vertex appears only once and each keyword only needs constant space cost, the space cost of keeping such an index is $O(\hat{l} \cdot n)$, where \hat{l} denotes the average size of $W(v)$ over V . Thus, the space cost is linear to the size of G .

5.2 Index Construction

To build the CL-tree index, we propose two methods, **basic** and **advanced**, as presented in Section 5.2.1 and 5.2.2.

5.2.1 The Basic Method

As k -cores of a graph are nested naturally, it is straightforward to build the CL-tree recursively in a top-down manner. Specifically, we first generate the root node for 0-core, which is exactly the entire graph. Then, for each k -core of 1-core, we generate a child node for the root node. After that, we only remain vertices with core numbers being 0 in the root node. Then for each child node, we can generate its child nodes in the similar way. This procedure is executed recursively until all the nodes are well built.

Algorithm 1 Index construction: **basic**

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1: function BUILDINDEX( $G(V, E)$ )
2:    $core_G[ ] \leftarrow k$ -core decomposition on  $G$ ;
3:    $k \leftarrow 0$ ,  $root \leftarrow (k, V)$ ;
4:   BUILDNODE( $root$ , 0);
5:   build an inverted list for each tree node;
6:   return  $root$ ;
7: function BUILDNODE( $root$ ,  $k$ )
8:    $k \leftarrow k + 1$ ;
9:   if  $k \leq k_{max}$  then
10:    obtain  $U_k$  from  $root$ ;
11:    compute the connected components for the induced graph on  $U_k$ ;
12:    for each connected component  $C_i$  do
13:      build a tree node  $p_i \leftarrow (k, C_i.vertexSet)$ ;
14:      add  $p_i$  into  $root.childList$ ;
15:      remove  $C_i$ 's vertex set from  $root.vertexSet$ ;
16:      BUILDNODE( $p_i$ ,  $k$ );

```

Algorithm 1 illustrates the pseudocodes. We first do k -core decomposition using the linear algorithm [2], and obtain an array $core_G[]$ (line 2), where $core_G[i]$ denotes the core number of vertex i in G . We denote the maximal core number by k_{max} . Then, we initialize the root node by the core number $k=0$ and V (line 3). Next, we call the function **BUILDNODE** to build its child nodes (line 4). Finally, we build an inverted list for each tree node and obtain a well built CL-tree (lines 5-6).

In **BUILDNODE**, we first update k and obtain the vertex set U_k from $root.vertexSet$, which is a set of vertices with core numbers being at least k . Then we find all the connected components from the subgraph induced by U_k (lines 8-11). Since each connected component C_i corresponds to a k -core, we build a tree node p_i with core number k and the vertex set of C_i , and then link it as a child of $root$ (lines 12-14). We also update $root$'s vertex set by removing vertices (line 15), which are shared by C_i . Finally, we call the **BUILDNODE** function to build p_i 's child nodes recursively until all the tree nodes are created (line 16).

Complexity analysis. The k -core decomposition can be done in $O(m)$. The inverted lists of each node can be built in $O(\hat{l} \cdot n)$. In function **BUILDNODE**, we need to compute the connected components with a given vertex set, which costs $O(m)$ in the worst case. Since the recursive depth is k_{max} , the total time cost is $O(m \cdot k_{max} + \hat{l} \cdot n)$. Similarly, the space complexity is $O(m + \hat{l} \cdot n)$.

5.2.2 The Advanced Method

While the **basic** method is easy to implement, it meets efficiency issues when both the given graph size and its k_{max} value are large. For instance, when given a clique graph with n vertices (*i.e.*, edges exist between every pair of nodes), the value of k_{max} is $n-1$. Therefore, the time complexity of the **basic** method could be $O((m + \hat{l}) \cdot n)$, which may lead to low efficiency for large-scale graphs. To enable more efficient index construction, we propose the **advanced** method, whose time and space complexities are almost linear with the size of the input graph.

The **advanced** method builds the CL-tree level by level in a bottom-up manner. Specifically, the tree nodes corresponding to larger core numbers are created prior to those with smaller core numbers. For ease of presentation, we divide the discussion into two main steps: creating tree nodes and creating tree edges.

1. Creating tree nodes. We observe that, if we acquire the vertices with core numbers at least c and denote the induced subgraph on the vertices as T_c , then the connected components of T_c have one-to-one correspondence to the c -cores. A simple algorithm would be, searching connected components for $T_c (0 \leq c \leq k_{max})$ independently, followed by creating one node for each distinct component. This algorithm apparently costs $O(k_{max} \cdot m)$ time, as computing connected components takes linear time.

However, we can do better if we can incrementally update the connected components in a level by level manner (*i.e.*, maintain the connected components of T_{c+1} from those of T_c). We note that, such a node

creation process is feasible by exploiting the classical *union-find forest* [1]. Generally speaking, the union-find forest enables efficient maintenance of connected components of a graph when edges are incrementally added. Using union-find forest to maintain connected components follows a process of edge examination. Initially, each vertex is regarded as a connected component. Then, edges are examined one by one. During the examine process, two components are merged together when encounters an edge connecting them. To achieve an efficient merge of components, the vertices in the component form a tree. The tree root acts as the representative vertex of the component. As such, merging two components is essentially linking two root vertices together. To guarantee the CL-tree nodes are formed in a bottom-up manner, we assign an examine priority to each edge. The priority is defined by the larger value of the two core numbers corresponding to the two end vertices of an edge. The edges associated to vertices with larger core numbers are examined first.

2. Creating tree edges. Tree edges are also inserted during the graph edge examination process. In particular, when we examine a vertex v with a set, B , of its neighbors, whose core numbers are larger than $\text{core}_G[v]$, we require that the tree node containing v should link to the tree node containing the vertex, whose core number is the smallest among all the vertices in B . Nevertheless, the classical union-find forest is not able to maintain such information. To address this issue, we thus propose an auxiliary data structure, called **Anchored Union-Find** (details of AUF are in Appendix D), based on the classical union-find forest. We first define *anchor vertex*.

Definition 3 (Anchor vertex) Given a connected subgraph $G' \subseteq G$, the anchor vertex is the vertex with core number being $\min\{\text{core}_G[v] | v \in G'\}$.

The AUF is an extension of union-find forest, in which each tree has an anchor vertex, and it is attached to the root node. In CL-tree, for any node p with corresponding k -*core* C_k , its child nodes correspond to the k -*cores*, which are contained by C_k and have core numbers being the most close to the core number of node p . This implies that, when building the CL-tree in a bottom-up manner, we can maintain the anchor vertices for the k -*cores* dynamically, and they can be used to link nodes with their child nodes. In addition, we maintain a vertex-node map, where the key is a vertex and the value is the tree node contains this vertex, for locating tree nodes.

Algorithm 2 presents the **advanced** method. Similar with **basic** method, we first conduct k -decomposition (line 2). Then, for each vertex, we initialize an AUF

Algorithm 2 Index construction: advanced

```

1: function BUILDINDEX( $G(V, E)$ )
2:    $\text{core}_G[] \leftarrow k$ -core decomposition on  $G$ ;
3:   for each  $v \in V$  do MAKESET( $v$ );
4:   put vertices into sets  $V_0, V_1, \dots, V_{k_{max}}$ ;
5:    $k \leftarrow k_{max}$ ,  $map \leftarrow \emptyset$ ;
6:   while  $k \geq 0$  do
7:      $V' \leftarrow \emptyset$ ;
8:     for each  $v \in V_k$  do  $V'.add(FIND(v))$ ;
9:     compute connected components for  $V_k \cup V'$ ;
10:    for each component with vertex set  $C_i$  do
11:      create a node  $p_i$  using  $(k, \{C_i - V'\})$ ;
12:      for each  $v \in \{C_i - V'\}$  do
13:         $map.add(v, p_i)$ ;
14:        for each  $u \in v$ 's neighbor vertices do
15:          if  $\text{core}_G[u] \geq \text{core}_G[v]$  then
16:            UNION( $u, v$ );
17:          if  $\text{core}_G[u] > \text{core}_G[v]$  then
18:             $uRoot \leftarrow FIND(u)$ ;
19:             $uAnchor \leftarrow uRoot.anchor$ ;
20:             $p' \leftarrow map.get(uAnchor)$ ;
21:            add  $p'$  to  $p$ 's child List;
22:             $vRoot \leftarrow FIND(v)$ ;
23:            if  $\text{core}_G[vRoot.anchor] > \text{core}_G[v]$  then
24:              UPDATEANCHOR( $vRoot, core_G[], v$ );
25:    $k \leftarrow k - 1$ ;
26:   build the root node  $root$ ;
27:   build an inverted list for each tree node;
28:   return  $root$ .

```

tree node (line 3). We group all the vertices into sets (line 4), where set V_k contains vertices with core numbers being exactly k (line 5). Next, we initialize k as k_{max} and the vertex-node map map , where the key is a vertex and the value is a CL-tree node whose vertex set contains this vertex. In the while loop (lines 6-25), we first find the set V' of the representatives for vertices in V_k , then compute the connected components for vertex set $V_k \cup V'$ (lines 7-9). Next, we create a node p_i for each component (lines 10-11). For each vertex $v \in \{C_i - V'\}$, we add a pair (v, p_i) to the map (lines 12-13). Then for each of v 's neighbor, u , if its core number is at least $\text{core}_G[v]$, we link u and v together in the AUF by a UNION operation (lines 14-16), and find p_i 's child nodes using the anchor of the AUF tree (lines 17-21). After vertex v has been added into the CL-tree, we update the anchor (lines 22-24). Then we move to the upper level in next loop (line 25). After the while loop, we build the root node of the CL-tree (line 26). Finally, we build the inverted list for each tree node and obtain the built index (lines 27-28).

Complexity analysis. In Algorithm 2, lines 1-3 can be completed in $O(m)$ (We assume $m \ll n$). In the while loop, the number of operations on each vertex and its neighbors are constant, and each can be done in $O(\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function. The keyword inverted lists of all the tree nodes can be

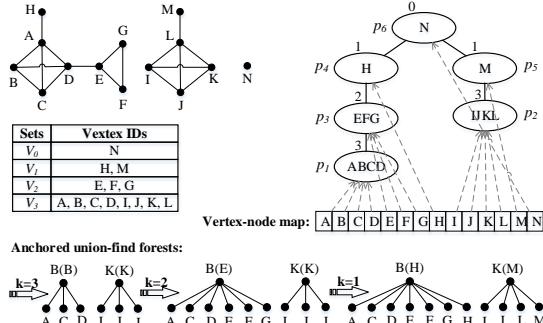


Fig. 5 An index built by **advanced** method.

computed in $O(n \cdot \hat{l})$. Therefor, the CL-tree can be built in $O(m \cdot \alpha(n) + n \cdot \hat{l})$. The space cost is $O(m + n \cdot \hat{l})$, as maintaining an AUF takes $O(n)$.

Example 3 Figure 5 depicts an example graph with 14 vertices A, \dots, N . V_i denotes the set of vertices whose core numbers are i . When $k=3$, we first generate two leaf nodes p_1 and p_2 , then update the AUF, where roots' anchor vertices are in the round brackets. When $k=2$, we first generate node p_3 , then link it to p_1 , and then update the AUF forest. When $k=1$, we first generate nodes p_4 and p_5 . Specifically, to find the child nodes of p_4 , we first find its neighbor A , then find A 's parent B using current AUF forest. Since the anchor vertex of B is E and E points to p_3 in the inverted array, we add p_3 into p_4 's child List. When $k=0$, we generate p_6 and finish the index construction.

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6 Index Maintenance

We now discuss an index maintenance algorithm which can handle the insertion or deletion of keywords and graph edges, without rebuilding the CL-tree from scratch. To insert (or remove) a keyword of a vertex, we just need to update the *invertedList* of the CL-tree node containing the vertex. To insert (or remove) a graph edge, we can first borrow the results from [19], which discusses how to maintain a k -core. Then we update our CL-tree index.

6.1 Keyword Update

Recall in the **Advanced** index construction method (Sec 5.2.2), we have built a vertex-node map, where each vertex is mapped to a CL-tree node. And the keyword update process mainly relies on this data structure. Note that in the **basic** index construction method, we can construct the map by traversing the CL-tree once.

Keyword update can be done in an online manner. In detail, to insert a new keyword for a vertex v , we can first locate the CL-tree node which contains v using the vertex-node map, and then insert the keyword and vertex ID into the associated *invertedList*. To remove a keyword of a vertex, we can do the process in the similar way.

Note that only the node that contains v needs to be considered. Since the CL-tree is compressed, only the *invertedList* of a single CL-tree node containing v needs to be updated.

6.2 Edge Insertion

As aforementioned, inserting an edge may trigger the updates of several CL-tree nodes as well as their structures. We illustrate this by Example 4.

Example 4 Consider the graph in Figure 5. If we insert an edge (H, G) as shown in Figure 6(a), the core number of vertex H increases to 2 and we need to move it down to a node in the lower level. If we insert an edge (G, I) , the connectivity of some vertices changes as shown in Figure 6(b) and thus the corresponding subtrees are merged as a new one.

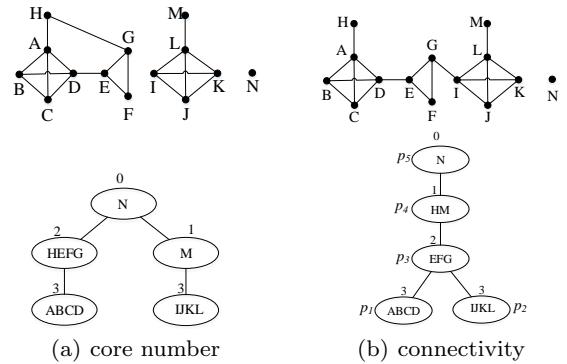


Fig. 6 The core number and connectivity change.

To maintain the CL-tree for inserting an edge, we propose an algorithm called `insertEdge`. The main idea is that, we first find vertices whose core numbers change, then change their positions in the CL-tree, and merge some subtrees. Let V^+ be the set of vertices whose core numbers increase after inserting an edge (u, v) . We summarize the main steps of `insertEdge` as follows.

- **Step 1:** Compute V^+ ;
- **Step 2:** Move down vertices of V^+ ;
- **Step 3:** Merge subtrees.

We now elaborate these steps one by one.

Step 1: Compute V^+ . A recent k -core maintenance algorithm [19] states that inserting an edge only affects the core numbers of a small number of vertices. We first give a definition, a theorem and a lemma proposed in this paper.

Definition 4 ([19]) Given a graph G and a vertex v , the induced core subgraph of v , denoted as G_v , is a connected subgraph containing v and the core numbers of all vertices in G_v equal to $\text{core}_G[v]$.

Notice that, the sets of vertices in G_u (G_v) are actually subsets of vertices in $p_u.\text{vertexSet}$ ($p_v.\text{vertexSet}$), where p_u , p_v denote the nodes that contain u , v .

Theorem 1 (k-core update theorem[19]) Given a graph G and two vertices u and v . After inserting or deleting an edge (u,v) in G , we have that,

- If $\text{core}_G[u] > \text{core}_G[v]$, only the core numbers of vertices in G_v may need to be updated.
- If $\text{core}_G[u] < \text{core}_G[v]$, only the core numbers of vertices in G_u may need to be updated.
- If $\text{core}_G[u] = \text{core}_G[v]$, only the core numbers of vertices in the union of G_u and G_v , i.e., $G_{u\cup v}$ may need to be updated.

Lemma 2 ([19]) After inserting (deleting) an edge, the core number of any vertex in G increases (decreases) by at most 1.

By above theorem and lemma, we can conclude that only a small number of vertices need to change their core numbers. In specific, we can first find node p_u (p_v) and then compute the vertex set V^+ in which vertices's core numbers increase by 1 using the algorithm in [19].

Step 2: Move down vertices of V^+ . Let p be the node containing V^+ and $c=\min\{\text{core}_G[u], \text{core}_G[v]\}$. Since the core numbers of vertices in V^+ increase by 1 (from c to $c+1$), we need move them down to nodes in the lower level. During the moving down process, we may also need to reorganize p 's child nodes. Let us illustrate this by Example 5.

Example 5 Consider a graph in Figure 7(a) and its CL-tree in Figure 7(b). Let us insert an new edge $(8, 11)$. We first get $V^+=\{8, 11, 23\}$ and $c=2$. Next, we move them down from r_1 to r_3 . Besides, we have to merge r_2 into r_3 and place r_4 as r_3 's child node, since their connectivity changes after the insertion. The updated CL-tree is depicted in Figure 7(c).

Clearly, moving down vertices of V^+ from p to p' may change the connectivity of p 's child nodes. Consider a specific vertex $a \in V^+$ and we initialize two empty sets B_1 and B_2 . For each of a 's

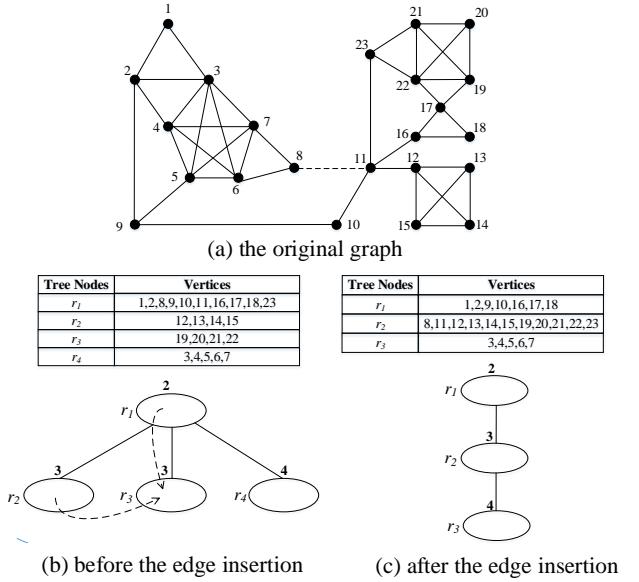


Fig. 7 An example of the tree index update.

neighbor b whose $\text{core}_G[b] > c$, we first find the node p_b containing b , and then trace it up from p_b along the CL-tree until a child node of p , denoted by o_b . If o_b has a core number of $c+1$, we put it into B_1 ; if it has a core number of $c+2$ or more, we put it into B_2 . Then, after moving down vertices of V^+ , nodes in B_1 should be merged into p' and nodes in B_2 will be child nodes of p' .

Algorithm 3 presents `moveDown`. If $V^+ \neq \emptyset$, we first initialize a node set P (line 3). Then, we remove V^+ from $p.\text{vertexSet}$ and update $p.invertedList$ (line 4). $\forall a \in V^+$, we enumerate a 's neighbor b whose $\text{core}_G[b] > c$, locate p_b , trace up from p_b to find p_b 's ancestor node o_b , which is a child node of p , and put o_b into P (lines 5-9). Let the node which has the largest size with core number being $c+1$ in P be p_{max} (line 10). Next, if $p_{max} = null$, we need to create a new child node of p (lines 11-14); otherwise, we merge and reorganize p 's child nodes (lines 15-22). Finally we return node p' (line 26), which will be used later.

Step 3: Merge subtrees. Recall in Figure 6(b), after inserting (G, I) , the corresponding subtrees, which correspond to the k -cores containing G and I are merged into one subtree. The process of merging subtrees starts from the tree nodes which contain G and I , and ends at their common ancestor node. The merging process is guaranteed by Lemmas 3 and 4.

Lemma 3 After inserting an edge between two vertices, the maximum numbers of disconnected k -cores which need to be merged is two.

Proof. It can be proved directly by contradiction. \square

Algorithm 3 move down vertices: `moveDown`

```

1: function MOVEDOWN( $V^+$ ,  $p$ )
2:   if  $V^+ = \emptyset$  then return  $p$ ;
3:    $P \leftarrow \emptyset$ ;
4:   update  $p$  using  $V^+$ ;
5:   for each  $a \in V^+$  do
6:     for each  $b \in a$ 's neighbor vertices do
7:       if  $\text{core}_G[b] > c$  and  $b \notin V^+$  then
8:         locate node  $p_b$ ;
9:         run TRACE( $p_b$ ), and update  $P$ ;
10:     $p_{max} \leftarrow$  a node of  $P$ , which has a core number of  $c+1$ 
      and its  $\text{vertexSet}$  is the largest among all nodes of  $P$ ;
11:    if  $p_{max} = \text{null}$  then
12:      create a new node  $p'$ ;
13:      update  $p'$ ;
14:      add  $P$  to  $childList$  of  $p'$ ;
15:    else
16:      add  $V^+$  to  $p_{max}.\text{vertexSet}$ ;
17:      for each  $p_i \in P$  do
18:        if  $p_i.\text{coreNum} = c+1$  then
19:          merge  $p_i$  to  $p_{max}$ ;
20:        else
21:          add  $p_i$  to  $childList$  of  $p_{max}$ ;
22:     $p' \leftarrow p_{max}$ ;
23:    update vertex-node map;
24:    if  $p.\text{vertexSet} = \emptyset$  then
25:      add  $\{p.\text{childList} - P\}$  to  $childList$  of  $p.\text{father}$ ;
26:  return  $p'$ ;

```

Lemma 4 In the process of merging subtrees, the maximum number of nodes which need to be merged in each level is two.

Proof. It can be proved directly by contradiction. \square

By Lemmas 3 and 4, we conclude that, to merge the subtrees, we can first trace two paths starting from p_u and p_v until their common ancestor in the CL-tree, and then merge the pairs of nodes on the paths, if their core numbers are the same.

Algorithm 4 presents the overall steps of `insertEdge`. Following Theorem 1, we first compute V^+ , and invoke `moveDown` to update these nodes in CL-tree (lines 2-16). Next, if p'_u and p'_v belong to two disconnected k -cores, we need to merge the subtrees (lines 17-19). In detail, we first trace two paths starting from p'_u and p'_v up until one common ancestor. Then, for each pair of nodes on the paths, if their core numbers are equal, we merge them as a single node. Finally, the tree index is updated. Note that during the above process, the elements of nodes and vertex-node map are also updated.

6.3 Edge Deletion

Similar to the edge insertion, deleting an edge may trigger the updates of CL-tree nodes as well as their structures. We illustrate this by Example 6.

Algorithm 4 index update algorithm: `insertEdge`

```

1: function INSERTEDGE( $p_u, p_v$ )
2:   if  $p_u.\text{coreNum} = p_v.\text{coreNum}$  then
3:     compute  $V_1^+$  in  $p_u.\text{vertexSet}$ ;
4:      $p'_u \leftarrow \text{MOVEDOWN}(V_1^+, p_u)$ ;
5:      $p'_v \leftarrow p_v$ ;
6:     if  $p_u \neq p_v$  then
7:       compute  $V_2^+$  in  $p_v.\text{vertexSet}$ ;
8:        $p'_v \leftarrow \text{MOVEDOWN}(V_2^+, p_v)$ ;
9:     else if  $p_u.\text{coreNum} < p_v.\text{coreNum}$  then
10:      compute  $V^+$  in  $p_u.\text{vertexSet}$ ;
11:       $p'_u \leftarrow \text{MOVEDOWN}(V^+, p_u)$ ;
12:       $p'_v \leftarrow p_v$ ;
13:    else
14:      compute  $V^+$  in  $p_v.\text{vertexSet}$ ;
15:       $p'_v \leftarrow \text{MOVEDOWN}(V^+, p_v)$ ;
16:       $p'_u \leftarrow p_u$ ;
17:    if  $p'_u$  and  $p'_v$  are in two disconnected  $k$ -cores then
18:      trace two paths starting from  $p'_u$  and  $p'_v$  up until
      a common ancestor;
19:      merge pairs of nodes with the same core number
      on the paths;

```

Example 6 Consider the graph in Figure 6. If we delete an edge (H, G) of the graph in Figure 5, the core number of vertex H decreases to 1. Thus we need to create a new node with core number being 1 and then move H up to the new node. If we delete an edge (G, I) , the connectivity of some vertices changes as shown in Figure 5 and thus the corresponding subtree has to be split to two new ones.

To maintain the CL-tree for deleting an edge, we propose an algorithm called `deleteEdge`. Let V^- be the set of vertices whose core numbers decrease after deleting an edge (u, v) . We summarize the main steps of `deleteEdge` as follows.

- **Step 1:** Compute V^- ;
- **Step 2:** Split nodes in a path;
- **Step 3:** Move up vertices of V^- .

We now elaborate these steps one by one.

Step 1: Compute V^- . By Lemma 2, the core numbers of vertices in G decrease by at most 1 after deleting an edge. We compute V^- using the algorithm in [19].

Step 2: Split nodes in a path. Similar to edge insertion, the connectivity of vertices may change after deleting an edge.

Let p be the node containing vertex u if $\text{core}_G[u] \leq \text{core}_G[v]$; or the node containing v if $\text{core}_G[v] < \text{core}_G[u]$. From the second example in Example 6, we conclude that after deleting an edge (u, v) , we may have to split p and its ancestor nodes. To enable efficient splitting, we first build a *vertex-tree* map for p . In this map, the key is a vertex v_{key} , which is a neighbor of a vertex in p and is in a descendant node of p ; the value of v_{key} is a child node of p , whose subtree contains v_{key} . The

vertex-tree map can be built simply by traversing the subtree. We illustrate the vertex-tree map via Example 7.

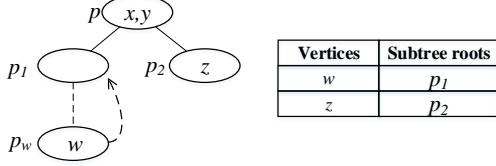


Fig. 8 Illustrating the vertex-tree map.

Example 7 Figure 8 depicts a subtree rooted at p . Suppose p only has one vertex x which has two neighbors y and z . Vertices y and z are in the descendant nodes of p . Then, in the vertex-tree map, there are two keys y and z , and their values are p_1 and p_2 respectively.

Next, we regroup vertices of p using the vertex-tree map. Specifically, we consider each vertex $a \in p.vertexSet$. For each neighbor b of a , if it has $core_G[b] > core_G[a]$, we locate the child node of p which contains b using the vertex-tree map. As a result, each vertex corresponds to a list of child nodes of p . Then, we partition vertices of p into groups such that:

- For each vertex a in a group g , there is at least another vertex in g that has the same corresponding child node or neighbors a .
- If there are two groups g_1 and g_2 , then their corresponding child nodes should be completely different.

Essentially, each group corresponds to a $k\text{-}\widehat{\text{core}}$. Since deleting an edge in a $k\text{-}\widehat{\text{core}}$ can result in at most two $k\text{-}\widehat{\text{core}}$, vertices in p can be partitioned into at most two groups. After regrouping, we can split the nodes in the path from p to all its ancestor nodes as follows:

- If there is only one group, we do not split p ; otherwise we split p into two nodes, each of which contains a group of vertices and links to a set of child nodes that its vertices correspond to.
- If p remains unsplit, and each child node of p is still linked as a child node of p , we stop; otherwise, we link the unvisited child node to p 's father node and perform these two steps for p 's father node.

Clearly, the splitting process is recursively performed on p 's ancestor nodes and thus we split nodes in a path. We give Example 8 to illustrate the process.

Example 8 Consider the graph in Figure 6(b). We show the splitting process for deleting (G, I) in Figure 9. We first locate node p_3 containing G , regroup vertices of p_3 and find that p_2 cannot be linked as a child node p_3 . So we link p_2 to p_3 's father node p_4 and perform splitting on p_4 . After regrouping vertices in p_4 , we split it to

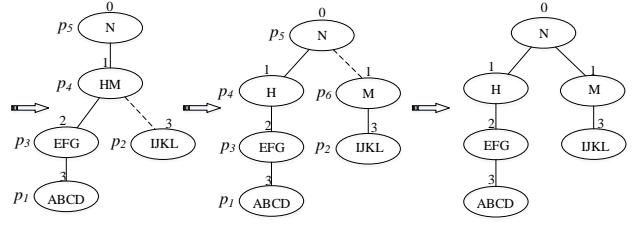


Fig. 9 The process of splitting nodes in a path.

two nodes because p_2 and p_3 are respectively shared by vertices H and M . Now since each child node of p_5 is still linked as a its child node, we stop the splitting process.

Step 3: Move up vertices of V^- .

After computing the vertex set V^- and modifying the structure of the tree index, the third step of our edge deletion algorithm is moving up vertices of V^- to an upper level. We give the `moveUp` algorithm in Algorithm 5.

As outlined in Algorithm 5, we first initialize two node sets P, P' (line 3). We need to remove V^- from $vertexSet$ of p and update $invertedList$ of p (line 4). Then if the core number of $father$ is $c - 1$, we add V^- to it; If the core number of $father$ is less than $c - 1$, we create a new node and join it to the tree (lines 5-10). Next, we collect the left vertices of p in set , reorganize and split them to nodes and update the vertex-node map (lines 11-13). We also need to update $childList$ and $invertedList$ of $p.father$ (lines 15-16). If there exist child nodes that are not visited in line 12, we re-link them to $p.father$ because these nodes are traceable only by vertices of V^- (lines 16-18).

Algorithm 5 move up vertices: `moveUp`

```

1: function MOVEUP( $V^-$ ,  $p$ )
2:   if  $V^- \neq \emptyset$  then
3:      $P, P' \leftarrow \emptyset$ ;
4:     update  $p$ ;
5:     if  $(p.father).getCore = c-1$  then
6:       add  $V^-$  to  $father$ ;
7:     else
8:       create new node  $newFather$ ;
9:       add  $V^-$  to  $newFather$ ;
10:      join  $newFather$  to the tree;
11:       $set \leftarrow p.vertexSet$ ;
12:       $P \leftarrow$  reorganize vertices of  $set$  and split to nodes;
13:      update vertex-node map;
14:      link each  $p_i \in P$  to  $p.father$ ;
15:      update  $invertedList$  of  $p.father$ ;
16:       $P' \leftarrow$  get child nodes which are not visited by the
        re-organize step;
17:      if  $P' \neq \emptyset$  then
18:        re-link each  $p \in P'$  to  $p.father$ ;

```

We outline Algorithm 6 for the edge deletion. Similar to edge insertion, we have three cases to handle separately. In these three cases, we first compute vertex set V^- (lines 3,7,11). Then we split the subtree and get the updated node p'_u (p'_v or p'_u and p'_v) (lines 4,8,12). Next we apply `moveUp` to update vertices of V^- (lines 5,9,14). In $p_u.coreNum = p_v.coreNum$ case, if vertices of V^- belongs to two disconnected $k-\widehat{\text{cores}}$, we should separate V^- to two sets and invoke `moveUp` accordingly (lines 16-18).

Algorithm 6 index algorithm: `deleteEdge`

```

1: function DELETEDGE( $p_u, p_v$ )
2:   if  $p_u.coreNum > p_v.coreNum$  then
3:     compute  $V^-$  in  $p_v.vertexSet$ ;
4:      $p'_v \leftarrow$  split the subtree;
5:     MOVEUP( $V^-, p'_v$ );
6:   else if  $p_u.coreNum < p_v.coreNum$  then
7:     compute  $V^-$  in  $p_u.vertexSet$ ;
8:      $p'_u \leftarrow$  split the subtree;
9:     MOVEUP( $V^-, p'_u$ );
10:  else
11:    compute  $V^-$  in  $p_u.vertexSet$ ;
12:     $p'_u, p'_v \leftarrow$  split the subtree;
13:    if  $p'_u = p'_v$  then
14:      MOVEUP( $V^-, p'_u$ );
15:    else
16:       $V_u^-, V_v^- \leftarrow$  separate  $V^-$ ;
17:      MOVEUP( $V_u^-, p'_u$ );
18:      MOVEUP( $V_v^-, p'_v$ );

```

]

7 Query Algorithms

In this section, we present three query algorithms based on the CL-tree index. Based on how we verify the candidate keyword sets, we divide our algorithms into incremental algorithms (from examining smaller candidate sets to larger ones) and decremental algorithm (from examining larger candidate sets to smaller ones). We propose two incremental algorithms called **Inc-S** (Incremental Space efficient) and **Inc-T** (Incremental Time efficient), to trade off between the memory consumption and the computational overhead. The decremental algorithm is called **Dec** (Decremental). Our interesting finding is that, while **Dec** seems not intuitive, it ranks as the most efficient one. **Inc-S** and **Inc-T** are presented in Section 7.1. **Dec** is introduced in Section 7.2.

7.1 The Incremental Algorithms

While the high-level idea of incremental algorithms resembles the basic solutions (see Section 4), **Inc-S** and **Inc-T** advance them with the exploitation of the CL-tree. Specifically, they can always verify the existence of $G_k[S']$ within a subgraph of G instead of the entire graph G . More interestingly, the subgraph for such verifications shrinks when the candidate set S' expands. Therefore, a large sum of redundant computation is cut off during the verification. We present **Inc-S** and **Inc-T** in Sections 7.1.1 and 7.1.2.

7.1.1 Inc-S Algorithm

We first introduce a new concept, called **subgraph core number**, which is geared to the main idea of **Inc-S**.

Definition 5 (Subgraph core number) The core number of a subgraph G' of G , $\text{core}_G[G']$, is defined as $\min\{\text{core}_G[v] \mid v \in G'\}$.

Inc-S follows the two-step framework (*verification* and *candidate generation*) introduced in Section 4. With the CL-tree, we improve the verification step as follows.

- **Core-based verification.** For each newly generated size- $(c+1)$ candidate keyword set S' expanded from size- c sets S_1 and S_2 , mark S' as a qualified set if $G_k[S']$ exists in a subgraph of core number $\max\{\text{core}_G[G_k[S_1]], \text{core}_G[G_k[S_2]]\}$.

The core-based verification guarantees that, with the expansion of the candidate keyword sets, the verification becomes faster as it only needs to examine the existence of $G_k[S']$ in a smaller $k-\widehat{\text{core}}$ (Recall that cores with large core numbers are nested in the cores with small core numbers). The correctness of such shrunk verification range is guaranteed by the following lemma.

Lemma 5 Given two subgraphs $G_k[S_1]$ and $G_k[S_2]$ of a graph G , for a new keyword set S' generated from S_1 and S_2 (i.e., $S' = S_1 \cup S_2$), if $G_k[S']$ exists, then it must appear in a $k-\widehat{\text{core}}$ with core number at least

$$\max\{\text{core}_G[G_k[S_1]], \text{core}_G[G_k[S_2]]\}. \quad (1)$$

The verification process can be further accelerated by checking the numbers of vertices and edges, as indicated by Lemma 6.

Lemma 6 Given a connected graph $G(V, E)$ with $n=|V|$ and $m=|E|$, if $m-n < \frac{k^2-k}{2} - 1$, there is no $k-\widehat{\text{core}}$ in G .

This lemma implies that, for a connected subgraph G' , whose edge and vertex numbers are m and n , if $m - n < \frac{k^2 - k}{2} - 1$, then we cannot find $G_k[S']$ from G' .

We present **Inc-S** in Algorithm 7. The input is a CL-tree rooted at $root$, a query vertex q , a positive integer k and a keyword set S . We apply **core-locating** on the CL-tree to locate the internal nodes whose corresponding k -cores contain q (line 2). Note that their core numbers are in the range of $[k, \text{core}_G[q]]$, as required by the structure cohesiveness. Then, we set $l=0$, indicating the sizes of current keyword sets, and initialize a set Ψ of $\langle S', c \rangle$ pairs, where S' is a set containing a keyword from S and c is the initial core number k (line 3). Note that we skip those keywords, which are in S , but not in $W(q)$. In the while loop (lines 4-18), for each $\langle S', c \rangle$ pair, we first perform **keyword-checking** to find $G[S']$ using the keyword inverted lists of the subtree rooted at node r_c . If we cannot ensure that $G[S']$ does not contain a k -core by Lemma 6, we then find $G_k[S']$ from $G[S']$ (lines 8-9). If $G_k[S']$ exists, we put S' with its core number into the set Φ_l (lines 10-11). Next, if Φ_l is nonempty, we generate new candidates by calling **GENECAND**(Φ_l), which is detailed in Appendix C. For each candidate S' in Ψ , we compute the core number using Lemma 5 and update it as a pair in Ψ (lines 12-17); otherwise, we stop (line 18). Finally, we output the communities of the latest verified keyword sets (line 19).

Algorithm 7 Query algorithm: **Inc-S**

```

1: function QUERY( $G, root, q, k, S$ )
2:   find subtree root nodes  $r_k, r_{k+1}, \dots, r_{\text{core}_G[q]}$ ;
3:   initialize  $l=0, \Psi$  using  $S$ ;
4:   while true do
5:      $l \leftarrow l + 1; \Phi_l \leftarrow \emptyset;$ 
6:     for each  $\langle S', c \rangle \in \Psi$  do
7:       find  $G[S']$  under the root  $r_c$ ;
8:       if  $G[S']$  is not pruned by Lemma 6 then
9:         find  $G_k[S']$  from  $G[S']$ ;
10:        if  $G_k[S']$  exists then
11:           $\Phi_l.add(\langle S', \text{core}_G[G_k[S']] \rangle);$ 
12:        if  $\Phi_l \neq \emptyset$  then
13:           $\Psi \leftarrow \text{GENECAND}(\Phi_l);$ 
14:          for each  $S'$  in  $\Psi$  do
15:            if  $S'$  is generated from  $S_1$  and  $S_2$  then
16:               $c \leftarrow \max\{\text{core}_G[G_k[S_1]], \text{core}_G[G_k[S_2]]\};$ 
17:               $\Psi.update(S', \langle S', c \rangle);$ 
18:            else break;
19:   output the communities of keyword sets in  $\Phi_{l-1}$ ;

```

Example 9 Consider the graph in Figure 3(a) and its index in Figure 4(b). Let $q=A$, $k=1$ and $S=\{w, x, y\}$. By Algorithm 7, we first find 3 root nodes r_1 , r_2 and r_3 . In the first while loop, we find 2 qualified keyword sets $\{x\}$ and $\{y\}$ with core numbers being 3 and 1. By

Lemma 5, we only need to verify the new candidate keyword set $\{x, y\}$ under node r_3 .

7.1.2 Inc-T Algorithm

We begin with a lemma which inspires the design of **Inc-T**.

Lemma 7 *Given two keyword sets S_1 and S_2 , if $G_k[S_1]$ and $G_k[S_2]$ exist, we have*

$$G_k[S_1 \cup S_2] \subseteq G_k[S_1] \cap G_k[S_2]. \quad (2)$$

This lemma implies, if S' is generated from S_1 and S_2 , we can find $G_k[S']$ from $G_k[S_1] \cap G_k[S_2]$ directly. Since every vertex in $G_k[S_1] \cap G_k[S_2]$ contains both S_1 and S_2 , we do not need to consider the keyword constraint again when finding $G_k[S']$.

Based on Lemma 7, we introduce a new algorithm **Inc-T**. Different from **Inc-S**, **Inc-T** maintains $G_k[S']$ rather than $\text{core}_G[G_k[S']]$ for each qualified keyword set S' . As we will demonstrate later, **Inc-T** is more effective for shrinking the subgraphs containing the ACs, and thus more efficient. As a trade-off for better efficiency, **Inc-T** consumes more memory as it needs to store a list of subgraph $G_k[S']$ in memory.

Algorithm 8 presents the steps of **Inc-T**. We first apply **core-locating** to find the k -core containing q from the CL-tree (line 2). Then, we set $l=0$, indicating the sizes of current keyword sets, and initialize a set Ψ of $\langle S', \widehat{G} \rangle$ pairs, where S' is a set containing a keyword from S and \widehat{G} is the k -core. The while loop (lines 4-18) is similar with that of **Inc-S**. The main differences are that: (1) for each qualified keyword set S' , **Inc-T** keeps $G_k[S']$ in memory (line 11); and (2) for each candidate keyword set S' generated from S_1 and S_2 , **Inc-T** finds $G_k[S']$ from $G_k[S_1] \cap G_k[S_2]$ directly without further keyword verification (lines 6-9, 16).

Example 10 Continue the graph and query ($q=A$, $k=1$, $S=\{w, x, y\}$) in Example 9. By **Inc-T**, we first find $G_1[\{x\}]$ and $G_1[\{y\}]$, whose vertex sets are $\{A, B, C, D\}$ and $\{A, C, D, E, F, G\}$. Then to find $G_1[\{x, y\}]$, we only need to search it from the subgraph, induced by the vertex set $\{A, C, D\}$.

7.2 The Decremental Algorithm

The decremental algorithm, denoted by **Dec**, differs from previous query algorithms not only on the generation of candidate keyword sets, but also on the verification of candidate keyword sets.

1. Generation of candidate keyword sets. **Dec** exploits the key observation that, if S' ($S' \subseteq S$) is a

Algorithm 8 Query algorithm: **Inc-T**

```

1: function QUERY( $G$ ,  $root$ ,  $q$ ,  $k$ ,  $S$ )
2:   find the  $k$ - $\widehat{core}$ , which contains  $q$ ;
3:   initialize  $l=0$ ,  $\Psi$  using  $S$ ;
4:   while true do
5:      $l \leftarrow l + 1$ ;  $\Phi_l \leftarrow \emptyset$ ;
6:     for each  $< S', \widehat{G} > \in \Psi$  do
7:       find  $G[S']$  from  $\widehat{G}$ ;
8:       if  $G[S']$  is not pruned by Lemma 6 then
9:         find  $G_k[S']$  from  $G[S']$ ;
10:        if  $G_k[S']$  exists then
11:           $\Phi_l.add(< S', G_k[S'] >)$ ;
12:        if  $\Phi_l \neq \emptyset$  then
13:           $\Psi \leftarrow \text{GENECAND}(\Phi_l)$ ;
14:          for each  $S' \in \Psi$  do
15:            if  $S'$  is generated from  $S_1$  and  $S_2$  then
16:               $\widehat{G} \leftarrow G_k[S_1] \cap G_k[S_2]$ ;
17:               $\Psi_l.\text{update}(S', < S', \widehat{G} >)$ ;
18:            else break;
19:   output the communities of keyword sets in  $\Phi_{l-1}$ ;

```

qualified keyword set, then there are at least k of q 's neighbors containing set S' . This is because every vertex in $G_k[S']$ must has degree at least k . This observation implies, we can generate all the candidate keyword sets directly by using the query vertex q and q 's neighbors, without touching other vertices.

Specifically, we consider q and q 's neighbor vertices. For each vertex v , we only select the keywords, which are contained by S and at least k of its neighbors. Then we use these selected keywords to form an itemset, in which each item is a keyword. After this step, we obtain a list of itemsets. Then we apply the well studied frequent pattern mining algorithms (e.g., Apriori [12] and FP-Growth [13]) to find the frequent keyword combinations, each of which is a candidate keyword set. Since our goal is to generate keyword combinations shared by at least k neighbors, we set the minimum support as k . In this paper, we use the well-known FP-Growth algorithm [13].

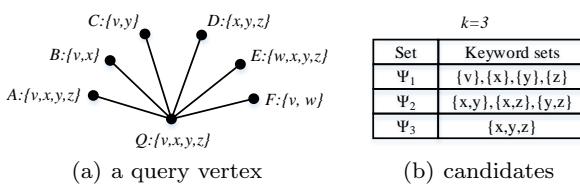


Fig. 10 An example of candidate generation.

Example 11 Consider a query vertex Q ($k=3$, $S=\{v, x, y, z\}$, with 6 neighbors in Figure 10(a), where the selected keywords of each vertex are listed in the curly braces. By FP-Growth, 8 candidate keyword sets) are gener-

ated, as shown in Figure 10(b). Ψ_i denotes the set of keyword sets with sizes being i .

2. Verification of candidate keyword sets. As candidates can be obtained using S and q 's neighbors directly, we can verify them either incrementally as that in **Inc-S**, or in a decremental manner (larger candidate keyword sets first and smaller candidate keyword sets later). In this paper, we choose the latter manner. The rationale behind is that, for any two keyword sets $S_1 \subseteq S_2$, the number of vertices containing S_2 is usually smaller than that of S_1 , which implies S_2 can be verified more efficiently than S_1 .

Algorithm 9 Query algorithm: **Dec**

```

1: function QUERY( $G$ ,  $root$ ,  $q$ ,  $k$ ,  $S$ )
2:   generate  $\Psi_1, \Psi_2, \dots, \Psi_h$  using  $S$  and  $q$ 's neighbors;
3:   find the subtree root node  $r_k$ ;
4:   create  $R_1, R_2, \dots, R_h$  by using subtree rooted at  $r_k$ ;
5:    $l \leftarrow h$ ;  $Q \leftarrow \emptyset$ ;
6:    $\widehat{R} \leftarrow R_h \cup \dots \cup R_1$ ;
7:   while  $l \geq 1$  do
8:     for each  $S' \in \Psi_l$  do
9:       find  $G[S']$  from the subgraph induced on  $\widehat{R}$ ;
10:      find  $G_k[S']$  from  $G[S']$ ;
11:      if  $G_k[S']$  exists then  $Q.add(G_k[S'])$ ;
12:    if  $Q = \emptyset$  then
13:       $l \leftarrow l - 1$ ;
14:       $\widehat{R} \leftarrow \widehat{R} \cup R_l$ ;
15:    else break;
16:   output communities in  $Q$ ;

```

Based on the above discussions, we design **Dec** as shown in Algorithm 9. We first generate candidate keyword sets using S and q 's neighbors by FP-Growth algorithm (line 2). Then, we apply **core-locating** to find the root (with core number k) of the subtree from CL-tree, whose corresponding k - \widehat{core} contains q (line 3). Next, we traverse the subtree rooted at r_k and find vertices which share keywords with q (line 4). R_i denote the sets of vertices sharing i keywords with q . Then, we initialize l as h (line 5), as we verify keyword sets with the largest size h first. We maintain a set \widehat{R} dynamically, which contains vertices sharing at least l keywords with q (line 6). In the while loop, we examine candidate keyword sets in the decremental manner. For each candidate $S' \in \Psi_l$, we first try to find $G[S']$, then find $G_k[S']$, and put $G_k[S']$ into Q if it exists (lines 8-11). Finally, if we have found at least one qualified community, we stop at the end of this loop and output Q ; otherwise, we examine smaller candidate keyword sets in next loop.

[Yankai:

8 Variants of ACQ Problem

In this section, we formally define two typical variants and attach the index based query algorithm afterwards.

8.1 Variant 1

Definition 6 Given a graph G , a positive integer k , a vertex $q \in V$, a predefined keyword set S , and a threshold $\theta \in [0,1]$, return a subgraph G_q , the following properties hold:

1. **Connectivity.** $G_q \subseteq G$ is connected and $q \in G_q$;
2. **Structure cohesiveness.** $\forall v \in G_q, \deg_{G_q}(v) \geq k$;
3. **Keyword cohesiveness.** $\forall v \in G_q$, it has at least $|S| \times \theta$ keywords in S .

In Definition 6, the keyword cohesiveness is relaxed. This can be applied for cases when the keyword information is weak. We illustrate Variant 1 in Example 12.

Example 12 Consider Figure 3(a). Let $q=A$ and $k=2$. For Variant 1, if the predefined keyword set is $\{x, y\}$ and the threshold is 50%, then the vertex set $\{A, B, C, D, E\}$ forms the target AC.

8.1.1 Query Algorithm

In line with Problem 1, we first introduce the basic solutions without index, which are extended naturally from **basic-g** and **basic-w**, and are denoted by **basic-g-v1** and **basic-w-v1** respectively. Their detail pseudocodes are presented as Algorithms 16 and 17 in Appendix E.

Algorithm 10 presents the index based algorithm, **SWT**, for Variant 1. We first apply **core-locating** to find node r_k , whose corresponding k -**core** contains q , from CL-tree (line 1). Then we traverse the subtree rooted at r_k , and collect a set V' of vertices containing at least $|S| \times \theta$ keywords by applying **keyword-checking**. Next, we find $G[S]$ from the subgraph induced by vertices in V' (line 3), and find $G_k[S]$ from $G[S]$ (line 4). Finally, we output $G_k[S]$ as the target AC, if it exists (line 5).

8.2 Variant 2

Definition 7 Given a graph G , a positive integer k , a vertex set $Q \subseteq V$, and a predefined keyword set S , return a subgraph G_Q if exists, the following properties hold:

Algorithm 10 Query algorithm: **SWT**

```

1: function QUERY( $G, root, q, k, S$ )
2:   find the node  $r_k$  from the CL-tree index;
3:   traverse the subtree rooted at  $r_k$  and collect a set
    $V'$  of vertices containing at least  $|S| \times \theta$  keywords by
   intersecting the inverted lists;
4:   find  $G[S]$  from the subgraph induced by  $V'$ ;
5:   find  $G_k[S]$  from  $G[S]$ ;
6:   output  $G_k[S]$  as the target AC.

```

1. **Connectivity.** $G_Q \subseteq G$ is connected and $Q \subseteq G_Q$;
2. **Structure cohesiveness.** $\forall v \in G_Q, \deg_{G_Q}(v) \geq k$;
3. **Keyword cohesiveness.** The size of $L(G_q, S)$ is maximal, where $L(G_q, S) = \cap_{v \in G_q} (W(v) \cap S)$ is the set of keywords shared in S by all vertices of G_q .

Variant 2 can be applied for cases when querying communities which contain the query set Q and share the maximum keywords. We illustrate Variant 2 in Example 13.

Example 13 Consider Figure 3(a). Let $Q=\{A, C\}$ and $k=2$. For Variant 2, if the predefined keyword set is $\{w, x, y, z\}$, then the vertex set $\{A, C, D\}$ forms the target AC whose shared keyword set is $\{x, y\}$.

8.2.1 Query Algorithm

The two algorithms to answer the Variant 2 without index are denoted by **basic-g-v2** and **basic-w-v2**, which are extended from **basic-g** and **basic-w**. Detailed pseudocodes are attached in Algorithm 18 and Algorithm 19 in Appendix E. **MDec** is based on the CL-tree index structure and adapted from the decremental algorithm **Dec**.

The basic idea of three algorithms is: We first get intersection between S and the keyword set of every vertex in Q . Then, we denote the first vertex of Q by q and take q as the query vertex. Next, we conduct the main query part. Finally, we find and output the qualified communities containing Q if they exist.]

9 Experiments

We now present the experimental results. Section 9.1 discusses the setup. We discuss the results in Sections 9.2 and 9.3. [**Yankai:** The experimental results of variants are presented in Sections 9.4.]

9.1 Setup

We consider four real datasets. For *Flickr*⁴ [29], a vertex represents a user, and an edge denotes a “follow”

⁴ <https://www.flickr.com/>

Algorithm 11 Query algorithm: MDec

```

1: function QUERY( $G$ ,  $root$ ,  $Q$ ,  $k$ ,  $S$ )
2:    $S = (\bigcap_{i=0}^{n-1} W(q_i)) \cap S$ ;
3:    $q \leftarrow$  first vertex  $q_0$  in  $Q$ ;
4:   generate  $\Psi_1, \Psi_2, \dots, \Psi_h$  using  $S$  and  $q$ 's neighbors;
5:   find the subtree root node  $r_k$ ;
6:   create  $R_1, R_2, \dots, R_{h'}$  by using subtree rooted at  $r_k$ ;
7:    $l \leftarrow h$ ;  $Q_c \leftarrow \emptyset$ ;
8:    $\hat{R} \leftarrow R_h \cup \dots \cup R_{h'}$ ;
9:   while  $l \geq 1$  do
10:    for each  $S' \in \Psi_l$  do
11:      find  $G[S']$  from the subgraph induced on  $\hat{R}$ ;
12:      find  $G_k[S']$  from  $G[S']$ ;
13:      if  $G_k[S']$  exists then  $Q_c.add(G_k[S'])$ ;
14:      if  $Q_c = \emptyset$  then
15:         $l \leftarrow l - 1$ ;
16:         $\hat{R} \leftarrow \hat{R} \cup R_l$ ;
17:      else break;
18:    find target ACs which contains Q from communities
     in  $Q_c$ ;

```

relationship between two users. For each vertex, we use the 30 most frequent tags of its associated photos as its keywords. For *DBLP*⁵, a vertex denotes an author, and an edge is a co-authorship relationship between two authors. For each author, we use the 20 most frequent keywords from the titles of her publications as her keywords. In the *Tencent* graph provided by the KDD contest 2012⁶, a vertex is a person, an organization, or a microblog group. Each edge denotes the friendship between two users. The keyword set of each vertex is extracted from a user's profile. For the *DBpedia*⁷, each vertex is an entity, and each edge is the relationship between two entities. The keywords of each entity are extracted by the Stanford Analyzer and Lemmatizer. Table 3 shows the number of vertices and edges, the k_{max} value, a vertex's average degree \hat{d} , and its keyword set size \hat{l} .

Table 3 Datasets used in our experiments.

Dataset	Vertices	Edges	k_{max}	\hat{d}	\hat{l}
Flickr	581,099	9,944,548	152	17.11	9.90
DBLP	977,288	3,432,273	118	7.02	11.8
Tencent	2,320,895	50,133,369	405	43.2	6.96
DBpedia	8,099,955	71,527,515	95	17.66	15.03

To evaluate ACQs, we set the default value of k to 6. The input keyword set S is set to the whole set of keywords contained by the query vertex. For each dataset, we randomly select 300 query vertices with core numbers of 6 or more, which ensures that there is a k -core

⁵ <http://dblp.uni-trier.de/xml/>⁶ <http://www.kddcup2012.org/c/kddcup2012-track1>⁷ <http://dbpedia.org/datasets>

containing each query vertex. Each data point is the average result for these 300 queries. We implement all the algorithms in Java, and run experiments on a machine having a quad-core Intel i7-3770 3.40GHz processor, and 32GB of memory, with Ubuntu installed. We present the effectiveness and efficiency results in Sections 9.2 and 9.3.

9.2 Results on Effectiveness

We now study the effectiveness of ACQ, and compare it with existing CD and CS methods. We then discuss a case study.

9.2.1 ACQ Effectiveness

We first define two measures, namely CMF and CPJ, for evaluating the keyword cohesiveness of the communities. Let $C(q) = \{C_1, C_2, \dots, C_L\}$ be the set of L communities returned by an algorithm for a query vertex $q \in V$ (Note that $S = W(q)$).

- **Community member frequency (CMF):** this is inspired by the classical document frequency measure. Consider a keyword x of q 's keyword set $W(q)$. If x appears in most of the vertices (or members) of a community C_i , then we regard C_i to be highly cohesive. The CMF uses the occurrence frequencies of q 's keywords in C_i to determine the degree of cohesiveness. Let $f_{i,h}$ be the number of vertices of C_i whose keyword sets contain the h -th keyword of $W(q)$. Then, $\frac{f_{i,h}}{|C_i|}$ is the relative occurrence frequency of this keyword in C_i . The CMF is the average of this value over all keywords in $W(q)$, and all communities in $C(q)$:

$$CMF(C(q)) = \frac{1}{L \cdot |W(q)|} \sum_{i=1}^L \sum_{h=1}^{|W(q)|} \frac{f_{i,h}}{|C_i|} \quad (3)$$

Notice that $CMF(C(q))$ ranges from 0 to 1. The higher its value, the more cohesive is a community.

- **Community pair-wise Jaccard (CPJ):** this is based on the similarity between the keyword sets of any pair of vertices of community C_i . We adopt the Jaccard similarity, which is commonly used in the IR literature. Let $C_{i,j}$ be the j -th vertex of C_i . The CPJ is then the average similarity over all pairs of vertices of C_i , and all communities of $C(q)$:

$$CPJ(C(q)) = \frac{1}{L} \sum_{i=1}^L \left[\frac{1}{|C_i|^2} \sum_{j=1}^{|C_i|} \sum_{k=1}^{|C_i|} \frac{|W(C_{i,j}) \cap W(C_{i,k})|}{|W(C_{i,j}) \cup W(C_{i,k})|} \right] \quad (4)$$

The $CPJ(C(q))$ value has a range of 0 and 1. A higher value of $CPJ(C(q))$ implies better cohesiveness.

1. Effect of common keywords. We examine the impact of the AC-label length (i.e., the number of keywords shared by all the vertices of the AC) on keyword cohesiveness. We collect ACs containing one to five keywords, and then group the ACs according to their AC-label lengths. The average CMF and CPJ value of each group is shown in Figure 11. For all the datasets, when the AC-label lengths increase, both CMF and CPJ value rises. This justifies the use of the maximal AC-label length as the criterion of returning an AC in our ACQ Problem.

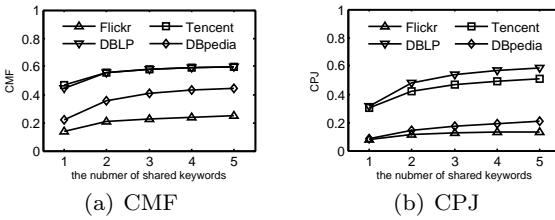


Fig. 11 AC-label length.

2. Comparison with existing CD methods As mentioned ahead, the existing CD methods for attributed graph can be adapted for community search. We choose to adapt **CODICIL** [25] for comparison. The main reasons are: (1) it has been tested on the ever reported largest attributed graph (vertex number: 3.6M); (2) it identifies communities of comparable or superior quality than those of many existing methods like [23, 33]; and (3) it runs faster than many existing methods. Since **CODICIL** needs users to specify the number of clusters expected, we set the numbers as 1K, 5K, 10K, 50K and 100K. The corresponding adapted algorithms are named as **Cod1K**, ..., **Cod100K** respectively. Other parameter settings are the same as those in [25]. We first run these algorithms offline to obtain all the communities. Given a query vertex q , they return communities containing q as the results.

We consider both keyword and structure for evaluating community quality. (1) *Keyword*: Figures 12(a) and (b) show that **ACQ** (implemented by **Dec**) always performs the best, in terms of CMF and CPJ. (2) *Structure*: As **CODICIL** has no guarantee on vertices' minimum degrees, it is unfair to compare them using this metric. We intuitively compare their structure cohesiveness by reporting the average degree of the vertices in the communities and the percentage of vertices having degrees of 6 or more. When the number of clusters in **CODICIL** is too large or too small, the structure cohesiveness becomes weak, as shown in Figures 12(c) and (d). **ACQ** always performs better than **CODICIL**, even

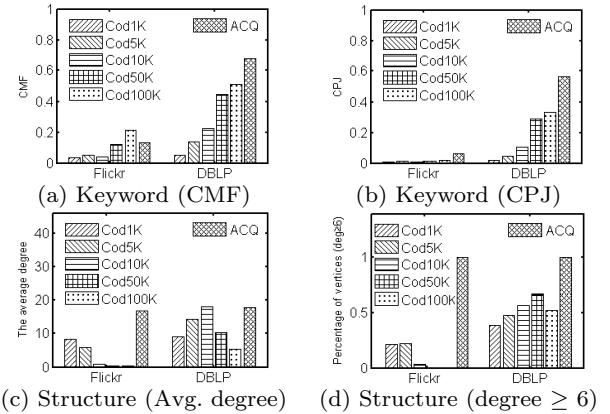


Fig. 12 Comparing with community detection method.

when its number of cluster is well set (e.g., **Cod10K** and **Cod50K** on DBLP dataset).

3. Comparison with existing CS methods. The existing methods mainly focus on non-attributed graphs. We implement two state-of-the-art methods: **Global** [28] and **Local** [5]. Both of them use the metric minimum degree, we thus focus on the keyword cohesiveness. Figure 13 shows the CMF and CPJ values for the four datasets. We can see that the keyword cohesiveness of **ACQ** is superior to both **Global** and **Local**, because **ACQ** considers vertex keywords, while **Global** and **Local** do not.

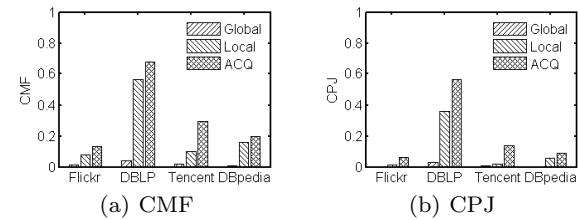


Fig. 13 Comparing with community search methods.

9.2.2 A Case Study

We next perform a case study on the DBLP dataset, in which we consider two renowned researchers in database and data mining: Jim Gray and Jiawei Han. We use $k = 4$ here. We use **Cod50K** to represent **CODICIL** for further analysis. We mainly consider the input query keyword set S , keywords and sizes of communities.

1. Effect of S . Figure 14 shows two ACs of Jiawei (AC-labels are shown in the captions), where the query keyword set S are set as {analysis, mine, data, information, network} and {mine, data, pattern, database} respectively. These two groups of Jiawei's collaborators

are involved in graph analysis (Figure 14(a)) and pattern mining (Figure 14(b)). Although these researchers all have close co-author relationship with Jiawei, the use of the input keyword set S enables the identification of communities with different research themes. For Jim, we can obtain similar results as discussed in Section 1 (Figure 2). While for CODICIL, it is not clear how to consider the keyword set S , and we thus do not show the results.

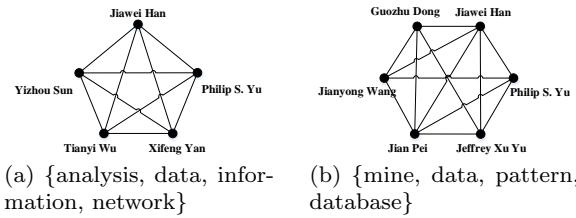


Fig. 14 Jiawei Han’s ACs.

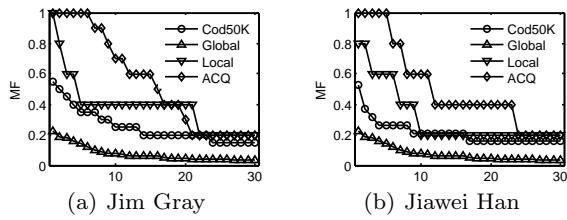


Fig. 15 Frequency distribution of keywords.

Table 4 # distinct keywords of communities.

Researcher	Cod50K	Global	Local	ACQ
Jim Gray	134	139,881	60	44
Jiawei Han	140	139,881	58	54

2. Keyword analysis. We analyze the frequency distribution of keywords in their communities. Specifically, given a keyword w_h , we define the member frequency (MF) of w_h as: $MF(w_h, C(q)) = \frac{1}{\mathcal{L}} \sum_{i=1}^{\mathcal{L}} \frac{f_{i,h}}{|C_i|}$. The MF measures the occurrence of a keyword in $C(q)$. For each C_q generated by an algorithm, we select 30 keywords with the highest MF values. We report the MF of each keyword in descending order of their MF values in Figure 15. We see that ACQ has the highest MF values for the top 20 keywords. Thus, the keywords associated with the communities generated by ACQ tend to repeat among the community members.

The number of distinct keywords of ACQ communities is also the fewest, as shown in Table 4. For example,

the k -core returned by **Global** has over $139K$ distinct keywords, about 2,300 times more than that returned by ACQ (less than 60 keywords). While the semantics of the k -core can be difficult to understand, the small number of distinct keywords of AC makes it easier to understand why the community is so formed. We further report the keywords with the 6 highest MF values in Jim and Jiawei’s communities in Tables 5 and 6. We can see that, words “sloan”, “digital”, “sky”, “survey”, and “sdss” reflect that the community is likely about the SDSS project led by Jim. The top-6 keywords of Jiawei’s AC are related to heterogenous networks. In contrast, the keywords of **Global** and **Local** tend to be less related to the query keyword set, and thus they cannot be used to characterize the communities specifically related to Jiawei. Note that the top-6 keywords of **Global** are the same for both Jim and Jiawei, as they are in the same k -core. The overall results show that, ACQ performs better than other methods.

Table 5 Top-6 keywords (Jim Gray).

Algo.	Keywords
Cod50K	server, archive, sloan, digital, database
Global	use, system, model, network, analysis, data
Local	database, system, multipetabyte, data, lsst, story
ACQ	sloan, digital, sky, data, sdss, server

Table 6 Top-6 keywords (Jiawei Han).

Algo.	Keywords
Cod50K	information, mine, data, cube, text, network
Global	use, system, model, network, analysis, data
Local	scalable, topical, text, phrase, corpus, mine
ACQ	mine, analysis, data, information, network, heterog

3. Effect of k on community size. We vary the value of k and report the average size of communities in Figure 16. We can see that the communities returned by **Global** are extremely large (more than 10^5), which can make them difficult for a query user to analyze. The community size of **Local** increases sharply when $k=8$. In this situation, **Local** returns the same community as **Global**. The size of an AC is relatively insensitive to the change of k , as AC contains around a hundred vertices for a wide range of values of k .

9.3 Results on Efficiency

For each dataset, we randomly select 20%, 40%, 60% and 80% of its vertices, and obtain four subgraphs in-

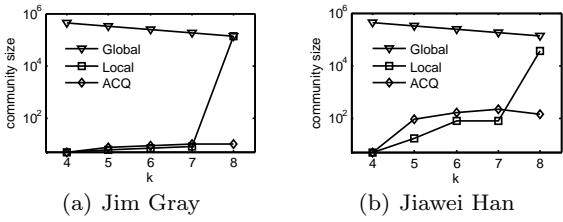


Fig. 16 Community size.

duced by these vertex sets. For each vertex, we randomly select 20%, 40%, 60% and 80% of its keywords, and obtain four keyword sets.

1. Index construction. Figures 17(a)-17(d) compare the efficiency of **Basic** and **Advanced**. We study their main parts, which build the tree without considering keywords. We denote them by **Basic-** and **Advanced-**. Notice that **Advanced** performs consistently faster, and scales better, than **Basic**. When the subgraph size increases, the performance gap between **Advanced** and **Basic** is enlarged. Similar results can be observed between **Advanced-** and **Basic-**. In addition, we also run the CD method **CODICIL**, which takes 32 mins, 2 mins, 1 day, and 3+ days (we stop it after running 3 days) to cluster the vertices of Flickr, DBLP, Tencent and DBpedia offline respectively.

[Yankai]:

2. Index Maintenance. In this section, we give the results of conducting experiments on both static and dynamic datasets. We first compare the performance between rebuilding the CL-tree and applying our algorithms for four static datasets in Figure 18(a)-18(h). We first separately conduct experiments in the edge insertion case and the edge deletion case. In two cases, we observe the performance of our algorithms with the core numbers of vertices increasing from 5 to 25. Then we randomly insert and delete 500 edges in four real datasets.

We conduct experiments on two dynamic datasets (Flickr and Youtube [21], [22]) and attach the results in Figure 18(i)-18(j). Flickr dynamic dataset has 2,585,569 vertices and 45,676,553 edges, and Youtube dynamic dataset has 1,881,147 vertices and 9,142,046 edges. We record 100 daily snapshots of Flickr graph and Youtube graph are taken to record the evolving process of the orginal graphs. We divide these daily records to 5 periods and randomly select 200 records in each period to operate in the tree index. Note that Youtube dynamic dataset only records the edge insertion time [21]. And for Flickr dynamic dataset, both the edge insertion and deletion records are included [22].

We can observe that the index maintenance algorithms are always much more efficient than rebuilding

the CL-tree. In four static datasets, the curves of four real datasets are slightly different as the core number increases. For the edge insertion case, as shown in the Figure 18(a)-18(d), dynamically updating the tree index is 10² to 10⁵ times faster than rebuilding the CL-tree. Note that our algorithms follows two main steps: (1) Compute vertices which may increase (decrease) the core numbers; (2) Update the tree index. And the reason for the phenomenon that it takes more time to as the core number increases, as shown in BDLP Figure 18(b), is that the number of vertices whose core numbers range from 10 to 20 is relatively larger than others. As a result of that, we need to recursively compute the vertices which may increase the core numbers and, if necessary, update corresponding subtrees afterwards.

For the edge deletion case, our index maintenance algorithm is around 10² times faster than rebuilding. This is because besides computing vertices whose core numbers may decrease, after deleting an edge, the connectivity of vertices in one node is unknown, and therefore we have to reorganize all vetics in this node no matter what the core number is.

In summary, as shown in the Figure 18, dynamically updating edges is more efficient than rebuilding the tree from scratch. The trends of these curves reflect the scale and cohesiveness of vertices in real datasets. After the edge insertion or deletion, most parts of the tree remain unaffected and unchanged, and therefore the index dynamic maintenance is more effective and efficient.

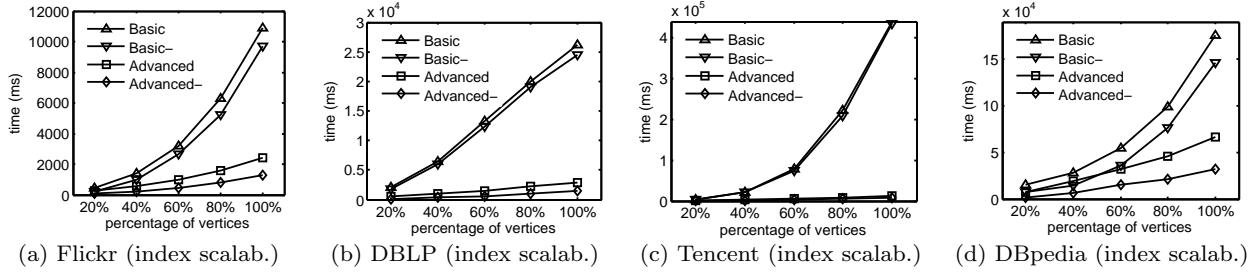
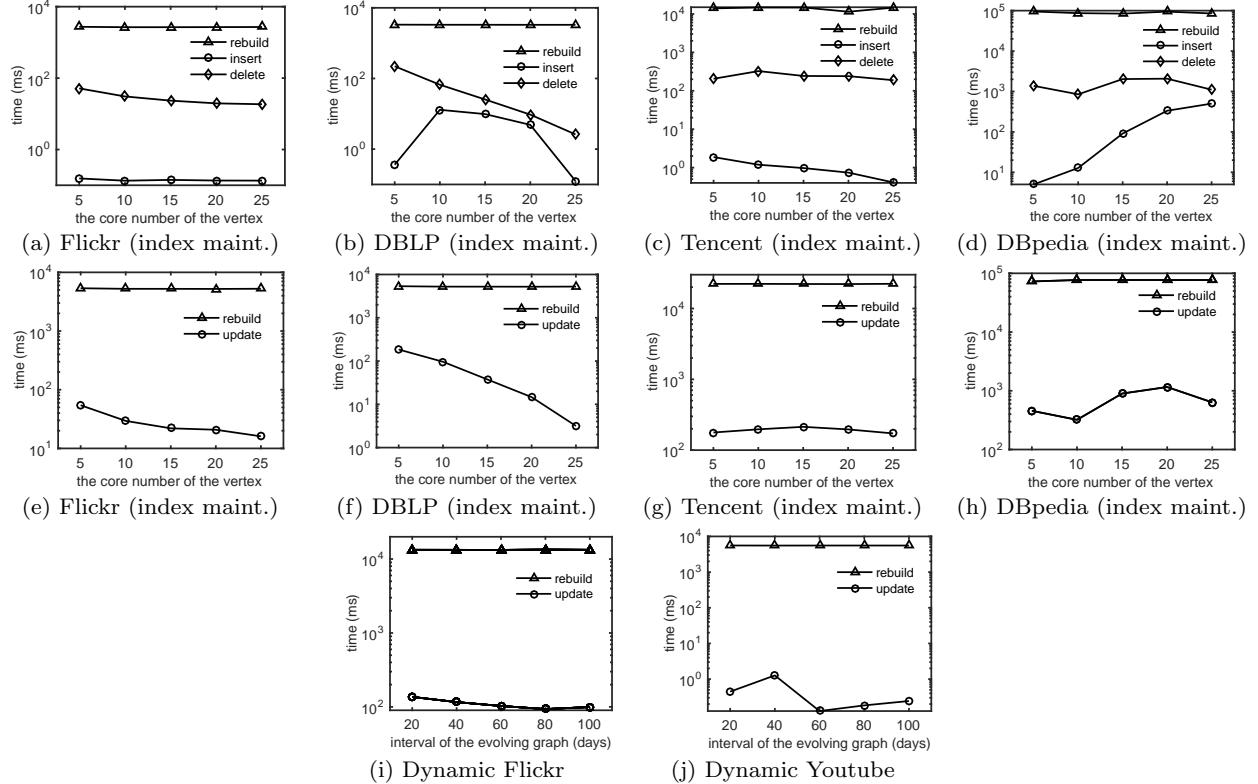
]

3. Efficiency of CS methods. Figures 19(a)-19(d) compares our best algorithm **Dec** with existing CS methods. We see that **Local** performs faster than **Global** for most cases. Also, **Dec**, which uses the CL-tree index, is the fastest.

4. Effect of k . Figures 19(e)-19(h) compare the query efficiency under different k . A lower k renders a larger subgraph, so as the time costs, for all the algorithms. Note that **basic-g** performs faster than **basic-w**, but are slower than index-based algorithms. **Inc-T** performs better than **Inc-S**, and **Dec** performs the best. The performance gaps decrease as k increases.

5. ACQ scalability w.r.t. keyword. Figures 19(i)-19(l) examine scalability over the fraction of keywords for each vertex. All the vertices are considered. The running times of the algorithms increase as more keywords are involved. **Dec** performs the best.

6. ACQ scalability w.r.t. vertex. Figures 19(m)-19(p) report the scalability over different fraction of vertices. All the keywords of each vertex are considered. Again, **Dec** scales the best.

**Fig. 17** Efficiency results of index construction.**Fig. 18** Efficiency results of index maintenance.

7. Effect of size of S . For each query vertex, we randomly select 1, 3, 5, 7 and 9 keywords to form the query keyword set S . As **Dec** performs better than **Inc-S** and **Inc-T**, we mainly compare **Dec** with the baseline solutions. Figures 19(q)-19(t) show that the cost of all algorithms increase with the $|S|$. Also, **Dec** is 1 to 3 order-of-magnitude faster than **basic-g** and **basic-w**.

8. analysis of Dec. To analyze why **Dec** performs the best, in Figures 20, we plot the average number of vertices, that share different numbers of keywords with the query vertices. Note that for each dataset, we vary the percentage of vertices from 20% to 100% (all the keywords of each vertex are kept). Recall that, **Dec** verifies candidate keyword sets in a decremental manner, i.e., larger keyword sets are examined before smaller

ones. We can see that, the number of vertices decreases very fast as the number of shared keywords increases. So if we verify larger keyword sets first, we can find the ACs from the subgraph induced by a smaller number of vertices. Hence, **Dec** performs fast.

9. Effect of invertedList. To test the importance of **invertedList**, we have implemented **Inc-S*** and **Inc-T***, which are respective variants of **Inc-S** and **Inc-T**, but without the **invertedList** structure at each CL-tree node. Figure 21 shows the results. We see that **Inc-S** (**Inc-T**) is 1 to 2 order of magnitude faster than **Inc-S*** (**Inc-T***) on all the four datasets in our experiments. The reason is that the keyword-checking operation mentioned in the above example is frequently performed in the ACQ search process. Thus, the **invertedList**, which im-

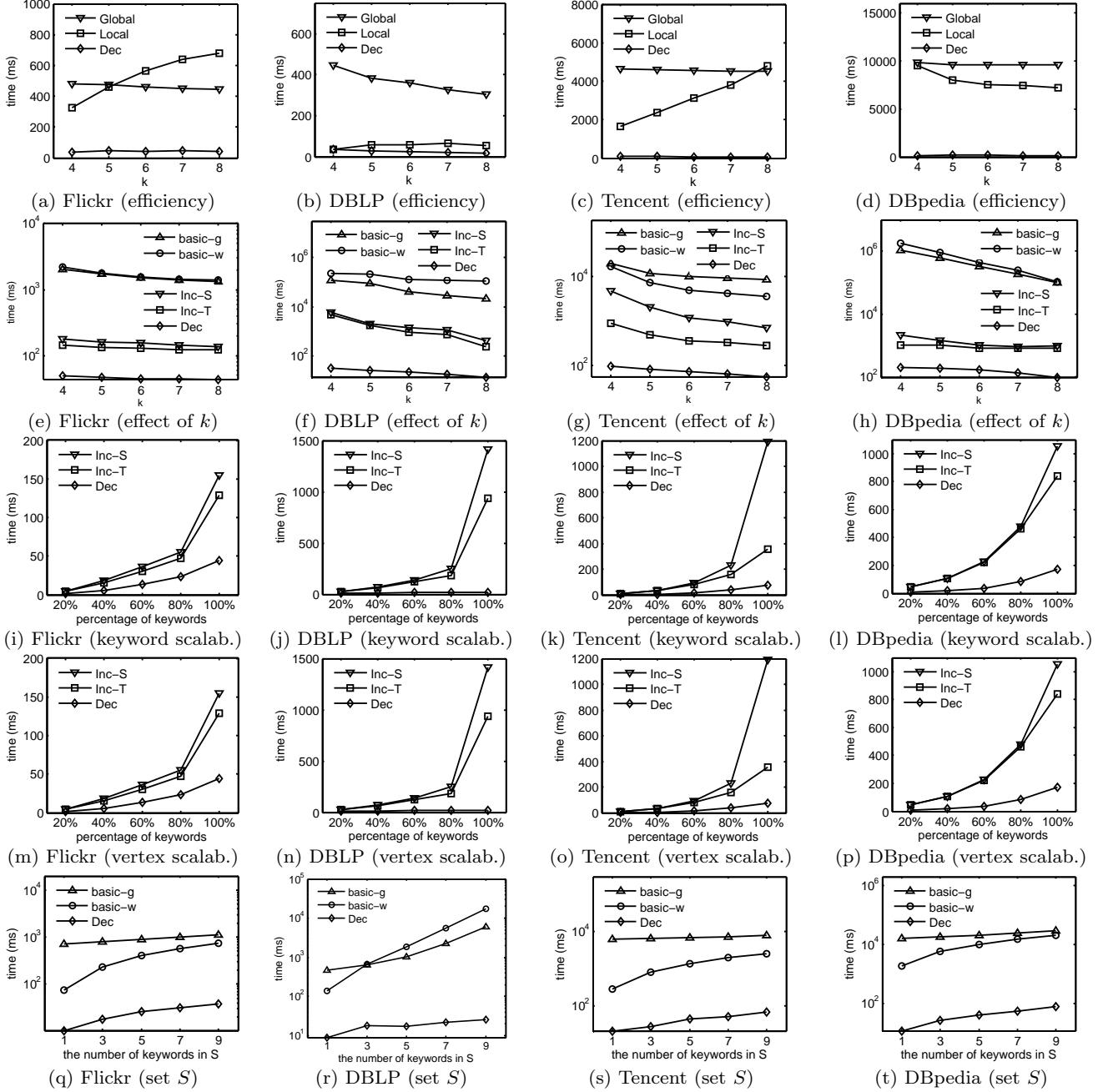


Fig. 19 Efficiency results of community search.

proves the performance of this operation, allows the ACQ search to be conducted more efficiently.

10. Non-attributed graphs. We have tested **Dec** and **Local** on non-attributed graphs. This is done by running these algorithms on our datasets, without using any of their associated keyword sets. As shown in Figure 22, for Flickr, Tencent and DBpedia, **Dec** is consistently faster than **Local**. In **Dec**, cores are organized into the CL-tree structure. Because the height of the CL-tree is not very high (lower than 405 for all datasets), the core-locating operation can be done quickly. For

DBLP, **Dec** is also faster than **Local**, except when $k=4$. In this dataset, a paper often has few (around 3 to 5) co-authors. Since an author may be closely related to a few co-authors, finding a 4-core in **Local** can be done efficiently through local expansion. From these experiments, we conclude that **Dec** can also be efficiently executed on non-attributed graphs.

[Yankai:

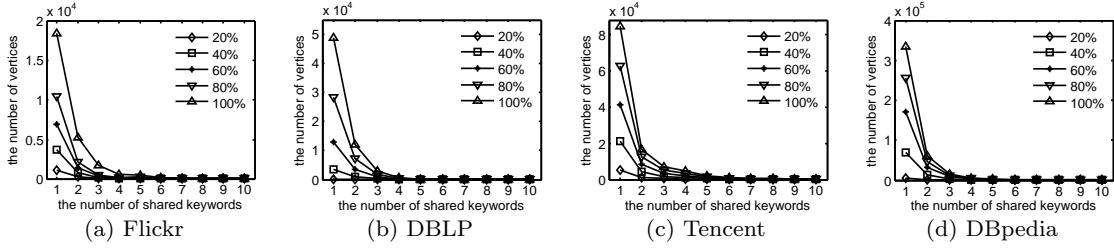


Fig. 20 The shared keywords.

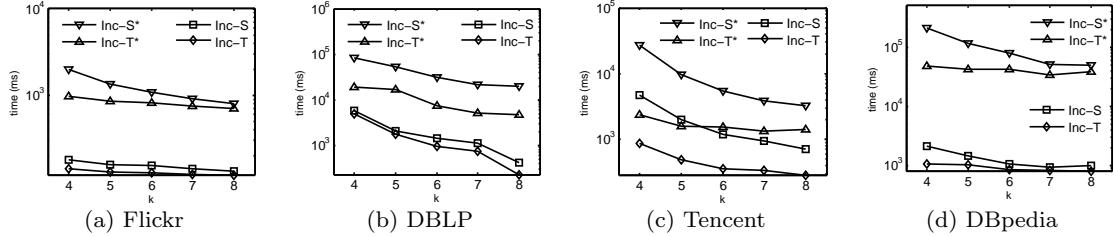


Fig. 21 Effect of InvertedList for Inc-S and Inc-T.

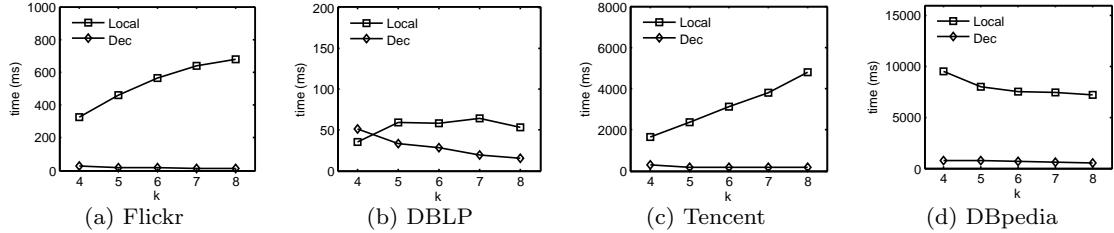


Fig. 22 Results on non-attributed graphs.

9.4 Experiments on Variants

[**Yankai:** The experimental results of variants are presented here and the basic algorithms for these variants are attached in Appendix E.]

1. Case study of Variants 1 and 2. We search Jiawei Han's communities with explicit AC-label constraints in Variants 1 and 2. We consider two keyword sets, *i.e.*, $S_1 = \{\text{stream, classification}\}$ and $S_2 = \{\text{cube, information, cluster}\}$. Note that the captions indicate the shared keywords of the communities. While for Variant 1 ($\theta=0.6$), we can obtain two communities (see Figures 24(a) and 24(b)). This implies that, for Variant 1, although we cannot find a community with AC-label being exactly S_2 , we still can find a community, in which each member contains at least 60% percentage of keywords of the input query keyword set. And for Variant 2, if the query set is $\{\text{Jiawei Han, Jing gao}\}$, and the keyword set is $S_1 = \{\text{stream, classification}\}$, we will also obtain the community in Figures 24(a).

2. Effect of θ in Variant 1. For each query vertex, we randomly select 10 keywords to form set S , set θ as 0.2, 0.4, 0.6, 0.8 and 1.0, and answer the query of Variant 1 using **basic-g-v1**, **basic-w-v1** and **SWT**. Fig-

ures 23(a)-23(d) show their efficiency. We can see that **SWT** outperforms the basic solutions consistently, as it uses the CL-tree index.

[**Yankai:** **3. Effect of $|Q|$ in Variant 2.** We limit query vertices with no more than 6 keywords. We randomly select 2, 3, 4, 5, 6 vertices to form the query set, and answer the query of Variant 2 using **basic-g-v2**, **basic-w-v2** and **MDec**. Figures 23(e)-23(h) show their efficiency. Similar with Variant 2, we can see that **MDec** performs the best by using the CL-tree index.]

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10 Conclusions

An AC is a community that exhibits structure and keyword cohesiveness. To facilitate ACQ evaluation, we develop the CL-tree index and its query algorithms. Our experimental results show that ACs are easier to interpret than those of existing community detection/search methods, and they can be “personalized”. Our solutions are also faster than existing community search algorithms.

We will study the use of other measures of structure cohesiveness (e.g., k -truss, k -clique) and keyword

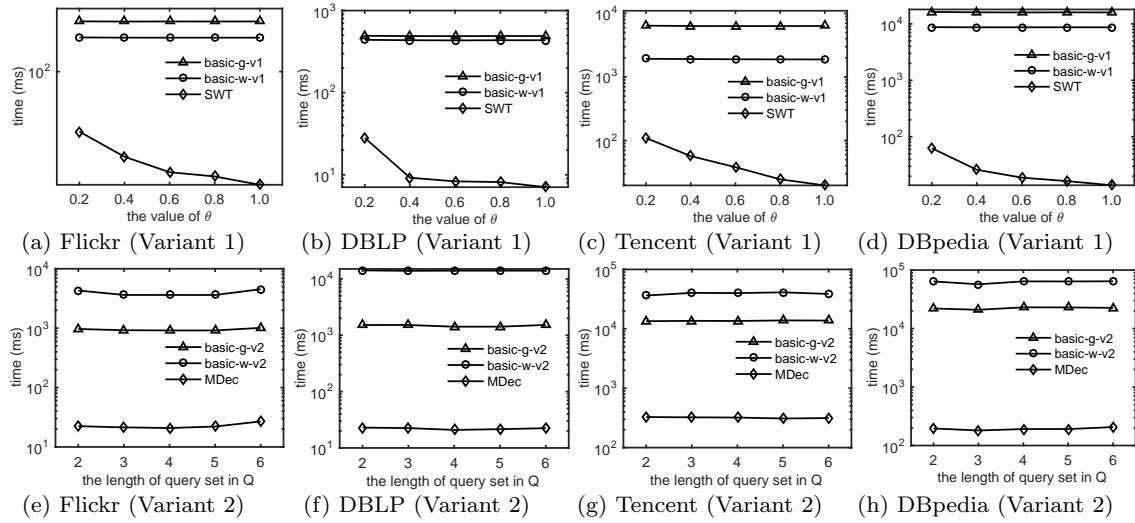


Fig. 23 Efficiency results of Variants of ACQ problem.

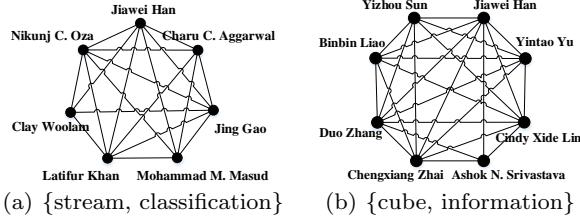


Fig. 24 Jiawei Han's communities on Variants 1 and 2.

cohesiveness (e.g., Jaccard similarity and string edit distance) in the ACQ definition. We will also investigate how the directions of edges will affect the formation of an AC. We will examine how graph pattern matching techniques [30, 8, 9] can be extended to find ACs. An interesting research direction is to study how to automatically generate a meaningful graph pattern that reflects a real community, and how to use these patterns to find ACs.

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A Proofs of Lemmas

Lemma 4 (Anti-monotonicity) *Given a graph G , a vertex $q \in G$ and a set S of keywords, if there exists a subgraph $G_k[S]$, then there exists a subgraph $G_k[S']$ for any subset $S' \subseteq S$.*

Proof. Based on the definition of $G_k[S]$, each vertex of $G_k[S]$ contains S . Consider a new keyword set $S' \subseteq S$. We can easily conclude that, each vertex of $G_k[S]$ contains S' as well. Also, note that $q \in G_k[S]$. These two properties imply that there exists one subgraph of G , namely $G_k[S]$, with core number at least k , such that it contains q and every vertex of it contains keyword set S' . It follows that there exists such a subgraph with maximal size (*i.e.*, $G_k[S']$). \square

Proposition 1 *For any keyword set S , and vertex q , if $G_k[S]$ exists, then $G_k[S] \subseteq G_k[S']$ for any subset $S' \subseteq S$.*

Proof. Since $G_k[S]$ contains vertex q and every vertex in $G_k[S]$ contains S' (due to $S' \subseteq S$), then $G_k[S] \cup G_k[S']$ also contains vertex q and every vertex in it contains S' . In addition, the core numbers of $G_k[S]$ and $G_k[S']$ are at least k , it follows that the core number of $G_k[S] \cup G_k[S']$ is at least k . Based on the definition of $G_k[S']$, we have $G_k[S] \cup G_k[S'] \subseteq G_k[S']$. It follows that $G_k[S] \subseteq G_k[S']$. \square

Lemma 5 *Given two subgraphs $G_k[S_1]$ and $G_k[S_2]$ of a graph G , for a new keyword set S' generated from S_1 and S_2 (*i.e.*, $S' = S_1 \cup S_2$), if $G_k[S']$ exists, then it must appear in a k -core with core number at least*

$$\max\{\text{core}_G[G_k[S_1]], \text{core}_G[G_k[S_2]]\}. \quad (3)$$

Proof. Since S' is generated from S_1 and S_2 , then $S_1 \subseteq S'$ and $S_2 \subseteq S'$. Based on Proposition 1, we have $G_k[S'] \subseteq G_k[S_1]$. With such a containment relationship, it follows that $\min\{\text{core}_G[v] \mid v \in G_k[S_1]\} \leq \min\{\text{core}_G[v] \mid v \in G_k[S']\}$. Hence, the core number of $G_k[S']$ is at least the core number of $G_k[S_1]$. Formally, $\text{core}_G[G_k[S_1]] \leq \text{core}_G[G_k[S']]$. For the same reason, $\text{core}_G[G_k[S_2]] \leq \text{core}_G[G_k[S']]$. It directly follows the lemma. \square

Lemma 6 *Given a connected graph $G(V, E)$ with $n=|V|$ and $m=|E|$, if $m - n < \frac{k^2-k}{2} - 1$, there is no k -core in G .*

Proof. From Definition 1, we can easily conclude that, for any specific k , a k -core has at least $k+1$ vertices. Since each vertex in a specific k -core has at least k edges, the minimum number of edges in a k -core is $\frac{(k+1)k}{2}$.

Consider a connected graph, which contains a k -core and has the minimum number of edges, where the k -core contains only $k+1$ vertices and all the rest $n-(k+1)$ vertices are connected with this k -core. The total number of edges is

$$\frac{(k+1)k}{2} + [n - (k+1)] = m \quad (4)$$

By simple transformation, we can conclude that, if $m - n < \frac{k^2-k}{2} - 1$, there is no k -core in G . \square

Lemma 7 *Given two keyword sets S_1 and S_2 , if $G_k[S_1]$ and $G_k[S_2]$ exist, we have*

$$G_k[S_1 \cup S_2] \subseteq G_k[S_1] \cap G_k[S_2]. \quad (5)$$

Proof. Based on Proposition 1 and $S_1 \subseteq S_1 \cup S_2$, we have $G_k[S_1 \cup S_2] \subseteq G_k[S_1]$. For the same reason we have $G_k[S_1 \cup S_2] \subseteq G_k[S_2]$. It directly follows the lemma. \square

B Details of Basic Solutions

Algorithms 12 and 13 present **basic-g** and **basic-w** respectively. The input of **basic-g** is a graph G , a query vertex q , an integer k and a set S . It first generates a set, Ψ , of candidate keyword sets, each of which contains a single keyword of S (line 2). Then, it finds the k -core, C_k , containing q from the graph G . In the while loop (lines 4-14), it first initializes an empty set Φ (line 5), which is used to collect all the qualified keyword sets. Then for each candidate keyword set $S' \in \Psi$, it finds $G[S']$ from C_k by considering the keyword constraint. After that, it finds $G_k[S']$ from $G[S']$ (lines 7-8), and put it into Φ if $G_k[S']$ exists (lines 9-10). After checking all the candidate keyword sets in Ψ , if there are at least one qualified keyword sets in Φ , it generates a new set Ψ' of larger candidate keyword sets by calling $\text{GENECAND}(\Phi)$ (see Appendix C) and continues to checking longer candidate keyword sets in next loop; otherwise, it stops and outputs all the communities of the latest verified keyword sets as the target ACs.

Algorithm 12 Basic solution: basic-g

```

1: function QUERY( $G, q, k, S$ )
2:   init  $\Psi$  using  $S$ ;
3:   find the  $k$ -core,  $C_k$ , containing  $q$  from  $G$ ;
4:   while true do
5:      $\Phi \leftarrow \emptyset$ ;
6:     for each  $S' \in \Psi$  do
7:       find  $G[S']$  from  $C_k$ ;
8:       find  $G_k[S']$  from  $G[S']$ ;
9:       if  $G_k[S']$  exists then
10:         $\Phi.add(S');$ 
11:      if  $\Phi \neq \emptyset$  then
12:         $\Psi \leftarrow \text{GENECAND}(\Phi);$ 
13:      else
14:        break;
15:   output the communities of keyword sets in  $\Phi$ ;
```

Algorithm 13 presents the pseudocodes of **basic-w**. It follows the main steps of **basic-g**, except that for each candidate keyword set S' , it finds $G[S']$ from G directly, rather than from C_k .

Algorithm 13 Basic solution: basic-w

```

1: function QUERY( $G, q, k, S$ )
2:   init  $\Psi$  using  $S$ ;
3:   while true do
4:      $\Phi \leftarrow \emptyset$ ;
5:     for each  $S' \in \Psi$  do
6:       find  $G[S']$  from  $G$ ;
7:       find  $G_k[S']$  from  $G[S']$ ;
8:       if  $G_k[S']$  exists then
9:          $\Phi.add(S');$ 
10:      if  $\Phi \neq \emptyset$  then
11:         $\Psi \leftarrow \text{GENECAND}(\Phi);$ 
12:      else
13:        break;
14:   output the communities of keyword sets in  $\Phi$ ;
```

C Candidate Generation

Given a set Φ of qualified keyword sets, Algorithm 14 generates new candidate keyword sets incrementally by linking each pair of keyword sets. We first initialize Ψ as an empty set (line 2). Then for each pair, S_i and S_j , of keyword sets in Φ , we sort their keywords. If they differ only at the last keyword, then we generate a new keyword set $S' = S_i \cup S_j$, by a union operation (lines 3-6). According to Lemma 1, if any subset of S' does not appear in Φ , we prune S' ; otherwise, we regard it as a candidate and add it into Ψ (lines 7-8). Finally, we return Ψ as the results (line 9).

Algorithm 14 Generate candidate keyword sets

```

1: function GENECAND( $\Phi$ )
2:    $\Psi \leftarrow \emptyset$ ;
3:   for each  $S_i \in \Phi$  do
4:     for each  $S_j \in \Phi$  do
5:       if  $S_i$  and  $S_j$  differs at the last keyword then
6:         initialize  $S' = S_i \cup S_j$ ;
7:         if  $S'$  cannot be pruned by Lemma 1 then
8:            $\Psi.add(S')$ ;
9:   return  $\Psi$ ;

```

D Anchored Union-find

Algorithm 15 presents the four functions of the anchored union-find (AUF) data structure.

Algorithm 15 Functions on the AUF data structure

```

1: function MAKESET( $x$ )
2:    $x.parent \leftarrow x$ ;
3:    $x.rank \leftarrow 0$ ;
4:    $x.anchor \leftarrow x$ ;
5: function FIND( $x$ )
6:   if  $x.parent=x$  then
7:      $x.parent \leftarrow \text{FIND}(x.parent)$ ;
8:   return  $x.parent$ ;
9: function UNION( $x, y$ )
10:   $xRoot \leftarrow \text{FIND}(x)$ ;
11:   $yRoot \leftarrow \text{FIND}(y)$ ;
12:  if  $xRoot=yRoot$  then return ;
13:  if  $xRoot.rank < yRoot.rank$  then
14:     $xRoot.parent \leftarrow yRoot$ ;
15:  else if  $xRoot.rank > yRoot.rank$  then
16:     $yRoot.parent \leftarrow xRoot$ ;
17:  else
18:     $yRoot.parent \leftarrow xRoot$ ;
19:     $xRoot.rank \leftarrow xRoot.rank + 1$ ;
20: function UPDATEANCHOR( $x, core_G[ ], y$ )
21:   $xRoot \leftarrow \text{FIND}(x)$ ;
22:  if  $core_G[xRoot.anchor] > core_G[y]$  then
23:     $xRoot.anchor \leftarrow y$ ;

```

The functions FIND and UNION are exactly the same as that of the classical union-find data structure [1]. For function MAKESET, the only change made on the classical MAKESET

is that, it adds a line of code for initializing $x.anchor$ as x (line 4). The function UPDATEANCHOR is used to update the anchor vertex of x 's representative vertex. It first finds x 's representative vertex by calling FIND (line 21). Then, if the core number of x 's representative vertex is larger than that of the current input vertex y , it updates the anchor vertex of x 's representative vertex as y .

Complexity analysis. The time complexities of functions FIND and UNION are $O(\alpha(n))$ [1], where $\alpha(n)$ is less than 5 for all practical values of n . In function MAKESET, since initializing $x.anchor$ can be done in $O(1)$, the time complexity of MAKESET is still $O(1)$. In function UPDATEANCHOR, as FIND can be completed in $O(\alpha(n))$ and updating anchor can be completed in $O(1)$, the total time cost of function UPDATEANCHOR is $O(\alpha(n))$.

E Basic Algorithm of Variants

[Yankai: 1. Variant 1. Algorithms with out index denoted as **basic-g-v1** and **basic-w-v1**. Their detail pseudocodes are presented in Algorithms 16 and 17.]

Algorithm 16 Query algorithm: **basic-g-v1**

```

1: function QUERY( $G, q, k, S$ )
2:   find the  $k$ - $\widehat{\text{core}}$ ,  $\mathcal{C}_k$ , containing  $q$  from  $G$ ;
3:   collect a set  $V'$  of vertices containing share at least
4:    $|S| \times \theta$  keywords from  $\mathcal{C}_k$ ;
5:   find  $G[S]$  from the subgraph induced by  $V'$ ;
6:   find  $G_k[S]$  from  $G[S]$ ;
7:   output  $G_k[S]$  as the target AC.

```

Algorithm 17 Query algorithm: **basic-w-v1**

```

1: function QUERY( $G, q, k, S$ )
2:   collect a set  $V'$  of vertices containing share at least
3:    $|S| \times \theta$  keywords from  $G$ ;
4:   find  $G[S]$  from the subgraph induced by  $V'$ ;
5:   find  $G_k[S]$  from  $G[S]$ ;
6:   output  $G_k[S]$  as the target AC.

```

[2. Variant 2. Algorithm **basic-g-v2**, **basic-w-v2** are extended from **basic-g** and **basic-w**. Detailed pseudocodes are attached in Algorithm 18 and Algorithm 19.]

]

Algorithm 18 Query algorithm: **basic-g-v2**

```

1: function QUERY( $G, Q, k, S$ )
2:    $S = (\bigcap_{i=0}^{n-1} W(q_i)) \cap S$ ;
3:    $q \leftarrow$  first vertex  $q_0$  in  $Q$ ;
4:   init  $\Psi$  using  $S'$ ;
5:   find the  $k$ -core,  $C_k$ , containing  $q$  from  $G$ ;
6:   while true do
7:      $\Phi \leftarrow \emptyset$ ;
8:     for each  $S' \in \Psi$  do
9:       find  $G[S']$  from  $C_k$ ;
10:      find  $G_k[S']$  from  $G[S']$ ;
11:      if  $G_k[S']$  exists then
12:         $\Phi.add(S')$ ;
13:      if  $\Phi \neq \emptyset$  then
14:         $\Psi \leftarrow \text{GENECAND}(\Phi)$ ;
15:      else
16:        break;
17:   find the target ACs which contains  $Q$  from communities of keyword sets in  $\Phi$ ;

```

Algorithm 19 Query algorithm: **basic-w-v2**

```

1: function QUERY( $G, Q, k, S$ )
2:    $S = (\bigcap_{i=0}^{n-1} W(q_i)) \cap S$ ;
3:    $q \leftarrow$  first vertex  $q_0$  in  $Q$ ;
4:   init  $\Psi$  using  $S'$ ;
5:   while true do
6:      $\Phi \leftarrow \emptyset$ ;
7:     for each  $S' \in \Psi$  do
8:       find  $G[S']$  from  $G$ ;
9:       find  $G_k[S']$  from  $G[S']$ ;
10:      if  $G_k[S']$  exists then
11:         $\Phi.add(S')$ ;
12:      if  $\Phi \neq \emptyset$  then
13:         $\Psi \leftarrow \text{GENECAND}(\Phi)$ ;
14:      else
15:        break;
16:   find the target ACs which contains  $Q$  from communities of keyword sets in  $\Phi$ ;

```
