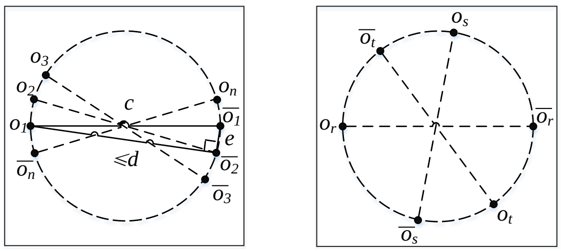

(a) Variables transformation. (b) A satisfied clause.

Figure 15: Illustrating the NP-hard proof.

1. Proof of Lemma 3.4.

PROOF. We prove the lemma by a reduction from the 3-SAT problem. An instance of the 3-SAT problem consists of ϕ=$C_1 \wedge C_2 \wedge \cdots \wedge C_n$, where each clause C_j=$x_j \vee y_j \vee z_j$(j=1,2,$\cdots,m$) and $\{x_j, y_j, z_j\} \subset \{u_1, \bar{u_1}, \cdots, u_n, \bar{u_n}\}$. The decision problem is to determine whether we can assign a value (true or false) to each variable $u_i, i = (1, 2, \cdots, n)$, such that ϕ is true. To transform an instance of 3-SAT problem to an instance of the SPM problem, we first describe how to transform the variables into spatial objects, and then discuss how to associate the keywords.

We assume all the spatial objects are in a unit square $[0, 1]^2$ data space. Let d be a value in $(0, 1)$. We consider a circle, whose center is at the center of the data space and has a diameter c=$d + \sigma$, where σ is a small positive value. We will discuss how to set the value of σ later. For variable u_1, we randomly place a corresponding spatial object o_1 on the circle. For its negation $\bar{u_1}$, we place an object $\bar{o_1}$ diametrically opposite on the circle, which implies $|o_1, \bar{o_1}|$=c. For the rest variables $u_2, u_3, \cdots, u_n$ and their negations, we place objects on the circle in the same way, such that $|o_1, o_2|$=$|o_2, o_3|$ =$\cdots$= $|o_{n-1}, o_n|$= $|o_n, \bar{o_1}|$ =$\cdots$=$|\bar{o_n}, o_1|$=e, where e can be computed using the cosine theorem:

$$e = \sqrt{\frac{c^2}{4} + \frac{c^2}{4} - 2 \cdot \frac{c^2}{4} \cdot cos\frac{180°}{n}} = \sqrt{\frac{c^2}{2}(1 - cos\frac{180°}{n})}. \quad (5)$$

We denote the set of all the placed $2n$ objects by Λ. Figure 15(a) illustrates the placement of objects in Λ. We now prove that it is possible to set a positive value of σ, such that for any object $o_i(\bar{o_i})$, the distance from it to any object in Λ, except $\bar{o_i}(o_i)$, is at most d. Let us consider o_1. Since the object furthest away from it in $\Lambda - \{\bar{o_1}\}$ is $\bar{o_2}$, we need to have $|o_1, \bar{o_2}| \leq d$. Notice that $o_1, \bar{o_1}$ and $\bar{o_2}$ form a right triangle. By Pythagorean theorem, we have

$$|o_1, \bar{o_2}| = \sqrt{|o_1, \bar{o_1}|^2 - |\bar{o_1}, \bar{o_2}|^2} = \sqrt{c^2 - e^2} \leq d. \quad (6)$$

Considering Eq (5), we have $\sqrt{\frac{1}{2}(d + \sigma)^2(1 + cos\frac{180°}{n})} \leq d$. Hence, to ensure the distance from o_1 to any object in $\Lambda - \{\bar{o_1}\}$ being at most d, we can set σ as

$$0 < \sigma \leq \frac{d}{\sqrt{\frac{1}{2}(1 + cos\frac{180°}{n})}} - d. \quad (7)$$

Next, we discuss how to associate keywords. For each pair of objects o_i and $\bar{o_i}$, we create one keyword w_i for them (i=1,2,$\cdots,n$). In other words, o_i and $\bar{o_i}$ share keyword w_i, and the only holders of w_i are o_i and $\bar{o_i}$. In addition, for each clause C_j in the instance ϕ of 3-SAT problem, we create one keyword v_j (j=1,2,$\cdots,m$) and associate it to the three objects corresponding to the three variables in C_j. Thus, given a 3-SAT instance ϕ, we have a spatial pattern P, in which (1) there are $(n + m)$ vertices (distinct keywords); (2) each pair of vertices has an edge with a distance interval $[0, d]$; and (3) each pair of vertices is with the mutual inclusion.

Next, to complete the proof, we need to prove that: (1) a satisfying assignment of the 3-SAT instance ϕ determines a set of spatial objects matched with the spatial pattern P; (2) if there exists a feasible solution to the SPM problem, i.e., a set of objects match with the pattern P, then there also exists a satisfying assignment of ϕ. We first show that (1) holds. A satisfying assignment of ϕ means that, for any pair of variables u_i and $\bar{u_i}$ one of them must be true, and any clause is also true. By selecting all the objects, which correspond to variables with true values, we can get a set of objects. Clearly, this set of objects is a match of P, because the distance and keyword requirements on each edge of P are well satisfied. In the following, we focus on proving (2).

Assume that we have a set Ψ of spatial objects matched with P, where objects in Ψ contains all the keywords $w_1, \cdots, w_n$ and $v_1, \cdots, v_m$, and the distance of each pair of objects is at most d. Consider any specific clause C_k=$u_r \vee u_s \vee u_t$ in ϕ, where 1$\leq r, s, t \leq n$ and 1$\leq k \leq m$. Since Ψ contains an object with keyword w_r, one of u_r and $\bar{u_r}$ must be assigned to be true, although we do not know which one is true. Similarly, we have this for u_s and $\bar{u_s}$, u_t and $\bar{u_t}$, respectively. It is easy to observe that, if any one of u_r, u_s, and u_t is true, then the value of C_k is true. The only assignment which makes the value of C_k false is when all the values of u_r, u_s, and u_t are assigned to be false. Next, we prove this case, however, cannot happen by contradiction. We show six objects corresponding to variables of C_k in Figure 15(b), where objects o_h and $\bar{o_h}$ ($h \in \{r, s, t\}$) correspond to variables u_h and $\bar{u_h}$ respectively.

Suppose above case happens, which implies Ψ contains objects $\bar{o_r}$, $\bar{o_s}$, and $\bar{o_t}$. Since Ψ contains an object with keyword v_k, whose only holders are objects o_r, o_s, and o_t, Ψ must contain at least one of them. As a result, Ψ contains at least one pair of the three pairs of objects: o_r and $\bar{o_r}$, o_s and $\bar{o_s}$, and o_t and $\bar{o_t}$. However, $|o_r, \bar{o_r}|$=$|o_s, \bar{o_s}|$ =$|o_t, \bar{o_t}|$=$c \geq d$, which implies that Ψ is not a valid match since the distance requirement is not satisfied. Thus, the case that u_r, u_s, and u_t are assigned to be false cannot happen.

Therefore, we conclude that if there exists a feasible solution to the SPM problem, then there also exists an assignment of ϕ, which makes it true, and (2) holds. Hence, the proof is complete. □