Noise-tolerant Localization From Incomplete Range Measurements for Wireless Sensor Networks

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***Abstract*—Accurate and sufficient range measurements are essential for range-based localization in wireless sensor networks. However, noise and data missing are inevitable in distance ranging, which may degrade localization accuracy drastically. Existing localization approaches often degrade in terms of accuracy in the co-existence of incomplete and corrupted range measurements. To address this challenge, a noise-tolerant localization algorithm called NLIRM is presented. By utilizing the natural low rank property of Euclidean distance matrix, the reconstruction of partially sampled and noisy distance matrix is formulated as a norm-regularized matrix completion problem, where Gaussian noises and outliers are smoothed by Frobenius-norm and L_1 norm regularization, respectively. As far as we are aware of, this is the first scheme that can recover the missing range measurements and explicitly sift Gaussian noise and outlier simultaneously. Simulation results demonstrate that, compared with traditional algorithms, NLIRM achieves better localization performance under the same experiment setting. In addition, our algorithm provides an accurate prediction of outlier positions, which is the prerequisite for malfunction diagnosis in WSN.**

I. INTRODUCTION

Localization is an important research problem in Wireless Sensor Networks (WSN) as the location information is critical for many WSN applications, such as environment monitoring, geographical routing, and data gathering, etc [1]. Due to the restrictions of economic cost, node energy and deployment condition, only a few so-called anchor nodes can acquire their global positions by equipping GPS devices. The purpose of localization algorithms is to determine the positions of all unknown nodes based on anchor nodes and inter-node measurements. Existing localization algorithms in WSN can be classified into two major categories: range-based and range-free [2]. Range-based algorithms collect Euclidean distance among sensor nodes by using ranging techniques such as Radio Signal Strength Indicator (RSSI), Time difference of Arrival (TDOA), and so on, while range-free algorithms generally rely on the connectivity information. The former category results in higher localization accuracy at the cost of additional hardware supplement, large communication traffic, and high computation burden. In contrast, the latter one is cost-effective but only applicable for coarse-grained localization.

We focus on the range-based localization algorithms, which can be described as follows: only a few anchor nodes know their global locations, while locations of the rests can be estimated based on inter-node Euclidean distance measurements and global locations of anchor nodes. Main stream researches assume that the distances between nodes are accurately and sufficiently measured. In practice however, data missing and noise are inevitable during distance ranging.

Distance missing. The relative positions of all nodes in WSN can be estimated accurately under the assumption that all inter-node distances are available [3]. However, in practical situations, this assumption may not realistically hold, and only a fraction of inter-node distances can be transmitted to the sink node due to limited communication range, energy constraints, and environmental affects [4]. Furthermore, for the purpose of reducing energy consumption of nodes and extending network lifetime, researchers intend to collect distance measurements according to some selective strategy. Thus only a small fraction of inter-node distances is transmitted to the sink node while remainder distances are missing, which increases the difficulty in localization implementation.

Ranging noise. In general, there are two kinds of ranging noise in practical localization system, normal noise and outlier [5]. Coming from the limitations of hardware and computation precision, normal noise tends to be moderate and follow the Gaussian distribution, we call it Gaussian noise. By contrast, outlier means anomalous distance measurements far beyond the normal range, which indicates severe error. Outlier may be caused by hardware malfunction, multipath effect, non-line-of-sight (NLOS) and malicious attacks, so it is difficult to predict. Ignoring the existence of ranging noises is not a good choice. It has been demonstrated that even a small number of outliers can degrade localization accuracy drastically [5].

Aforementioned facts explain that only a subset of inter-node distances, corrupted by random noises and outliers, can be used by localization algorithms. As a consequence, if we construct the Euclidean Distance Matrix (EDM) of sensor nodes, a part of entries will be left unknown or noisy. Existing solutions based on matrix completion theory often ignore to deal with outliers and accordingly suffer from largely reduced accuracy in case of heavy noises and outliers.

To address this challenge, we propose a noise-tolerant localization algorithm for incomplete range measurements. By utilizing the natural low rank property of EDM, the reconstruction of partially sampled and noisy distance matrix is formulated as a Norm-Regularized Matrix Completion (NRMC) problem, where Gaussian noises and outliers are smoothed by Frobenius-norm and L_1 norm regularization,

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respectively. We design an efficient algorithm based on alternating direction method of multiplier (ADMM) [6] to solve the NRMC problem and further apply the Multi-Dimension scaling (MDS) method to localize unknown nodes. Simulation results demonstrate that compared with traditional algorithms, NLIRM achieves better localization performance with a small fraction of noisy range measurements.

The main contributions of this work are summarized as follows:

- We propose a novel localization algorithm based on extensive matrix completion. As far as we are aware of, this is the first scheme that can recover the missing range measurements and explicitly sift Gaussian noise and outlier simultaneously. Experimental results demonstrate that the proposed algorithm achieves an accurate and robust localization.
- We design an efficient algorithm based on ADMM to solve the EDM reconstruction problem. By utilizing the separate structure of this optimization problem, our approach blends the benefits of dual decomposition and augmented Lagrangian methods so that it holds the property of superior convergence.
- In addition to localize sensor nodes, the proposed NLIRM algorithm is able to recognize the positions of outliers accurately, which provides the basis for node fault diagnosis, scheduling policy, and topology adjustment in WSN.

The rest of this paper is organized as follows. Related work is summarized in Section Ⅱ. Section Ⅲ introduces the proposed localization method, including problem formulation, EDM reconstruction via norm-regularized matrix completion, optimization using ADMM and the localization algorithm. Section Ⅳ represents experimental results and demonstrates the performance improvement by using the proposed algorithm. Finally, we conclude this paper in Section Ⅴ.

II. Related Work

A. Range-based Localization

In recent years, rang-based localization has been a subject of intense study and many efforts have been made, such as trilateration-based [7], MDS-MAP [3], [8] and SDP [9], [10]. The trilateration-based algorithm employs the distances from unknown nodes to anchor nodes and the positions of unknown ones are estimated by performing trilateration operation. Its performance is deteriorated by inaccurate distance measurements and error accumulation. In [7], a novel iterative localization method is proposed to sequentially merge the elements in network to finalize localization and tolerate ranging noises. The trilateration-based algorithm only uses distances between the anchor nodes and unknown nodes. As an alternative, MDS-MAP [3], [8] utilizes all distances simultaneously based on the multidimensional scaling (MDS) during one operation. This algorithm maps inter-node distances into a low-dimensional space and a relative map is generated. With sufficient knowledge of anchor node positions, absolute coordinates of all nodes can be determined. However, this algorithm requires sufficient and relatively accurate distance measurements. With respect to [9], [10], sensor network localization problem is converted to a Semi-Definite Programming (SDP) problem by relaxing the nonconvex constraints in problem formulation. This approach uses the distance measurements as constraints and reduces the quadratic constraints in linear forms by introducing slack variables. Under some assumptions on sensor node distances, it is proven that the converted problem by SDP relaxation produces a solution to the original localization problem. However, SDP algorithm can only handle small-scale problem due to its high computational complexity.

In practice however, the existence of ranging noises is a fact that cannot be neglected for localization algorithms. As mentioned above, ranging noises in general include Gaussian noise and outlier. The former tends to be moderate and predictable and lots of efforts have been made on how to deal with it [11], [12]. However, detecting and sifting outliers is more meaningful but much more difficult. Existing approaches are mainly based on graph embedding and rigidity theory. For example, a localization approach with outlier detection is proposed in [13]. By defining verifiable edges and deriving the conditions for an edge to be verifiable, this approach can explicitly detect and eliminate outliers before location computation. Based on graph rigidity theory, Xiao [14] proposes a robust patch merging operation that rejects outliers for both multilateration and patch merging, and a robust network localization algorithm is further developed.

Another challenge for range-based localization is that it is almost impractical to obtain distance information of all pair-wise nodes in network. Feng et al. [15] first introduce the matrix completion theory into location finding problem in WSN. In [15], localization from a small fraction of random entries of EDM is formulated as a low rank matrix recovery problem and the nuclear norm is applied as a convex surrogate of the nonconvex matrix rank function, which has strong theoretical guarantees. However, nuclear norm may not be a good approximation to the rank function. Specifically, nuclear norm adds the singular values together and treat them differently while non-zero singular values in rank function have the equal contribution. Moreover, this algorithm simply treats the ranging noises as Gaussian random noise, while outlier is ignored. This approach suffers high level localization error from the outliers of range measurements. All such limitations motivate us to design an accurate and robust approach to localize sensor nodes from incomplete range measurements for WSN. Based on the regularization technique, we introduce L_1 norm and Frobenius norm regularization factors to handle the outlier and Gaussian noise respectively, which improves the localization accuracy under the interference of various noises.

B. Matrix Completion

Matrix completion, derived from the famous Compressive Sensing (CS) theory [16], has attracted significant attention. As the extension of compressive sensing in matrix space, matrix completion utilizes the low rank property of matrix, which is equivalent to the sparsity of singular values, to reconstruct the original matrix from a fraction of all entries.

A standard matrix completion problem can be formulated as follows:

$$\min_{X\in\mathbb{R}^{m\times n}} rank(X) \quad s.t.\, P_{\Omega}(M) = P_{\Omega}(X) \tag{1}$$

where $\Omega \subseteq [n_1]\times[n_2]$ ($[n_1]=\{1,2,\cdots,m\}$, $[n_2]=\{1,2,\cdots,n\}$)

denotes the support set of subscripts of sampled entries, and $P_\Omega(\cdot)$ is the orthogonal projection operator defined as:

$$[P_\Omega(M)]_{ij} = \begin{cases} M_{ij}, & if\,(i,j)\in\Omega \\ 0, & otherwise \end{cases} \tag{2}$$

Unfortunately, the above rank minimization problem (1) is NP-hard due to the non-convexity and discontinuous nature of the rank function. Inspired by compressive sensing theory, Candes and Recht [17] introduce the nuclear norm, i.e. the sum of singular values of a matrix, to replace the rank function. As a result, problem (1) can be relaxed as:

$$\min_{X\in\mathbb{R}^{m\times n}} \|X\|_* \quad s.t.\, P_\Omega(M)=P_\Omega(X) \tag{3}$$

In real application, the sampled entries might be noisy, and the equality constraint in (3) will be too strict, resulting in over-fitting. Therefore, the following relaxed form of (3) is often considered for matrix completion with noise:

$$\min_{X\in\mathbb{R}^{m\times n}} \lambda\|X\|_* + \frac{1}{2}\|P_\Omega(M-X)\|_F^2 \tag{4}$$

where the parameter λ controls the rank of X and the selection of λ should depend on the noise level.

Many algorithms have been proposed to solve the matrix completion problem, such as Singular Value Thresholding (SVT) algorithm [18], Fixed Point Continuation with Approximate SVD (FPCA) algorithm [19], and Truncated Nuclear Norm Regularization (TNNR) algorithm [20], etc. Among these algorithms, both the SVT and FPCA employ the convex nuclear norm to approximate the matrix rank. However, they obtain suboptimal performance in real applications since the nuclear norm is not a good approximation to the rank function [20]. In order to address this problem, Hu et al employ the truncated nuclear norm to approximate the matrix rank and propose a novel TNNR method [20]. Different from nuclear norm based approaches which minimize the summation of all the singular values, the TNNR only minimize the smallest $\min(m,n)-l$ singular values since the rank of matrix only corresponds to the first l nonzero singular values. In this way, the TNNR gets a more accurate and robust approximation to the rank function. The abovementioned matrix completion algorithms can reconstruct the original matrix accurately under the condition that sampled entries are deterministic or corrupted by moderate Gaussian noise, however, performance of these algorithms decreases drastically if the sampled matrix contains outliers.

III. Robust Localization with Incomplete and Noisy Distance Information

A. Problem Formulation

In a typical wireless sensor network, n sensor nodes are randomly deployed in a region. A small number of anchor nodes are location-known and localization algorithm aims at obtaining the positions of unknown nodes. In range-based algorithms, distance measurements between nodes can be acquired by ranging techniques, such as TDOA and RSSI. However, due to energy constraints and noise interference, what can be acquired is a subset of noisy distance measurements.

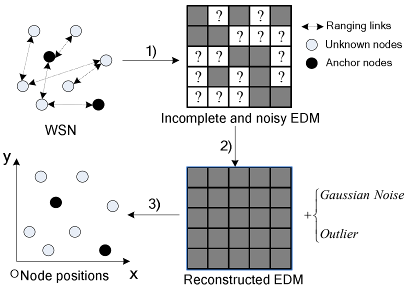


Fig.1. Localization procedure in WSN

Let $\{x_1,x_2,\cdots,x_n\}\in\mathbb{R}^d$ represent the coordinates of n nodes in d dimensional space. $X=[x_1,x_2,...,x_n]$ is the matrix of all node locations. We further define Euclidean Distance Matrix $D\in\mathbb{R}^{n\times n}$, where $D_{ij}=\|x_i-x_j\|^2$, $i,j\in\{1,2,\cdots,n\}$. As mentioned above, we cannot acquire a complete matrix D but an incomplete and noisy sampled matrix, denoted as M. By decomposing the partially sampled matrix into the original EDM and noise matrices, the relationship between D and M can be given by:

$$P_\Omega(M)=P_\Omega(D+Z+G) \tag{5}$$

where Z is a sparse outlier matrix, and G is a dense Gaussian noise matrix.

The relative positions of all nodes in WSN can be estimated accurately simply by applying MDS method under the assumption that matrix D is fully known. Furthermore, with the knowledge of at least $d+1$ anchor nodes, the relative positions can be transformed to absolute positions.

Unfortunately, as mentioned above, the EDM tends to be partially known and corrupted by noise, so that the reconstruction is prerequisite and significant. Localization procedure of this paper can be described as the following steps showing in Fig. 1.

1) A small fraction of distance measurements are collected as the incomplete and noisy sampled EDM.
2) Original EDM is reconstructed from the incomplete and noisy EDM via NRMC.
3) Apply MDS-based method to determine the absolute positions of all unknown nodes.

In this paper, we focus on reconstructing matrix D from the incomplete and noisy distance matrix M to achieve an accurate and robust localization.

B. EDM Reconstruction via Norm Regularized Matrix Completion (NRMC)

In order to reconstruct the original EDM from a fraction of noisy entries, we introduce matrix completion theory, which has been discussed in section II briefly. It is well known that the prerequisite of matrix completion is the low rank property of the object matrix to be completed. The following theorem shows that the EDM is of low rank.

Theorem 1 [21]. Given a matrix $X=[x_1,x_2,\cdots,x_n]\in\mathbb{R}^{d\times n}$, where x_i denotes the coordinate of the ith node in d dimensional space, let the matrix D be the corresponding Euclidean Distance Matrix, i.e.,

$$D_{ij}=\|x_i-x_j\|^2 = x_i^{\mathrm{T}}x_i + x_j^{\mathrm{T}}x_j - 2x_i^{\mathrm{T}}x_j \tag{6}$$

Then the rank of matrix D is at most $d+2$.

In this work, we focus on localization of 2D sensor networks (it can be extended to 3D case in a same way), so that $d = 2$ and the rank of D is at most 4. In fact, the rank of D can be regarded as 4 in normal networks. Therefore, for the sampled matrix without noise or only with moderate Gaussian noise, we can utilize the existing matrix completion methods to reconstruct the Euclidean distance matrix D. However, as stated before, the sampled matrix usually involves the sparse but large outlier noise and the dense but small Gaussian noise. In order to efficiently smooth these noises, the regularization technique is introduced into the standard matrix completion method. Specially, we employ the Frobenius-norm and L_1-norm to smooth Gaussian noise and outlier, respectively. Hence the reconstruction of EDM can be modeled as the Norm Regularized Matrix Completion (NRMC) problem:

$$\begin{aligned}&\min_{D,Z,G\in\mathbb{R}^{n\times n}} rank(D)+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &\quad s.t.\ \ P_\Omega(M)=P_\Omega(D+Z+G)\end{aligned}\tag{7}$$

where $\|Z\|_1=\sum_{i=1}^{m}\sum_{j=1}^{n}|Z_{ij}|$, $\|G\|_F^2=\sum_{i=1}^{n}\sum_{j=1}^{n}|G_{ij}|^2$.

However, the above NRMC problem is NP-hard due to the nonconvexity of rank function. Following [20], we relax matrix rank to the truncated nuclear norm, so the problem (7) can be reformulated as:

$$\begin{aligned}&\min_{D,Z,G\in\mathbb{R}^{n\times n}} \|D\|_l+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &\quad s.t.\ \ P_\Omega(D+Z+G-M)=0\end{aligned}\tag{8}$$

where l is the real/estimated rank of the object matrix, and $\|D\|_l=\sum_{i=l+1}^{n}\sigma_i(D)$ denotes the truncated nuclear norm of the matrix D, $\sigma_i(D)$ is the ith largest singular value of matrix D.

In order to efficiently solve the problem (8), we first introduce a very useful theorem and its corollary as follows.

Theorem 2 [20]. Given a matrix $X\in\mathbb{R}^{m\times n}$ with rank r, suppose $U\Sigma V^T$ is the Singular Value Decomposition of X, where $U=(u_1,\cdots,u_r)$, $\mathrm{V}=(v_1,\cdots,v_r)$, $\Sigma=diag\{\sigma_1,\sigma_2,\cdots,\sigma_r\}$, if let $\tilde{A}=(u_1,\cdots,u_l)^T$, $\tilde{B}=(v_1,\cdots,v_l)^T$, then

$$(\tilde{A},\tilde{B})=\arg\max_{AA^T=I_{l\times l},BB^T=I_{l\times l}}\mathrm{Tr}(AXB^T)\tag{9}$$

and

$$\max_{AA^T=I_{l\times l},BB^T=I_{l\times l}}\mathrm{Tr}(AXB^T)=\sum_{i=1}^{l}\sigma_i(X)\tag{10}$$

Corollary 1. For any matrix $X\in\mathbb{R}^{m\times n}$ and each nonnegative integer $l\le\min(m,n)$, we have

$$\|X\|_l=\|X\|_*-\max_{AA^T=I_{l\times l},BB^T=I_{l\times l}}\mathrm{Tr}(AXB^T)\tag{11}$$

Based on Corollary 1, the NRMC problem can be rewritten as follows:

$$\begin{aligned}&\min_{D,Z,G\in\mathbb{R}^{n\times n}}\|D\|_*-\max_{AA^T=I_{l\times l},BB^T=I_{l\times l}}\mathrm{Tr}(ADB^T)+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &\quad s.t.\ \ P_\Omega(D+Z+G-M)=0\end{aligned}\tag{12}$$

Obviously, according to theorem 2, we can design a simple but efficient iterative scheme to solve the problem (12). Specifically, we set $D_0=\mathrm{P}_\Omega(M)$, in the kth iteration, we first fix D_k and compute A_k and B_k according to (9) based on the SVD of D_k, and then we fix A_k and B_k to update D_{k+1}. Therefore, the key to solve the problem (12) is how to solve the following problem:

$$\begin{aligned}&\min_{D,Z,G\in\mathbb{R}^{n\times n}}\|D\|_*-\mathrm{Tr}(A_kDB_k^T)+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &\quad s.t.\ \ P_\Omega(D+Z+G-M)=0\end{aligned}\tag{13}$$

C. Optimizing NRMC Using ADMM

In this section, we will introduce the popular alternating direction method of multipliers (ADMM) for solving the problem (13). Before elaborating the specific optimization method, we first briefly review some well-known theorems that are useful for the subsequent analysis.

Theorem 3 [22]. For each $\tau\ge 0$ and $Y\in\mathbb{R}^{m\times n}$, the solution of the following problem

$$\arg\min_{X\in\mathbb{R}^{m\times n}}\left\{\tau\|X\|_1+\frac{1}{2}\|X-Y\|_F^2\right\}$$

is given by the shrinkage operator $\boldsymbol{\mathcal{S}}_\tau(Y)\in\mathbb{R}^{m\times n}$, which is defined component-wisely as

$$[\boldsymbol{\mathcal{S}}_\tau(Y)]_{ij}=sign(Y_{ij})\cdot\max(0,|Y_{ij}|-\tau)\tag{14}$$

where $sign(\cdot)$ is the sign function.

Theorem 4 [18]. For each $\tau\ge 0$ and $Y\in\mathbb{R}^{m\times n}$ with rank r, the solution of the following problem

$$\arg\min_{X\in\mathbb{R}^{m\times n}}\left\{\tau\|X\|_*+\frac{1}{2}\|X-Y\|_F^2\right\}$$

is given by the soft-thresholding operator $\boldsymbol{\mathcal{D}}_\tau(Y)\in\mathbb{R}^{m\times n}$, which is defined as

$$\boldsymbol{\mathcal{D}}_\tau(Y)=U\boldsymbol{\mathcal{S}}_\tau(\Sigma)V^T\tag{15}$$

where $U\in\mathbb{R}^{m\times r}$, $V\in\mathbb{R}^{n\times r}$ and $\Sigma\in\mathbb{R}^{r\times r}$ are obtained by the singular value decomposition (SVD) of Y

$$Y=U\Sigma V^T\ \text{ and }\ \Sigma=diag(\{\sigma_i\,|\,1\le i\le r\})$$

where σ_i is the ith largest singular value of Y.

In order to apply ADMM to our NRMC problem, we first rewrite the constraints of problem (13) as a linear form:

$$\begin{aligned}&\min_{D,Z,G\in\mathbb{R}^{n\times n}}\|D\|_*-\mathrm{Tr}(A_kDB_k^T)+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &\quad s.t.\ \ D+Z+G+E-M=0,\ \ P_\Omega(E)=0\end{aligned}\tag{16}$$

ADMM can be viewed as an attempt to blend the benefits of dual decomposition and augmented Lagrangian methods for constrained optimization. The augmented Lagrangian function corresponding to problem (16) is

$$\begin{aligned}L_\rho(D,Z,G,E,Y)&=\|D\|_*-\mathrm{Tr}(A_kDB_k^T)+\lambda\|Z\|_1+\mu\|G\|_F^2\\ &+\langle Y,D+Z+G+E-M\rangle+\frac{\rho}{2}\|D+Z+G+E-M\|_F^2\end{aligned}\tag{17}$$

where Y is the Lagrange multiplier of the linear constraint, $\rho>0$ is the penalty parameter for the violation of the linear constraint, and $\langle\cdot\rangle$ denotes the standard trace inner product.

Applying the ADMM to (17), given the initial setting $Z_0=G_0=E_0=Y_0=0$, we have the following iterative scheme:

$$\begin{cases} D^{i+1} = \arg\min\limits_{D\in\mathbb{R}^{n\times n}} L_\rho(D, Z^i, G^i, E^i, Y^i) \\ Z^{i+1} = \arg\min\limits_{Z\in\mathbb{R}^{n\times n}} L_\rho(D^{i+1}, Z, G^i, E^i, Y^i) \\ G^{i+1} = \arg\min\limits_{G\in\mathbb{R}^{n\times n}} L_\rho(D^{i+1}, Z^{i+1}, G, E^i, Y^i) \\ E^{i+1} = \arg\min\limits_{P_\Omega(E)=0} L_\rho(D^{i+1}, Z^{i+1}, G^{i+1}, E, Y^i) \\ Y^{i+1} = Y^i + \rho(D^{i+1} + Z^{i+1} + G^{i+1} + E^{i+1} - M) \end{cases} \tag{18}$$

Step 1: Update D.

$$\begin{aligned} D^{i+1} = \arg\min_{D\in\mathbb{R}^{n\times n}} & \|D\|_* - \mathrm{Tr}(A_k D B_k^T) + \lambda\|Z^i\|_1 + \mu\|G^i\|_F^2 \\ & + \langle Y^i, D + Z^i + G^i + E^i - M\rangle \\ & + \frac{\rho}{2}\|D + Z^i + G^i + E^i - M\|_F^2 \end{aligned} \tag{19}$$

Ignoring constant terms, this can be rewritten as

$$\begin{aligned} D^{i+1} = \arg\min_{D\in\mathbb{R}^{n\times n}} & \|D\|_* - \mathrm{Tr}(A_k D B_k^T) + \langle Y^i, D\rangle \\ & + \frac{\rho}{2}\|D + Z^i + G^i + E^i - M\|_F^2 \end{aligned} \tag{20}$$

Note that $\mathrm{Tr}(A_k D B_k^T) = \langle D, A_k^T B_k\rangle$, we have

$$\begin{aligned} D^{i+1} &= \arg\min_{D\in\mathbb{R}^{n\times n}} \|D\|_* + \langle D, Y^i - A_k^T B_k\rangle + \frac{\rho}{2}\|D + Z^i + G^i + E^i - M\|_F^2 \\ &= \arg\min_{D\in\mathbb{R}^{n\times n}} \|D\|_* + \frac{\rho}{2}\left\|D + Z^i + G^i + E^i - M + \frac{1}{\rho}(Y^i - A_k^T B_k)\right\|_F^2 \end{aligned} \tag{21}$$

Based on Theorem 4, D_{i+1} can be calculated as follows:

$$D^{i+1} = \boldsymbol{D}_{\frac{1}{\rho}}(M - Z^i - G^i - E^i - \frac{1}{\rho}(Y^i - A_k^T B_k)) \tag{22}$$

Step 2: Update Z.

$$Z^{i+1} = \arg\min_{Z\in\mathbb{R}^{n\times n}} \lambda\|Z\|_1 + \frac{\rho}{2}\left\|Z + D^{i+1} + G^i + E^i - M + \frac{1}{\rho}Y^i\right\|_F^2 \tag{23}$$

According to Theorem 3, we get

$$Z^{i+1} = \boldsymbol{S}_{\frac{\lambda}{\rho}}(M - D^{i+1} - G^i - E^i - \frac{1}{\rho}Y^i) \tag{24}$$

Step 3: Update G.

$$\begin{aligned} G^{i+1} &= \arg\min_{G\in\mathbb{R}^{n\times n}} \mu\|G\|_F^2 + \langle Y^i, G\rangle + \frac{\rho}{2}\|G + D^{i+1} + Z^{i+1} + E^i - M\|_F^2 \\ &= \frac{\rho(M - D^{i+1} - Z^{i+1} - E^i) - Y^i}{2\mu + \rho} \end{aligned} \tag{25}$$

Step 4: Update E.

$$\begin{aligned} E^{i+1} &= \arg\min_{P_\Omega(E)=0} \langle Y^i, E\rangle + \frac{\rho}{2}\|E + D^{i+1} + Z^{i+1} + G^{i+1} - M\|_F^2 \\ &= \mathrm{P}_{\overline{\Omega}}(M - D^{i+1} - Z^{i+1} - G^{i+1} - \frac{1}{\rho}Y^i) \end{aligned} \tag{26}$$

where $\overline{\Omega}$ means the complementary set of Ω.

Step 5: Update Y.

$$Y^{i+1} = Y^i + \rho(D^{i+1} + Z^{i+1} + G^{i+1} + E^{i+1} - M) \tag{27}$$

Based on the aforementioned analysis, we summarize the whole procedure in Algorithm 1 called as Norm Regularized Matrix completion using ADMM (NRMC-ADMM).

Algorithm 1: NRMC-ADMM

Input: Incomplete noisy distance matrix M, the support set Ω, the estimated rank l of objective matrix, and the parameters ρ, λ, μ.

Output: Objective matrix D^{opt}, outlier matrix Z^{opt}.

1: Initialize $D_0 = \mathrm{P}_\Omega(M)$, $Z_0 = G_0 = E_0 = Y_0 = 0$, $k = 0$;
2: **while** not converged do
3: $[U, \Sigma, V] = svd(D_k)$, $A_k = U(:, 1:l)^T, B_k = V(:, 1:l)^T$;
4: $Z_{k+1}^0 = Z_k, G_{k+1}^0 = G_k, E_{k+1}^0 = E_k, Y_{k+1}^0 = Y_k, i = 0$;
5: **while** not converged do
6: $D_{k+1}^{i+1} = \boldsymbol{D}_{\frac{1}{\rho}}(M - Z_{k+1}^i - G_{k+1}^i - E_{k+1}^i - \frac{1}{\rho}(Y_{k+1}^i - A_k^T B_k))$;
7: $Z_{k+1}^{i+1} = \boldsymbol{S}_{\frac{\lambda}{\rho}}(M - D_{k+1}^{i+1} - G_{k+1}^i - E_{k+1}^i - \frac{1}{\rho}Y_{k+1}^i)$;
8: $G_{k+1}^{i+1} = \dfrac{\rho(M - D_{k+1}^{i+1} - Z_{k+1}^{i+1} - E_{k+1}^i) - Y_{k+1}^i}{2\mu + \rho}$;
9: $E_{k+1}^{i+1} = P_{\overline{\Omega}}(M - D_{k+1}^{i+1} - Z_{k+1}^{i+1} - G_{k+1}^{i+1} - \frac{1}{\rho}Y_{k+1}^i)$;
10: $Y_{k+1}^{i+1} = Y_{k+1}^i + \rho(D_{k+1}^{i+1} + Z_{k+1}^{i+1} + G_{k+1}^{i+1} + E_{k+1}^{i+1} - M)$
11: $i = i + 1$;
12: **end while**
13: $D_{k+1} = D_{k+1}^i, Z_{k+1} = Z_{k+1}^i, G_{k+1} = G_{k+1}^i$;
14: $E_{k+1} = E_{k+1}^i, Y_{k+1} = Y_{k+1}^i$;
15: $k = k + 1$;
16: **end while**
17: **Output** $D^{opt} \leftarrow D_{k+1}$, $Z^{opt} \leftarrow Z_{k+1}$.

In fact, the inner loop 5-12, which attempts to find the exact solution of problem (16), is dispensable. We find that updating D, Z, G, E only once is adequate to make the algorithm converge to the optimal solution. Based on this, a simpler and faster algorithm called as Fast-NRMC-ADMM is proposed.

Algorithm 2: Fast-NRMC-ADMM

Input: Incomplete noisy distance matrix M, the support set Ω, the estimated rank l of objective matrix, and the parameters ρ, λ, μ.

Output: Objective matrix D^{opt}, outlier matrix Z^{opt}.

Initialize $D_0 = \mathrm{P}_\Omega(M)$, $Z_0 = G_0 = E_0 = Y_0 = 0$, $k = 0$;
while not converged do
$[U, \Sigma, V] = svd(D_k)$, $A_k = U(:, 1:l)^T, B_k = V(:, 1:l)^T$;
$G_{k+1} = \dfrac{\rho(M - D_k - Z_k - E_k) - Y_k}{2\mu + \rho}$;
$E_{k+1} = P_{\overline{\Omega}}(M - D_k - Z_k - G_{k+1} - \frac{1}{\rho}Y_k)$;
$Z_{k+1} = \boldsymbol{S}_{\frac{\lambda}{\rho}}(M - D_k - G_{k+1} - E_{k+1} - \frac{1}{\rho}Y_k)$;
$D_{k+1} = \boldsymbol{D}_{\frac{1}{\rho}}(M - Z_{k+1} - G_{k+1} - E_{k+1} - \frac{1}{\rho}(Y_k - A_k^T B_k))$;
$Y_{k+1} = Y_k + \rho(D_{k+1} + Z_{k+1} + G_{k+1} + E_{k+1} - M)$;
$k = k + 1$;

10: **end while**
11: **Output** $D^{opt} \leftarrow D_{k+1},\ Z^{opt} \leftarrow Z_{k+1}$.

Remark 1. From Algorithm 2, we can see that the main computation dominating each iteration is to perform the SVD twice which costs $O(n^3)$ operations. However, for step 3, only the first l singular values and their corresponding singular vectors must be computed, while for step 7, according to the definition of soft-thresholding operator, we only need those singular values greater than a threshold and their corresponding singular vectors. Hence, partial SVD can be implemented to save considerately the computation of SVD. In the experiments later, as the same as SVT and FPCA, we use the well-known PROPACK package (http://sun.stanford.edu/~rmunk/PROPACK/) to accomplish the partial SVD.

Remark 2. For the Fast-NRMC-ADMM algorithm, the reason we decide to perform the alternating tasks in the order of $G_{k+1} \to E_{k+1} \to Z_{k+1} \to D_{k+1}$ is that we actually allow partial SVD in the computation of D_{k+1}. Hence, to prevent the error resulted by the partial SVD from affecting the later solution, we do not follow conventional alternating order: $D_{k+1} \to Z_{k+1} \to G_{k+1} \to E_{k+1}$. We must also point out that full SVD is valid no matter which alternating order among variables $\{D_{k+1}, Z_{k+1}, G_{k+1}, E_{k+1}\}$ is used.

D. Robust Localization Based on the Reconstructed EDM

Based on the reconstructed EDM derived from NRMC, all pair-wise range measurements between nodes are available, and the MDS-based method can be applied to determine node locations that fit the range measurements. For our problem, the MDS-based method can be described as two steps. We first apply MDS to the reconstructed EDM to generate a relative map which represents the relative coordinates of all nodes, and then the relative coordinates are mapped to the absolute ones with the assistance of anchor nodes.

MDS is used to find a placement of points in low dimensional space, where distances between points resemble the original similarities. When it comes to localization, the similarities are treated as Euclidean distances [3]. By applying MDS method to the reconstructed EDM, the relative positions (up to rotation, reflection and translation) of all nodes are generated. Then position alignment is needed to map the relative coordinates to absolute ones based on three or more anchor nodes. Let R_i and $T_i (i = 1, 2, \cdots, n)$ denotes the relative coordinates and absolute coordinates of ith node in 2D space, respectively. It is clear that R_i can be mapped to T_i through a linear transformation including translation, rotation and reflection. Assuming nodes $1, 2, \cdots, k$ $(k \geq 3)$ are anchor nodes, we have

$$\left[T_2 - T_1, T_3 - T_1, \cdots, T_k - T_1\right] = Q_1 Q_2 \left[R_2 - R_1, R_3 - R_1, \cdots, R_k - R_1\right] \quad (28)$$

where Q_1 and Q_2 represent the rotation matrix and reflection matrix, respectively. Applying the transformation to all the unknown nodes $k+1, k+2, \cdots, n$ we can obtain the absolute coordinates of them easily.

Based on the EDM reconstruction and above MDS-based method, the proposed Noise-tolerant Localization from Incomplete Range Measurements (NLIRM) is summarized in Algorithm 3.

Algorithm 3: NLIRM

Input: Incomplete noisy distance matrix M, the support set Ω, the estimated rank l of objective matrix, and the positions of anchor nodes $\{T_1, T_2, \cdots, T_k\}$ $(k \geq 3)$.

Output: Positions of unknown nodes $\{T_i, i = k+1, k+2, \cdots n\}$.

//EDM reconstruction

1: Reconstruct the distance matrix D from $P_\Omega(M)$ by Fast-NRMC-ADMM algorithm:

$$\begin{aligned} \min_{D,Z,G \in \mathbb{R}^{n \times n}} \ & rank(D) + \lambda \|Z\|_1 + \mu \|G\|_F^2 \\ s.t.\ \ & P_\Omega(M) = P_\Omega(D + Z + G) \end{aligned};$$

//MDS-based method

2: Double center the reconstructed matrix D:
$W = -\frac{1}{2} JDJ$, where $J = I - \frac{1}{n} 1 \cdot 1^{\mathrm{T}}$ and I is identity matrix;

3: Decompose matrix W by SVD:
$$[U, \Lambda, V] = svd(W);$$

4: Generate the relative coordinate matrix:
$$R = \left[R_1, R_2, \cdots R_n\right] = \Lambda_2^{1/2} U_2^{\mathrm{T}}$$
where $R_i \in \mathbb{R}^{2\times 1}$, $\Lambda_2 = \Lambda(1:2, 1:2)$, and $U_2 = U(:, 1:2)$;

5: Compute the coordinate-transform matrix:
$Q = Q_1 Q_2 = \left[T_2 - T_1, T_3 - T_1, \cdots, T_k - T_1\right] / \left[R_2 - R_1, R_3 - R_1, \cdots, R_k - R_1\right]$;

6: Obtain the absolute positions of unknown nodes:
$$\left\{T_i = Q \cdot (R_i - R_1) + T_1, \quad i = k+1, k+2, \cdots n\right\}$$

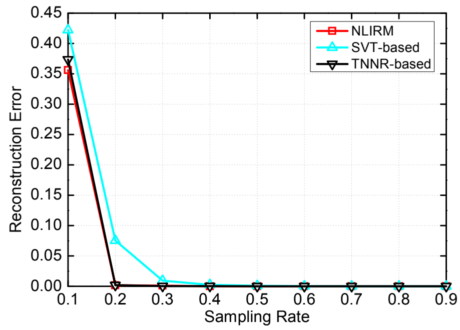
7: **Output** $\{T_i, i = k+1, k+2, \cdots n\}$.

IV. Experimental Evaluation

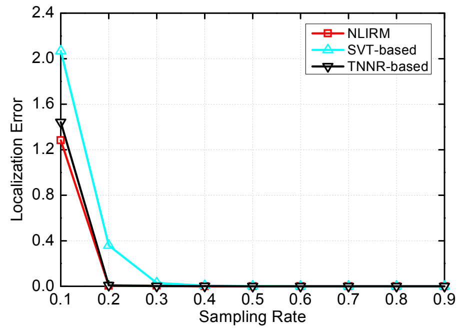
In this section, we evaluate the performance of the proposed NLIRM algorithm from incomplete EDM in various range noise conditions. We generate networks of 100 nodes randomly distributed in a $100m \times 100m$ square area with 6 of them are anchor nodes. Let $X \in \mathbb{R}^{2\times 100}$ be the coordinate matrix of all nodes and $D \in \mathbb{R}^{100\times 100}$ denote the EDM. Corresponding noisy EDM D_noise is generated by introducing noise to matrix D, and then a fraction of entries in D_noise are sampled randomly as the known incomplete noisy range measurements. To evaluate the performance of Algorithm 3, the reconstruction error and localization error are chosen as the evaluation metrics, which are defined as follows:

1) EDM reconstruction error:
$$e_c = \|D' - D\|_F / \|D\|_F \quad (29)$$
where D' represents the reconstructed EDM.

2) localization error:
$$e_l = \|X' - X\|_F / n \quad (30)$$
where X' denotes the corresponding coordinate matrix obtained by Algorithm 3.

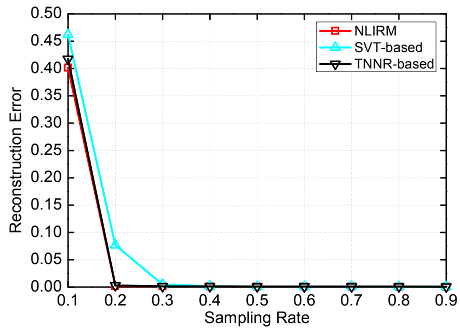


(a) EDM reconstruction error

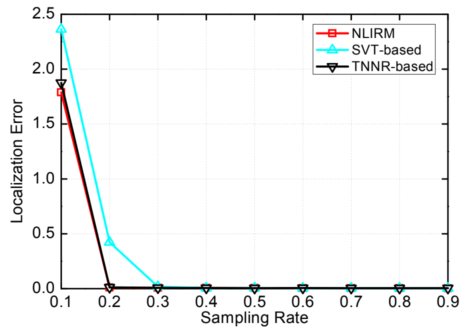


(b) Localization error

Fig.2. Performances evaluation in ideal conditions without noise



(a)EDM reconstruction error



(b) Localization error

Fig.3. Performances evaluation under the interference of Gaussian noises

For an extensive performance evaluation of NLIRM under the interference of noise, four categories of localization experiments under the following types of range noise conditions are carried out to compare with the SVT-based localization algorithm proposed in [15] and the TNNR method proposed in [20]. For each setting, we integrate results from 100 network instances.

- **NGNO:** no Gaussian noise, no outlier.
- **WGNO:** with Gaussian noise, no outlier.
- **NGWO:** no Gaussian noise, with outlier.
- **WGWO:** with Gaussian noise, with outlier.

A. Localization under NGNO

This set of experiments assumes that all sampled distance measurements are deterministic. Fig.2 shows the reconstruction error and localization error against sampling rates. The vertical axis in Fig.2 (a) and Fig.2 (b) represents the EDM reconstruction error and localization error, respectively.

With the increase of sampling rate, the reconstruction error and localization error of each algorithm decline rapidly and maintain at a low level finally. Fig.2 (a) shows that NLIRM can reconstruct EDM with smaller error compared with SVT-based algorithm. A close to zero reconstruction error is achieved by NLIRM when the sampling rate reaching 20%, while that of SVT-based algorithm is about 0.08. TNNR-based algorithm achieves a similar performance with NLIRM in EDM reconstruction as it introduces truncated nuclear norm to approximate the rank of EDM accurately. From Fig.2 (a) and Fig.2 (b) we observe that the localization error and reconstruction error have the same trend and NLIRM performs better than SVT-based algorithm in terms of localization error, while the localization error of TNNR-based algorithm and NLIRM is at the same level.

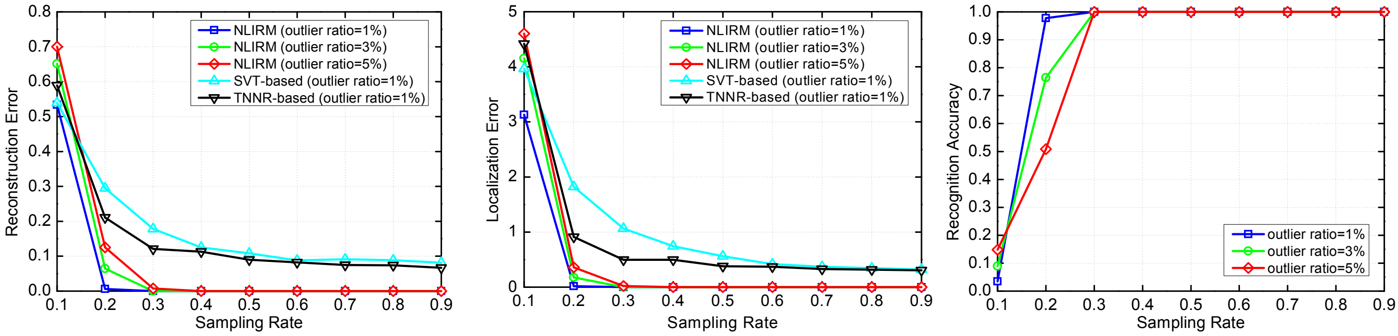
B. Localization under WGNO

This set of experiments assumes that the range measurements are affected by Gaussian noise with zero mean and variance of 100. Fig.3 illustrates the reconstruction error and localization error when the sampling rate enlarges. All the three algorithms are capable of handling the Gaussian noise but NLIRM and TNNR-based algorithm achieve better performance, especially when the sampling rate is below 30%. The performance of NLIRM and TNNR-based algorithm is at the same level but the difference is that NLIRM separates the Gaussian noise explicitly, while TNNR-based algorithm smoothes it implicitly.

C. Localization under NGWO

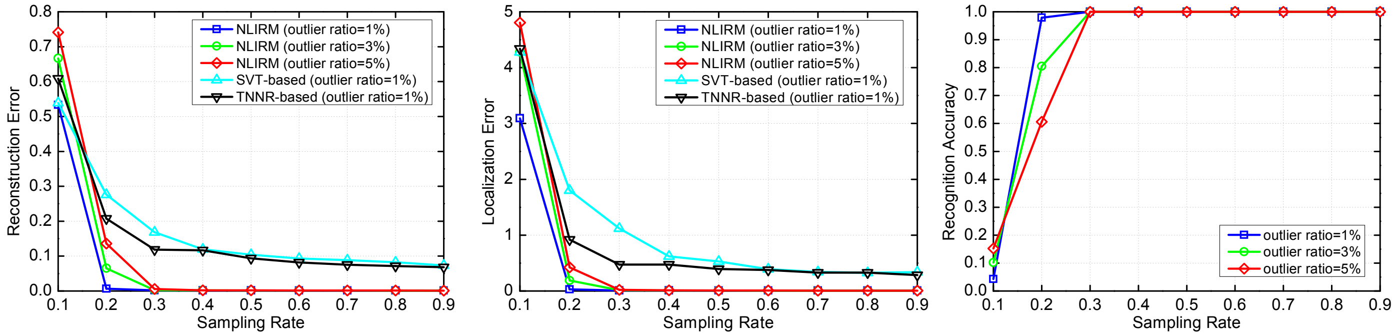
This set of experiments is based on the assumption that the range measurements are affected by outliers. The value of each outlier is randomly generated from 5000 to 10000. The ratio of outliers in NLIRM is set to be 1%, 3% and 5%, while the ratio in SVT-based algorithm and TNNR-based algorithm are fixed at 1%.

The performance gaps between the above three algorithms are highlighted in Fig.4 (a) and (b). We observe that NLIRM is still effective under the interference of outliers, while performances of SVT-based algorithm and TNNR-based algorithm are relatively poor. As shown in Fig.4 (a), the EDM reconstruction error is at a high level (about 0.1) even when the sampling rate reaches 90%, indicating that SVT-based algorithm and TNNR-based algorithm are incapable of handling outliers. On the contrary, the reconstruction error by using NLIRM decreases quickly as the sampling rate increases. When the sampling rate reaches 30%, the error tends to close to zero. Furthermore, from Fig.4 (b) we can see that when the sampling rate is 30%, the localization error of SVT-based



(a)EDM reconstruction error (b) Localization error (c) Outlier recognition accuracy

Fig.4. Performances evaluation under the interference of outliers



(a) EDM reconstruction error (b) Localization error (c) Outlier recognition accuracy

Fig.5. Performances evaluation under the interference of mixed noise

algorithm and TNNR-based algorithm is more than 1m and 0.5m respectively, while that of NLIRM is close to zero.

Furthermore, NLIRM is able to recognize the positions of outliers accurately when the sampling rate reaches a certain bound value. This ability is beneficial to estimate the state and surrounding environment of sensor nodes, which provide the basis for node fault diagnosis, scheduling policy and topology adjustment. We define the outlier recognition accuracy p_r as:

$$p_r = \frac{2 \cdot precision \cdot recall}{precision + recall}$$
$$precision = \frac{m\_true}{m\_all}, recall = \frac{m\_true}{m} \tag{31}$$

where m_all and m_true represent the number of outliers recognized by NLIRM and the number of true outliers among them, respectively. m denotes the actual number of outliers. Fig. 4(c) illustrates the change of recognition accuracy with regard to different sampling rates. As shown in Fig. 4(c), the recognition accuracy reaches up to 100% when the sampling rate is 30%, demonstrating that all outliers have been recognized accurately at this sampling rate.

D. Localization under WGWO

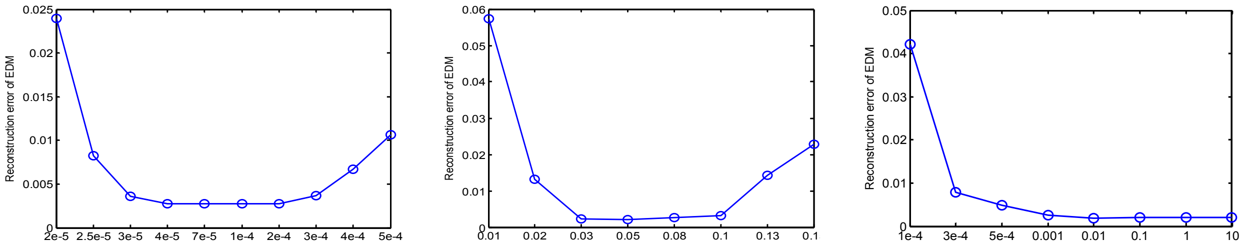
This set of experiments assumes that the range measurements are affected by both Gaussian noise and outliers. The Gaussian noise owns a mean value of zero and variance of 100, while the value of each outlier is generated from 5000 to 10000 randomly. The ratio of outliers in NLIRM is set to be 1%, 3% and 5%, respectively, while SVT-based algorithm and TNNR-based algorithm use a fixed ratio of 1%.

As can be seen from Fig. 5(a), SVT-based algorithm and TNNR-based algorithm are unable to reconstruct the EDM well under the interference of mixed noise even when the sampling rate approaching to 90%. As a result, both SVT-based algorithm and TNNR-based algorithm suffer a high level of localization error. In contrast, NLIRM performs well in handling the mixed noise. Both reconstruction error and localization error are reduced to close to zero when the sampling rate reaches 30%. In addition, the positions of all outliers can be predicted accurately when the sampling rate is more than 30%, as shown in Fig.5 (c).

E. Sensitivity Analysis of NRMC-ADMM to the parameters setup

Next, we investigate the sensitivity of the proposed Fast-NRMC-ADMM algorithm, the basis of our NLIRM, to the parameters setup. As shown in function (17), there are three different parameters that need to be determined in this algorithm (i.e., ρ, λ and μ). Since the rank of the output matrix D is mainly determined by parameter ρ , we set it to a small value in order to achieve a low rank. As the coefficients of L_1 norm and Frobenius norm regularized factors, parameter λ and μ should be determined by the degree of outlier and Gaussian noise. It is clear that λ and μ need to be reduced with the increase of noises. Conversely, if all noises are not exist, we should set λ and μ to relatively large values. Hence, in order to understand how these parameters affect the algorithm performance, we conduct three sets of experiments under WGWO. Without loss of generality, the sampling rate and outlier ratio are fixed at 30% and 5%, and the mean value of Gaussian noise is zero with variance 100.

1) Fixing λ=0.05, $\mu = 0.01$, and varying ρ from 2e-5 to 5e-4;



(a) EDM reconstruction error with varying ρ (b) EDM reconstruction error with varying λ (c) EDM reconstruction error with varying μ

Fig.6. Performance of Fast-NRMC-ADMM with varying ρ, λ and μ

2) Fixing $\rho = 1e-4$, $\mu = 0.01$, and varying λ from 0.01 to 0.15;
3) Fixing $\rho = 1e-4, \lambda = 0.05$, varying μ from 1e-4 to 10;

The performance of Fast-NRMC-ADMM algorithm for the three sets of experiments is shown in Fig.6. First, from Fig. 6(a) we can observe that overall the performance is improved as we increase ρ, but it starts to decline when $\rho > 2e-4$. Second, the performance of the algorithm is insensitive to these parameters if they fall in a certain range ($\rho \in [4e{-}5, 2e{-}4]$, $\lambda \in [0.03, 0.1]$, $\mu > 0.01$), and the performance deteriorates significantly when they are outside the range. Based on the above observations, we set $\rho = 1e-4, \lambda = 0.05$ and $\mu = 0.01$.

V. CONCLUSION

This paper focuses on the problem of robust localization in WSN. We proposed a noise-tolerant localization from incomplete range measurements algorithm called NLIRM. The reconstruction of partially sampled noisy EDM is formulated as a norm-regularized matrix completion problem and can be solved by alternating direction method of multiplier, and then the classical MDS-based method is used to obtain the positions of unknown nodes. Simulation results demonstrate that with a small fraction of noisy distance measurements, the proposed algorithm achieves a promising accuracy improvement compared with traditional algorithms.

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