# CSE 2331 HOMEWORK 2

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1.

$$\begin{aligned} T(n) &= \sum_{i=27}^{n^4} \sum_{j=111}^{\lfloor i^{2/3}(\log_7(i))^2 \rfloor} c = \sum_{i=27}^{n^4} i^{2/3}(\log_7(i))^2 c = \sum_{i=1}^{n^4} \left( i^{2/3}(\log_7(i))^2 c \right) + c_1 \\ &= \sum_{i=1}^{(n^4/2)-1} \left( i^{2/3}(\log_7(i))^2 c \right) + \sum_{i=n^4/2}^{n^4} \left( i^{2/3}(\log_7(i))^2 c \right) + c_1. \end{aligned} \tag{1}$$

We find an upper bound:

$$\begin{aligned} T(n) &\leq \sum_{i=1}^{n^4} \left( (n)^{8/3}(\log_7(n^4))^2 c \right) + c_1 = \sum_{i=1}^{n^4} \left( n^{8/3} 16(\log_7(n))^2 c \right) + c_1 \\ &= n^4 \left( n^{8/3} 16(\log_7(n))^2 c \right) + c_1 = \boxed{2^4 c n^{20/3}(\log_7(n))^2} + c_1. \end{aligned} \tag{2}$$

And a lower bound:

$$\begin{aligned} T(n) &\geq \sum_{i=n^4/2}^{n^4} \left( (n^4/2)^{2/3}(\log_7(n^4/2))^2 c \right) = \sum_{i=n^4/2}^{n^4} \left( (n^4/2)^{2/3}(4\log_7(n) - \log_7(2))^2 c \right) \\ &= \frac{n^4}{2} \left( (n^4/2)^{2/3}(4\log_7(n) - \log_7(2))^2 c \right) = \left( \frac{n^4}{2} \right)^{5/3} (4\log_7(n) - \log_7(2))^2 c \\ &= n^{20/3} \, (1/2)^{5/3} \, (4\log_7(n) - \log_7(2))^2 c \\ &= c n^{20/3} \, (1/2)^{5/3} \, (16(\log_7(n))^2 - 8\log_7(n)\log_7(2) + (\log_7(2))^2) \\ &\geq \boxed{2^{7/3} c n^{20/3} (\log_7(n))^2} - c n^{20/3} 2^{4/3} \log_7(n) \log_7(2)). \end{aligned} \tag{3}$$

Thus $T(n) = kn^{20/3}(\log_7(n))^2$, where $2^{7/3}c \leq k \leq 2^4 c$.

2.

$$\begin{aligned} T(n) &= \sum_{i=6}^{\lfloor n\log_3(n) \rfloor} \sum_{k=1}^{\lfloor (i^3-i)/14 \rfloor} c = \frac{c}{14} \left( \sum_{i=6}^{\lfloor n\log_3(n) \rfloor} i^3 - \sum_{i=6}^{\lfloor n\log_3(n) \rfloor} i \right) \\ &= \frac{c}{14} \left( \sum_{i=1}^{\lfloor n\log_3(n) \rfloor} (i^3) + k_1 - \sum_{i=1}^{\lfloor n\log_3(n) \rfloor} (i) + k_2 \right) \\ &= \frac{c}{14} \left( \frac{(n\log_3(n))^2(n\log_3(n)+1)^2}{4} + k_1 - \frac{1}{2} n\log_3(n)(n\log_3(n)+1) + k_2 \right) \\ &= \frac{c}{14} \left( \frac{1}{4}(n\log_3(n))^2(n\log_3(n)+1)^2 + k_1 - \frac{1}{2} n\log_3(n)(n\log_3(n)+1) + k_2 \right) \\ &= \Theta(n^4(\log_3(n))^4). \end{aligned} \tag{4}$$

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3.

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\begin{aligned}
T(n) &= \sum_{i=n/2}^{n\log_7(n)} \sum_{j=i}^{n\log_7(n)} \sum_{k=6j^2}^{6j^2+\sqrt{n}} c = \sum_{i=n/2}^{n\log_7(n)} \sum_{j=i}^{n\log_7(n)} c(\sqrt{n}+1) \\
&= \Theta\left(\sum_{i=n/2}^{n\log_7(n)} \sum_{j=i}^{n\log_7(n)} c\sqrt{n}\right) = \Theta\left(\sum_{i=n/2}^{n\log_7(n)} c\sqrt{n}(n\log_7(n)-i)\right) \\
&= \Theta\left(\sum_{i=n/2}^{n\log_7(n)} c\sqrt{n}(n\log_7(n)-i)\right) \\
&= \Theta\left(\sum_{i=1}^{n\log_7(n)} c\sqrt{n}(n\log_7(n)-i) - \sum_{i=1}^{n/2} c\sqrt{n}(n\log_7(n)-i)\right) \\
&= \Theta\left(\sum_{i=1}^{n\log_7(n)} c\sqrt{n}n\log_7(n) - \sum_{i=1}^{n\log_7(n)} c\sqrt{n}i - \sum_{i=1}^{n/2} c\sqrt{n}n\log_7(n) + \sum_{i=1}^{n/2} c\sqrt{n}i\right) \\
&= \Theta\left(cn^{5/2}(\log_7(n))^2 - cn^{3/2}\frac{1}{2}(n(\log_7(n))^2+\log_7(n)) - c\frac{n^{5/2}}{2}\log_7(n) + \frac{n^{3/2}}{4}(n/2+1)\right) \\
&= \Theta\left(cn^{5/2}\frac{1}{2}(\log_7(n))^2 - c\frac{n^{5/2}}{2}\log_7(n) + \frac{n^{5/2}}{8}\right) \\
&= \Theta\left(n^{5/2}(\log_7(n))^2\right).
\end{aligned} \tag{5}
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4.

$$
\begin{aligned}
T(n) &= \sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \sum_{j=1}^{i^2} \sum_{k=j}^{i^2} c = \Theta\left(\sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \sum_{j=1}^{i^2} (i^2-j)c\right) \\
&= \Theta\left(\sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \left(\sum_{j=1}^{i^2} ci^2 - \sum_{j=1}^{i^2} jc\right)\right) = \Theta\left(\sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \left(ci^4 - \frac{1}{2}i^2(i^2+1)c\right)\right) \\
&= \Theta\left(\sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \left(ci^4 - \frac{c}{2}(i^4+i^2)\right)\right) = \Theta\left(\sum_{i=(\log_8(n))^2}^{n^{3/2}\log_9(n)} \left(\frac{c}{2}i^4\right)\right) \\
&= \Theta\left(\sum_{i=1}^{n^{3/2}\log_9(n)} \left(\frac{c}{2}i^4\right) - \sum_{i=1}^{(\log_8(n))^2} \left(\frac{c}{2}i^4\right)\right) = \Theta\left(\left((n^{3/2}\log_9(n))^5\right) - \left((\log_8(n))^{10}\right)\right) \\
&= \Theta\left((n^{3/2}\log_9(n))^5\right).
\end{aligned} \tag{6}
$$

Note: we used the fact that $\sum_{i=1}^{n} i^4 \in \Theta(n^5)$.

5.

$$
\begin{aligned}
T(n) &= \Theta\left(\sum_{k=1}^{\log_6(n^3-3)} \sum_{l=1}^{n^{1/2}-n^{-1/2}} c\right) = \Theta\left(\sum_{k=1}^{\log_6(n^3-3)} (n^{1/2}-n^{-1/2})c\right) \\
&= \Theta\left(\log_6(n^3-3)(n^{1/2}-n^{-1/2})c\right) = \Theta\left(3\log_6(n)(n^{1/2}-n^{-1/2})c\right) \\
&= \Theta\left(n^{1/2}\log_6(n)\right).
\end{aligned} \tag{7}
$$

6.

$$\begin{aligned} T(n) &= \Theta\left(\sum_{k=1}^{\log_5(n^{5/4}-8)} \sum_{l=1}^{\frac{i^2-12}{7}} c\right) = \Theta\left(\sum_{k=1}^{\log_5(n^{5/4}-8)} \frac{i^2-12}{7} c\right) \\ &= \Theta\left(\sum_{k=1}^{\log_5(n^{5/4}-8)} i^2 c\right) = \Theta\left(\sum_{k=1}^{(1/2)\log_5(n^{5/4}-8)-1} i^2 c + \sum_{k=(1/2)\log_5(n^{5/4}-8)}^{\log_5(n^{5/4}-8)} i^2 c\right). \end{aligned} \tag{8}$$

We find an upper bound, letting $i = n^{5/4}$:

$$T(n) \le \Theta\left(\sum_{k=1}^{\log_5(n^{5/4}-8)} n^{5/2} c\right) = \Theta\left(\sum_{k=1}^{\frac{5}{4}\log_5(n)} n^{5/2} c\right) = \Theta\left(\frac{5c}{4} n^{5/2}\log_5(n)\right). \tag{9}$$

And we find a lower bound, letting $i = \frac{1}{2}n^{5/4}$:

$$\begin{aligned} T(n) &\ge \Theta\left(\sum_{k=(1/2)\log_5(n^{5/4}-8)}^{\log_5(n^{5/4}-8)} \frac{1}{4} n^{5/2} c\right) = \Theta\left(\frac{1}{4} c n^{5/2} (1/2)\log_5(n^{5/4}-8)\right) \\ &= \Theta\left(\frac{c}{8} n^{5/2}\log_5(n)\right). \end{aligned} \tag{10}$$

So since the upper and lower bounds have the same time complexity up to a constant, we know $t \in \Theta\left(n^{5/2}\log_5(n)\right)$.

7. We find the number of iterations $k$ using the equation $\lfloor \frac{i^3}{8^k} \rfloor = \lfloor \sqrt{i} \rfloor + 1$. Thus the inner while loop executes $\lfloor \log_8(\frac{i^3}{\lfloor \sqrt{i} \rfloor + 1}) \rfloor$ times, and hence takes $k\log(i^{2.5})$ time for some constant $k$. The outer loop executes $\lfloor \frac{n^{3/2}-5}{12} \rfloor$ times, and thus takes $k_1 n^{3/2}$ time for some constant $k_1$. Hence we have

$$T(n) = \sum_{l=1}^{\lfloor k_1 n^{3/2} \rfloor} k\log(i^{2.5}). \tag{11}$$

And since $6 \le i < n^{3/2}$, we use $n^{3/2}$ as an upper bound for $i$, and we use $n^{3/2}/2$ as a lower bound. Thus

$$\begin{aligned} T(n) &\le \sum_{l=1}^{\lfloor k_1 n^{3/2} \rfloor} k\log(n^{15/4}) = k_1 k n^{3/2}\log(n^{15/4}) \in O\left(n^{3/2}\log(n)\right). \\ T(n) &\ge \sum_{l=1}^{\lfloor k_1 n^{3/2} \rfloor} k\log(n^{15/4}/2) = k k_1 n^{3/2}(\log(n^{15/4}) - \log 2) \in \Omega\left(n^{3/2}\log(n)\right). \end{aligned} \tag{12}$$

Hence we know $T(n) \in \Theta\left(n^{3/2}\log(n)\right)$.

8. Note that the inner while loop executes $\lfloor \frac{n^2-5}{i} \rfloor$ times, which takes $cn^2/i$ time.

$$\begin{aligned} T(n) &= \frac{cn^2}{5} + \frac{cn^2}{5\cdot 7} + \frac{cn^2}{5\cdot 7^2} + \cdots + \frac{cn^2}{4n} \\ &= \frac{cn^2}{5}\left(1 + \frac{1}{7} + \frac{1}{7^2} + \cdots + \frac{1}{4n}\right) \\ &\le \frac{cn^2}{5}\left(1 + \frac{1}{7} + \frac{1}{7^2} + \cdots\right) \\ &= \frac{cn^2}{5}\left(\frac{1}{1-\frac{1}{7}}\right) = \frac{cn^2}{5}\left(\frac{7}{6}\right) \in O\left(n^2\right). \end{aligned} \tag{13}$$

And

$$\begin{aligned} T(n) &= \frac{cn^2}{5} + \frac{cn^2}{5 \cdot 7} + \frac{cn^2}{5 \cdot 7^2} + \cdots + \frac{cn^2}{4n} \\ &\geq \frac{cn^2}{5} \in \Omega(n^2). \end{aligned} \tag{14}$$

Hence $T(n) \in \Theta(n^2)$.

9. The inner loop executes $\lfloor \sqrt{i} \rfloor$ times, and so takes $c\sqrt{i}$ time for some constant $c$. Thus we have

$$\begin{aligned} T(n) &= c\sqrt{2} + c\sqrt{2 \cdot 4} + c\sqrt{2 \cdot 4^2} + \cdots + c\sqrt{n^5 - 1} \\ &= c\sqrt{2}\left(1 + 2 + 2^2 + \cdots + \sqrt{\frac{n^5-1}{2}}\right) \\ &= c\sqrt{2}\left(1 + \frac{2(1 - 2^{\sqrt{\frac{n^5-1}{2}}})}{1-2}\right) \\ &= c\sqrt{2}\left(2^{\sqrt{\frac{n^5-1}{2}}+1} - 1\right) \\ &= c\sqrt{2}\left(2 \cdot (2^{\sqrt{n^5-1}})^{1/\sqrt{2}} - 1\right) \in \Theta\left(\left(2^{\frac{\sqrt{2}}{2}}\right)^{n^{2.5}}\right). \end{aligned} \tag{15}$$

10. The innermost while loop executes $\lfloor \log_{1.5}(i^2/12) \rfloor + 1$ times, and thus takes $c\log(i)$ time for some constant $c$. The next while loop executes $\lfloor \log_{13}\left(\lfloor i^{5/2} \rfloor / 8\right) \rfloor$ times, and takes $c_1 \log(i)$ time for some constant $c_1$. Hence our running time is

$$\begin{aligned} T(n) &= \sum_{i=1}^{n^3} cc_1(\log(i))^2 \\ &= \sum_{i=1}^{(n^3)/2-1} cc_1(\log(i))^2 + \sum_{i=n^3/2}^{n^3} cc_1(\log(i))^2. \end{aligned} \tag{16}$$

And thus we have

$$\begin{aligned} T(n) &\leq \sum_{i=1}^{n^3} cc_1(\log(n^3))^2 \\ &= 9cc_1 n^3 (\log(n))^2 \in O\left(n^3(\log(n))^2\right), \\ T(n) &\geq \sum_{i=n^3/2}^{n^3} cc_1(\log(\frac{n^3}{2}))^2 = \frac{n^3}{2} cc_1(3\log(n) - \log(2))^2 \\ &= \frac{n^3}{2} cc_1(9(\log(n))^2 - 6\log(n)\log(2) + (\log(2))^2) \in \Omega\left(n^3(\log(n))^2\right). \end{aligned} \tag{17}$$

So we conclude $T(n) \in \Theta\left(n^3(\log(n))^2\right)$.