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Math 5522H Homework 9

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VIII.5.9 If a non-constant, entire function f obeys the condition $|f(z)| = 1$ whenever $|z| = 1$. show that f must be of the type $f(z) = cz^m$ where $m \in \mathbb{N}$ and c is a constant with $|c| = 1$.

Proof. We first show that $f(z)$ takes on the form given by Exercise VIII.5.8 in the entirety of $\mathbb{C}$ using analytic continuation. Then, we will show that f in fact has the desired form $f(z) = cz^m$. Fix f as above. Now we know f is analytic in $\mathbb{D}(0,1)$ and continuous on $\overline{\mathbb{D}}(0,1)$ by virtue of being entire. We are also given that $|f(z)| = 1$ $\forall z \in \partial\mathbb{D}(0,1)$, and that f is non-constant. Then by the result of Exercise VIII.5.8 that f has the form:

$$f(z) = c\prod_{k=1}^{r}\left(\frac{z - a_k}{1 - \overline{a}_k z}\right)^{m_k}$$

$\forall z \in \overline{\mathbb{D}}(0,1)$.

Then let $g(z) : \mathbb{C} \setminus \{a_k\} \to \mathbb{C}$ be given by:

$$g(z) = c\prod_{k=1}^{r}\left(\frac{z - a_k}{1 - \overline{a}_k z}\right)^{m_k}$$

$\forall z \in \mathbb{C} \setminus \{a_k\}$.

We see that g is analytic in $\mathbb{C}\setminus\{a_k\}$ since it is a quotient of analytic functions. Then let the domain set of g be our domain for analytic continuation. Since f is entire, we know that f and g are analytic in $\mathbb{C} \setminus \{a_k\}$. Also, f and g agree on the unit disk, and the unit disk has a limit point in $\mathbb{C} \setminus \{a_k\}$ (any point on the boundary will do). Then by analytic continuation, we have that f and g agree on $\mathbb{C} \setminus \{a_k\}$.

From the proof of the stated exercise, we recall that r is the number of distinct zeroes of the function in $\mathbb{D}(0,1)$, which is finite, but at least 1. Since we wish to show $f(z) = cz^m$, we will prove that f entire $\Rightarrow r = 1$ in $\mathbb{D}(0,1)$, and $a_1 = 0$. Let $A = \{a_k\} \subset \mathbb{D}(0,1)$. Suppose for contradiction that $\exists a_{k_1} \in A$ s.t. $a_{k_1} \neq 0$. Then since f entire, we know f is analytic at $\overline{a}_{k_1}^{-1} \Rightarrow$:

$$f(\overline{a}_{k_1}^{-1}) = c\prod_{k=1}^{r}\left(\frac{\overline{a}_{k_1}^{-1} - a_k}{1 - \overline{a}_k \cdot \overline{a}_{k_1}^{-1}}\right)^{m_k}$$

since this formula now holds $\forall z \in \mathbb{C}\backslash\{a_k\}$. But when $k = k_1$, the denominator of f vanishes $\Rightarrow f$ has a singularity at $\overline{a}_{k_1}^{-1} \Rightarrow f$ is not entire. This is a contradiction which tells us our supposition that $\exists a_{k_1} \in A$ s.t. $a_{k_1} \neq 0$ must be false. Hence we must have $a_k = 0 \ \forall k \Rightarrow f$ has the form:

$$f(z) = c\prod_{k=1}^{r} z^{m_k}$$

But recall that $|c| = 1$ and r is the number of zeroes of f in $\mathbb{D}(0,1)$. Then we must have $r = 1$ since z^{m_k} has its only zero at 0. Thus f has the form:

$$f(z) = cz^m$$

in $\mathbb{D}(0,1)$. $\square$

We prove the following exercise in order to make use of its result in Exercise VIII.5.11.

V.8.57 Suppose that a function f is analytic and free of zeroes in a domain D. Under the assumption that a branch g of the pth-root function of f exists in D, prove that there are exactly p distinct branches of the pth-root of f in D, each having the form cg for some pth-root of unity c.

Proof. Let c be a pth root of unity. Then we know by definition that $c^p = 1 \Rightarrow [cg(z)]^p = c^p[g(z)]^p = f(z)$ by definition of g. Thus we see that each element of $\{cg(z)\}$ defines a branch of the pth root of f. We show that these are the only possible branches. Let $h(z)$ be an arbitrary branch of the pth root of f in D. Since g and h are both pth roots of f, we know they are both zero-free in D, since if not, we would have $z_0 \in D$ s.t. $g(z_0) = 0 \Rightarrow [g(z_0)]^p = 0^p = f(z_0)$, which is a contradiction since f is free of zeroes (likewise for h). Then we define $q(z) = \frac{h(z)}{g(z)}$ in D s.t. :

$$[q(z)]^p = \frac{[h(z)]^p}{[g(z)]^p} = \frac{f(z)}{f(z)} = 1$$

$\forall z \in D$. Since h and g are analytic and g is free of zeroes in D, we know q is analytic $\Rightarrow$

$$p[q(z)]^{p-1}q'(z) = 0.$$

Now since h is also zero-free in D, we know q must be zero-free in $D \Rightarrow q'(z) = 0$ $\forall z \in D$. Thus $q(z) = k$ a constant. Then since $[q(z)]^p = k^p = 1$, we have that k is a pth root of unity. Hence $h(z) = kg(z) \Rightarrow h \in \{cg(z)\}$, which means that cg describes the set of all possible pth root functions. Since there are p pth roots of unity, the claim holds. $\square$

VIII.5.11 Suppose that a function f is analytic in a domain D. Under the assumption that a branch g of the pth-root function of f exists in D, prove that there are exactly p distinct branches of the pth-root of f in D, each having the form cg for some pth-root of unity c.

Proof. Let c be a pth root of unity. Then we know by definition that $c^p = 1 \Rightarrow [cg(z)]^p = c^p[g(z)]^p = f(z)$ by definition of g. Thus we see that each element of $\{cg(z)\}$ defines a branch of the pth root of f. We show that these are the only possible branches. Let $h(z)$ be an arbitrary branch of the pth root of f in D. Now from the discrete mapping theorem, we know since f is analytic, the set of zeroes of f in D consists entirely of isolated points. If this set is empty, the problem reduces to that of Exercise V.8.57, so we treat the case where f has at least one zero in D. Let $A = \{z_1, ..., z_k\}$ be the set of zeroes of f. The by definition of a branch of the pth root of f, we have the equality:

$$[g(z)]^p = f(z) = [h(z)]^p$$

$\forall z \in D$. Now let $z_j \in A \Rightarrow$

$$f(z_j) = 0 = [g(z_j)]^p = [h(z_j)]^p.$$

Thus we must have $g(z_j) = 0 = h(z_j)\ \forall j \in \{1, ..., k\}$. And by the same reasoning, we know A is precisely the set of all zeroes for g, h. We define $q(z) = \frac{h(z)}{g(z)}$ in $D \setminus A$ s.t. :

$$[q(z)]^p = \frac{[h(z)]^p}{[g(z)]^p} = \frac{f(z)}{f(z)} = 1$$

$\forall z \in D \setminus A$. Then by the proof of Exercise V.8.57, we know that $h(z) = \alpha g(z)$ $\forall z \in D \setminus A$ for some pth root of unity α. But at points in A, we know $g(z_j) = 0 = h(z_j) \Rightarrow h(z) = \alpha g(z)\ \forall z \in D \Rightarrow h(z) \in \{cg(z)\}$. Hence there are exactly p distinct branches of the pth root of f given by $\{cg(z)\}$. $\square$

VIII.5.12 Let $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ be a Taylor series with a finite radius of convergence $\rho > 0$, and let f denote the function that is its sum in the disk $\mathbb{D}(z_0, \rho)$. Show that there must be at least one point ζ of the circle $K(z_0, \rho)$ about which the following is true: for no $r > 0$ can f be extended to a function that is analytic in the set $\mathbb{D}(z_0, \rho) \cup \mathbb{D}(\zeta, r)$.

Proof. We assume $\nexists \zeta$ s.t. the given statement is true. Then $\exists g(z)$ analytic in $\mathbb{D}(z_0, \rho')$ for some $\rho' > \rho$ s.t. $f(z) = g(z)\ \forall z \in \mathbb{D}(z_0, \rho')$. Then since g is analytic in $\mathbb{D}(z_0, \rho')$, we know g has a Taylor series expansion in $\mathbb{D}(z_0, \rho')$, and furthermore that this Taylor series is unique. But since f coincides with g in $\mathbb{D}(z_0, \rho')$, we must have that $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ is the Taylor series for $g \Rightarrow$ this series converges absolutely and normally in $\mathbb{D}(z_0, \rho')$ since g is analytic. Hence the radius of convergence of this series is ρ'. But this is a contradiction since we know the radius of convergence of $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ is $\rho \Rightarrow$ we must have that $\exists$ some ζ on $K(z_0, \rho)$ s.t. for no $r > 0$ can f be extended to a function that is analytic in the set $\mathbb{D}(z_0, \rho) \cup \mathbb{D}(\zeta, r)$. $\square$