CSE 5522 HOMEWORK 1

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1. (a) We compute $P(x) = \int_{\infty}^{\infty} P(x,y)dy$. From the definition of the multivariate Gaussian distribution, we have:

$$P(x,y) = \frac{1}{(2\pi)|C|^{1/2}} e^{\frac{-1}{2}(x-\mu)^T C^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \mu\right)},$$

where C the covariance matrix is given by:

$$C = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

We compute:

$$|C| = \det(C) = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho^2).$$

And:

$$\begin{aligned} C^{-1} &= \frac{1}{|C|}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \\ &= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}. \end{aligned} \tag{1}$$

And:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} - \mu\right) = \begin{pmatrix} x - \mu_1 \\ y - \mu_2 \end{pmatrix}.$$

So we have:

$$\begin{aligned} &(x-\mu)^T C^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \mu\right) \\ =&(x-\mu_1, y-\mu_2)\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}\begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix} \\ =&(x-\mu_1, y-\mu_2)\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} \sigma_2^2(x-\mu_1) - \rho\sigma_1\sigma_2(y-\mu_2) \\ -\rho\sigma_1\sigma_2(x-\mu_1) + \sigma_1^2(y-\mu_2) \end{pmatrix} \\ =&\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \cdot \sigma_2^2(x-\mu_1)^2 - \rho\sigma_1\sigma_2(x-\mu_1)(y-\mu_2) \\ -&\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \cdot \rho\sigma_1\sigma_2(x-\mu_1)(y-\mu_2) + \sigma_1^2(y-\mu_2)^2 \\ =&\frac{(x-\mu_1)^2}{\sigma_1^2(1-\rho^2)} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2(1-\rho^2)} + \frac{(y-\mu_2)^2}{\sigma_2^2(1-\rho^2)} \\ =&\frac{1}{1-\rho^2}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right). \end{aligned} \tag{2}$$

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So all together:

$$\begin{aligned} P(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Z} \\ &= Ae^{-\frac{1}{2}Z}. \end{aligned} \tag{3}$$

for $A = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$, where:

$$\begin{aligned} Z &= \frac{1}{1-\rho^2}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right) \\ &= \frac{1}{1-\rho^2}\left(\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2 + (1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right) \\ &= \frac{1}{1-\rho^2}\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 \\ &= B + \left(\frac{x-\mu_1}{\sigma_1}\right)^2. \end{aligned} \tag{4}$$

So we have:

$$\begin{aligned} B &= \frac{1}{1-\rho^2}\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2 \\ &= \frac{1}{1-\rho^2}\left(\frac{(y-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(y-\mu_2)(x-\mu_1)}{\sigma_2\sigma_1} + \rho^2\frac{(x-\mu_1)^2}{\sigma_1^2}\right) \\ &= \frac{1}{1-\rho^2}\left(\frac{y^2}{\sigma_2^2} - \frac{2y\mu_2}{\sigma_2^2} - \frac{2y\rho(x-\mu_1)}{\sigma_1\sigma_2} + \frac{\mu_2^2}{\sigma_2^2} + \frac{2\mu_2\rho(x-\mu_1)}{\sigma_1\sigma_2} + \frac{\rho^2(x-\mu_1)^2}{\sigma_1^2}\right) \\ &= \frac{y^2 - 2y\mu_2 - 2y\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1) + \mu_2^2 + 2\mu_2\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1) + \rho^2\frac{\sigma_2^2}{\sigma_1^2}(x-\mu_1)^2}{(1-\rho^2)\sigma_2^2} \\ &= \frac{y^2 - 2y\mu_2 - 2y\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1) + (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))^2}{(1-\rho^2)\sigma_2^2} \\ &= \frac{\left(y - (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))\right)^2}{(1-\rho^2)\sigma_2^2} \\ &= \frac{(y-g(x))^2}{(1-\rho^2)\sigma_2^2}, \end{aligned} \tag{5}$$

where $g(x) = (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1))$. So we have:

$$\begin{aligned} \int_{-\infty}^{\infty} P(x,y)dy &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}B} \\ &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(y-g(x))^2}{(1-\rho^2)\sigma_2^2}}, \end{aligned} \tag{6}$$

So we let $\sigma' = \sqrt{1-\rho^2}\sigma_2$. Thus:

$$\begin{aligned}\int_{-\infty}^{\infty} P(x,y)dy &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}B} \\ &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \sqrt{2\pi}\sigma' \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma'} e^{-\frac{1}{2}\frac{(y-g(x))^2}{\sigma'^2}}.\end{aligned} \tag{7}$$

But this integrand on the right is exactly a univariate Gaussian distribution in y with mean $g(x)$ and variance σ'^2. So it integrates to 1, and we have:

$$\begin{aligned}\int_{-\infty}^{\infty} P(x,y)dy &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}B} \\ &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \sqrt{2\pi}\sigma' \\ &= Ae^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \sqrt{2\pi}\sqrt{1-\rho^2}\sigma_2 \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} \cdot \sqrt{2\pi}\sqrt{1-\rho^2}\sigma_2 \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}}.\end{aligned} \tag{8}$$

Note from the definition of univariate Gaussian distribution that

$$P(x) = \int_{-\infty}^{\infty} P(x,y)dy$$

is itself Gaussian with mean μ_1 and variance σ_1^2.

(b) The mean is μ_1 and the variance is σ_1^2.

(c) *Assume $\rho = 0$. Prove that $P(x,y) = P(x)P(y)$.*
Proof. Note by the same excruciating derivation from part (a) we know:

$$P(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\frac{(y-\mu_2)^2}{\sigma_2^2}}.$$

So recall:

$$P(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Z}. \tag{9}$$

And also recall:

$$Z = \frac{1}{1-\rho^2}\left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 \tag{10}$$

Now since $\rho = 0$ we have:

$$Z = \left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 \tag{11}$$

And so:

$$\begin{aligned} P(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)} \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)}, \end{aligned} \tag{12}$$

again since $\rho = 0$. But:

$$\begin{aligned} P(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}} e^{-\frac{1}{2}\frac{(y-\mu_2)^2}{\sigma_2^2}} \\ &= P(x) \cdot P(y). \end{aligned} \tag{13}$$

□

2. (a) *Prove that if A, B, C are mutually independent, then A, B are conditionally independent given C.*

   Remark 1. These are really easy, know how to do these if you got them wrong.

   Proof. Let A, B, C be mutually independent. Then we know by definition:

$$\begin{aligned} P(A,B,C) &= P(A)P(B)P(C) \\ P(A,B) &= P(A)P(B) \\ P(B,C) &= P(B)P(C) \\ P(A,C) &= P(A)P(C) \end{aligned} \tag{14}$$

   Recall from the definition of conditional independence that A and B are conditionally independent given C if and only if we have:

$$P(A|B,C) = P(A|C).$$

   So using product rule, we write:

$$P(A|B,C) = \frac{P(A,B,C)}{P(B,C)} = \frac{P(A)P(B)P(C)}{P(B)P(C)} = P(A).$$

   But note:

$$P(A|C) = \frac{P(A,C)}{P(C)} = \frac{P(A)P(C)}{P(C)} = P(A).$$

   So we have the desired equality: A, B are conditionally independent given C by definition. □

   (b) We give a simple example JPT to illustrate part (a).

TABLE 1. Joint probability table.

	A		¬ A	
	B	¬ B	B	¬ B
C	0.125	0.125	0.125	0.125
¬C	0.125	0.125	0.125	0.125

Note we have $P(A) = P(B) = P(C) = 0.5$. Note also

$$\begin{aligned} &P(A, B) = P(B, C) = P(A, C) = 0.25 \\ =&P(A)P(B) = P(A)P(C) = P(B)P(C). \end{aligned} \tag{15}$$

And finally note that $P(A, B, C) = 0.125 = P(A)P(B)P(C) = 0.5^3$. So we have mutual independence. And we see that $P(A|B, C) = \frac{P(A,B,C)}{P(B,C)} = \frac{0.125}{0.25} = \frac{1}{2}$. And $P(A|C) = \frac{P(A,C)}{P(C)} = \frac{0.25}{0.5} = 0.5$. So they are equal and we have A, B conditionally independent given C.

(c) *Prove that if A and B, C are conditionally independent given D, then A, B are conditionally independent given D.*

Proof. Since A and B, C are conditionally independent given D, we know:

$$P(A|B, C, D) = P(A|D).$$

We want to show:

$$P(A|B, D) = P(A|D).$$

Note that $P(B) = P(B, C|P(C) = 1)$. So we have:

$$P(A|B, D) = P(A|B, C, D|P(C) = 1) = P(A|D).$$

So we have the desired result. □

Remark 2. Know this is as well, it's really easy, just d-separation. If there's any single directed path that isn't d separated, then they're dependent.

3. (a) Given that E is empty, fanbelt broken is independent of **alternator broken** since it is d-separated by no charging; **battery age, battery dead, and battery meter** since they are d-separated by battery flat; and **no oil, no gas, fuel line blocked, starter broken, and dipstick** since they are all d-separated by one of lights, oil light, gas gauge, or car won't start.

   (b) Given E is empty, battery meter is independent of **alternator broke, fanbelt broken, and no charging** since they are d-separated by battery flat. It is also independent of **no oil, no gas, fuel line blocked, starter broken, and dipstick** since they are all d-separated by one of lights, oil light, gas gauge, or car won't start.

   (c) Given $E =$ battery flat, battery age is conditionally independent of **no oil, no gas, fuel line blocked, starter broken, lights, oil light, gas gauge, car won't start, and dipstick** since d-separated by battery flat, which is observed.

   (d) Given E = battery dead, no charging, we know battery flat is conditionally independent of **battery age, alternator broke, and fanbelt broken** since they are d separated by one of the nodes in E. And it is also independent of **no oil, no gas, fuel line blocked, starter broken, and dipstick** since they are all d-separated by one of lights, oil light, gas gauge, or car won't start.

4. (a)

$$P(b|j,m) = \alpha P(b) \sum_E \left(P(E) \sum_A \left(P(A|b,E) P(j|A) P(m|A) \right) \right)$$
$$= \alpha P(b) \sum_E \left(P(E \left(P(a|b,E) P(j|a) P(m|a) + P(\neg a|b,E) P(j|\neg a) P(m|\neg a) \right) \right) \quad (16)$$
$$= \alpha P(b) [P(e) \left(P(a|b,e) P(j|a) P(m|a) + P(\neg a|b,e) P(j|\neg a) P(m|\neg a) \right)$$
$$+ P(\neg e) \left(P(a|b,\neg e) P(j|a) P(m|a) + P(\neg a|b,\neg e) P(j|\neg a) P(m|\neg a) \right)].$$

(b)

$$\alpha^{-1} = \sum_B \sum_E \sum_A P(B,E,A,j,m)$$
$$= \alpha^{-1} \sum_B \left(P(B) \sum_E \left(P(E) \sum_A \left(P(A|B,E) P(j|A) P(m|A) \right) \right) \right) \quad (17)$$

Assume B is true. Then we have:

$$\alpha_b^{-1} = 0.001 \sum_E \left(P(E) \sum_A \left(P(A|b,E) P(j|A) P(m|A) \right) \right)$$
$$= 0.001 \cdot 0.002 \sum_A \left(P(A|b,e) P(j|A) P(m|A) \right)$$
$$+ 0.001 \cdot 0.998 \sum_A \left(P(A|b,\neg e) P(j|A) P(m|A) \right)$$
$$= 0.001 \cdot 0.002 \cdot 0.95 \cdot 0.90 \cdot 0.7 \quad (18)$$
$$+ 0.001 \cdot 0.998 \cdot 0.94 \cdot 0.90 \cdot 0.7$$
$$+ 0.001 \cdot 0.002 \cdot 0.05 \cdot 0.05 \cdot 0.01$$
$$+ 0.001 \cdot 0.998 \cdot 0.06 \cdot 0.05 \cdot 0.01$$
$$\approx 0.0005922.$$

Now assume B is false. Then we have:

$$\alpha_{\neg b}^{-1} = 0.001 \sum_E \left(P(E) \sum_A \left(P(A|b,E) P(j|A) P(m|A) \right) \right)$$
$$= 0.999 \cdot 0.002 \sum_A \left(P(A|b,e) P(j|A) P(m|A) \right)$$
$$+ 0.999 \cdot 0.998 \sum_A \left(P(A|b,\neg e) P(j|A) P(m|A) \right)$$
$$= 0.999 \cdot 0.002 \cdot 0.29 \cdot 0.90 \cdot 0.7 \quad (19)$$
$$+ 0.999 \cdot 0.998 \cdot 0.001 \cdot 0.90 \cdot 0.7$$
$$+ 0.999 \cdot 0.002 \cdot 0.71 \cdot 0.05 \cdot 0.01$$
$$+ 0.999 \cdot 0.998 \cdot 0.999 \cdot 0.05 \cdot 0.01$$
$$\approx 0.00149.$$

So $\alpha^{-1} = \alpha_b^{-1} + \alpha_{\neg b}^{-1} = 0.00208$. Thus $\alpha = 479.8234$.

(c)

$$\begin{aligned}
P(b|j,m) &= 479.8234P(b)\sum_E \left(P(E)\sum_A \left(P(A|b,E)P(j|A)P(m|A)\right)\right) \\
&= 479.8234P(b)\sum_E \left(P(E\left(P(a|b,E)P(j|a)P(m|a) + P(\neg a|b,E)P(j|\neg a)P(m|\neg a)\right)\right) \\
&= 479.8234P(b)[P(e)\left(P(a|b,e)P(j|a)P(m|a) + P(\neg a|b,e)P(j|\neg a)P(m|\neg a)\right) \\
&+ P(\neg e)\left(P(a|b,\neg e)P(j|a)P(m|a) + P(\neg a|b,\neg e)P(j|\neg a)P(m|\neg a)\right)] \\
&= 0.28417.
\end{aligned} \tag{20}$$

(d)

$$P(j,m) = \alpha^{-1} = 0.00208. \tag{21}$$

5.

Remark 3. Ask for solutions.

(a) *Proof.* Let (V, E) be an undirected path from $X \to W$. Then since this path must contain U for some node $U \in Blanket(X)$, which we know by definition of the Markov blanket: the path must pass through one of the parents, children, or children's parents of X. Thus it satisfies the first d-separation criterion, and hence is conditionally independent. □

(b) *Proof.* Assume W is not X and outside markov blanket. Then assume W is not a descendant of X, and assume there exists an undirected path to it from X. Given the parents of X, we know W is d separated from the grandparents of X by the first d separation criterion since we are assuming we observe its parents. And it is d-separated from its siblings by the second d separation criterion (common cause) since we observe its parents. And since any undirected path to X from a node which is not its descendant must pass through its grandparents or siblings, we know W is conditionally independent of X. □

Remark 4. Ask for solution. You should know how to write joint probabilities from Markov networks. for problem 7

7.

$$\begin{aligned}
\frac{P(x_1 = 1 \vee Y = \{\, y_1, ..., y_n \,\})}{P(x_1 = -1 \vee Y = \{\, y_1, ..., y_n \,\})} &= e^{-E(x_+,y)+E(x_-,y)} \\
&= e^{-(h+h\sum_{i>1} x_i - 2\beta - \beta\sum_{i,j\neq 1,2;1,3} x_i x_j - \eta\sum_j x_i y_i)} \\
&\times e^{(-h+h\sum_{i>1} x_i - 2\beta - \beta\sum_{i,j\neq 1,2;1,3} x_i x_j - \eta\sum_j x_i y_i)}.
\end{aligned} \tag{22}$$