MATH 5591H HOMEWORK 4

BRENDAN WHITAKER

Section 11.2 Exercises

11. *Let φ be a linear transformation from the finite dimensional vector space V to itself such that $\varphi^2 = \varphi$.*
   (a) *Prove that $Image(\varphi) \cap ker\varphi = 0$.*
   Proof. Note that $\varphi : V \to V$. Let $I = Image(\varphi)$ and let $K = ker\varphi$. Let $a \in K$. Then $\varphi(a) = 0$. Then let $a \in I$. Then there exists $b \in V$ s.t. $\varphi(b) = a$. But then note:
$$\varphi^2(b) = \varphi(\varphi(b)) = \varphi(a) = 0 = \varphi(b) = a.$$
   So $a = 0$, hence $K \cap I = 0$. □
   (b) *Prove that $V = Image(\varphi) \oplus ker\varphi$.*
   Proof. We prove that $V = Image\varphi + ker\varphi$. Since $Image\varphi \subseteq V$ and $ker\varphi \subseteq V$, we know that if $v \in Image\varphi$ and $w \in ker\varphi$, then $v, w \in V$, so $v + w \in V$. So $Image\varphi + ker\varphi \subseteq V$. We prove the other inclusion. Now let $a \in V$. If $a \in ker\varphi$ then we are done. So let $a \notin ker\varphi$. Then $\varphi(a) = b \neq 0 \in V$. Then we have:
$$\begin{aligned} \varphi(b - a) &= \varphi(b) - \varphi(a) = \varphi(b) - \varphi^2(a) \\ &= \varphi(b) - \varphi(\varphi(a)) = \varphi(b) - \varphi(b) = 0. \end{aligned} \tag{1}$$
   So we know that $b - a \in ker\varphi$. So then $a - b \in ker\varphi$ since φ is a linear transformation. Now note:
$$\varphi(a) + (a - b) = b + a - b = a.$$
   and since $\varphi(a) \in Image(phi)$ and $a - b \in ker\varphi$, we have shown $V \subseteq Image\varphi + ker\varphi$. Thus $V = Image\varphi + ker\varphi$, and since we showed they have zero intersection in the last part, we have proved $V = Image\varphi \oplus ker\varphi$. □
   (c) *Prove that there is a basis of V s.t. the matrix of φ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.*
   Proof. Let $A = \{ v_1, ..., v_k \}$ be a basis for $\varphi(V)$. Then let $B = \{ v_{k+1}, ..., v_n \}$ be a basis for $ker\varphi$. We know this basis must have $n - k$ elements since $A \cup B$ must be a basis for V since we proved the direct sum in the last part. Now recall that the coefficient matrix of φ with respect to any basis C is given by (a_{ij}) where $\varphi(c_i) = \sum_j a_{ij} c_j$. So we find the matrix of φ with respect to $A \cup B$. Let $v_i \in A \cup B$. Suppose $v_i \in A$. Then $v_i = \varphi(w)$ for some $w \in V$. So we have $\varphi(v_i) = \varphi^2(w) = \varphi(w) = v_i$. So the i-th column of the i-th row must be a 1 and all other entries in that column are zero. And since $v_i \in A$, we know that $i \leq k$. Now let $v_i \in B$. Remember they are disjoint by part (a). Then $\varphi(v_i) = 0$, so the i-th column is all zeroes. Thus we have constructed the matrix of φ with respect to the basis $A \cup B$, and it is a diagonal matrix with only ones and zeroes along the diagonal. □

Section 11.3 Exercises

3. *Let S be any subset of V^* for some finite dimensional space V. Define $Ann(S) = \{ v \in V : f(v) = 0, \forall f \in S \}$. ($Ann(S)$ is called the annihilator of S in V.*
   (a) *Prove that $Ann(S)$ is a subspace of V.*
   Proof. Recall Definition **??**. Let $v, w \in Ann(S)$. Then $f(v) = f(w) = 0 \ \forall f \in S \subseteq Hom(V, F)$, where V is a vector space over the field F. Then $f(v + w) = f(v) + f(w) = 0 + 0 = 0$ since f is a homomorphism. So $v + w \in Ann(S)$. Now let $r \in F$. Then $f(rv) = rf(v) = r \cdot 0 = 0$ since again f is a homomorphism. So $rv \in Ann(S)$. Thus $Ann(S)$ is a subspace by definition. □

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(b) *Let W_1 and W_2 be subspaces of V^*. Prove that $Ann(W_1 + W_2) = Ann(W_1) \cap Ann(W_2)$ and $Ann(W_1 \cap W_2) = Ann(W_1) + Ann(W_2)$.*

Proof. Recall:

$$Ann(W_1 + W_2) = \{ v \in V : (f+g)(v) = 0, \forall f + g \in W_1 + W_2 \} .$$

So let $v \in Ann(W_1+W_2)$. Then with $g = 0$, we have $(f+g)(v) = f(v) = 0$, for all $f \in Ann(W_1)$. Now let $f = 0$, by same argument, $g(v) = 0$ for all $g \in W_2$, so $v \in Ann(W_1)$, so $v \in Ann(W_1) \cap Ann(W_2)$. Now let $v \in Ann(W_1) \cap Ann(W_2)$. Then $f(v) = 0$ and $g(v) = 0$ for all $f \in W_1, g \in W_2$. Then for arbitrary $f + g \in W_1 + W_2$. We have $(f + g)(v) = f(v) + g(v) = 0 + 0 = 0$. So $v \in Ann(W_1 + W_2$. So we have proved both inclusions: $Ann(W_1+W_2) = Ann(W_1) \cap Ann(W_2)$. Now we prove the second equality: recall:

$$Ann(W_1 \cap W_2) = \{ v \in V : f(v) = 0, \forall f \in W_1 \cap W_2 \} .$$

Let $u \in Ann(W_1)$ and $v \in Ann(W_2)$. Then for any $f \in W_1 \cap W_2$, $f(u) = 0$ and $f(v) = 0$, so $f(u+v) = f(u) + f(v) = 0 + 0 = 0$, since f is a homomorphism. So $u + v \in Ann(W_1 \cap W_2)$, so $Ann(W_1) + Ann(W_2) \subseteq Ann(W_1 \cap W_2)$.

Now we apply the result of part (c). We want to show $Ann(W_1 \cap W_2) \subseteq Ann(W_1) + Ann(W_2)$. By this result we know this is equivalent to showing:

$$Ann(Ann(W_1 \cap W_2)) = W_1 \cap W_2 \subseteq Ann(Ann(W_1) + Ann(W_2)).$$

So let B_V be a basis for V, and let B_{V^*} be a basis for V^*. Then let $\{ f_1, ..., k \}$ be a basis for W_1 and define $\{ f_l, ..., f_m \}$ as basis for W_2, without loss of generality, where $m, k \leq n = \dim V = \dim V^*$. Then by part (f) we know $Ann(W_1) = F(B_V \setminus \{ f_1, ..., f_k \})$ and $Ann(W_2) = F(B_V \setminus \{ f_l, ..., f_m \})$. So $Ann(W_1) + Ann(W_2) = A = F(B_V \setminus (\{ f_l, ..., f_m \} \cap \{ f_1, ..., f_k \}))$. And by part (f) again we know $Ann(A) = F(B_{V^*} \setminus (B_{V^*} \setminus F(\{ f_l, ..., f_m \} \cap \{ f_1, ..., f_k \}))) = W_1 \cap W_2$. So we have proved the other inclusion, and we are done. □

(c) *Let W_1 and W_2 be subspaces of V^*. Prove that $W_1 = W_2$ if and only if $Ann(W_1) = Ann(W_2)$.*

Proof. Let $\{ g_1, ..., g_n \}$ be a basis of V^{**}. And we have the natural isomorphism which sends $g_i \mapsto v_i \in B_V$, the basis of V. So $V \cong V^{**}$. So V^* must have a basis $\{ f_1, ..., f_n \}$ and let $\{ f_1, ..., f_k \}$ be a basis for W_1. By part (f), we know $Ann(Ann(W_1)) = Ann(F \{ v_{k+1}, ..., v_n \})$. But again by part F and since $v_i(f_j) = f_j(v_i) = 0, \forall i \neq j$, we know $Ann(F \{ v_{k+1}, ..., v_n \}) = F \{ f_1, ..., f_k \}$. But this is exactly W_1, so $Ann(Ann(W)) = W$, and so since $Ann(W_1) = Ann(W_2)$, we know $Ann(Ann(W_1)) = Ann(Ann(W_2)) \Rightarrow W_1 = W_2$. □

(d) *Prove that the annihilator of S is the same as the annihilator of the subspace of V^* spanned by S.*

Proof. Note $Ann(S) = \{ v \in V : f(v) = 0, \forall f \in S \}$. And note that

$$Ann(FS) = \{ v \in F : f(v) = 0, \forall f \in FS \} .$$

Now V^* is finite dimensional since we know how to generate the dual basis, and the dimension of V^* is the same as the dimension of V. So S has a finite maximal linearly independent set $B_S = \{ f_1, ..., f_k \}$. Let $v \in Ann(FS)$. Then since $1 \in F$, we know $S \subseteq FS$, so $f(v) = 0, \forall f \in S$, so $Ann(FS) \subseteq Ann(S)$.

Now let $v \in Ann(S)$. Then since $B_S \subseteq S$, we know $v \in Ann(B_S)$. Then

$$FS \subseteq FB_S = F \{ f_1, ..., f_k \} = \{ r_1 f_1 + \cdots + r_k f_k : r_i \in F, f_i \in B_S \} .$$

Then $f_i(v) = 0$ for all i since they are in B_S, and $r_i \cdot 0 = 0$, so $v \in Ann(FB_S) \subseteq Ann(FS)$ since $FS \subseteq FB_S$. Hence $Ann(S) \subseteq Ann(FS)$, and so they are equal. □

(e) *Assume V is finite dimensional with basis $v_1, ..., v_n$. Prove that if $S = \{ v_1^*, ..., v_k^* \}$ for some $k \neq n$, then $Ann(S)$ is the subspace spanned by $\{ v_{k+1}, ..., v_n \}$.*

Proof. Note that S is some subset of the dual basis, so let's change notation to be consistent with lecture. Let $S = \{ v_1^*, ..., v_k^* \} = \{ f_1, ..., f_k \}$. Note since $k \neq n$, $\{ v_{k+1}, ..., v_n \}$ is nonempty. Let $v = r_1v_1 + \cdots + r_nv_n \in Ann(S)$. Then $f_i(v) = 0$, $1 \leq i \leq k$. We want to show $v \in F\{ v_{k+1}, ..., v_n \}$. Suppose $v \notin F\{ v_{k+1}, ..., v_n \}$, then since $v \in V$, we know there exists $i \leq k$ s.t. the coefficient of v_i in $r_1v_1 + \cdots + r_nv_n$ is nonzero. But if this is true, we would have $f_i(r_1v_1 + \cdots + r_nv_n) \neq 0$ since each of the basis vectors is linearly independent. This is a contradiction, since $f_i(v) = 0$ for all $v \in Ann(S)$. So we must have that $v \in F\{ v_{k+1}, ..., v_n \}$. And hence $Ann(S) \subseteq F\{ v_{k+1}, ..., v_n \}$.
Now let $v \in F\{ v_{k+1}, ..., v_n \}$. Then $v = r_{k+1}v_{k+1} + \cdots + r_nv_n$. Chose arbitrary $f_i \in S$. Then $i \leq k$, so $f(r_jv_j) = r_jf(v_j) = f_j \cdot 0 = 0$ for all $j > k$, by definition of f_i, since $i \neq j$. Thus $f_j(v) = 0$ since $j > k$ for all $v_j \in \{ v_{k+1}, ..., v_n \}$. So since this holds for all $f_j \in S$, $v \in Ann(S)$, so $F\{ v_{k+1}, ..., v_n \} \subseteq Ann(S)$. ☐

(f) *Assume V is finite dimensional. Prove that if W^* is any subspace of V^*, then* $\dim Ann(W^*) = \dim V - \dim W^*$.

Proof. We have a basis of $\{ v_1, ..., v_n \}$ of V. Let $\{ f_1, ..., f_n \}$ be the corresponding basis of the finite dimensional V^* (since V is finite dimensional), and without loss of generality, let $\{ f_1, ..., f_k \}$ be a basis for W^*, which we know has a basis since it is a subspace. By the previous exercise, $Ann(W^*) = F\{ v_{k+1}, ..., v_n \}$. So it has dimension $n - k$, and since $\dim V = n$ and $\dim W^* = k$, we are done. ☐