MATH 5591H HOMEWORK 6

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Section 11.4 Exercises

2. *Let F be a field and let $A_1, A_2, ..., A_n$ be (column) vectors in F^n. Form the matrix A whose i-th column is A_i. Prove that these vectors form a basis of F^n if and only if* $\det A \neq 0$.

Proof. Recall Corollary 27 from Dummit and Foote, which states that if R is an integral domain, then $\det A \neq 0$ for $A \in M_n(R)$ if and only if the columns of A are R-linearly independent as elements of the free R-module of rank n.

Now since F^n is a vector space, we know that if we have a set of n linearly independent vectors, it must be a basis. So let the column vectors A_i form a basis of F^n. Then they must be linearly independent. So by the corollary, we know $\det A \neq 0$. Now let $\det A \neq 0$. Then by the corollary, we know A_i are linearly independent over F as elements of F^n, since F is field, thus an integral domain. So then since F^n is a vector space of $\dim F^n = n$, they must form a basis, since if they didn't, we would need some other linearly independent vector to generate the missing elements of F^n, which would contradict the fact that $\dim F^n = n$. □

3. *Let R be any commutative ring with 1, let V be an R-module and let $x_1, ..., x_n \in V$. Assume that for some $A \in M_{n\times n}(R)$,*

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Prove that $(\det A)x_i = 0$, *for all* $i \in \{ 1, 2, ..., n \}$.

Proof. Recall Theorem 30 from Dummit and Foote, which states that if B is the transpose of the matrix of cofactors of A, then $AB = BA = (\det A)I$. So note:

$$0 = B0 = BA \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\det A)I \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\det A) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

And this is zero if and only if $(\det A)x_i = 0$ for all i. □

Section 11.5 Exercises

5. *Prove that if M is a free R-module of rank n, then $\Lambda^k(M)$ is a free R-module of rank $\binom{n}{k}$ for* $k = 0, 1, 2, ...$ Let $B = \{ u_1, ..., u_n \}$ be a basis in M. Equivalently, we claim:

Lemma 1. *The basis in $\Lambda^k(M)$ is:*

$$\Lambda^k(B) = \{ u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k} : i_1 < i_2 < ... < i_k \}.$$

Note this set has $\binom{n}{k}$ elements since we are choosing k from n, since $|B| = n$.

Proof. Note that $\Lambda^k(M)$ has a universal property: If $\Phi : M^k \to N$ is a k-linear alternating mapping, then there is a hom-sm $\beta : \Lambda^k(M) \to N$ such that:

$$\beta(v_1 \wedge \cdots \wedge v_k) = \Phi(v_1, ..., v_k), \forall v_i \in M.$$

And since this basis is obtained from the natural projection of B, we know $\Lambda^k(B)$ generates $\Lambda^k(M)$.

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Let $i_1 < i_2 < ... < i_k$. Define a k-linear mapping from $M^k \to R$ by sending:

$$\Phi(u_{j_1}, ..., u_{j_k}) = \begin{cases} sign(\sigma) & \text{if } (j_1, ..., j_k) = \sigma(i_1, ..., i_k) \text{ for some } \sigma \in S_k \\ 0 & \text{otherwise} \end{cases}$$

So we have basis vectors $u_{j_1} \wedge \cdots \wedge u_{j_k}$ and we want to send them to $sign(\sigma)$ only if they are some permutation of our i's. Then Φ induces a homomorphism $\beta : \Lambda^k(M) \to R$ such that:

$$\beta(v_1 \wedge \cdots \wedge v_k) = \Phi(v_1, ..., v_k), \forall v_i \in M.$$

And $\beta(u_{j_1} \wedge \cdots \wedge u_{j_k}) = 0$, $\forall j_1 < ... < j_k$ if $\neq (i_1, ..., i_k)$. So we only define Φ on basis vectors and expand it to the whole space by k-linearity. Now we have a hom-sm which maps our chosen vector to $sign(\sigma)$ and all other vectors to zero. So suppose

$$s = r_1 v_1 + \cdots + r(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k}) + \cdots + r_{\binom{n}{k}} v_{\binom{n}{k}} = 0.$$

Then $\beta(s) = \beta(r(u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_k})) = 0$, since β maps all other vectors to zero. So we must have $r \neq 0$ since our basis vector is nonzero. This implies that $u_{i_1} \wedge \cdots \wedge u_{i_k}$ is not a linear combination of other vectors from $\Lambda^k(B)$. So what we proved is that any element from $\Lambda^k(B)$ is not a linear combination of the others, so we proved this set is linearly independent and thus a basis. □

12. (a) *Prove that if $f(x, y)$ is an alternating bilinear map on V (i.e. $f(x, x) = 0$ for all $x \in V$) then $f(x, y) = -f(x, y)$ for all $x, y \in V$.*

Proof. Observe:

$$\begin{aligned} 0 &= f(x+y, x+y) = f(x+y, x) + f(x+y, y) \\ &= f(x, x) + f(y, x) + f(x, y) + f(y, y) = f(y, x) + f(x, y). \end{aligned} \tag{1}$$

So adding $-f(x, y)$ to both sides we have:

$$-f(x, y) = f(y, x).$$

□

(b) *Suppose that $-1 \neq 1$ in F. Prove that $f(x, y)$ is an alternating bilinear map on V (i.e. $f(x, x) = 0$ for all $x \in V$) if and only if $f(x, y) = -f(y, x)$ for all $x, y \in V$.*

Proof. The forward direction follows from part (a). For the second direction, assume $f(x, y) = -f(y, x)$. So we have $f(x, x) = -f(x, x)$. Since $-1 \neq 1$, we know $1 + 1 = r \neq 0 \in F$. So we have:

$$rf(x, x) = 0.$$

Suppose $f(x, x) \neq 0 \in W$, where W is the vector space which f maps to. Then since $r \neq 0$ we have a contradiction since $\{ f(x, x) \}$ is linearly independent. So $f(x, x) = 0$ for all $x \in V$. □

(c) *Suppose that $-1 = 1$ in F. Prove that every alternating bilinear form $f(x, y)$ on V is symmetric (i.e. $f(x, y) = f(y, x)$ for all $x, y \in V$). Prove that there is a symmetric bilinear map on V that is not alternating. [One approach: show that $C^2(V) \subseteq \mathcal{A}^2(V)$ and $C^2(V) \neq \mathcal{A}^2(V)$ by counting dimensions. Alternatively, construct an explicit symmetric map that is not alternating.]*

Proof. For the first part, we use part (a), so we know :

$$f(x, y) = -f(y, x) = f(y, x),$$

since $1 = -1$. For the second part, consider $f : V \to F$ given by $f(x, y) = x \cdot y$, the dot product. It is symmetric since addition in F is abelian, but it is not alternating. Note $f(x, x) = 0$ if and only if $x = 0$. □