MATH 5591H HOMEWORK 1

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Section 10.1 Exercises

8. *An element m of the R-module M is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted:*

$$Tor(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) *Prove that if R is an integral domain, then $Tor(M)$ is a submodule of M (called the torsion sobumodle of M).*

Proof. We know $\mathrm{Tor}(M)$ is a subset of M by its definition. We first prove it is an additive subgroup. Let $m \in \mathrm{Tor}(M)$. Then $\exists r \in R$, $r \neq 0$ s.t. $rm = 0$. Then consider $-m \in M$. From exercise 1 we know $-m = (-1)m$, so we have:

$$r(-m) = r(-1)m = (-1)rm = (-1)0 = 0,$$

since R is commutative. So we have that $-m \in \mathrm{Tor}(M)$ as well, hence we have additive inverses. We check that it has additive closure. Let $m, n \in \mathrm{Tor}(M)$. Then we have $r, s \in R$, neither being zero, s.t. $rm = 0, sn = 0$. Now consider $m + n$. We have:

$$rs(m + n) = rsm + rsn = srm + rsn = s0 + r0 = 0.$$

Since we have no zero divisors, since R is an integral domain, we know $rs \neq 0$, so $m + n \in \mathrm{Tor}(M)$, we have additive closure, and $\mathrm{Tor}(M)$ is a subgroup of M. Now we need only check that it is closed under the left action of R. So let $r \in R$ and $m \in \mathrm{Tor}(M)$. Then consider rm. We assume $r \neq 0$, since otherwise $rm = 0$ which is in our subgroup. And we know $\exists s \in R$, $s \neq 0$ s.t. $sm = 0$. Now we have $srm = rsm = r0 = 0$, so rm is in $\mathrm{Tor}(M)$. So it's a submodule. □

(b) *Give an example of a ring R and an R-module M such that $Tor(M)$ is not a submodule (consider the torsion elements in the R-module R).*

So from the previous exercise, we know we must choose some R which is not an integral domain. We consider the torsion elements in the R-module R, which are:

$$\mathrm{Tor}(R) = \{r \in R : sr = 0 \text{ for some nonzero } s \in R\},$$

but these are exactly the right zero divisors of R. We consider the ring $R = \mathbb{Z}_6 \cong \mathbb{Z}/6\mathbb{Z}$, and the module of R over itself. Note that in R, $2 \cdot 3 = 6 = 0$, $4 \cdot 3 = 12 = 0$, and $1, 5$ are not zero divisors, so we have:

$$\mathrm{Tor}(R) = \{0, 2, 3, 4\}.$$

So note that $2, 3 \in \mathrm{Tor}(R)$ and $1 \in R$, but $2 + 1 \cdot 3 = 5 \notin \mathrm{Tor}(R)$, so by the submodule criterion, it is not a submodule.

(c) *If R has zero divisors, show that every nonzero R-module has nonzero torsion elements.*

Proof. Suppose R has zero divisors. So $\exists r, s \in R$ nonzero such that $rs = 0$. Now let M be an R-module. We wish to show that $\exists m \in M$ s.t. $m \neq 0$, $tm = 0$ for some nonzero $t \in R$. Let $n \in M$ s.t. $n \neq 0$. Now consider $sn \in M$ and $r \in R$. Now note that $rsn = 0$ and that r and sn are both nonzero, so sn is a nonzero torsion element. □

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9. *If N is a submodule of M, the annihilator of N in R is defined to be:*

$$Ann_R(N) = \{r \in R : rn = 0 \text{ for all } n \in N\}.$$

Prove that the annihilator of N in R is a two-sided ideal of R.

Proof. Let $A = \text{Ann}_R(N)$. We first show that A is an additive subgroup of R. We know it is nonempty since $0 \in A$, and it is a subset of R by construction. Now let $x, y \in A$. Consider $x(-y) = -xy$. Note $-xyn = -x(yn) = -x0 = 0\ \forall n \in N$, so by the subgroup criterion, A is a subgroup. Let $r \in R$, $n \in N$, and $a \in A$. Observe:

$$ran = r(an) = r0 = 0,$$

$$arn = a(rn) = 0,$$

since a annihilates n, and N is closed under the action of R, so $rn \in N$, and hence a also annihilates (rn). Since our n was arbitrary, this holds for all $n \in N$. Thus $ra \in A$ and $ar \in A$, and thus $RA \subseteq A$ and $AR \subseteq A$, so since it's also an additive subgroup, A is a two-sided ideal. □

10. *If I is a right ideal of R, the annihilator of I in M is defined to be:*

$$Ann_M(I) = \{m \in M : am = 0 \text{ for all } a \in I\}.$$

Prove that the annihilator of I in M is a submodule of M.

Proof. Since I is a right ideal, we know $Ir \subseteq I\ \forall r \in R$. Let $A = \text{Ann}_M(I)$ which we know is nonempty since $0 \in M$ since it is an abelian group, and $a0 = 0\ \forall a \in I$. Let $m, n \in A$, let $a \in I$, and let $r \in R$. Observe:

$$a(m + rn) = am + arn = 0 + arn = (ar)n = 0,$$

since $a \in I \Rightarrow ar \in I$ (I is right ideal), hence n annihilates (ar). Thus $(m + rn) \in A$. Then by the submodule criterion, since this holds for arbitrary $m, n \in A$, $r \in R$, and A is nonempty, we know A is a submodule of M. □

Section 10.2 Exercises

9. *Let R be a commutative ring. Prove that $Hom_R(R, M)$ and M are isomorphic as left R-modules. [Show that each element of $Hom_R(R, M)$ is determined by its value on the identity of R.]*

Proof. Recall:

$$H = \text{Hom}_R(R, M) = \{\phi : R \to M\},$$

where R and M are R-modules. Let $\phi \in H$. Recall that from the definition of H, we know:

$$\phi(rs + t) = r\phi(s) + \phi(t),$$

for all $r, s, t \in R$. So note that $\forall r \in R$, we have:

$$\phi(r) = r\phi(1_R),$$

hence ϕ is complete determined by its value on 1_R. Also observe that $\phi(1_R) \in M$, so define a map $\Phi : M \to H$ by $\Phi(m) = \phi_m$, where we define $\phi_m(1_R) = m$. We prove this map is an R-module isomorphism. We first prove it is an R-module homomorphism. So let $m, n \in M$, then we have:

$$\Phi(m) + \Phi(n) = \phi_m + \phi_n$$

Now we prove surjectivity. So let $\psi \in H$, then $\psi(1_R) = m$ for some $m \in M$, so we know $\psi = \phi_m$. Then note that $\Phi(m) = \phi_m$, so Φ is surjective. □

Section 10.3 Exercises

7. *Let N be a submodule of M. Prove that if both M/N and N are finitely generated, then so is M.*

Proof. Suppose M is not finitely generated. Then we have:

$$M/N = RA,$$

where $A = \{x_1 + N, ..., x_n + N\}$. And since N is also finitely generated, we know $N = RA_N$, and $M - N$ is not finitely generated. Now we know $x_i \in M - N$ since otherwise we would have $x_i + N = N$. So then since M is not finitely generated, we know $\exists y \in M - N$ s.t. $y \notin R\{x_i\}$, hence $y + N \notin RA = \{(rx_1) + N, ..., (rx_n) + N\}$, but since $y \in M - N$ we know $y + N \neq N$, hence $y + N \in M/N$. But we said $M/N = RA$, so this is a contradiction, so we must have that M is finitely generated. □

12. *Let R be a commutative ring and let A, B, and M be R-modules. Prove the following isomorphisms of R-modules:*

(a) $Hom_R(A \times B, M) \cong Hom_R(A, M) \times Hom_R(B, M)$.

Proof. Let $H = \text{Hom}_R(A \times B, M)$, $H_A = \text{Hom}_R(A, M)$, and $H_B = \text{Hom}_R(B, M)$. Let $\Phi : H_A \times H_B \to H$ be given by $\Phi((\phi, \psi)) = \phi + \psi$, where $\phi \in H_A, \psi \in H_B$. We prove this is an isomorphism of R-modules.

Homomorphism: Observe:

$$\begin{aligned}\Phi((\phi_1, \psi_1) + (\phi_2, \psi_2)) &= \Phi((\phi_1 + \phi_2, \psi_1 + \psi_2)) = \phi_1 + \psi_1 + \phi_2 + \psi_2 \\ &= \Phi((\phi_1, \psi_1)) + \Phi((\phi_2, \psi_2)).\end{aligned} \tag{1}$$

In the above expression, the first equality comes from the definition of addition in $H_A \times H_B$. The second and third equalities comes from the definition of Φ. And we also know:

$$\Phi(r(\phi, \psi)) = \Phi((r\phi, r\psi)) = r\phi + r\psi = r(\phi + \psi) = r\Phi((\phi, \psi)),$$

hence Φ preserves mult. by R, by the definition of scalar multiplication on the R-module $H_A \times H_B$, and the definition of Φ.

Surjectivity: Let $\varphi \in H$. Then $\varphi : A \times B \to M$. So let $\phi \in H_A$ be given by $\phi(a) = \varphi(a, 0)$, and let $\psi \in H_B$ be given by $\phi(b) = \varphi(0, b)$. Then we have: $\Phi((\phi, \psi)) = \varphi$. Then Φ is surjective.

Injectivity: Let $\Phi((\phi_1, \psi_1)) = \phi_1 + \psi_1 = \phi_2 + \psi_2 = \Phi((\phi_2, \psi_2)) \in H_A \times H_B$. Then note that

$$(\phi_1 + \psi_1)(a, 0) = \phi_1(a) = \phi_2(a) = (\phi_2 + \psi_2)(a, 0),$$

and the same holds when we let $a = 0$, and use an arbitrary b value, so we get that $\psi_1 = \psi_2$ as well. Hence Φ is injective. And thus it is an isomorphism. □

(b) $Hom_R(M, A \times B) \cong Hom_R(M, A) \times Hom_R(M, B)$.

Proof. Let $H = \text{Hom}_R(M, A \times B)$, $H_A = \text{Hom}_R(M, A)$, and $H_B = \text{Hom}_R(M, B)$. Let $\Phi : H_A \times H_B \to H$ be given by $\Phi((\phi, \psi)) = (\phi, \psi) \in H$, where $\phi \in H_A$, and $\psi \in H_B$. We prove this map is an isomorphism.

Homormorphism: Observe:

$$\begin{aligned}\Phi((\phi_1, \psi_1) + (\phi_2, \psi_2)) &= \Phi((\phi_1 + \phi_2, \psi_1 + \psi_2)) = (\phi_1 + \phi_2, \psi_1 + \psi_2) \\ &= (\phi_1, \psi_1) + (\phi_2, \psi_2) = \Phi((\phi_1, \psi_1)) + \Phi((\phi_2, \psi_2)).\end{aligned} \tag{2}$$

The first equality follows from addition in the R-module $H_A \times H_B$, the second comes from the definition of Φ, the third comes from addition in H, and the last again comes from the definition of Φ. And we also know:

$$\Phi(r(\phi, \psi)) = \Phi((r\phi, r\psi)) = (r\phi, r\psi) = r(\phi, \psi) = r\Phi((\phi, \psi)),$$

by the definition of scalar mult. in H, hence since Φ preserves addition and scalar multiplication, we know it is a homomorphism.

Surjectivity: Let $\varphi \in H$, then we know $\varphi : M \to A \times B$. Then the image of any element of M under φ is a two dimensional vector whose first component lives in A, and whose second component lives in B. So let $\phi : M \to A$ be given by $\phi(m) = \varphi(m)_1$, the first component of $\varphi(m)$. and let $\psi(m) = \varphi(m)_2$. Then $\Phi((\phi, \psi)) = (\phi, \psi) = \varphi$. Hence Φ is surjective.

Injectivity: Let $\Phi((\phi_1, \psi_1)) = (\phi_1, \psi_1) = (\phi_2, \psi_2) = \Phi((\phi_2, \psi_2))$. Then we must have $\phi_1 = \phi_2$, and $\psi_1 = \psi_2$, since otherwise we do not have equality of these ordered pairs of hom-sms in H. But then we have shown that the arguments of Φ are equal in this case, so Φ must be injective. $\square$