CSE 6331 HOMEWORK 2

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Theorem 5. *If $T(n)$ is asymptotically nondecreasing and $f(n)$ is smooth, then $T(n) = O(f(n)|n$ is a power of $b)$ implies $T(n) = O(f(n))$.*

1. *Show that Theorem 5 would not hold if $T(n)$ is not asymptotically nondecreasing.*
   Let $f(n) = n^2$, and let $b = 2$. We define:
$$T(n) = \begin{cases} n^2 & n = 2^k, k \in \mathbb{N} \\ n^n & otherwise \end{cases}.$$
   Observe that $f(n) = n^2$ is smooth: $f(2n) = (2n)^2 = 4n^2 \leq 4 \cdot n^2, \forall n \in \mathbb{N}$, so $f(2n) \in O(n^2)$. Also $T(n) \in O(n^2|n$ is a power of 2), since $T(n) = n^2$ when n is a power of 2. We show $T(n) \notin O(n^2)$. Suppose for contradiction that there was a positive constant $c \in \mathbb{N}$ and $N \in \mathbb{N}$ s.t. $\forall n > N$, $T(n) \leq cn^2$. Then this must hold for all powers of 3, so we restrict ourselves to the case when $n \in B = \{n : n = 3^k, k \in \mathbb{N}\}$. Then n is not a power of 2, so $T(n) = n^n$. So we have $n^n \leq cn^2, \forall n > N, n \in B$. Then we would have:
$$n^2 \cdot n^{n-2} \leq cn^2,$$
$$n^{n-2} \leq c,$$
   for all $n > N, n \in B$, which is a contradiction, since c is a constant. So we must have $T(n) \notin O(n^2)$.

2. *Show that Theorem 5 would not hold if $f(n)$ is nondecreasing but not smooth (even if $T(n)$ is asymptotically nondecreasing).*
   Let $f(n) = 2^n$. Let:
$$T(n) = 2^{2^{\lceil log_2 n \rceil}}.$$
   When n is a power of 2, $n = 2^k$, we have $\lceil log_2 n \rceil = k$, so $T(n) = 2^n$, so $T(n) \in O(2^n)$ when n is a power of 2. But is $T(n) \in O(2^n)$ in general? Let $n = 2^k + 1$, so n runs through the positive integers which are one more than a power of 2. Then for $k \in \mathbb{N}$ we have $T(n) = 2^{2^{k+1}} = 2^{2 \cdot n} \notin O(2^n)$, so Theorem 5 does not hold in this case.
3. *Prove **Theorem 6:** If $T(n)$ is asymptotically nondecreasing and $f(n)$ is smooth, then $T(n) = \Omega(f(n)|n$ is a power of $b)$ implies $T(n) = \Omega(f(n))$.*

   Proof. We know $T(n) \geq c_1 f(n)$ for n sufficiently large and a power of b. And by definition of smooth, we know $f(bn) \in O(f(n)) \Rightarrow f(n) \in \Omega(f(bn))$. So we have $f(n) \geq c_2 f(bn)$ for sufficiently large n, and $\forall n$ there is $k \in \mathbb{N}$ s.t. $b^k \leq n < b^{k+1}$. When n is sufficiently large, we have:
$$T(n) \geq T(b^k) \geq c_1 f(b^k) \geq c_1 c_2 f(b \cdot b^k) \geq c_1 c_2 f(b^{k+1}) \geq c_1 c_2 f(n).$$
   Thus we know $T(n) \in \Omega(f(n))$. □

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