Bonus Problems

AU17

Week 1

Exercise 1. *Give an example of a countable infinite abelian group which doesn not have proper infinite subgroups.*

Exercise 2. *Prove that $\sqrt{2}$ is irrational using the fact that $(\sqrt{2}-1)^n \to 0$.*

Exercise 3. *Suppose you know $\exists c_1, c_1$ s.t. $c_1 < \frac{p_n}{n \log n} < c_n \ \forall n > 1$. Prove from this that $\sum \frac{1}{p_n} = \infty$.*

Exercise 4. *Prove that $\pi(n) \sim \frac{n}{n \log n} \Leftrightarrow p_n \sim n \log n$, where $\pi(n)$ is the number of primes $\leq n$.*

Exercise 5. *Prove that any real number $x > 0$ is a limit of the form $x = \lim_{n \to \infty} \frac{k_n^2}{m_n^2}$, where $k_n, m_n \in \mathbb{N}$.*

Exercise 6. *Prove that the set of algebraic numbers is countable.*

Exercise 7. *Prove that if $x \in (0,1) \subset \mathbb{Q}$, then its simple continued fraction expansion is finite.*

Exercise 8. *Find the simple continued fraction expansions for $\sqrt{2}, \sqrt{3}$.*

Exercise 9. *Is there some $k \in \mathbb{N}$ s.t. every $n \in \mathbb{N}$ is a sum of k cubes?*

Exercise 10. *The Waring Problem:* *Is it true that for any $k \in \mathbb{N}$, $k \geq 2$, there exists $C(k)$ s.t. any $n \in \mathbb{N}$ is a sum of $C(k)$ k-th powers of non-negative integers?*

Exercise 11. *Prove that the set of non-normal integers is uncountable.*

Exercise 12. *Prove that Champernowne is normal.*

Exercise 13. *If you replace the first index of each square in the Champernowne number with 17, is it still normal?*

Exercise 14. *If you replace each square in the Champernowne number with 17, is it still normal?*

Exercise 15. *Wall's Theorem:* *If (x_n) is normal, then $\forall a, b \in \mathbb{N}$ (x_{an+b} is also normal.*

Exercise 16. *Prove that every real number $x > 0$ is a limit of the form $x = \lim_{n \to \infty} \frac{p_n}{q_n}$, where $p_n, q_n \in \mathbb{P}$.*

Exercise 17. *Find the simple continued fraction expansions for $\sqrt{n^2 + 1} \ \forall n \in \mathbb{N}$.*