MATH 5591H HOMEWORK 2

BRENDAN WHITAKER

Section 10.2 Exercises

6. *Prove that* $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n,m)\mathbb{Z}$.

   Proof. Let $H = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$, and let $K = \mathbb{Z}/(n,m)$. Also, let $l = \gcd(n,m)$. Then $K = \mathbb{Z}_l$. So let $\phi \in H$. Then $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$. We note here that ϕ is completely determined by where it sends $1 \in \mathbb{Z}_n$, since we must have $\phi(n \cdot 1) = \phi(0) = 0$ by the definition of a group homomorphism, thus we must have that $n\phi(1) = 0 \in \mathbb{Z}_m$. In order to have $n\phi(1) = 0$, we need $\phi(1)$ to be a multiple of m. So we need $\phi(1)$ to be a multiple of m/l, since every prime factor in l is also in the factorization of n, so we need only the prime factors of m which are not in l, hence $\phi(1)$ must be a multiple of m/l. Now note there are exactly l multiples of m/l in $\mathbb{Z}_m$. We denote these $a_0, ..., a_{l-1}$. So we have exactly l distinct homomorphisms in H, so we denote these $\phi_0, ..., \phi_{l-1}$, where $\phi_i(1) = a_i = im/l \in \mathbb{Z}_m$. Then let $\Phi : H \to K$ be given by:

   $$\Phi(\phi_i) = i \in \mathbb{Z}_l.$$

   We prove this map is an isomorphism. **Homomorphism:** Observe:

   $$\Phi(\phi_i + \phi_j) = \Phi(\phi_{i+j \mod l}) = i + j = \Phi(\phi_i) + \Phi(\phi_j) \in \mathbb{Z}_l.$$

   The first equality is by the additive operation on the $\mathbb{Z}$-module H, and the other equalities follow from the definition of Φ and the additive operation on $\mathbb{Z}_l$. Since ϕ_i is a homomorphism of R-modules, it preserves multiplication by scalars, so we have $z\phi_i(1) = \phi_i(z) = za_i$, and since $\{a_i\} \cong \mathbb{Z}_l$ as a group, we know $za_i = a_{zi \mod l}$. So we have:

   $$\Phi(z\phi_i) = zi = z\Phi(\phi_i) \in \mathbb{Z}_l.$$

   So Φ preserves scalar mult, and hence it is a homomorphism.
   Surjectivity: Let $i \in \mathbb{Z}_l$. Then consider $\psi \in H$ s.t. $\psi(1) = im/l$, but this is exactly how we defined ϕ_i, so we know $\phi_i = \psi$, and then $\Phi(\psi) = \Phi(\phi_i) = i$. So Φ is surjective.
   Injectivity: Let:

   $$\Phi(\psi) = \Phi(\xi),$$

   then since we enumerated all the elements of H, we know we must have $\psi = \phi_i$ and $\xi = \phi_j$ for some $0 \leq i, j \leq l-1$. Then we have:

   $$\Phi(\phi_i) = i = j = \Phi(\phi_j) \in \mathbb{Z}_l,$$

   so $i \equiv j \mod l$, but since both these numbers are between 0 and $l-1$, we know $i = j$, so $\psi = \xi$, and Φ is injective. Hence it is an isomorphism. □

11. *Let* $A_1, A_2, ..., A_n$ *be* R*-modules and let* B_i *be a submodule of* A_i *for each* $i = 1, 2, ..., n$. *Prove that:*

   $$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$

   Proof. So let $A = (A_1 \times \cdots \times A_n)$, $B = (B_1 \times \cdots \times B_n)$, and $C = (A_1/B_1) \times \cdots \times (A_n/B_n)$. Note that:

   $$A/B = \{ (a_1, ..., a_n) + B \}.$$

   We know B is a submodule of A since it is clearly a subset since each component b_i of $(b_1, ..., b_n)$ is also in A_i. Also:

   $$(b_1, ..., b_n) + r(d_1, ..., d_n) = (b_1, ..., b_n) + (rd_1, ..., rd_n) = (b_1 + rd_1, ..., b_n + rd_n),$$

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because of how we defined add. and mult. by R in the R -module B, and because each B_i is a submodule of A_i. Then we know A/B is an R-module since we may factorize by any submodule of A. , so we let $\phi : A/B \to C$ be given by

$$\phi((a_1, a_2, ..., a_n) + B) = (a_1 + B_1, a_2 + B_2, ..., a_n + B_n).$$

We prove that ϕ is an isomorphism.

Homomorphism: Let $(x_1, x_2, ..., x_n) + B, (y_1, y_2, ..., y_n) + B \in A/B$, then

$$\begin{aligned} \phi(((x_1, x_2, ..., x_n) + B) + ((y_1, y_2, ..., y_n) + B)) &= \phi(((x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)) + B) \\ &= \phi((x_1 + y_1, x_2 + y_2, ..., x_n + y_n) + B) \\ &= (x_1 + y_1 + B_1, x_2 + y_2 + B_2, ..., x_n + y_n + B_n) \\ &= (x_1 + B_1, x_2 + B_2, ..., x_n + B_n) \\ &+ (y_1 + B_1, y_2 + B_2, ..., y_n + B_n) \\ &= \phi((x_1, x_2, ..., x_n) + B) + \phi((y_1, y_2, ..., y_n) + B), \end{aligned} \tag{1}$$

by the direct product operation on A/B and C. And for multiplication, we have:

$$\begin{aligned} \phi(r((x_1, ..., x_n) + B)) &= \phi(r(x_1, ..., x_n) + B) \\ &= \phi(rx_1, ..., rx_n) + B) \\ &= (rx_1 + B, ..., rx_n + B) \\ &= r(x_1 + B, ..., x_n + B) \\ &= r\phi((x_1, ..., x_n) + B), \end{aligned} \tag{2}$$

so ϕ is a homomorphism.

Injection: Let $(x_1, x_2, ..., x_n) + B, (y_1, y_2, ..., y_n) + B \in A/B$, and let

$$\begin{aligned} \phi((x_1, x_2, ..., x_n) + B) &= \phi((y_1, y_2, ..., y_n) + B) \\ \Rightarrow (x_1 + B_1, x_2 + B_2, ..., x_n + B_n) &= (y_1 + B_1, y_2 + B_2, ..., y_n + B_n). \end{aligned} \tag{3}$$

So then we have that $x_i + B_i = y_i + B_i$ for all i, thus

$$\begin{aligned} (y_1, y_2, ..., y_n) + B = (y_1, y_2, ..., y_n) + (B_1 \times B_2 \times \cdots \times B_n) &= (y_1 + B_1 \times y_2 + B_2 \times \cdots \times y_n + B_n) \\ = (x_1 + B_1 \times x_2 + B_2 \times \cdots \times x_n + B_n) &= (x_1, x_2, ..., x_n) + B \end{aligned} \tag{4}$$

by the direct product operation, so ϕ is in injective.

Surjection: Let $(a_1 + B_1, a_2 + B_2, ..., a_n + B_n) \in C$. Then we must have that $a_i \in A_i$ for all i by definition of C and the quotient modules A_i/B_i, so $(a_1, a_2, ..., a_n) \in A \Rightarrow (a_1, a_2, ..., a_n) + B \in A/B$, and $\phi((a_1, a_2, ..., a_n) + B) = (a_1 + B_1, a_2 + B_2, ..., a_n + B_n)$, so ϕ is surjective by definition. Hence ϕ is an isomorphism, and $A/B \cong C$. □

Section 10.3 Exercises

15. *An element $e \in R$ is called a* ***central idempotent*** *if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R, prove that $M = eM \oplus (1 - e)M$.*

Proof. So we wish to show that M is the direct sum of the two specified submodules. Note that we know that these sets are both submodules by Exercise 14 of Section 1, which tells us that zM is a submodule for any z in the center of R. We know e is in the center since it is a central idempotent. And $(1 - e)r = r - er = r - re = r(1 - e)$. So it is also in the center. Now we need only show that $M = eM + (1 - e)M$, and that $eM \cap (1 - e)M = 0$.

Let $m \in M$. Then $m = em+(1-e)m = em+m-em$, where $em \in eM$, and $(1-e)m \in (1-e)M$, so $m \in eM+(1-e)M$. Now let $em+(1-e)n \in eM+(1-e)M$. Then we have $em+n-en = n+e(m-n)$. So we know $M = eM + (1 - e)M$. So let $m \in eM \cap (1 - e)M$. Then $m = en_1 = (1 - e)n_2$ for some $n_1, n_2 \in M$. Then we have:

$$m = en_1 = (1 - e)n_2 = e^2n_1 = e(1 - e)n_2 = (e - e^2)n_2 = (e - e)n_2 = 0,$$

so we have shown that if $m \in eM \cap (1 - e)M$, $m = 0$, so $eM \cap (1 - e)M = 0$. And thus $M = eM \oplus (1 - e)M$ by definition. □

22. *Let R be a Principal Ideal Domain, let M be a torsion R-module, and let p be a prime in R (do not assume M is finitely generated, hence it need not have a nonzero annihilator). The* ***p-primary component of*** *M is the set of all elements of M that are annihilated by some positive power of p.*
   (a) *Prove that the p-primary component is a submodule.*
   Proof. Let N denote the p-primary component of M. Note that:
$$N = \{ m \in M : \exists k \in \mathbb{N}, p^k m = 0 \} .$$
   We apply the submodule criterion. Note that $N \neq \varnothing$ since $0 \in N$. Let $x, y \in N$, and let $r \in R$. Then we know $\exists k, l \in \mathbb{N}$ s.t. $p^k x = p^l y = 0$. Observe:
$$p^k p^l (x + ry) = p^l p^k x + rp^k p^l y = p^l 0 + rp^k 0 = 0,$$
   so we know $x + ry \in N$, hence by the submodule criterion, N is a submodule of M. □
   (b) *Prove that this definition of p-primary component agrees with the one given in Exercise 18 when M has a nonzero annihilator.*
   Proof. Assume M has a nonzero annihilator a, and this is the minimal such element. Then let p^α be a prime power factor in the prime factorization of a. Let:
$$N = \{ m \in M : \exists k \in \mathbb{N}, p^k m = 0 \} .$$
   In Exercise 18, the definition given for the annihilator of p^α is:
$$A = Ann_M(p^\alpha) = \{m \in M : p^\alpha m = 0\}.$$
   So clearly any element of A is in N; just let $k = \alpha$. So let $m \in N$. Then $\exists k \in \mathbb{N}$ s.t. $p^k m = 0$. Suppose $k > \alpha$. Then since $am = 0$, we must have some other product of primes $r = r_1 \cdots r_l \mid a$ s.t. $r \nmid p^\alpha$. But since we proved that N is a submodule in part (a), we know $Ann(N) = \{ r \in R : rm = 0, \forall m \in N \}$ is an ideal in R. Note then that $r, p^k \in Ann(N)$. But since $p^k \nmid r$ since otherwise we would have $p^k \mid a$, which is impossible since we said $r > \alpha$. So then $r \notin Rp^k$, hence $Ann(N)$ is not a principal ideal, but this is impossible, since we are in a PID, so we must have $k \leq \alpha$. Hence $m \in A$, and thus $N \subseteq A$, and the definitions are equivalent, because the sets are equal.
   □
   (c) *Prove that M is the (possibly infinite) direct sum of its p-primary components $\{ M_i \}$, as p runs over all primes of R.*
   Proof. Let $\{ p_i \}$ be all the primes in R. $\forall i$, let $a_i = \prod_{j \neq i} p_j^{r_j}$. Then $a_i M \subseteq M_i$, since $p_i^{r_i}(a_i M) = \prod_{j=1}^{\infty} p_j^{r_j} M = 0$ (since M is a torsion module, and hence $\forall m \in M$ there exists a nonzero $r \in R$ s.t. $rm = 0$, and the prime decomposition of r is in $\prod_{j=1}^{\infty} p_j^{r_j}$). Then:
$$gcd(a_1, a_2, ...) = 1,$$
   so there exists $c_1, c_2, ... \in R$ not necessarily all nonzero s.t. $c_1 a_1 + \cdots = 1$. So $\forall u \in M$,
$$u = \sum_{i=1}^{\infty} c_i a_i \in M_1 + M_2 + \cdots .$$
   Now let $u \in M_i \cap (\sum_{j \neq i} M_j)$. Then $p_i^{r_i}, a_i \in Ann(u)$. So, $(p_i^{r_i}) = (1) \subseteq Ann(u)$, so $u = 0$. So $\forall i, M_i \cap (\sum_{j \neq i} M_j) = 0$. So since we know $M = M_1 + M_2 + \cdots$, and the pairwise intersection of each of these is 0, we know that $M = M_1 \oplus M_2 \oplus \cdots$. □

0. *Let M be an R-module and let I, J be ideals in R.*
   (a) *Prove that $Ann(I + J) = Ann(I) \cap Ann(J)$.*
   Proof. Let $m \in Ann(I + J)$. Then $(i + j)m = 0$ for all $i \in I, j \in J$. Then letting $i = 0$, we know $m \in Ann(J)$, and letting $j = 0$, we know $m \in Ann(I)$. So $Ann(I + J) \subseteq Ann(I) \cap Ann(J)$. Now let $m \in Ann(I) \cap Ann(J)$. Then $im = 0, \forall i \in I$, and $jm = 0, \forall j \in J$. Then we have:
$$(i + j)m = im + jm = 0 + 0 = 0,$$
   by he definition of an R-module. So $Ann(I) \cap Ann(J) \subseteq Ann(I + J)$. Hence they are equal. □
   (b) *Prove that $Ann(I) + Ann(J) \subseteq Ann(I \cap J)$.*

Proof. Let $m \in Ann(I) + Ann(J)$. Then $m = n + k$ for some $n \in Ann(I), k \in Ann(J)$. Let $i \in I \cap J$. Then we know:

$$im = i(n + k) = in + ik = 0 + 0 = 0,$$

by the distributivity of the action of R on M, and since $i \in I$, and $i \in J$, and since n, k are in the respective annihilators. Thus $m \in Ann(I \cap J) \Rightarrow Ann(I) + Ann(J) \subseteq Ann(I \cap J)$. □

(c) *Give an example where the inclusion in part (b) is strict.*
Let R be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$. Note this is not an integral domain since we can construct zero divisors in the form of a pair piecewise functions, one of which is zero on half the interval, and the other being zero on the other half. We consider the R-module of R over itself. Then let I be the ideal of functions which are zero on $[0, 1/2]$, and J be the ideal of functions which are zero on $[1/2, 1]$. Now note that $I + J \neq R$ since $f(x) = 1$ is in R, but not in $I+J$, since all functions in $I+J$ are zero at $1/2$. But $I \cap J = 0$, since these functions must be zero across both halves, and so $Ann(I \cap J) = R$, and so $Ann(J) + Ann(I) = I + J \subsetneq R = Ann(I \cap J)$.

(d) *If R is commutative and unital and I, J are comaximal, prove that $Ann(I \cap J) = Ann(I) + Ann(J)$.*
Proof. Assume R is commutative and unital, and I, J are comaximal. Let $m \in Ann(I + J) = Ann((1)) = Ann(R)$ since I, J are comaximal, and R is commutative and unital. So $rm = 0$ for all $r \in R$. So then $m \in Ann(I)$, and since $0 \in Ann(J)$, we may write $m = m + 0$, so $m \in Ann(I) + Ann(J)$. And thus $Ann(I + J) \subseteq Ann(I) + Ann(J)$. So they are equal by the result of part (b). □