MATH 5576H FINAL REVIEW

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Exercise. *Prove that* $\forall m, n \in \mathbb{N}$, $a = m^2 - n^2$, $b = 2mn$, *and* $c = m^2 + n^2$ *give a Pythagorean triple, and that all primitive triples are of this form.*

Proof. Trivial to prove it is a triple.

One of a, b is odd, else all 3 are even, not primitive.

One of a, b is even, else $a^2, b^2 \equiv 1 \mod 4$.

Then $c^2 \equiv 2 \mod 4$, impossible since even squares are $0 \mod 4$.

Choose b even. Write:

$$b^2 = c^2 - a^2 = (c-a)(c+a).$$

Since a, c are both odd, $(a-c), (a+c)$ are both even.

We divide both sides by 4:

$$\frac{b^2}{4} = \left(\frac{b}{2}\right)^2 = \frac{(c-a)}{2}\frac{(c+a)}{2}.$$

$\frac{(c-a)}{2}, \frac{(c+a)}{2}$ are relatively prime, else their sum and difference a, c would have a common factor, which they don't because triple is primitive.

Since these two numbers are relatively prime and their product is a square, unique factorization (FTA) gives us:

$$\frac{(c-a)}{2} = n^2,$$
$$\frac{(c+a)}{2} = m^2,$$

for positive integers m, n. And m, n relatively prime.

Adding and subtracting the two above equations, we get:

$$m^2 - n^2 = a,$$
$$m^2 + n^2 = c,$$

So we have:

$$\left(\frac{b}{2}\right)^2 = n^2m^2,$$
$$b^2 = 4n^2m^2,$$
$$b = 2mn.$$

$\square$

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