MATH 5591H HOMEWORK 3

BRENDAN WHITAKER

Section 10.4 Exercises

9. *Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $Q \otimes_R N$ be the module obtained from N by extension of scalars from R to Q. Prove that the kernel of the R-module homomorphism $\iota : N \to Q \otimes_R N$ is the torsion submodule of N. [Exercise 10.1.**??**,Exercise 10.4.8]*

   Proof. Recall that the torsion submodule is defined as:
$$Tor(N) = \{n \in N : rn = 0 \text{ for some nonzero } r \in R\}.$$
   And recall that $\iota(n) = 1 \otimes n$. Let $n \in Tor(N)$. Then $\iota(n) = 1 \otimes n$. Since $n \in Tor(N)$, there exists $r \neq 0$ such that $rn = 0$, and we also have $1/r \in Q$. So we have:
$$1 \otimes n = 1(1 \otimes n) = \frac{1}{r} r(1 \otimes n) = \frac{1}{r}(1 \otimes rn) = \frac{1}{r}(1 \otimes 0) = 0.$$
   Thus $n \in ker\iota$, and $Tor(N) \subseteq ker\iota$. Now let $n \in ker\iota$. Then
$$\iota(n) = 1 \otimes n = 0 = 1 \otimes 0.$$
   So we must have that there exists $r \neq 0$ s.t. $rn = 0$. And by the result of Exercise 10.4.8(c), we know that $(1/d) \otimes n = 0$ if and only if there exists $r \in R$ s.t. $rn = 0$. Hence we know $n \in Tor(N)$. ☐

10. *Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, ..., e_n$.*

    Recall the definition of a free module of rank n:

Definition 1. A **free module** is a direct sum of finitely or infinitely many copies of R,
$$F_\Lambda = \bigoplus_{\alpha \in \Lambda} R = \{a_{\alpha_1} + \cdots + a_{\alpha_k} : k \in \mathbb{N}, \alpha_i \in \Lambda, a_{\alpha_i} \in R\},$$
where the sum of u's above is a formal sum. We can also define it as:
$$F_\Lambda = \{(a_\alpha)_{\alpha \in \Lambda} : a_\alpha \in R, \forall \alpha, a_\alpha = 0 \text{ for all but finitely many } \alpha\}.$$
Note R is unital here.

(a) *For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$, then $m_i = 0$ for $i = 1, ..., n$.*

Proof. Let $t = a_1(u_1 \otimes v_1) + \cdots + a_l(u_l \otimes v_l) \in M \otimes N$. And for each $v_i \in N$ we have:
$$v_i = r_1 e_1 + \cdots + r_n e_n,$$
with $r_j \in R$ uniquely by the definition of our standard basis. Then we may write:
$$\begin{aligned} t &= a_1(u_1 \otimes (r_{1,1}e_1 + \cdots + r_{1,n}e_n)) + \cdots + a_l(u_l \otimes (r_{l,1}e_1 + \cdots + r_{l,n}e_n)) \\ &= (a_1u_1 \otimes (r_{1,1}e_1 + \cdots + r_{1,n}e_n)) + \cdots + (a_lu_l \otimes (r_{l,1}e_1 + \cdots + r_{l,n}e_n)) \\ &= ((a_1u_1 \otimes r_{1,1}e_1) + \cdots + (a_1u_1 \otimes r_{1,n}e_n)) + \cdots + ((a_lu_l \otimes r_{l,1}e_1) + \cdots + (a_lu_l \otimes r_{l,n}e_n)) \\ &= ((a_1r_{1,1}u_1 \otimes e_1) + \cdots + (a_1r_{1,n}u_1 \otimes e_n)) + \cdots + ((a_lr_{l,1}u_l \otimes e_1) + \cdots + (a_lr_{l,n}u_l \otimes e_n)) \\ &= ((a_1r_{1,1}u_1 \otimes e_1) + \cdots + (a_lr_{l,1}u_l \otimes e_1)) + \cdots + ((a_1r_{1,n}u_1 \otimes e_n) + \cdots + (a_lr_{l,n}u_l \otimes e_n)) \\ &= ((a_1r_{1,1}u_1 \cdots + a_lr_{l,1}u_l) \otimes e_1) + \cdots + ((a_1r_{1,n}u_1 + \cdots + a_lr_{l,n}u_l) \otimes e_n). \end{aligned} \tag{1}$$
And since our expression for each v_i was unique by definition of a basis, this expression for t is unique.

Date: SP18.

So letting $m_i = (a_1 r_{1,i} u_1 \cdots + a_l r_{l,i} u_l)$, we set:

$$t = \sum_{i=1}^{n} m_i \otimes e_i = 0,$$

where $m_i \in M$. But note we also have:

$$\sum_{i=1}^{n} 0 \otimes e_i = 0.$$

So since we just proved the representation above is unique, we know we must have $m_i = 0$ $\forall i$. $\square$

(b) *Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent, then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]*

Proof. Note that now we relax the assumption that our elements from R^n generate R^n. So now they are only linearly independent. We have:

$$1 \otimes 2 = 2 \otimes 1 = 0 \otimes 1 = 0,$$

but $1 \neq 0 \in \mathbb{Z}/2\mathbb{Z}$, and 2 is just a single element of some R module over R, so it is linearly independent. So we have found a counterexample. $\square$

16. *Suppose R is commutative and let I and J be ideals of R, so $R/I, R/J$ are naturally R-modules .*

(a) *Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$.*

Proof. Let:

$$t = a_1(b_1 \mod I \otimes c_1 \mod J) + \cdots + a_l(b_l \mod I \otimes c_l \mod J) \in R/I \otimes_R R/J,$$

with $a_i, b_i, c_i \in R$. Then we have:

$$\begin{aligned} t &= a_1 b_1 (1 \mod I \otimes c_1 \mod J) + \cdots + a_l b_l (1 \mod I \otimes c_l \mod J) \\ &= (1 \mod I \otimes a_1 b_1 c_1 \mod J) + \cdots + (1 \mod I \otimes a_l b_l c_l \mod J) \\ &= 1 \mod I \otimes (a_1 b_1 c_1 + \cdots + a_l b_l c_l) \mod J, \end{aligned} \tag{2}$$

so since $(a_1 b_1 c_1 + \cdots + a_l b_l c_l) \in R$, we have written t as a simple tensor. $\square$

(b) *Prove that there is an R module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$ mapping $(r \mod I) \otimes (r' \mod J)$ to $rr' \mod (I+J)$.*

Proof. Let $\phi : R/I \otimes_R R/J \to R/(I+J)$ be given by $\phi((r \mod I) \otimes (r' \mod J)) = rr' \mod (I+J)$. We prove this is an isomorphism. Since we proved that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$, we need only to check elements of this form.

Homomorphism: We have:

$$\begin{aligned} &\phi((1 \mod I) \otimes (r \mod J) + (1 \mod I) \otimes (s \mod J)) \\ =&\phi((1 \mod I) \otimes (r + s \mod J)) \\ =&r + s \mod (I+J) \\ =&r \mod (I+J) + s \mod (I+J) \\ =&\phi((1 \mod I) \otimes (r \mod J)) + \phi((1 \mod I) \otimes (s \mod J)). \end{aligned} \tag{3}$$

So addition is preserved, and for $a \in R$, we also have:

$$\begin{aligned} \phi(a((1 \mod I) \otimes (r \mod J))) &= \phi(((a \mod I) \otimes (r \mod J))) \\ &= ar \mod (I+J) \\ &= a(r \mod (I+J)) \\ &= a\phi((1 \mod I) \otimes (r \mod J)). \end{aligned} \tag{4}$$

So ϕ is an R-module homomorphism.

Injectivity: Observe:

$$\phi((1 \mod I) \otimes (r \mod J)) = \phi((1 \mod I) \otimes (s \mod J)), \tag{5}$$

which gives us:

$$r \mod (I+J) = s \mod (I+J), \tag{6}$$

thus we know $r - s \in (I+J)$. So $r - s = j \mod I$ for some $j \in J$. So we have:

$$\begin{aligned} &(1 \mod I) \otimes (r \mod J) - (1 \mod I) \otimes (s \mod J) \\ =&(1 \mod I) \otimes (r-s \mod J) \\ =&(r-s \mod I) \otimes (1 \mod J) \\ =&(j \mod I) \otimes (1 \mod J) \\ =&(1 \mod I) \otimes (j \mod J) \\ =&0. \end{aligned} \tag{7}$$

So ϕ must be injective.

Surjectivity: Let $r \mod (I+J) \in R/(I+J)$. Then $\phi((1 \mod I) \otimes (r \mod J)) = r \mod (I+J)$, so ϕ is surjective. Hence ϕ is an isomorphism. □

20. *Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.*

Proof. Define $t = 2 \otimes 2 + x \otimes x$. We first express t as a simple tensor in R. We define $\beta : \mathbb{Z}[x] \times \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes \mathbb{Z}[x]$ given by $\beta((p(x), q(x)) = p(x) \otimes q(x)$. We also define $\gamma : \mathbb{Z}[x] \times \mathbb{Z}[x] \to \mathbb{Z}[x]$ given by $\gamma((p(x), q(x)) = p(x)q(x)$. This map is bilinear, so we have an induced homomorphism $\varphi : \mathbb{Z}[x] \otimes \mathbb{Z}[x] \to \mathbb{Z}[x]$, so altogether, we have:

$$\begin{tikzcd} & \mathbb{Z}[x] \times \mathbb{Z}[x] \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\ \mathbb{Z}[x] \otimes \mathbb{Z}[x] \arrow[rr, "\varphi"] & & \mathbb{Z}[x] \end{tikzcd}.$$

Then we would have:

$$p \otimes q = 2 \otimes 2 + x \otimes x,$$

for some $p, q \in \mathbb{Z}[x]$. But we also know:

$$2 \otimes 2 + x \otimes x = 4(1 \otimes 1) + x \otimes x = 4(1 \otimes 1) + x^2(1 \otimes 1) = (4 + x^2)(1 \otimes 1) \in \mathbb{Z}[x]$$

But $(4 + x^2)$ is a prime in $\mathbb{Z}[x]$. To write t as a simple tensor in $\mathbb{Z}[x]$, we must have $4 + x^2 = ab$ for some $a, b \in \mathbb{Z}[x]$, so that we may write:

$$ab(1 \otimes 1) = a \otimes b \in \mathbb{Z}[x].$$

So let $4 + x^2 = ab$, and since it is a prime and we are in $\mathbb{Z}[x]$, without loss of generality, we must have $b = 1$, but note that $1 \notin I$, so it is impossible to write t as a simple tensor in $I \otimes_R I$, since under the same bilinear map γ, we have:

$$\begin{tikzcd} & I \times I \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\ I \otimes I \arrow[rr, "\phi"] & & I^2 \end{tikzcd},$$

from which we see that the image $u \otimes v \mapsto uv$ of any simple tensor is reducible. □

21. *Suppose R is commutative, and let I and J be ideals of R.*
   (a) *Show that there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $i \otimes j$ to the element ij.*
   Proof. Let $\phi : I \otimes_R J \to IJ$ be given by:

$$\phi(r_1(i_1 \otimes j_1) + \cdots + r_n(i_n \otimes j_n)) = r_1 i_1 j_1 + \cdots + r_n i_n j_n.$$

We show that ϕ is a surjective homomorphism of R-modules. Observe:

$$\begin{aligned} &\phi((r_1(i_1 \otimes j_1) + \cdots r_n(i_n \otimes j_n)) + (s_1(i'_1 \otimes j'_1) + \cdots s_m(i'_m \otimes j'_m))) \\ =&\phi(r_1(i_1 \otimes j_1) + \cdots r_n(i_n \otimes j_n) + s_1(i'_1 \otimes j'_1) + \cdots s_m(i'_m \otimes j'_m)) \\ =&r_1 i_1 j_1 + \cdots + r_n i_n j_n + s_1 i'_1 j'_1 + \cdots + s_m i'_m j'_m \\ =&\phi((r_1(i_1 \otimes j_1) + \cdots r_n(i_n \otimes j_n)) + \phi((s_1(i'_1 \otimes j'_1) + \cdots s_m(i'_m \otimes j'_m))). \end{aligned} \tag{8}$$

So ϕ preserves addition. Additionally:

$$\begin{aligned} \phi(r(i \otimes j)) &= \phi((ri \otimes j)) \\ &= rij \\ &= r\phi((i \otimes j)). \end{aligned} \tag{9}$$

So ϕ also preserves scalar multiplication for simple tensors and thus for general tensors as well. Now we show that ϕ is surjective. Let $r \in IJ$. Then

$$r = \sum_{k=1}^{n} i_k j_k,$$

for $i_k \in I, j_k \in J$. Then $\phi(i_1 \otimes j_1 + \cdots + i_n \otimes j_n) = r$, because we already proved ϕ is a homomorphism and hence preserves addition, so ϕ is surjective. □

(b) *Give an example to show that the map in (a) need not be injective [Exercise 10.4.17].*
Consider $I = (2, x)$ and $R = \mathbb{Z}[x]$. We define a map: $\phi : I \otimes_R I \to II = I$ given by $\phi(i \otimes j) = ij$. By part (a), we know it is a surjective homomorphism. Note:

$$\phi(2 \otimes x) = \phi(x \otimes 2) = 2x.$$

But from Exercise 10.4.17(c), we know that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.