MATH 5591H HOMEWORK 7

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12.1 Exercises

2. *$B = \{ x_1, ..., x_n \}$ be a maximal linearly independent set in M if and only if RB is free and M/RB is torsion module.*

   Proof. (a) *$\{ x_1, ..., x_n \}$ is linearly independent if and only if $R\{ x_1, ..., x_n \}$ is a free module with basis $\{ x_1, ..., x_n \}$.*

   Proof. Professor Leibman completed this proof in class. □

   (b) Let $\{ x_1, ..., x_n \}$ be a maximal linearly independent set. Let $y \in M$. Then $\exists a_1, ..., a_n, b$ s.t. $a_1x_1 + \cdots + a_nx_n + by = 0$ and not all of $a_1, ..., a_n, b$ are zero. If $b = 0$, then $a_1x_1 + \cdots + a_nx_n = 0$, this is impossible, since $x_1, ..., x_n$ are linearly independent. So $b \neq 0$, and $by = 0$ mod $R\{ x_1, ..., x_n \}$. So $b\overline{y} = 0 \in M/R\{ x_1, ..., x_n \}$. So $\forall \overline{y} \in M/R\{ ... \}$, $\exists b \neq 0$ s.t. $b\overline{y} = 0$. Now we prove in the other direction. Assume that $M/R\{ x_1, ..., x_n \}$ is a torsion module. Take $\forall y \in M$. Find $b \neq 0$ s.t. $b\overline{y} = 0$, that is, $by \in R\{ x_1, ..., x_n \}$. So $by = a_1x_1 + \cdots + a_nx_n$ for some a_i, so $y, x_1, ..., x_n$ are linearly dependent, so $\{ x_1, ..., x_n \}$ is a maximal linearly independent set. We know this since we proved we could not add any other linearly independent element without making the whole set dependent. So it's maximal.

   □

4. *Let R be an integral domain, let M be an R-module and let N be a submodule of M. Suppose M has rank n, N has rank r and the quotient M/N has rank s. Prove that $n = r + s$.* Use:

$$0 \to N \to M \to M/N \to 0.$$

Multiply tensor by field of fractions. Use $rank(M) = rank(N) + rank(M/N)$.

Proof. Let $A = \{ x_1, ..., x_s \}$, a set of elements in M whose images are a maximal independent set in M/N. And let $B = \{ x_{s+1}, ..., x_{s+r} \}$ be a maximal independent set in N. We prove A is independent in M. Suppose it weren't. Then there is $l \neq 0$ in R and $x_i \in A$ s.t. $lx_i = \sum_{j \neq i, \leq s} r_jx_j$. But then under the natural projection we would have a similar equality for $\overline{x_i}$ which would contradict the independence of $\overline{A}$.

We wish to show that $A \cup B$ is a maximal linearly independent set. We first show it is independent. Let $x_i \in A$. Suppose there exists a nonzero $l \in R$ s.t. $lx_i = r_{s+1}x_{s+1} + \cdots + r_{s+r}x_{s+r}$ for $r_i \in R$. Then under the natural projection $\pi : M \to M/N$, we have $\pi(lx_i) = l\pi(x_i) = 0 \in M/N$. But note $\pi(x_i)$ is in $\overline{A}$ which is an independent set in M/N so we must have $\pi(x_i) \neq 0$ and that $\nexists l \in R$ s.t. $l\pi(x_i) = 0$. This is a contradiction, so we must have that there exists no such l, so every element in A is independent of B. Now let $x_j \in B$ and suppose there exists a nonzero $l \in R$ s.t. $lx_j = r_1x_1 + \cdots + r_sx_s$. Then π maps this to $0 \in M/N$ since $lx_j \in N$, but then since $\overline{A}$ is independent in M/N, we must have $r_1 = \cdots = r_s = 0$. Then we have $lx_j = 0$ which is a contradiction since B cannot contain any torsion elements or it would not be independent. Then we have proved $A \cup B$ is independent.

Now we show $A \cup B$ is maximal. Let $y \in M$. Then since $\overline{A}$ is a maximal linearly independent set in M/N, we know there exist $c, c_1, ..., c_s$ not all zero such that:

$$c\overline{y} + c_1\overline{x_1} + \cdots + c_s\overline{x_s} = 0,$$

which implies:

$$cy + c_1x_1 + \cdots + c_sx_s = n \in N.$$

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Now since B is a maximal linearly independent set in N, we know that since $n \in N$, there exists $k, c_{s+1}, ..., c_{s+r} \in R$ not all zero s.t.

$$kn = k(cy + c_1x_1 + \cdots + c_sx_s) = c_{s+1}x_{s+1} + \cdots + c_{s+r}x_{s+r}.$$

But if $k = 0$, then we must have $c_{s+1}, .., c_{s+r} = 0$ since B is independent. So we must have $k \neq 0$, thus we can write:

$$kcy = \sum_{i=1}^{s} -kc_ix_i + \sum_{i=s+1}^{s+r} c_ix_i.$$

And since we know $c, c_1, ..., c_s$ are not all zero, we have found a nonzero $kc \in R$ (since we are in an ID) s.t. kcy is a linear combination of $x_1, ..., x_{s+r}$. So we have shown that $A \cup B$ is a maximal independent set in M, since for any $y \in M$ there is kc s.t. kcy is a combination of elements in $A \cup B$.

Now we wish to show that $rank(M) = n = r + s$. So we use part (b) of Exercise 2 above. Note that R^{r+s} is a submodule of M, since $x_1, ..., x_{s+r} = A \cup B$ is a maximal linearly independent set in M, and $R(A \cup B) = R^{r+s}$, and we have closure by ring action since M is an R-module.

Lemma 1. *If $\{ u_1, ..., u_n \}$ is a maximal linearly independent set, it doesn't have to generate M, but $M/R\{ u_1, ..., u_n \}$ is a torsion module, because otherwise we could add one more element to this set and it would still be linearly independent.*

Proof. Suppose $M/R\{ u_1, ..., u_n \}$ is not torsion. Then $\exists u' \in M/R\{ u_1, ..., u_n \}$ s.t. $ru' \neq 0 \in M/R\{ u_1, ..., u_n \}$ (i.e. $ru' \notin R\{ u_1, ..., u_n \}$) for all $r \in R$. But this is exactly the definition of linear independence, so then $\{ u_1, ..., u_n, u' \}$ is independent, which is a contradiction since we said $\{ u_1, ..., u_n \}$ was maximal. □

So by the above Lemma, we know M/R^{r+s} is torsion. Then by Exercise 2 part (b), we know $rank(M) = n = r + s$. □

5. *Consider $\mathbb{Z}[x] \sim F[x, y]$. Note $(2, x)$ is not principal.* Note M has rank 1, is torsion free, but not free. It has rank 1 because if you take one of these elements, something linearly dependent maybe, idk. Consider $M/(2)$ then x is a torsion element here since $2x = 0$. So it's a torsion module or something. And actually, it's true for any module over PID.
9. *Give an example of an integral domain R and a nonzero torsion R-module M such that $Ann(M) = 0$. Prove that if N is a finitely generated torsion R-module, then $Ann(N) \neq 0$.*

   Let $R = \mathbb{Z}$, an integral domain. Define:

   $$M = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}.$$

   Then $\forall a \in M$, $\exists k \in \mathbb{Z}$ such that:

   $$a = (a_1 + \mathbb{Z}/2\mathbb{Z}, ..., a_k + \mathbb{Z}/2^k\mathbb{Z}, 0, ...)$$

   for some $a_1, ..., a_k \in \mathbb{Z}$. Thus $2^k a = 0 \in M$, so M is a torsion module. We claim that $\mathrm{Ann}(M) = 0$. Suppose there exists a nonzero $r \in \mathbb{Z}$ s.t. $r \in Ann(M)$. Then choose $k \in \mathbb{Z}$ s.t. $r < 2^k$. Then define:

   $$a = (0, ..., 0, 1 + \mathbb{Z}/2^k\mathbb{Z}, 0, ...)$$

   where the nonzero entry is in the k-th position. Then since $ra = 0$, we must have $r = 0$ since r will not annihilate the nonzero entry of a since $r < 2^k$. This is a contradiction since we said $r \neq 0$. So we must have $\mathrm{Ann}(M) = 0$.

   Proof. Let R be a integral domain. Let N be finitely generated torsion R-module. Then $N \subseteq R\{ x_1, ..., x_n \}$. And since it is torsion, there exist $\{ r_1, ..., r_n \}$ s.t. $r_ix_i = 0$, where $r_i \neq 0\ \forall i$. Then since we have no zero divisors, $lcm(r_1, ..., r_n) \neq 0$, and this is in the annihilator by commutativity in R. □
11. *Let R be a PID, let a be a nonzero element of R and let $M = R/(a)$. For any prime p of R, prove that:*

   $$p^{k-1}M/p^kM \cong \begin{cases} R/(p) & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases},$$

   where n is the power of p dividing a in R.

Proof. We first treat the case where $p \nmid a$. Then since p is a prime in R, we know $\gcd(a, p) = 1$. So then we have $(p) \cap (a) = 0$. let $\pi : R \to R/(a) = M$. Then observe:

$$\pi((p)) = (p)/(a) \cong [(p) + (a)]/(a) \cong (p)/((p) \cap (a)) \cong (p)/(0) \cong (p).$$

But note that $(p)+(a) = (1) = R$, so we have shown $(p) = pM \cong R/(a) = M$, so $p^{k-1}M = p^kM = M$ for all k, and thus since $M/M \cong 0$, we have the desired result.

Now let $p \mid a$, and assume $k \leq n$. Then we have $a = p^n p_1^{c_1} \cdots p_l^{c_l}$, for some distinct primes p_i. Using the result of Exercise 12.1.7 and the Chinese remainder theorem, we have:

$$\begin{aligned}
\frac{p^{k-1}M}{p^kM} &= \frac{p^{k-1}R/(a)}{p^kR/(a)} \\
&\cong \frac{p^{k-1}R/(p^n)(p_1^{c_1})\cdots(p_l^{c_l})}{p^kR/(p^n)(p_1^{c_1})\cdots(p_l^{c_l})} \\
&\cong \frac{R/(p^{n-k+1})(p_1^{c_1})\cdots(p_l^{c_l})}{R/(p^{n-k})(p_1^{c_1})\cdots(p_l^{c_l})} \\
&\cong \frac{R/(p^{n-k+1}) \oplus R/(p_1^{c_1}) \oplus \cdots \oplus R/(p_l^{c_l})}{R/(p^{n-k}) \oplus R/(p_1^{c_1}) \oplus \cdots \oplus R/(p_l^{c_l})} \qquad (1) \\
&\cong (R/(p^{n-k+1}))/(R/(p^{n-k})) \oplus (R/(p_1^{c_1}))/(R/(p_1^{c_1})) \\
&\oplus \cdots \oplus (R/(p_l^{c_l}))/(R/(p_l^{c_l})) \\
&\cong (R/(p^{n-k+1}))/(R/(p^{n-k})) \oplus 0 \oplus \cdots \oplus 0 \\
&\cong (R/(p^{n-k+1}))/(R/(p^{n-k})) \\
&\cong R/(p).
\end{aligned}$$

Now suppose $k > n$. Then $a | p^{k-1} \Rightarrow p^{k-1}M \cong raR/(a) \cong 0$. □

12. *Let R be a PID and let p be a prime in R.*

(a) *Let M be a finitely generated torsion R-module. Use the previous exercise to prove that $p^{k-1}M/P^kM \cong F^{n_k}$ where F is the field $R/(p)$ and n_k is the number of elementary divisors of M which are powers p^α with $\alpha \geq k$.*

Proof. Recall that a module over a PID is free if and only if it is torsion free, so since M is not torsion free, it is not free, and by Theorem 6, we have:

$$M \cong R^r \oplus R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_l^{\alpha_l}),$$

where the primes are not necessarily distinct, and all the α's are positive. But then by Theorem 5, since M is torsion, we know $r = 0$. So we have:

$$M \cong R/(p_1^{\alpha_1}) \oplus \cdots \oplus R/(p_l^{\alpha_l}).$$

Define $a = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$. Now we apply the result of the previous exercise to each of these summands. Let s be the power of p dividing $p_i^{\alpha_i}$. We set $M' = R/(p_i^{\alpha_i})$. So we know:

$$p^{k-1}M'/P^kM' \cong \begin{cases} R/(p) & \text{if } k \leq s \\ 0 & \text{if } k > s \end{cases},$$

So we have that $k \leq s$ for exactly n_k of the elementary divisors $p_i^{\alpha_i}$, and so each of these summands is isomorphic to F, and the rest are zero. So we have:

$$M \cong F \oplus \cdots \oplus F \cong F^{n_k}.$$

□

(b) *Suppose M_1 and M_2 are isomorphic finitely generated torsion R-modules. Use (a) to prove that, for every $k \geq 0$, M_1 and M_2 have the same number of elementary divisors p^α with $\alpha \geq k$. Prove that this implies M_1 and M_2 have the same set of elementary divisors.*

Proof. Applying part (a), we have:

$$F^{n_{k_1}} \cong F^{n_{k_2}}.$$

which tells us $n_{k_1} = n_{k_2}$ since they are isomorphic vector spaces of those dimensions. And we are done, since we iterate over the list of primes p_i in the list of elementary divisors $\{ p_i^{\alpha_i} \}$, and also iterate over k from zero to α_i for each p_i, and observe that we have exactly the same elementary divisors for M_1 and M_2 by induction. $\square$