CSE 2321 HOMEWORK 10

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1. (a) $f(n) = 4096$; $g(n) = log_2(2048)$. Let $c_1 = c_2 = \frac{4096}{log_2(2048)}$. Then $c_1 g(n) = c_2 g(n) = 4096$. Hence $c_1 g(n) \leq f(n) \leq c_2 g(n)$ $\forall n \in \mathbb{N}$ since $f(n) = 4096$, hence $f(n) \in \Theta(g(n))$.

   (b) $f(n) = log_2(n^2)$; $g(n) = n^2 log_2(n)$. Note $f(n) = 2log_2(n)$. Let $c = 2$. Then note that $f(n) \leq cg(n)$ $\forall n \in \mathbb{N}$, since $2log_2(n) \leq 2n^2 log_2(n)$ $\forall n \in \mathbb{N}$. We prove this by induction.

   Proof. Let $n = 1$. Then $2log_2(1) = 0 \leq 0 = 2 \cdot 1^2 log_2(1)$. So the base case holds. Now assume for some fixed $k \in \mathbb{N}$ that $2log_2(k) \leq 2k^2 log_2(k)$. Then we must have that $1 \leq k^2$. Then note that $2log_2(k+1) \leq 2(k+1)^2 log_2(k+1) \Leftrightarrow 1 \leq (k+1)^2 = k^2 + 2k + 1$, but we know that $1 \leq k^2$, and $2k + 1 \geq 1$, so this must be true. Hence the claim holds. □
   Thus $f(n) \in O(g(n))$ by definition.

   (c) $f(n) = 3^{n+1}$; $g(n) = 3^n$. Note that $f(n) = 3^{n+1} = 3 \cdot 3^n = 3g(n)$. So since $g(n) \leq 3g(n) = f(n) \leq 6g(n)$ $\forall n \in \mathbb{N}$, we have that $f(n) \in \Theta(g(n))$.

   (d) $f(n) = 2^n$; $g(n) = 4^n$. Note $g(n) = 4^n = (2^2)^n = 2^{2n} = (2^n)^2 = (f(n))^2$. So $x \leq x^2$ for all positive values of x and $f(n) = 2^n$ is positive for all $n \in \mathbb{N}$, we must have that $f(n) \leq (f(n))^2 = g(n)$ $\forall n \in \mathbb{N} \Rightarrow f(n) \in O(g(n))$ by definition.

2. (a)

$$
\begin{aligned}
T(n) &= c + 3T(n/3) \\
&= c + 3c + 3^2 T(n/3^2) \\
&= c + 3c + 3^2 c + 3^3 T(n/3^3) \\
&= c + 3c + 3^2 c + 3^3 c + \cdots + 3^{log_3(n)-1} c + 3^{log_3(n)} T(1) \\
&= c + 3c + 3^2 c + 3^3 c + \cdots + 3^{log_3(n)-1} c + c' n \\
&= c + \sum_{i=1}^{\lfloor log_3(n) \rfloor - 1} 3^i c \\
&= c + 3 \frac{1 - 3^{log_3(n)-1}}{1-3} \\
&= c + \frac{n-3}{2} \in \Theta(n).
\end{aligned} \tag{1}
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(b)

$$
\begin{aligned}
T(n) &= cn^2 + T(n-1) \\
&= cn^2 + c(n-1)^2 + T(n-2) \\
&= cn^2 + c(n-1)^2 + c(n-2)^2 + T(n-3) \\
&= cn^2 + c(n-1)^2 + c(n-2)^2 + \cdots + c(n-(n-1))^2 + T(0) \\
&= c + c(2)^2 + c(3)^2 + \cdots + c(n-2)^2 + c(n-1)^2 + cn^2 + c' \\
&= c' + c\sum_{i=1}^{n} i^2 \\
&= c' + c\left(\sum_{i=1}^{n}(i^2)\right) \\
&= c' + c\left(\frac{n(2n+1)(n+1)}{6}\right) \\
&= c' + \frac{2cn^3 + 3cn^2 + cn}{6} \in \Theta(n^3).
\end{aligned} \tag{2}
$$

(c)

$$
\begin{aligned}
T(n) &= clog_2(n) + T(n-1) + T(n-4) \\
&\geq clog_2(n) + 2T(n-4) \\
&= clog_2(n) + 2clog_2(n-4) + 2^2T(n-8) \\
&= clog_2(n) + 2clog_2(n-4) + 2^2clog_2(n-8) + \cdots + 2^{\frac{n}{4}-1}2c + 2^{\frac{n}{4}}T(n-4\frac{n}{4}) \\
&\geq 2^{\frac{n}{4}}T(0)
\end{aligned} \tag{3}
$$

$c'2^{\frac{n}{4}} \in \Omega(2^{\frac{n}{4}})$.

So since $T(n) \in \Omega(2^{\frac{n}{4}})$, the running time has an exponential lower bound.

3. .

4. The inner for loop executes $n-i$ times and the outer for loop executes n times. Let the running time for the two for loops be given by $F(n)$. Then for some constant c, we have:

$$\begin{aligned} F(n) &= \sum_{i=1}^{n}\sum_{j=i}^{n-1} c = c\sum_{i=1}^{n}(n-i) = c\left(\sum_{i=1}^{n} n - \sum_{i=1}^{n} i\right) \\ &= c\left(n^2 - \frac{n^2+n}{2}\right) \in \Theta(n^2). \end{aligned} \tag{4}$$

Thus, for some constant k, the recurrence relation is:

$$\begin{aligned} T(n) &= kn^2 + T(n-5) \\ &= kn^2 + k(n-5)^2 + T(n-2\cdot 5) \\ &= kn^2 + k(n-5)^2 + k(n-10)^2 + T(n-3\cdot 5) \\ &= kn^2 + k(n-5)^2 + k(n-10)^2 + \cdots + k(n-(\frac{n}{5}-1)5)^2 + T(n-\frac{n}{5}5) \\ &= kn^2 + k(n-5)^2 + k(n-10)^2 + \cdots + k(5)^2 + T(0) \\ &= T(0) + \sum_{i=1}^{\lfloor n/5 \rfloor} k(5i)^2 \\ &= T(0) + 25k\frac{(n/5)((n/5)+1)(2(n/5)+1)}{6} \in \Theta(n^3). \end{aligned} \tag{5}$$