CSE 2331 HOMEWORK 3

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1. **This one seems sketchy.** The inner while loop executes $\lfloor \log_5(\lfloor n/2 \rfloor/7) \rfloor$ times, which takes $c\log(n)$ time for some constant c. The for loop executes $\lfloor n/2 \rfloor$ times, which takes $c_1 n$ time for some constant c_1. Thus steps 2-8 take $c_2 n \log(n)$ time for some constant c_2. So our recurrence relation is given by $T(n) = cn\log(n) + T(n-8)$. Thus

$$\begin{aligned}
T(n) &= cn\log(n) + T(n-8) \\
&= cn\log(n) + c(n-8)\log(n-8) + T(n-2\cdot 8) = \cdots \\
&= cn\log(n) + \cdots + c(n-n)\log(n-n) + T(1) \\
&= cn\log(n) + \sum_{k=1}^{\lfloor n/8 \rfloor} c(n-8i)\log(n-8i) + T(1) \\
&\leq cn\log(n) + \sum_{k=1}^{\lfloor n/8 \rfloor} cn\log(n) + T(1) \\
&= cn\log(n) + (n/8)cn\log(n) + T(1) \in O(n^2\log(n)), \\
T(n) &\geq cn\log(n) + \sum_{k=\lfloor n/16 \rfloor}^{\lfloor n/8 \rfloor} c(n-8i)\log(n-8i) + T(1) \\
&\geq cn\log(n) + \sum_{k=\lfloor n/16 \rfloor}^{\lfloor n/8 \rfloor} c(n-8(n/16))\log(n-8(n/16)) + T(1) \\
&= cn\log(n) + (n/16)c(n/2)\log(n/2) + T(1) \in \Omega(n^2\log(n)).
\end{aligned} \tag{1}$$

Thus $T(n) \in \Theta(n^2\log(n))$.

2. The for loop executes $\lfloor n/2 \rfloor$ times and thus takes cn time for some constant c. So our recurrence relation is given by $T(n) = cn + T(\frac{3}{5}n)$. So the running time is

$$\begin{aligned}
T(n) &= cn + T(\frac{3}{5}n) \\
&= cn + c\frac{3}{5}n + T(\left(\frac{3}{5}\right)^2 n) \\
&= cn + c\frac{3}{5}n + \cdots + c\left(\frac{3}{5}\right)^{\lfloor \log_{3/5}(n) \rfloor} n + T(1) \\
&= cn + \sum_{k=1}^{\lfloor \log_{3/5}(n) \rfloor} cn\left(\frac{3}{5}\right)^i + T(1) \\
&\leq cn + \sum_{k=1}^{\infty} cn\left(\frac{3}{5}\right)^i + T(1) \\
&= cn\frac{1}{1-\frac{3}{5}} + T(1) \in O(n), \\
T(n) &\geq cn \in \Omega(n).
\end{aligned} \tag{2}$$

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Hence $T(n) \in \Theta(n)$.

3. The inner for loop executes $n-10$ times and thus takes cn time for some constant cn. And the outer for loop executes 5 times and thus takes constant time for some constant $c_1 = 5$. So our recurrence relation is given by $T(n) = 5cn + 5T(\lfloor n/5 \rfloor)$. Thus our running time is given by

$$\begin{aligned}
T(n) &= 5cn + 5T\left(\lfloor n/5 \rfloor\right) \\
&= 5cn + 5cn + 5^2 T\left(\lfloor \frac{n}{5^2} \rfloor\right) \\
&= 5cn + 5cn + 5cn + 5^3 T\left(\lfloor \frac{n}{5^3} \rfloor\right) \\
&= \lfloor \log_5(n) \rfloor 5cn + 5^{\lfloor \log_5(n) \rfloor} T(1) \in \Theta(n \log(n)).
\end{aligned} \tag{3}$$

4. The first for loop executes $n-10$ times, and so takes cn time for a constant c. And since $\log_{3/2}(n)$ grows faster than $\log_{7/5}(n)$ to find the worst case running time, we assume the if statement always executes line 6, and so our recurrence relation is given by $T(n) = cn + T(2n/3)$. So the running time is:

$$\begin{aligned}
T(n) &= cn + T(2n/3) \\
&= cn + \left(\frac{2}{3}\right) cn + T\left(\frac{2^2 n}{3^2}\right) \\
&= cn + \left(\frac{2}{3}\right) cn + \left(\frac{2}{3}\right)^2 cn + \cdots + T\left(\log_{2/3}(n)\right) \\
&\leq cn \frac{1}{1-\frac{2}{3}} \in O(n). \\
T(n) &\geq cn \in \Omega(n).
\end{aligned} \tag{4}$$

Hence $T(n) \in \Theta(n)$.

5. The innermost for loop executes $\lfloor \sqrt{n} \rfloor$ times, the middle and outer for loops each execute $\lfloor n/3 \rfloor$ times, hence lines 2-8 take $cn^{2.5}$ time for some constant c. So our recurrence relation is given by $T(n) = cn^{2.5} + T(3n/4)$, and the running time is:

$$\begin{aligned}
T(n) &= cn^{2.5} + T(3n/4) \\
&= cn^{2.5} + \left(\frac{3}{4}\right)^{1\cdot 2.5} cn^{2.5} + T\left(\frac{3^2 n}{4^2}\right) \\
&= cn^{2.5} + \left(\frac{3}{4}\right)^{1\cdot 2.5} cn^{2.5} + \left(\frac{3}{4}\right)^{2\cdot 2.5} cn^{2.5} + \cdots + T\left(\log_{3/4}(n)\right) \\
&\leqslant cn^{2.5}\left(1 + \left(\frac{3}{4}\right)^{2.5} + \left(\frac{3}{4}\right)^{2\cdot 2.5} + \cdots\right) \\
&= cn^{2.5} \frac{1}{1-\left(\frac{3}{4}\right)^{2.5}} \in O(n^{2.5}). \\
T(n) &\geqslant cn^{2.5} \in \Omega(n^{2.5}).
\end{aligned} \tag{5}$$

Hence $T(n) \in \Theta(n^{2.5})$.

6. **I DON'T UNDERSTAND THIS ONE.** The recurrence relation is given by:

$$\begin{aligned}
T(n) &= cn + T(n-4) + T(n-10) + T(n-16) + \cdots + T(1) \\
&\geqslant T(n-4) + T(n-10) \geq 2T(n-10) \\
&\geqslant 2^2 T(n - 2\cdot 10) \geqslant 2^3 T(n - 3\cdot 10) \geqslant \cdots \geqslant 2^{\frac{n-1}{10}} T(n - \frac{n-1}{10} 10) \\
&= 2^{\frac{n-1}{10}} T(1) \in \Omega(2^{\frac{n}{10}}).
\end{aligned} \tag{6}$$

So the running time $T(n)$ has an exponential lower bound.

7. Lines 2-4 execute $n-8$ times and so take cn time for a constant c. The for loop executes 4 times and thus our recurrence relation is given by $T(n) = cn + 4T(n/2)$. So the running time is:
$$\begin{aligned} T(n) &= cn + 4T(n/2) \\ &= cn + \frac{4}{2}cn + 4T(n/2^2) \\ &= cn + \frac{4}{2}cn + \frac{4}{2^2}cn + 4T(n/2^3) \\ &= cn + \frac{4}{2}cn + \frac{4}{2^2}cn + \cdots + 4T(1) \\ &\leqslant 4cn\left(\frac{1}{1-\frac{1}{2}}\right) + 4T(1) \in O(n). \\ T(n) &\geqslant cn \in \Omega(n). \end{aligned} \tag{7}$$
Thus $T(n) \in \Theta(n)$.

8. The for loop executes $\lfloor n/2 \rfloor$ times, and so takes cn time for a constant c. And thus our running time is given by:
$$\begin{aligned} T(n) &= cn + T(n/6) + T(5n/6) \\ &\leq cn + 2T(5n/6) \\ &= cn + \frac{5}{6}cn + T(c5^2/6^2) \\ &= cn + \frac{5}{6}cn + \frac{5^2}{6^2}cn + T(c5^3/6^3) \\ &= cn + \frac{5}{6}cn + \frac{5^2}{6^2}cn + \cdots + T(1) \\ &\leq cn + \frac{5}{6}cn + \frac{5^2}{6^2}cn + \cdots \\ &= cn\frac{1}{1-\frac{5}{6}} \in O(n). \\ T(n) &\geq cn + T(n/6) \\ &= cn + \frac{1}{6}cn + T(c/6^2) \\ &= cn + \frac{1}{6}cn + \frac{1}{6^2}cn + T(c/6^3) \\ &= cn + \frac{1}{6}cn + \frac{1}{6^2}cn + \cdots + T(1) \\ &\geq cn \in \Omega(n). \end{aligned} \tag{8}$$
Hence $T(n) \in \Theta(n)$.

9. The inner for loop executes $\lfloor n/2 \rfloor$ times and takes cn time for a constant c. The while loop executes $\lfloor \log_2(n-1/9) \rfloor$. Thus the recurrence relation is given by:
$$\begin{aligned} T(n) &= cn\left(T(n-9) + T(n-18) + T(n-36) + \cdots + T(n-9\cdot 2^k)\right) \\ &\geq T(n-9) + T(n-18) \geq T(n-18) + T(n-18) = 2T(n-18) \\ &= 2^2T(n-36) = 2^3T(n-3\cdot 18) \\ &= 2^{n/18}T(1) \in \Omega(2^{n/18}). \end{aligned} \tag{9}$$
Since $T(n) \in \Omega(2^{n/18})$, the running time has an exponential lower bound.

10. The inner for loop executes $\lfloor n/2 \rfloor$ times and takes cn time for a constant c. Our recurrence relation is $T(n) = 5cn + 5T(n/6)$. Hence the running time is:

$$\begin{aligned}
T(n) &= 5cn + \frac{5^2}{6}cn + 5^2 T(n/6^2) \\
&= 5cn + \frac{5^2}{6}cn + \frac{5^3}{6^2}cn + 5^3 T(n/6^3) \\
&= 5cn + \frac{5^2}{6}cn + \frac{5^3}{6^2}cn + \cdots + 5^{\log_6(n)} T(1) \\
&= 5\left(cn + \frac{5}{6}cn + \frac{5^2}{6^2}cn + \cdots + 5^{\log_6(n)-1} T(1)\right) \\
&\leq 5\left(cn + \frac{5}{6}cn + \frac{5^2}{6^2}cn + \cdots\right) \\
&= 5cn\frac{1}{1-\frac{5}{6}} \in O(n). \\
T(n) &\geq 5cn \in \Omega(n).
\end{aligned} \tag{10}$$

Thus $T(n) \in \Theta(n)$.