# MATH 5590H FINAL THEOREMS

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**Theorem 1.** *$Inn(G) \cong G/Z(G)$.*

**Theorem 2.** *$F[x]$ is an ED.*

**Theorem 3.** *$F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.*

**Theorem 4.** *Ways to prove a group is abelian:*
1. *Show that the commutator $xyx^{-1}y^{-1}$ of any two elements is trivial.*
2. *Show the group is a direct product of abelian groups.*

**Theorem 5.** *If $P \cap Q = 1$, and $|PQ| = |G|$, then $PQ = G$.*

**Theorem 6.** *If $P \trianglelefteq G$, $P \cap Q = 1$, and $PQ = G$, then $P \rtimes Q = G$.*

**Theorem 7.** *If $P, Q$ are sylow $p, q$-subgroups of a group $G$ with only two distinct prime factors, and $n_p = 1$, then $P \rtimes Q = G$.*

**Theorem 8.** *If $\mathbb{Z}_n = \mathbb{Z}_m \times \mathbb{Z}_k$ and $m, k$ are relatively prime, then we must have $\mathbb{Z}_n = \mathbb{Z}_{mk}$, and if $(m, k) = 1$, then $\mathbb{Z}_{mk} \cong \mathbb{Z}_m \times \mathbb{Z}_k$.*

**Theorem 9.** *If $N \trianglelefteq G$ and both $N$ and $G/N$ are solvable, then $G$ is solvable.*

**Theorem 10.** *All $p$-groups are nilpotent.*

**Theorem 11.** *Any subring must be an additive subgroup.*

**Theorem 12.** *Any cyclic group of a cyclic group ($\mathbb{Z}$) is cyclic.*

**Theorem 13.** *A homomorphism is injective if and only if its kernel is $(0)$.*

**Theorem 14.** *The ideal $(1) = R$, the whole ring, and the ideal $(0) = \{0\}$ is just the ideal containing only the element $0$.*

**Theorem 15.** *Ways to show an ideal $M$ is maximal:*
1. *Show that if an ideal $I$ contains $M$ then $I = M$ or $I = R$, the whole ring.*
2. *Show that $R/M$ is a field.*

**Theorem 16.** *An ideal $P$ is prime if and only if the quotient ring $R/P$ is an integral domain.*

**Corollary 1.** *See page 685 for information on Noetherian rings, prime ideals, radicals, etc.*

**Theorem 17.** *If $x$ is nilpotent then $\phi(x)$ is nilpotent (Exercise 7.3.32).*

**Theorem 18.** *If $\phi$ is surjective, then the preimage of a maximal ideal is maximal.*

**Theorem 19.** *Any nonzero ring homomorphism from a field into a ring is injective (Corollary 7.4.10).*

**Theorem 20.** *If $\phi$ is surjective, the image of an ideal is an ideal.*

*Proof.* Let $\phi : R \to S$ be a surjective hom. Then let I be an ideal in $R$. COnsider $\phi(i)$. We want to show that $\phi(I)s \subseteq \phi(I)$ $\forall s \in S$. So since $\phi$ is sujective, $\exists r \in R$ s.t. $\phi(r) = s$. And note $Is \subseteq I$ since I is ideal. So we have $\phi(Is) = \phi(I)\phi(s) \subseteq \phi(I)$ which tells us that the image is indeed an ideal by definition. □

**Theorem 21.** *Any ideal in a commutative, unital ring is a subring.*

**Theorem 22.** *$\mathbb{Z}[i], \mathbb{Z}$ are EDs.*

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**Theorem 23.** *Maximal ideals are always prime.*

**Theorem 24.** *In a PID, every nonzero prime ideal is maximal.*

**Theorem 25.** *In UFDs, irreducible if and only if prime.*

**Theorem 26.** *Primes in the Gaussian integers. Note that the conjugate of any prime is also prime here. A Gaussian integer is prime if and only if: one of $a, b$ is zero and its absolute value is a prime of the form $4k + 3$, or both are nonzero and $a^2 + b^2$ is a prime number. Refer to Proposition 18 on page 291.*

**Theorem 27.** *Ideals can be principal but not maximal/prime in a PID. Consider $4\mathbb{Z}$. It is not prime in $\mathbb{Z}$ but it is principal.*

**Theorem 28.** *Every ideal is the kernel of some ring hom.*

**Theorem 29.** *Prime iff the quotient ring is an Integral domain.*

**Theorem 30.** *Maximal if and only if the quotient ring is a field.*