MATH 5590H HOMEWORK 6

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Proposition 1. *If $|x| = n < \infty$ then $|x^a| = \frac{n}{(n,a)}$.*

Proof. Write $n = (n,a)b$, $a = (n,a)c$. Then $(b,c) = 1$. We wish to show $|x^a| = b$, since $b = \frac{n}{(n,a)}$. Note

$$(x^a)^b = x^{ab} = x^{(n,a)cb} = x^{(n,a)bc} = x^{nc} = 1.$$

Since $|\langle x^a \rangle| = |x^a|$, we have that $|x^a| \mid b$. Let $|x^a| = k$. Th.en since $x^{ak} = 1$, we must have that $n \mid ak$ since $|\langle x \rangle| = n$. So $(n,a)b \mid (n,a)ck$. Thus $b \mid ck$. Then since $(b,c) = 1$, we must have that $b \mid k$, and so since $k \mid b$ we have that $b = k$. $\square$

Proposition 2. *If G is a finite, abelian group, and p is a prime dividing $|G|$, the G contains an element of order p.*

Proof. We induct on the order of G. We know $G > 1$, else no primes divide $|G|$. Then we have some non-identity element x. Let p be a prime dividing the order of G. We assume $|G| \neq p$, else the result is trivial by Lagrange. Assume $|G| > p$.

Suppose $p \mid |x|$, then write $|x| = pn$ for some $n \in \mathbb{Z}$. Then $|x^n| = \frac{pn}{(pn,n} = \frac{pn}{n} = p$. Then we again have an element of order p.

So assume $p \nmid |x|$. Let $N = \langle x \rangle$, and N is normal in G since G abelian. And $|G/N| = \frac{|G|}{|N|}$ by Lagrange. Since $N \neq 1$, we also have that $|G/N| < |G|$. Since $|N||G/N| = |G|$, $p \nmid |N| = |\langle x \rangle|$, and $p \mid |G|$, we must have that $p \mid |G/N|$. Since N is proper, the order of the factor group G/N must be less than $|G|$, so we have that there exists an element $\bar{y} = yN$ of G/N with order p. Now $yN \neq N$, and $y^pN = N$, so suppose $\langle y^p \rangle = \langle y \rangle$, then $\exists k \in \mathbb{Z}$ s.t. $y = (y^p)^k$, so $y \in N$ by closure of $\langle y^p \rangle$. But this is a contradiction, so we must have $|\langle y^p \rangle| \neq |\langle y \rangle|$. And we cannot have that $|\langle y^p \rangle| > |\langle y \rangle|$ since y^p is generated by y. So we have that $|\langle y^p \rangle| < |\langle y \rangle|$. So $|y^p| < |y|$, and we have $|y^p| = \frac{|y|}{(|y|,p)}$ by Proposition 1, so $(|y|,p) > 1 \Rightarrow p \mid |y|$. But this situation was treated in the previous paragraph, so we have an element of order p. $\square$

Exercise 3.4.2. *Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.*

Composition series for D_8:

- $1 \trianglelefteq \langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle sr^2 \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle sr \rangle \trianglelefteq \langle sr, r^2 \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle sr^3 \rangle \trianglelefteq \langle sr, r^2 \rangle \trianglelefteq D_8$
- $1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle sr, r^2 \rangle \trianglelefteq D_8$

The composition factors are all isomorphic to $\mathbb{Z}_2$.

Composition series for Q_8:

- $1 \trianglelefteq \langle -1 \rangle \trianglelefteq \langle i \rangle \trianglelefteq Q_8$
- $1 \trianglelefteq \langle -1 \rangle \trianglelefteq \langle j \rangle \trianglelefteq Q_8$
- $1 \trianglelefteq \langle -1 \rangle \trianglelefteq \langle k \rangle \trianglelefteq Q_8$

The composition factors are all isomorphic to $\mathbb{Z}_2$.

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Exercise 3.4.5. *Prove that subgroups and quotient groups of a solvable group are solvable.*

Proof. We first prove that the subgroups are solvable. Let G be solvable. Then there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G,$$

such that G_{i+1}/G_i is abelian for all i.

Let $H < G$, and let $H_i = G_i \cap H$. Then let $x \in H_i$, and $y \in H_{i+1}$. Then $yxy^{-1} \in H$ since $x, y \in H$, and $yxy^{-1} \in H_i$ since $G_i \triangleleft G_{i+1}$. Thus $H_i \triangleleft H_{i+1}$. So we have

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_s = H.$$

Since $G_i \triangleleft G_{i+1}$, by the second isomorphism theorem, we have

$$\frac{H_{i+1}}{H_i} = \frac{H_{i+1}}{H_{i+1} \cap G_i} \cong \frac{H_{i+1}G_i}{G_i} \leq \frac{G_{i+1}}{G_i}$$

since $H_{i+1} \leq G_{i+1}$ and $G_i \leq G_{i+1} \Rightarrow H_{i+1}G_i \leq G_i$ by closure of G_i. And $\frac{G_{i+1}}{G_i}$ is abelian, so $\frac{H_{i+1}}{H_i}$ is abelian. And thus H is solvable by definition.

Next we prove that the quotient groups are solvable. Let $N \trianglelefteq G$. Then consider the quotient groups G_iN/N. We know

$$G_0N/N = 1N/N = N/N = 1,$$

and

$$G_sN/N = GN/N = G/N.$$

We prove that $G_iN/N \triangleleft G_{i+1}N/N$. Let

$$\bar{y} = ynN = yN \in G_{i+1}N/N,$$

and let

$$\bar{x} = xnN = xN \in G_iN/N,$$

s.t. $y \in G_{i+1}$ and $x \in G_i$. Then we have

$$\bar{y}\bar{x}\bar{y}^{-1} = (yN)(xN)(y^{-1}N) = yxy^{-1}N \in G_iN/N,$$

since $yxy^{-1} \in G_i$ and $G_i \triangleleft G_{i+1}$ give us

$$yxy^{-1}N = yxy^{-1}nN \in G_iN/N,$$

for all $n \in N$. Hence $G_iN/N \triangleleft G_{i+1}N/N$ for all i.

Now we show that $A = \frac{G_{i+1}N/N}{G_iN/N}$ is abelian for all i. Let $\bar{x}G_iN/N = xn_1N(G_iN/N) = xN(G_iN/N) \in A$ and $\bar{y}G_iN/N = yn_2N(G_iN/N) = yN(G_iN/N) \in A$ s.t. $x, y \in G_i$ and $n_1, n_2 \in N$. Then

$$\begin{aligned}(\bar{x}G_iN/N)(\bar{y}G_iN/N) &= xN(G_iN/N)yN(G_iN/N) = xyN(G_iN/N) = yxN(G_iN/N) \\ &= yN(G_iN/N)xN(G_iN/N) = (\bar{y}G_iN/N)(\bar{x}G_iN/N),\end{aligned} \tag{1}$$

since G_{i+1}/G_i is abelian for all i, thus A is abelian for all i. Hence the chain

$$1 = G_0N/N \trianglelefteq G_1N/N \trianglelefteq \cdots \trianglelefteq G_sN/N = G/N$$

satisfies the condition for G/N to be solvable, so all quotient groups of G are solvable. □

Exercise 5.1.14. *Let $G = A_1 \times A_2 \times \cdots \times A_n$ and for each i let B_i be normal in A_i. Prove that $B = B_1 \times B_2 \times \cdots \times B_n \trianglelefteq G$ and that*

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n) = C.$$

Proof. Let $a = (a_1, a_2, ..., a_n) \in G$ and let $b = (b_1, b_2, ..., b_n) \in B$. Then

$$\begin{aligned}aba^{-1} &= (a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)(a_1, a_2, ..., a_n)^{-1} = (a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)(a_1^{-1}, a_2^{-1}, ..., a_n^{-1}) \\ &= (a_1b_1a_1^{-1}, a_2b_2a_2^{-1}, ..., a_nb_na_n^{-1}).\end{aligned} \tag{2}$$

But each of $a_ib_ia_i^{-1}$ is in B_i since $B_i \trianglelefteq A_i$ for all i. So $(a_1b_1a_1^{-1}, a_2b_2a_2^{-1}, ..., a_nb_na_n^{-1}) \in B \Rightarrow B \trianglelefteq G$.

Then we know G/B is a group, so we let $\phi : G/B \to C$ be given by

$$\phi((a_1, a_2, ..., a_n)B) = (a_1B_1, a_2B_2, ..., a_nB_n).$$

We prove that ϕ is an isomorphism.
Homomorphism: Let $(x_1, x_2, ..., x_n)B, (y_1, y_2, ..., y_n)B \in G/B$, then

$$\begin{aligned}\phi(((x_1, x_2, ..., x_n)B)((y_1, y_2, ..., y_n)B)) &= \phi(((x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n))B) \\ &= \phi((x_1y_1, x_2y_2, ..., x_ny_n)B) \\ &= (x_1y_1B_1, x_2y_2B_2, ..., x_ny_nB_n) \\ &= (x_1B_1, x_2B_2, ..., x_nB_n)(y_1B_1, y_2B_2, ..., y_nB_n) \\ &= \phi((x_1, x_2, ..., x_n)B)\phi((y_1, y_2, ..., y_n)B),\end{aligned} \tag{3}$$

By the direct product operation on G/B and C, so ϕ is a homomorphism.
Injection: Let $(x_1, x_2, ..., x_n)B, (y_1, y_2, ..., y_n)B \in G/B$, and let

$$\begin{aligned}\phi((x_1, x_2, ..., x_n)B) &= \phi((y_1, y_2, ..., y_n)B) \\ \Rightarrow (x_1B_1, x_2B_2, ..., x_nB_n) &= (y_1B_1, y_2B_2, ..., y_nB_n).\end{aligned} \tag{4}$$

So then we have that $x_iB_i = y_iB_i$ for all i, thus

$$\begin{aligned}(y_1, y_2, ..., y_n)B &= (y_1, y_2, ..., y_n)(B_1 \times B_2 \times \cdots \times B_n) = (y_1B_1 \times y_2B_2 \times \cdots \times y_nB_n) \\ &= (x_1B_1 \times x_2B_2 \times \cdots \times x_nB_n) = (x_1, x_2, ..., x_n)B\end{aligned} \tag{5}$$

by the direct product operation, so ϕ is in injective.
Surjection: Let $(a_1B_1, a_2B_2, ..., a_nB_n) \in C$. Then we must have that $a_i \in A_i$ for all i by definition of C and the quotient groups A_i/B_i so $(a_1, a_2, ..., a_n) \in G \Rightarrow (a_1, a_2, ..., a_n)B \in G/B$, and $\phi((a_1, a_2, ..., a_n)B) = (a_1B_1, a_2B_2, ..., a_nB_n)$, so ϕ is surjective by definition. Hence ϕ is an isomorphism, and $G/B \cong C$. □

Exercise 5.2.1(a). *Find the number of nonisomorphic abelian groups of order* 100*.*

We go about finding the invariant factors of abelian groups of order 100. We know the total number of nonisomorphic groups is the product of the partition numbers of each of the powers of the unique primes in the prime factorization of $n = 100$.

First, we find the prime decomposition of 100, which is $100 = 2^25^2$.

The partition number of 2 is 2 so there are $2 \cdot 2 = 4$ possible abelian groups of order 100.

Exercise 5.2.1(b). *Find the number of nonisomorphic abelian groups of order* 576*.*

The prime decomposition of 576 is $2^6 \cdot 3^2$. So again, we find the partition numbers. The partition number of 6 is 11 and the partition number of 2 is 2 so we have $2 \cdot 11 = 22$ different abelian groups.

Exercise 5.2.4(b). *Determine which pairs of*

$$\{2^2, 2 \cdot 3^2\}, \{2^2 \cdot 3, 2 \cdot 3, \}, \{2^3, 3^2\}, \{2^2 \cdot 3^2, 2\}$$

are isomorphic, where $\{a_1, ..., a_k\}$ denotes $\mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_k}$.

Let $G_1 = \{2^2, 2 \cdot 3^2\}$, then by decomposing into prime powers each factor in the direct product, we see that $G_1 \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9$. Similarly:

$G_2 = \{2^2 \cdot 3, 2 \cdot 3, \} \Rightarrow G_2 \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

$G_3 = \{2^3, 3^2\} \Rightarrow G_3 \cong \mathbb{Z}_8 \times \mathbb{Z}_9$.

$G_4 = \{2^2 \cdot 3^2, 2\} \Rightarrow G_4 \cong \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2$.

So $G_1 \cong G_4$ and none of the other pairs are isomorphic.

Exercise 6. *Prove that every p-group is solvable.*

Proof. Let G be a p-group. Then $|G| = p^k$ for some $k \in \mathbb{Z}$. We know that G is finite, and a finite group is solvable if and only if for each divisor n of $|G|$ such that $(n, \frac{|G|}{n}) = 1$, G has a subgroup of order n. We proceed by induction on k. Clearly groups of prime order are solvable, since if H has prime order, $1 \trianglelefteq H$, and $H/1 = H \cong \mathbb{Z}_{|H|}$, and indeed the only normal subgroups of H are $H, 1$. So the claim is true when $k = 1$. Assume the claim holds for all $i < k + 1$. We will prove G is solvable when $|G| = p^{k+1}$. We know each divisor n of $|G|$ is of the form p^i where $i < k + 1$. And the only n for which $(n, \frac{|G|}{n}) = (p^i, \frac{p^{k+1}}{p^i}) = 1$ holds are $n = 1$ or $n = p^{k+1}$ and in both of these cases, G has a subgroup of this order, namely 1 and G itself, so G is solvable for all k. □

Exercise 7. *Prove that if the orders of A, B are coprime, then any subgroup of $A \times B$ is of the form $H \times K$ where $H \leq A$ and $K \leq B$.*

Proof. If $H \times K$ is $1, A \times B$, then the result is trivial, so let $H \times K$ be a nontrivial, proper subgroup. Let $x = (h_1, k_1), y = (h_2, k_2) \in H \times K$ such that $xy^{-1} \neq 1 \Rightarrow x \neq 1 \neq y$. Then $(h_1, k_1)((h_2^{-1}, k_2^{-1}) = (h_1 h_2^{-1}, k_1 k_2^{-1}) \in H \times K$ since $H \times K$ is a subgroup, so $h_1 h_2^{-1} \in H$ and $k_1 k_2^{-1} \in K$. So we have inverses and closure for both H, K and we need only show that they are subsets of A, B, respectively. Since A, B have relatively prime orders, $A \cap B = 1$. Let $h = h_1 h_2^{-1}$ and $k = k_1 k_2^{-1}$. Since $|h| \mid |H|, |A|$, we must have that $|h| \nmid |K|, |B|$, else and similarly, $|k| \nmid |H|, |A|$ since otherwise, A, B do not have coprime orders. So $h \notin K$ and $k \notin H$, and $H \subset A, K \subset B \Rightarrow H \leq A, K \leq B$. $\square$