COMPUTER GRAPHICS

PART 1

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Contents

1 Graphic Adapters

$$\text{Graphic Adapters Overview} = \begin{cases} Vector \\ Raster \\ Accelerated \end{cases}$$

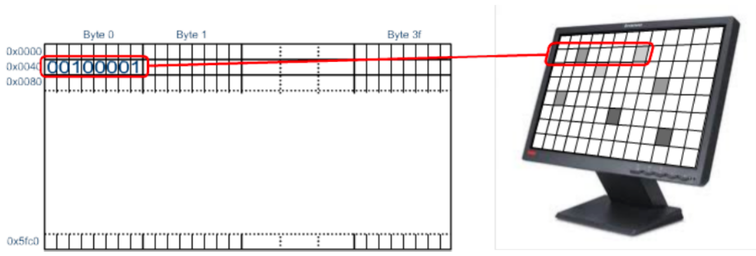
1.1 Vector Graphic Adapters

This type of adapters are old-fashioned and not used any more. The technology used is similar to the one in oscilloscopes : a moving , turned on/off **beam** in a CRT used to draw objects on a screen.
Used mainly in '70s in high-end visualisation tools , later in arcade gaming machines (ex: Atari Battlezone) and even used in the Vectrex , a home entrainment system.
Graphics are drawn as a **set of commands** sent from software to hardware (adapter). The commands are used to generate **3 analog** signals that control horizontal & vertical positions and the beam intensity.

1.2 Raster Graphic Adapters

Raster graphic adapters divide the screen into a **matrix** of individually assignable elements called **pixels** which are assigned **colors.** If the color comes from a spatial sampling of an image the adapter can reproduce it on the monitor.
Raster adapters have a special memory called Video Memory (**VRAM**) made of cells that contain information about the color of each pixel on the screen. A

component on the video card (RAMDAC for analog displays) converts the information to the signal required to transfer the image to the monitor.
Images on the screen can be written by setting specific values in the VRAM (`writeScreen()` function). Still in used but slowly dismissed due to reduction of



hardware costs of better technologies. Initially the memory was just enough to store a single screenshot.

1.3 Accelerated Graphic Adapters

Are a special kind of raster graphic adapters that have **much more memory** than the one required to store a single screenshot. Instead of writing **directly on the screen buffer** ,images are stored in different areas of the VRAM.
Commands that can be interpreted by the adapter include :

- Draw points,lines and other figs
- Write text
- Transfer raster images from VRAM to screen buffer
- 3D projections
- Deform + effects on images

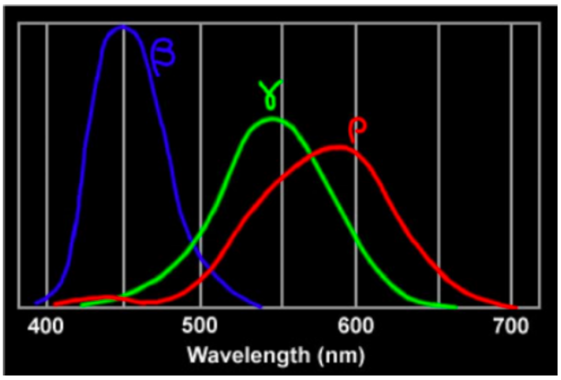
Used today , these adapters can perform complex tasks (multi - display screening, stereoscopic images ..)

1.4 Color vision

How is color on-screen encoded in bits? Commonly a system called **RGB** is used. In the following sections we'll find why.

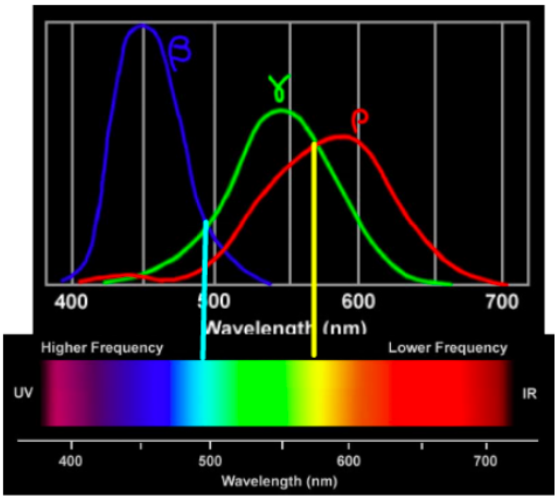
1.4.1 Human Vision

The color of the light is determined by the **wavelength** of the photons that transmit it. Visible light ranges from 400-700 nm wavelength.
Depending on the **light source** , lots of photons of **different wavelengths** are emitted. The photons then interact with the environment where objects, depending on their composition , **reflect or absorb** the various wavelengths at different intensities. The reflected photons are then focused on the retina where **rods** (sensible to light intensity) and **cones** (sensible to light color) transmit information to the brain through **nerves**.
There are 3 types of cones ρ, β, γ each sensible to a different portion of light spectrum.
By combing the stimuli of different cones the brain allows the vision of a given color.



1.4.2 Color reproduction

Color reproduction uses the inverse procedure of the human vision :
it associates a different **emitter** for each color the human cones can capture. Since the main wavelengths perceived by the cones are **red, green & blu** , different hues are constructed by mixing light of these three.
Mixing two of three primary colors **cyan, magenta and yellow** are obtained.
Mixing the three in different proportions **all** possible hues can be obtained.



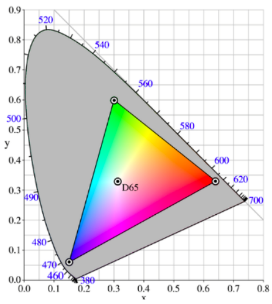
Color range

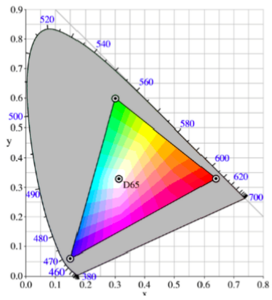
The **spectrum** that a monitor can produce is just a small portion of the entire spectrum the eye can see (grey area).
Color range of a monitor corresponds to a cube where the three colors are placed on the 3 axis.

Color synthesis

The levels of red green and blue are translated into three **electrical signals** whose intensity controls the light emitted by the screen for each of the primary colors.

As the system is digital **DACs** are used to **quantize** the signals. Quantisation further **reduces** the number of visible colors : quantisation levels are usually $2^{d/3}$ with **d** being the number of bits per pixel





1.5 Image Resolution

The resolution of an image defines the **density of pixels** that compose it. When dealing with **raster graphics on screen** the density is relative to the **monitor size** : the resolution defines thus the number of pixels displayed on the screen on the horizontal (**width**) and vertical (**height**
Pixels are not always **square shaped** so the horizontal resolution $\neq$ vertical resolution.

Memory of Images

Raster images require lots of memory : FullHD up to 6MB

$$\text{Memory} = w * h * d_{bits}$$

A first way to reduce image size was to reduce the **colors** using a **color palette** : a predefined set of colors that contains a **limited** number of entries.
The palette is encoded as an array of **RGB values** that are used to define the possible colors that can be used in an image.
If the **color depth d** (number of bits per pixel) is ≤ 8 a color palette is used (the palette contains 2^d colors).
If a user defined palette is used, the size of the palette must be added to the image size. With p = bits to encode a palette entry

$$\text{Memory} = w * h * d + p * 2^d_{bits}$$

2 2D Graphics

2D Graphics primitives are procedures that draw simple geometric shapes based on a **2D coordinates system** , a set of integer with unit the **pixel** $\rightarrow$ *pixel coordinates*.
The coordinate system is **Cartesian** , with the origin on the **left-top** ranging from

$$0 \leq x \leq s_w - 1$$

$$0 \leq y \leq s_h - 1$$

A **clipping** procedure avoids that coordinates go out of boundaries ,causing **wrap-arounds** or writing of **un-allocated memory space**

Figure 1: Axis disposition

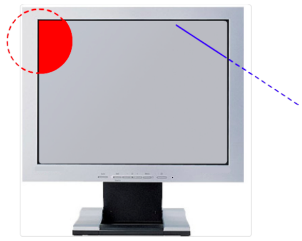


Figure 2: clipping

2.1 Point primitives

The **point** is the simplest 2D primitive which consists in setting a **pixel** in given **position & color**.
Generally the graphic primitive that draws the point is called `plot()` but the **actual** way in which a point is drawn is **hardware dependent** : every adapter has its own plot() algorithm.

2.1.1 Linear Interpolation

A very simple numerical way to compute **intermediate points** giving **two** known points is called **interpolation** :

$$I(x_0, x, x_1, y_0, y_1) = y = y_0 + (x - x_0)\frac{y_1 - y_0}{x_1 - x_0}$$

where $(x_0, y_0), (x_1, y_1)$ are the known values.
Interpolation can be used to find N-1 intermediate points, equally spaced among two points $(x_0, y_0), (x_N, y_N)$:

$$I(0, i, N, y_0, y_N) = y_i = y_0 + \frac{y_N - y_0}{N} i$$

.

Alternatively y_i can be found **recursively** starting from y_{i-1} :

$$y_i = y_0 + \frac{y_N - y_0}{N} i \text{ where } dy = \frac{y_N - y_0}{N} \rightarrow y_1 = y_{i-1} + dy$$

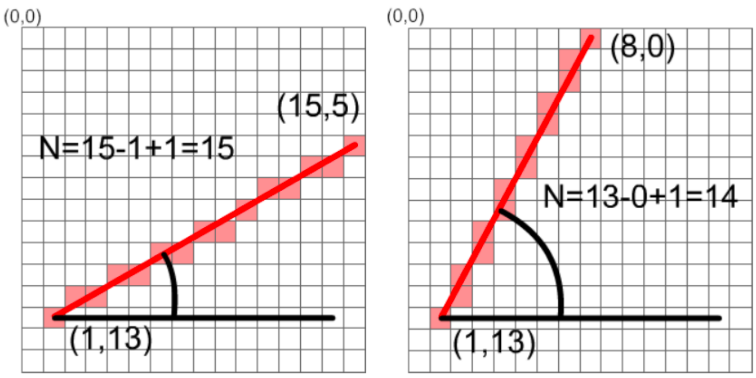
2.2 Line primitives

The **line primitives** connect **two points** $(x_0, y_0), (x_1, y_1)$ on screen with a straight segment. Each pixel that composes the line (except for p_0, p_1 must touch another pixel to keep the line continuous. The **number of pixels** involved depends on the **angle** between the line and the x-axis :

$$N = max(|x_1 - x_0|, |y_1 - y_0|) + 1$$

which corresponds to

$$\text{Number of pixels} = \begin{cases} \theta < 45 & |x_1 - x_0| + 1 \\ \theta > 45 & |y_1 - y_0| + 1 \end{cases}$$



After finding the number of required pixels, different algorithms perform the line drawing task. Two main algorithms are :

- Interpolation algorithm : floating points operations (good on modern hardware)
- Bresenham algorithm : integer operations (good on old or embedded/ special purpose hardware)

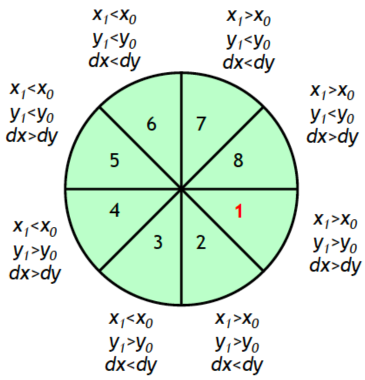
2.2.1 Interpolation Algorithm

A popular line drawing algorithm is the Interpolation algorithm.

```
{
   if( |x1-x0| >= |y1-y0|){ //Angle >45 o <45?
      if(x0 > x1){          // to get smallest value as index in for loop
         swap(x0,y0,x1,y1);
      }
      y=y0;
      dy = (y1-y0)/(x1-x0); //interpolation increment (can be negative!!)
      for(x=x0;x<=x1;x++){
         plot(x,round(y),c); //rounding to nearest int
         y += dy;
      }
   }else{
         if(y0 > y1){
         swap(x0,y0,x1,y1);
      }
      x=x0;
      dx = (x1-x0)/(y1-y0);
      for(y=y0;y<=y1;y++){
         plot(round(x),y,c);
         x += dx;
      }
}
```

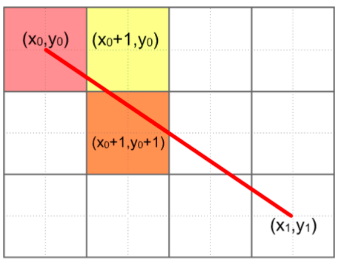
2.2.2 Bresenham Algorithm

The algorithm has 8 different implements depending on which **octant** the points lie :



Using octant 1 a example :

1. Step. : Find the number of pixels $N = max|x_1 - x_0|, |y_1 - y_0| + 1$
2. Step. : First pixel drawn in its position
3. Step. : Depending on the slope select the feasible pixels



4. Step. : Select the pixels whose center is closer to the line (orange pixel)
5. Step. : The process is repeated from step 2 until the end is reached. At each iteration y (in this case) **remains constant** or **increases** by one depending on whether the distance from the previous pixel is greater than 0.5. If the distance is greater y is increased and the distance is reset by one.

```
{
   dy = (y1-y0)/(x1-x0);
   dist = 0 ;
   y=y0;
   plot(x,y,c);
   for(x=x0+1;x<=x1;x++){
      dist += dy;
      if(dist > 0.5){
         y++;
         dist = dist-1;
      }
      plot(x,y,c);
   }
}
```

The algorithm above used **floats** fro computation which is not what we wanted. The integer version of the algorithm is obtained by **multiplying** all terms considering the distance times **2(x1-x0)**:

```
{
   dy = 2(y1-y0);
   dx = x1-x0;
   idist = 0 ;
   y=y0;
   plot(x,y,c);
   for(x=x0+1;x<=x1;x++){
      idist += dy;
      if(idist > dx){
         y++;
         idist -= 2dx;
      }
      plot(x,y,c);
   }
}
```

Obviously the algorithm changes slightly for the other octants.

1

```
dy = 2*(y1 - y0);
dx = x1 - x0;
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(x = x0+1; x <= x1; x++) {
    idist += dy;
    if(idist > dx) {
        y++;
        idist -= 2 * dx;
    }
    plot(x, y, c);
}
```

8

```
dy = 2*(y0 - y1);
dx = x1 - x0;
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(x = x0+1; x <= x1; x++) {
    idist += dy;
    if(idist > dx) {
        y--;
        idist -= 2 * dx;
    }
    plot(x, y, c);
}
```

4

```
dy = 2*(y1 - y0);
dx = x0 - x1;
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(x = x0-1; x >= x1; x--) {
    idist += dy;
    if(idist > dx) {
        y++;
        idist -= 2 * dx;
    }
    plot(x, y, c);
}
```

5

```
dy = 2*(y0 - y1);
dx = x0 - x1;
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(x = x0-1; x >= x1; x--) {
    idist += dy;
    if(idist > dx) {
        y--;
        idist -= 2 * dx;
    }
    plot(x, y, c);
}
```

2

```
dy = y1 - y0;
dx = 2*(x1 - x0);
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(y = y0+1; y <= y1; y++) {
    idist += dx;
    if(idist > dy) {
        x++;
        idist -= 2 * dy;
    }
    plot(x, y, c);
}
```

3

```
dy = y1 - y0;
dx = 2*(x0 - x1);
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(y = y0+1; y <= y1; y++) {
    idist += dx;
    if(idist > dy) {
        x--;
        idist -= 2 * dy;
    }
    plot(x, y, c);
}
```

7

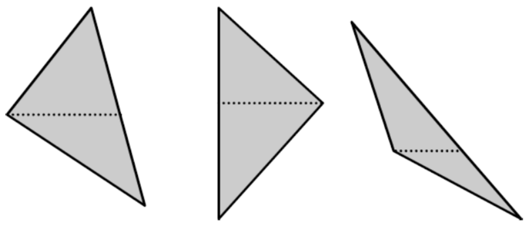
```
dy = y0 - y1;
dx = 2*(x1 - x0);
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(y = y0-1; y >= y1; y--) {
    idist += dx;
    if(idist > dy) {
        x++;
        idist -= 2 * dy;
    }
    plot(x, y, c);
}
```

6

```
dy = y0 - y1;
dx = 2*(x0 - x1);
idist = 0;
x = x0; y = y0;
plot(x, y, c);
for(y = y0-1; y >= y1; y--) {
    idist += dx;
    if(idist > dy) {
        x--;
        idist -= 2 * dy;
    }
    plot(x, y, c);
}
```

2.3 Triangle Primitives

Triangles are very important as they're the basis for **3D** computer graphics. Triangles having one **edge parallel to the horizontal** axis is the **easiest** to draw. Other triangles can be split in 2 to obtain easy to draw triangles.

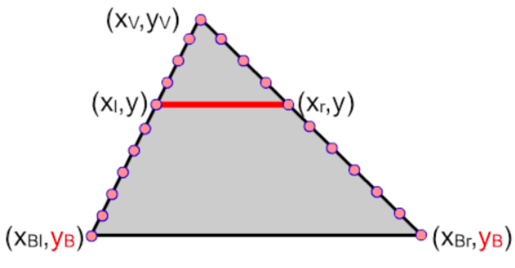


2.3.1 Triangles with edge // to x-axis

Triangles of this type are characterized by 5 values:

- x_v ,y_v coordinates of the vertex
- y_B vertical coordinates of the base
- x_{Bl}, x_{Br} horizontal coordinates of the base

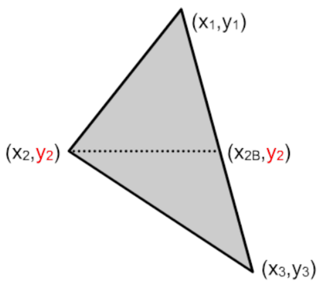
The two edges not parallel to x-axis can be considered as two lines.// The triangles is **filled** by drawing horizontal lines that connect pixels over the angled edges. The x_l, x_r coordinates can be found using **interpolation**.



```
{ /*since the two lines have diff. slope , store in
  dxr ,dxl the increments in the x direction */
  dxl = (xBl - xv)/(yB - yv);
  dxr = (XBr - xv)/(yB - yv);
  xl = xr = xv; // starting from vertex
  /* assumption yv < yB , otherwise change loop direction */
  for ( y = yv; y <=yB ; y++) {
     for(x=round(xl);x<=round(xr); x++){
        plot(x,y,c)
     }
     xl += dxl;
     xr += dxr;
  }
}
```

2.3.2 Triangle splitting

More complex triangles can be split in order to obtain two triangles with edges parallel to x-axis.



An easy way to find the middle point (x2,y2) is to take the three points and sort them through the y-axis. The one in the middle is the middle point.
To find the corresponding point on the opposing edge :

- **y-coordinate** is the same as in point (x2,y2)

- **x-coordinate** is obtained via **interpolation**:

$$x_{2B} = I(y_1, y_2, y_3, x_1, x_3)$$

2.4 Normalized coordinates

Current displays are available in different **resolutions** and **sizes**. Moreover in windowed operating systems applications must be **confined** only in a portion of the screen. When changing the resolutions/window size the applications still want to show the **same image** exploiting all the features of the display. A special coordinates system called **Normalized Screen Coordinates** is normally used to address points on screen in a device in an independent way.

NSC are Cartesian coordinate system where x and y range between to **canonical values** [OpenGL -1 $\rightarrow$ 1]. If the window/memory area resolution is known in s_w, s_h pixels, the coordinates system (x_s, y_s) can be derived from the normalized screen coordinates (x_n, y_n) :

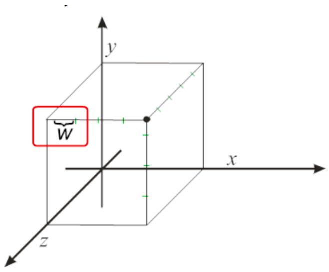
$$x_s = (s_w - 1) * (x_n + 1)/2$$

$$y_s = (s_h - 1) * (1 - y_n)/2$$

3 3D Graphics

To define a point in a 3D-space , 3 coordinates are typically used . However in computer graphics **4** coordinates x, y, z, w called **homogeneous coordinates** are used :

- x, y, z are used to define the **point in the 3D space**
- z defines a **scale**, the unit of measure used by the coordinates



Consequence of using 4 coordinates $\rightarrow$ **infinite** number of coordinates define the same point , in particular all tuples of four values that are **linearly dependent** represent the same point in 3D space :

$$(2, 2, 2.5, 0.5), (4, 4, 5, 1) : (4, 4, 5, 1) = 2(2, 2, 2.5, 0.5)$$

The **real position** of a point in 3D spcae is defined by $w = 1$. To obtain the real position a simple division of **w** is sufficient.

$$(x, y, z, w) \rightarrow (x', y', z') = (\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$$

3.1 Affine Transformations

The process of varying the coordinates of the points of an object in the space is called **transformation**.
Transformations can be very **complex** since all points should be repositioned in a 3D space .
A large **set of transformations** with a mathematical concept called **affine transformations**
Objects in 3D space are defined by the coordinates of their points.By applying affine transformations to the coordinates 4 different transformations can be done :

- **Translation**
- **Scaling**
- **Rotation**
- **Shear**

The new object is drawn using the new points and the corresponding **primitives**.

$$p = (x, y, z, w) \rightarrow p' = (x', y', z', w')$$

To express transformations $4x4$ matrices M can be used. So the new point p' can be obtained by multiplying p by the **transformation matrix** :

$$p' = M \cdot p^T$$

or

$$p' = p \cdot M^T$$

depending on the **convention**

3.1.1 Translation

Moves the points of the object while maintaining its **size & orientation**. Translation can be performed along the three axis : dx, dy, dz are the quantities that define how much the object is being moved :

$$x' = x + dx$$

$$y' = y + dy$$

$$z' = z + dz$$

By using the 4th coordinate the **translation matrix** can be obtained:

$$\begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + dx \\ y + dy \\ z + dz \\ 1 \end{bmatrix}$$

3.1.2 Scaling

Scaling modifies the **size of an object** while maintaining constant **position and orientation**.It can have different effects :

- **Enlarge**
- **Shrink**
- **Deform** ex:Sphere $\rightarrow$ rugby ball
- **Mirroring** ex: Object on the right $\rightarrow$ symmetrical on the left

Scale transformations have **a center** : a point that is **not moved** during the transformation. The center of transformation can be anywhere on the 3D space (also outside the object!).
Now we assume that the center corresponds to the **origin**.

- **Proportional scaling**
  Enlarges or shrinks the object of the same amount **s** in all the directions : this leaves the proportions intact.

  $$x' = s \cdot x$$

  $$y' = s \cdot y$$

  $$z' = s \cdot z$$

  If $s > 1 \rightarrow$ **enlarge**
  Else $0 < s < 1 \rightarrow$ **shrink**

- **Non proportional scaling**
  Deforms an object by using different scaling factors s_x, s_y, s_z that allows shrinking or enlarging in different directions.

  $$x' = s_x \cdot x$$

  $$y' = s_y \cdot y$$

  $$z' = s_z \cdot z$$

If $s_i > 1 \rightarrow$ **enlarge**
Else $0 < s_i < 1 \rightarrow$ **shrink**
A **scaling matrix** can be used to represent transformations :

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Mirroring**
  By using **negative scaling factors** mirroring can be obtained. Three different types exist in 3D space:

  1. **Planar**
     Creates a symmetric object with respect to a **plane** by assigning **-1** scaling factor to the axis **perpendicular to the plane**

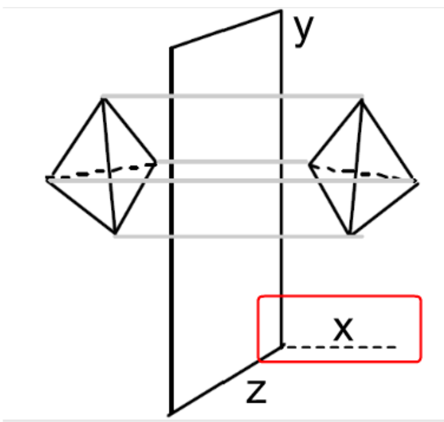


Figure 3: $s_x = -1, s_y = 1, s_z = 1$

2. **Axial**
   Creates a symmetric object with respect to a **axis** by assigning **-1** to all scaling factors **except** the one of the axis.

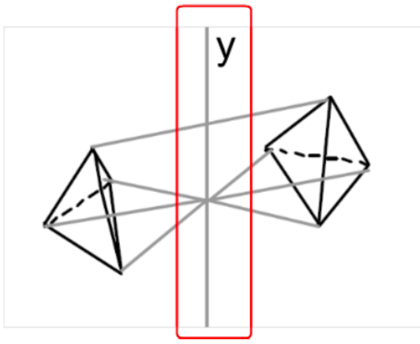


Figure 4: $s_x = -1, s_y = 1, s_z = -1$

3. **Central**
   Creates a symmetric object with respect to the **origin**.It is obtained by assigning **-1** to all scaling factors.

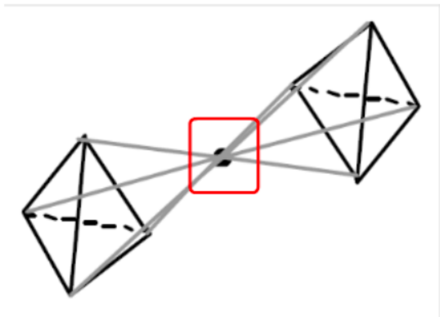


Figure 5: $s_x = -1, s_y = -1, s_z = -1$

Notice that if a scaling factor of **0** is chosen it **flattens** the image along that axis. This makes the transformation matrix **not invertible**

3.1.3 Rotation

Varies the objects **orientation** leaving unchanged its **position and size**. Rotation happens along a chosen axis , a line where points are **unaffected** by the

transformation.
Rotation can occur also on non conventional axis but we will mainly consider rotations along x,y,z axis passing through the origin.
A rotation of angle α about the z-axis :

$$x' = x \cdot cos\alpha - y \cdot sin\alpha$$

$$y' = x \cdot sin\alpha + y \cdot cos\alpha$$

$$z' = z$$

As the z-axis is the **axis of rotation** its points remain unchanged. A rotation of angle α about the y-axis :

$$x' = x \cdot cos\alpha + z \cdot sin\alpha$$

$$y' = y$$

$$z' = -x \cdot sen\alpha + z \cdot cos\alpha$$

A rotation of angle α about the x-axis :

$$x' = x$$

$$y' = y \cdot cos\alpha - z \cdot sin\alpha$$

$$z' = y \cdot sin\alpha + z \cdot cos\alpha$$

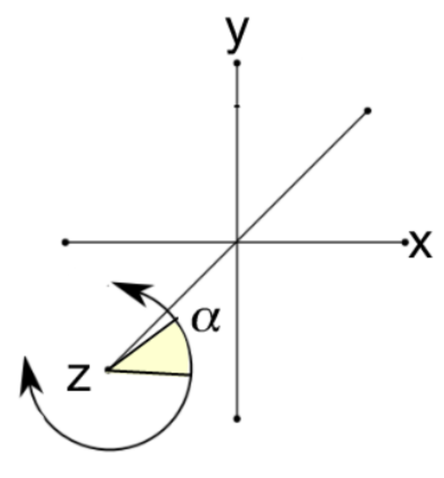


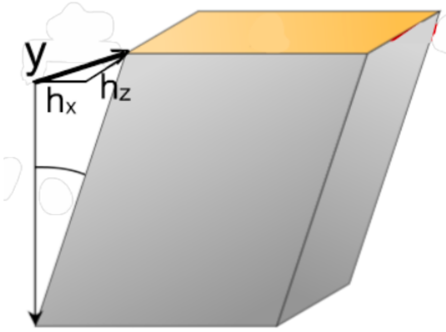
Figure 6: Z-Axis rotation

Again matrices can be used to express rotation :

$$R_z = \begin{bmatrix} cos\alpha & -sin\alpha & 0 & 0 \\ sin\alpha & cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_y = \begin{bmatrix} cos\alpha & 0 & sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -sin\alpha & 0 & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.2 Shear

Shear bends the objects in one direction. It has an **axis** and **center**. Considering axis y and the origin as center



as the values of y increase the object is bent following the direction of a vector defined by two values (h_x, h_z in this case) :

$$x' = x + y \cdot h_x$$

$$y' = y$$

$$z' = z + y \cdot h_z$$

Along the **x-axis**:

$$x' = x$$

$$y' = y + x \cdot h_y$$

$$z' = z + x \cdot h_z$$

Along the **z-axis**:

$$x' = x + z \cdot h_x$$

$$y' = y + z \cdot h_y$$

$$z' = z$$

Again matrices can be used to express shear :

$$H_x(h_y, h_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_y & 1 & 0 & 0 \\ h_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_y(h_x, h_z) = \begin{bmatrix} 1 & h_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h_z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_z(h_x, h_y) = \begin{bmatrix} 1 & 0 & h_x & 0 \\ 0 & 1 & h_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

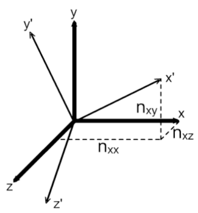
4 3D Transform

A general matrix representation can be derived starting from all the transformations found so far :

$$M = \left| \begin{array}{ccc:c} n_{xx} & n_{yx} & n_{zx} & d_x \\ n_{xy} & n_{yy} & n_{zy} & d_y \\ n_{xz} & n_{yz} & n_{zz} & d_z \\ \hdashline 0 & 0 & 0 & 1 \end{array} \right| = \left| \begin{array}{c:c} M_R & \mathbf{d}^T \\ \hdashline \mathbf{0} & 1 \end{array} \right|$$

- **Mr** : sub-matrix representing **rotation,scaling & shear**
- **dt** : translation
- **1** : to ensure that the w coordinate remains **unchanged**

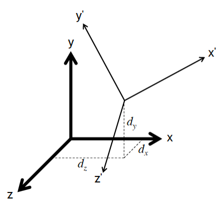
The columns of M_R represent **directions & sizes** of the new axes in the old reference system
Rotations maintain the size of the axis constant but change their direction.
Scalings maintain the direction of the axis constant , changing the size.



$$M_R = \left| \begin{array}{ccc} \boxed{\begin{array}{c} n_{xx} \\ n_{xy} \\ n_{xz} \end{array}} & \begin{array}{c} n_{yx} \\ n_{yy} \\ n_{yz} \end{array} & \begin{array}{c} n_{zx} \\ n_{zy} \\ n_{zz} \end{array} \end{array} \right|$$

Vector d^t represents the position of the origin of the new coordinate system in the old one

$$M = \left| \begin{array}{c:c} M_R & \begin{array}{c} d_x \\ d_y \\ d_z \end{array} \\ \hdashline \mathbf{0} & 1 \end{array} \right|$$



4.1 Inversion of transformations

To return an object to its **origina** state transformation can be reversed. **Matrix inversion** can be applied when using the matrices representation of transformations.

$$p' = (x', y', z', 1) \rightarrow p = (x, y, z, 1)$$

$$p = M^{-1}p$$

Matrix M^{-1} is **invertible** if its submatrix M_R is invertible. Generally M^{-1} is always invertible except when dealing axis degeneration (zero factor scaling for example).
Another method of inverting transformations is by using a reverse matrix :

$$\begin{vmatrix} 1 & 0 & 0 & -d_x \\ 0 & 1 & 0 & -d_y \\ 0 & 0 & 1 & -d_z \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Figure 7: Translation

$$\begin{vmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

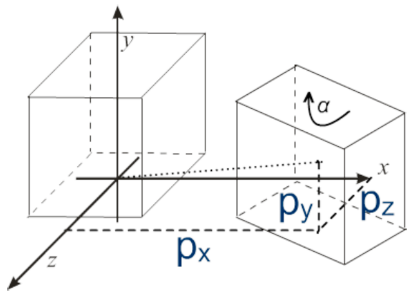
Figure 8: Scaling

$$R_x(-\alpha) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad R_y(-\alpha) = \begin{vmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad R_z(-\alpha) = \begin{vmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Figure 9: Reverse rotation

4.2 Composition

During the creation of scene an object is subject to **several** transformations. Applying a **sequence of transformations** is called **composition**. An example is the movement of a cube , sides parallel to x,y,z axis and with center in the origin.
- Translation of center to position p_x, p_y, p_z
- Rotation of angle α around y



- **Rotation** around y of $\alpha \rightarrow p' = R_y(\alpha) \cdot p$

$$R_y(\alpha) = \begin{bmatrix} cos\alpha & 0 & sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -sin\alpha & 0 & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Translation** in position $p'' = T(p_x, p_y, p_z) \cdot p'$

$$T(p_x, p_y, p_z) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{p'' = T(p_x, p_y, p_z) \cdot R_y(\alpha) \cdot p}$$

Matrices appear in **reverse** order wrt to the transformations they represent.

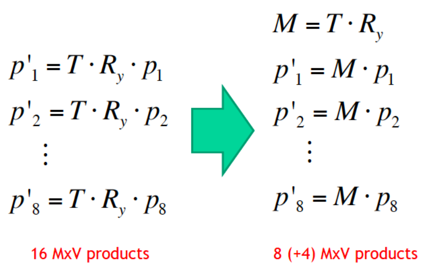
4.2.1 Properties of composition of transformations

Matrix-Matrix and Matrix-Vector products are **associative**:

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

$$A \cdot (B \cdot p) = (A \cdot B) \cdot p$$

Using the associative property we can obtain a **single matrix** corresponding to the product of **all the transformations**.
This is useful because usually the multiplication is done for many points ($10^4 \sim 10^6$, so having one matrix that sums up all transformation **improves performances**.



As in the figure instead of having 16 matrix-vector products we only have 12 (4 are to create the matrix M).
Inversion can be handled by considering :

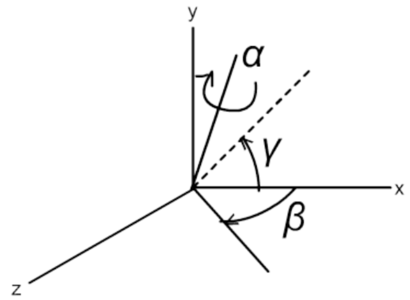
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

Example : $M = R_y(30°) \cdot T(1,2,3) \rightarrow M^{-1} = T(1,2,3)^{-1} \cdot R_y(30°)^{-1}$ Matrix products are **not commutative** : the order of the transformations is important , and transformations cannot be swapped without obtaining a **different** result.

4.3 Transformations around an arbitrary axis or center

Case : Rotation

Instead of rotating an object around the x,y or z axis we consider now a rotation of angle α around an arbitrary axis that passes through the origin. Depending on where the considered axis is it forms **two angles** with the the other axis.



In this case the two angles are γ, β :
- $\gamma \rightarrow$ how much the axis rises on the xz plane
- $\beta \rightarrow$ how much it rises on the xy plane

The angles and planes chosen are arbitrary, other angles and planes can be used to describe the same transformations.

1. $R_y(-\beta)$
   Considering a rotation of $-\beta$ along the y-axis ,the arbitrary axis now lies on the xy plane.
2. $R_z(-\gamma)$
   Then considering a rotation of $-\gamma$ along the z-axis ,the arbitrary axis now corresponds to the x-axis.
3. $R_x(\alpha)$
   As our chosen axis corresponds to the x-axis , the rotation of the object of angle α can be done around the x-axis.
4. $R_z(\gamma)$ and $R_y(\beta)$
   To restore the original axis position.

Final transformation composition :

$$\boxed{p' = R_y(\beta)R_z(\gamma)R_x(\alpha)R_Z(-\gamma)R_y(-\beta) \cdot p}$$

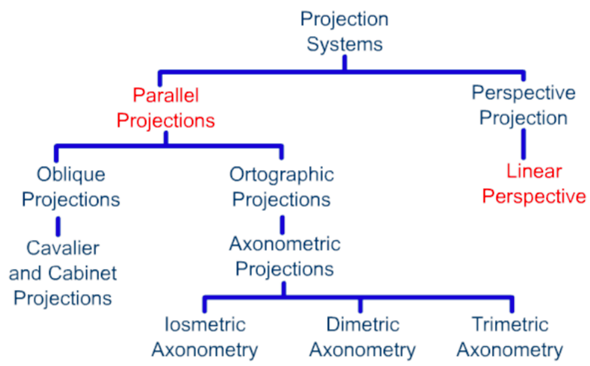
If the axis does not pass through the **origin** , a **translation T** must be applied of a point known through which the axis passes:

$$\boxed{p' = T(p_x, p_y, p_z)R_y(\beta)R_z(\gamma)R_x(\alpha)R_Z(-\gamma)R_y(-\beta)T(-p_x, -p_y, -p_z) \cdot p}$$

Similar procedures can be applied to :

- **Scaling**
- **Shear**

5 Projections



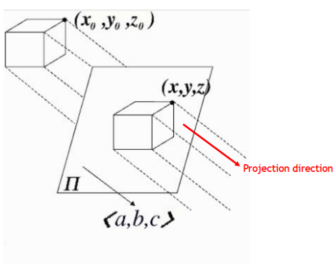
In 3D computer graphics the goal is to represent a three dimensional space on a screen with 2 dimensions:

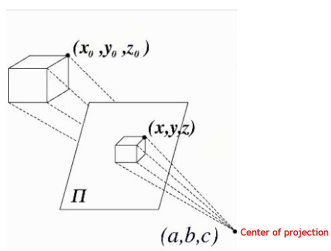
- The 3D graphics uses geometrical primitives defined in 3 dimensions
- 3D graphics produces a 2D representation of the scene to show on screen.

The second step is performed using **projections**. Key features :

- Projections of linear segments **remain** linear segments
- Projected segments connect the projections of the segment's end points

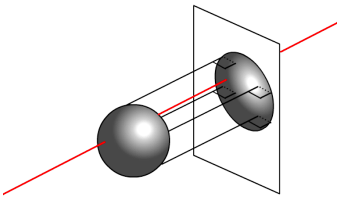
So to create a 2D projection of a 3D polyhedron it is sufficient to **project its vertices** and connect them.
In parallel projections all the rays are **parallel to the same direction**.
In perspective projections all the rays pass **through a point** called

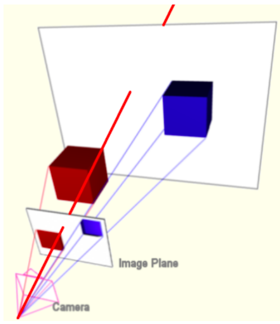




Doing a projection , we loose one coordinate so a point on screen corresponds to an **infinite** number of coordinates (consequence of moving from 3D $\rightarrow$ 2D) :

in both parallel & perspective projections any point on screen corresponds to **a line of points** in 3D. In parallel projections all points that pass through a line parallel to projections ray are mapped to the **same pixel**.
In perspective projections all points aligned with both projected pixel and the center of projection are mapped to the **same pixel**.

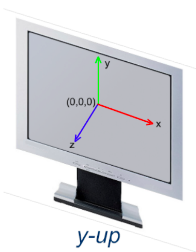


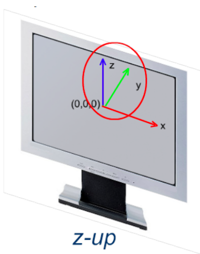


In 3D computer graphics the concept of projection becomes the **conversion** of 3D coordinates from one reference system to another.
World coordinates→ 3D Normalized Screen Coordinates

World coordinates

Coordinate system that describes the objects in the 3D space. It is a right-handed Cartesian coordinate system with the **origin** in the **center of the screen.** Some applications invert the z and y axis , with the y axis point inside the screen.

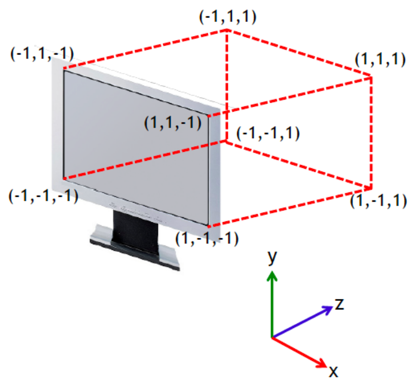




3D Normalized Screen Coordinates

Allow to specify the positions of points on screen (or window) in a device-independent way. 3D images must be characterized by a **distance** to allow ordering the surfaces and prevent the construction of unrealistic images.

3D Normalized coordinates have a **third** component ranging form the same extents (ex : -1,1). This way coordinates with a smaller z-value will be considered to be **closer** to the viewer.



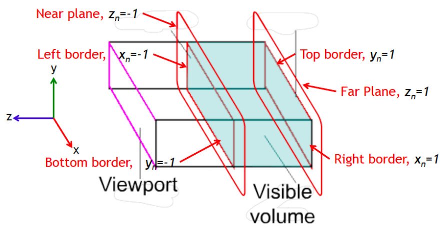
5.1 Parallel Projections

Orthogonal projections are projections where the plane is either xy,xz or yz and the **projections rays** are **perpendicular to it**.

Projection plane parallel to xy-plane

The projections are **perpendicular** to the **z-axis**. Limiting the range of a scene is important to avoid showing objects **behind the observer** or **too far away** :

- The plane with the **minimum z component** $\rightarrow$ **near plane**
- The plane with the **maximum z component** $\rightarrow$ **far plane**



Usually distance from viewport to near plane is very small. Things before the near plane and behind the far plane are **not shown** in the scene: only the visible volume will be seen.

Orthogonal projections can be implemented by **normalizing** the x,y,z coordinates of the projection box in the (-1,1) range. Then a **projection matrix** can be computed to find the normalized 3D coordinates :

$$p_N = P_{ort} \cdot p_W$$

How to find P_{ort}?
Coordinates l,r are the **x-coordinates** in the 3D space that will be displayed on the left and right borders of the screen. Everything on the left of l or right of r will be **cut**.
Similarly t,b are the **y-coordinates** of the top and bottom borders of the screen. Finally we call -n ,-f the **z-coordinates** of the near and far planes . Since the z-axis is oriented in the opposite direction the positive distance is used over the negative one. Also using this annotation means that $n > f$ even if n is closer than f! Bottom left front point will have coordinates (-1,-1,-1) while the top right back point has coordinates (1,1,1).
To create the P_{ort} matrix:

1. Move the center of the box to correspond to the center of the space
   The center of the box will have coordinates :
   $$c = (\frac{l+r}{2}, \frac{t+b}{2}, \frac{f+n}{2})$$
   So to align the center we must **inverse translate**
   $$c' = T^{-1}(\frac{l+r}{2}, \frac{t+b}{2}, \frac{f+n}{2}) \cdot c$$
   so that the center now corresponds to the origin.

$$T_{ort} = \begin{vmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{-f+(-n)}{2} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

2. Then normalise the coordinates

$$S_{ort} = \begin{vmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

3. Z goes to the viewer : points closer should have inverse Z coordinate
   Changing the sign of Z is done by mirroring :

$$M_{ort} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Which can be resumed by using a single combined matrix:

$$P_{ort} = M_{ort} \cdot S_{ort} \cdot T_{ort} = \begin{vmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{l+r}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Once the **normalized** screen coordinates are obtained for example for the vertices of a triangle we can apply the the usual drawing primitive procedure to fill up the whole triangle. By repeating this step for **every triangle** that composes the image we can build up a **2D view of a 3D object**.

5.1.1 Aspect Ratio

Normalized coordinates are able to support **non-square pixels** it wasn't able to deal with the proportion of the window where objects are drawn . The **aspect ratio** $a = \frac{D_x}{D_y}$ must be considered where D_x, D_y are the **horizontal, vertical** dimensions. Aspect ratios are usually 4:3 or 16:9.
Having a resolution of 2000 x 1000 with **rectangular pixels**. In this case is $a = \frac{2000}{1000} = 2$? No because the **aspect ratio** is defined using **metrical units**. Summing up:

- **Square pixels**
  The **aspect ratio** can be computed by dividing the the number of pixels on the horizontal and vertical directions.

- **Rectangular pixels**
  The **aspect ratio** must be computed by using the actual **physical** dimensions must be used.

Normalized screen coordinates does **not** take care of the aspect ratio : the **projection matrix** must take care of this adding a **scaling factor**.
Considering the viewport the **width** is r-l and the height is t-b so the ratio of the window is $\frac{r-l}{t-b}$ if :

$$\frac{r-l}{t-b} = a = \frac{D_x}{D_y}$$

then the image will not be **distorted**.

5.1.2 Projection matrices and aspect ratio

Usually the **projection box** is centred vertically and horizontally in the world. Using the half-width **w** from the center to the left/right border we can use less information to computed the matrix. The vertical equivalent of the w is computed using the **aspect ratio a** : $\frac{a}{w}$. This way we can compute the position of left,right, bottom and top only knowing the half-width and the aspect ratio.

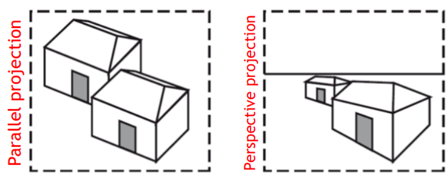
$$P_{ort} = \begin{vmatrix} \frac{1}{w} & 0 & 0 & 0 \\ 0 & \frac{a}{w} & 0 & 0 \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Note that the element in position (1,3)=(1,4)= 0 because the projection box is **already** centred in the origin.

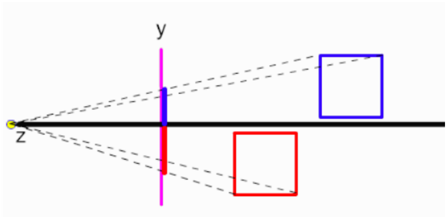
5.2 Perspective Projections

Parallel projections do **not** change the apparent size of an object with the distance from the observer. It is used mainly for drawings and is not that suitable for 3D computer graphics.

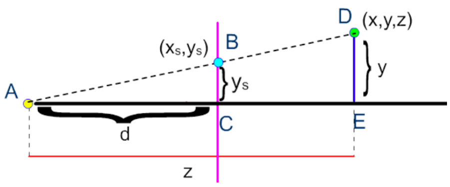
Prospective projection on the other hand represent an object with a different size depending on its **distance** from the **projection plane** : this makes it suitable for **immersive visualisations**. This is the result of all the projections rays **passing through the same point**.



Rays intersect the **projection plane** at different points depending on the distance of the object : if two objects have the same size but are at different distances than the ones **closer** to the plane have a **larger** projection.



As for the parallel projections also perspective projections make us of **normalized screen coordinates** to project objects on the projection plane. Given the space coordinates $(x, y, z) \rightarrow (x_s, y_s, z_s)$. Now we will focus only on x_s, y_s.



- y_s
  To simplify the computation the **center of projection** (yellow dot) corresponds to the origin $(0, 0, 0)$. The projection plane is located at distance d on the z-axis from the center of projection.
  Tracing the projection ray from point (x, y, z) to the center of projection we obtain two **similar** triangles ABC, ADE. It is easy to see that y_s is the height of ABC while y the height of ADE $\rightarrow y_s : d = y : z$ which leads to

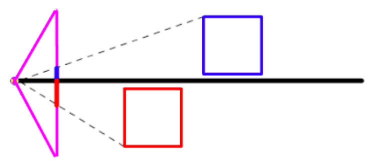
$$\boxed{y_s = \frac{d \cdot y}{z}}$$

- x_s
  The same computation as for y_s occur :

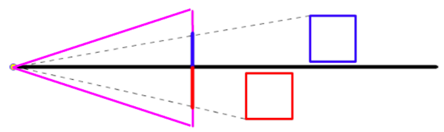
$$\boxed{x_s = \frac{d \cdot x}{z}}$$

The distance **d** from center of projection to projection plane plays an important role .It acts like the **camera lens** so changing parameter d has the effect of performing a **zoom**:

- **Short d**
  Like a wide lens, emphasizes the distances of objects from the plane. Allows to capture a larger number of objects producing smaller images



- **Long d**
  Like tele-lens, reducing the differences in size for objects at different distances. It reduces the number of objects visible in the scene producing **enlarged** objects.



- $D \to \infty$
  If distance d tends to infinity we obtain **parallel projections**

As for the parallel projections , also **perspective projections** can be obtained with a **matrix-vector product**.As the world coordinate system is oriented in the **opposite** direction of the z-axis , the z-coordinates are **negative** :

$$\boxed{x_s = \frac{d \cdot x}{-z}}$$

$$\boxed{y_s = \frac{d \cdot y}{-z}}$$

The projection matrix for perspective with **center in the origin** and projection plane at distance **d** on the z-axis:

$$P_{persp} = \begin{vmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

The last row is no longer $|0001|$ as per usual : the result is a vector which no longer has component **w =1**:

$$\boxed{[d \cdot x, d \cdot y, d \cdot z, -z]}$$

To obtain the equivalent Cartesian coordinates we must divide by the w component (-z) :

$$[x_s, y_s, -d, 1]$$

The z-coordinate is always **-d** regardless of the what the z-coordinate is, which means that all information about **distance** is lost $\rightarrow$ no proper 3D normalized screen coordinates can be defined.
The solution to not flat the z component is to **add** an element $= 1$ in the third row of the fourth column:

$$P_{persp} = \begin{vmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & \boxed{1} \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

which leads to normalized screen coordinates :

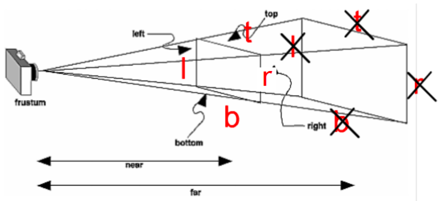
$$|x_s, y_s, -d - \frac{1}{z}, 1|$$

Now we have a third component depending on d but also on **z**. Depending on the distance:

- Negative but smaller when closer to the viewer.
- Negative but larger when farther away from the viewer.
- Tends to $-d$ as the distance goes to ∞.

5.2.1 Perspective matrix on screen

Now that we have basic tools for creating (after normalization) proper normalized screen coordinates that respect the distance of objects. We need to combine these new tools with transformations to be able to show correctly on screen the desired coordinates.
In the case of perspective projections instead of a view box like in parallel projections we have a **frustum**:



The frustum is defined by it's **near** and **far** plane and **top**,**bottom** ,**left** and **right** coordinates. The coordinates are **not constant** any more: now they depend on the **distance**.
By default t,b,l,r are defined on the **near plane** so the distance **d** corresponds to the value **n** of the near plane. The resulting projection matrix is:

$$U_{persp} = \begin{vmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$