Contents

1 Exam questions

1.1 23-02-2011

Q:Make the proof of equivalence of FPE and AIC criteria

The Find Prediction Error (FPE) and Akaike Information Criterion (AIC) are both estimation criteria used in model-order selection.

$$\text{FPE}(n_\theta) = \frac{N + n_\theta}{N - n_\theta} J_N(\hat{\theta}_N, n_\theta)$$

$$\text{AIC}(n_\theta) = 2\frac{n_\theta}{N} + ln(J_N(\hat{\theta}, n_\theta))$$

Where N = sample size , n_θ=order of the model and $J_N(\hat{\theta}_N, n_\theta)$= the performance index on its best parameter vector which is dependent on n_θ.

$$ln(FPE) = ln(\frac{N + n_\theta}{N - n_\theta} J_N(\hat{\theta}_N, n_\theta))$$

$$ln(FPE) = ln(\frac{1 + \frac{n_\theta}{N}}{1 - \frac{n_\theta}{N}} J_N(\hat{\theta}_N, n_\theta))$$

Remark

Remind that $ln(1 + x) \approx x$ when $x = 0$

If $n_\theta << N \rightarrow \frac{n_\theta}{N} \approx 0$ so :

$$ln(1 + \frac{n_\theta}{N}) - ln(1 - \frac{n_\theta}{N}) + ln(J_N(\hat{\theta}; n_\theta)) \approx \frac{n_\theta}{N} - (-\frac{n_\theta}{N}) + ln(J_N(\hat{\theta}; n_\theta))$$

$$2\frac{n_\theta}{N} + ln(J_N(\hat{\theta}; n_\theta)) = AIC(n_\theta)$$

So if $n << N$ (always true in predicted applications):

$$\boxed{ln(FPE) = AIC}$$

Notice that if f(x) has a minimum in x_0 then also $ln(f(x))$ has a minimum in $x_0 \rightarrow \frac{d}{dx}(ln(f(x))) = \frac{1}{f(x)} f'(x)$:

$$\boxed{argmin_\theta\{FPE(n_\theta\} = argmin_\theta\{AIC(n_\theta\}}$$

So FPE and AIC provide the **same optimal** value for n_θ

1.2 05-07-2011

Q:Write the expression of the "Error-to-Signal-Ratio"for a SSP, as function of prediction horizon k.Moreover list and explain the main properties of variance of the prediction error,as function of k

The Error to signal ratio is a useful prediction measure:

$$ESR(k) = \frac{var[y(t) - \hat{y}(t|t-k)]}{var[y(t)]}$$

The prediction error variance $var[\epsilon] = var[y(t) - \hat{y}(t|t-k)]$ has the following properties:

- $k = 0 \rightarrow var[\epsilon] = 0$
  Predicting at time step 0 means predicting the present y(t) so $var[\epsilon] = var[y(t) - y(t)] = 0$
- $k = 1 \rightarrow var[\epsilon] = \lambda^2$
  Predicting at time step-1 leads always to a predicting error equivalent to the white noise of the process. So $var[\epsilon] = var[e(t)] = \lambda^2$
- $k \rightarrow \infty \implies var[\epsilon] \rightarrow var[y(t)]$
  With an infinite large prediction horizon the prediction is equal to zero.So $var[\epsilon] = var[y(t) - 0] = var[y(t)]$
- The prediction error in function of k is a monotonic (not strictly) increasing function.

From the properties above we can derive that:

$$ESR(0) = 0$$

$$ESR(k \rightarrow \infty) = 1$$

So the higher the ESR the worse is the prediction.

1.3 05-09-2012

Q:Give the definition of the covariance function of stochastic process.Suppose then that the process is stationary.Which are then the properties of the covariance function?

The covariance is expected value of the product of two unbiased random variables v (defined at the same experiment S)at time instants t1, t2 :

$$\gamma(t_1, t_2) : E[(v(t_1, S) - m(t_1))(v(t_2, S) - m(t_2))]$$

If t1 = t2 = t the covariance degenerates in **variance**:

$$\gamma(t) = E[(v(t, S) - m(t))^2]$$

If the process is also stationary:
$\gamma(t_1, t_2)$ depends on $\tau = |t_1 - t_2|$
This means that the covariance depends on the **distance in time** and not on specific considered samples.
For example: $\gamma(t_1, t_2) = \gamma(t_3, t_4) \rightarrow |t_1 - t_2| = |t_3 - t_4|$
$\gamma(\tau) = E[(v(t) - m)(v(t - \tau) - m)]$ has properties :

- $\gamma(0) = E[(v(t) - m)^2] \rightarrow$ **variance**
- $|\gamma(\tau)| \leq \gamma(0)$
- $\gamma(\tau) = \gamma(-\tau)$
- if $m = 0$ the covariance function is equivalent to the correlation function.

Q:Explain what is a seasonal component of a signal and illustrate how it can be detected and removed so as to spot out the stochastic part of the signal

We assume a model of the raw dataset as follows:

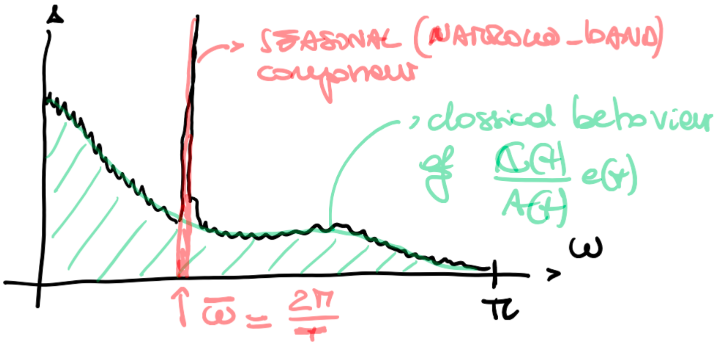
$$y(t) = \tilde{y}(t) + s(t), \text{ t=1,2,...N}$$

$$\tilde{y}(t) = \frac{C(Z)}{A(Z)} e(t)$$

$$s(t) \text{ is a periodic signal with period T} : s(t+kT) = s(t)$$

Remark

- T is usually a-priori known (a day, a week ,a year...)
- It is possible to have multiple seasonal behaviour overlapped, where each component can be dealt with independently.
- If T is **not** known a-priori, it is easy to detect it by a simple FFT of the raw signal



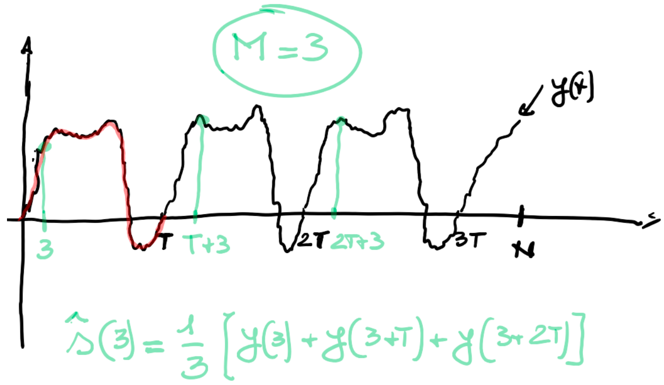
- s(t) is **not a trend** → the process (raw data) y(t) can have **both** a trend and a seasonal behaviour. **FIRST** remove the trend **THEN** the seasonal behaviour.
- If we don't remove a seasonal behaviour we end up with an ARMA model having a pair of complex conjugate poles at:

$$e^{\pm j\Omega}, \Omega = \frac{2\pi}{T}$$

The seasonal component of raw data can be estimated :

$$t = 1, 2, 3...N \text{ , } M \cdot T \leq N$$

$$\boxed{\frac{1}{M}\sum_{h=0}^{M-1} y(t+hT) \text{ , t=1,2,3...N}}$$



Once $\hat{s}$ is computed we can remove it from y(t).
After modelling $\tilde{y}(t)$ with an ARMA model the prediction will be:

$$\boxed{\hat{y}(t+1|t) = \hat{\tilde{y}}(t+1|t) + \hat{s}(|t+1|_T)}$$

1.4 02-20-2013

Q:Give the definition of spectral density of a stochastic process y(t) and list its properties

The spectral density describes the distribution of power into frequency components composing that signal.According to Fourier analysis any physical signal can be decomposed into a number of discrete frequencies, or a spectrum of frequencies over a continuous range. The statistical average of a certain signal or sort of signal (including noise) as analyzed in terms of its frequency content, is called its spectrum. The **power density / spectral density / spectrum** of a **SSP** y(t) :

$$\Gamma_y(w) = \sum_{\tau=-\infty}^{\infty} \gamma_y(\tau) e^{-jw\tau}$$

where $\Gamma_y(w)$ is the **Discrete Fourier Transform**.
Properties :

1. $\Gamma_y(w)$ is a **real** function of a **real** variable w which means that $Im\{\Gamma_y(w)\} = 0$

2. $\Gamma_y(w)$ is a **positive** function which means that $\Gamma_y(w) \geq 0, \forall w \in \Re$

3. $\Gamma_y(w)$ is an **even** function which means that $\Gamma_y(w) = \Gamma_y(-w)$

4. $\Gamma_y(w)$ is a **periodic** function of period 2π which means that $\Gamma_y(w) = \Gamma_y(w + k - 2\pi)$.

Q:Prove (the complete proof is required) that if $S \in M(\theta)$, a P.E.M. method can guarantee that the estimated model is the true model asymptotically.

Lets assume that the **real system S** that has generated the dataset is within the model class : $\mathbf{S} \in m(\theta) \rightarrow a\theta^0$ exists so that $m(\theta^0) = \mathbf{S}$.

$$\text{Is } \theta^0 = \bar{\theta}?$$

In other words , is the P.E.M performance index able to select the **true** parameter θ^0?

Proof

Consider the prediction error $\epsilon(t, \theta) = y(t) - \hat{y}(t|t-1, \theta)$. Add on both sides $-\hat{y}(t|t-1, \theta^0)$:

$$\epsilon(t, \theta) - \hat{y}(t|t-1, \theta^0) = y(t) - \hat{y}(t|t-1, \theta) - \hat{y}(t|t-1, \theta^0)$$

Where $y(t) - \hat{y}(t|t-1, \theta^0)$ is the **white noise** e(t) of the true system **S**.

$$\epsilon(t, \theta) = e(t) - (\hat{y}(t|t-1, \theta^0) - \hat{y}(t|t-1, \theta))$$

Square and apply expected value:

$$E[\epsilon(t, \theta)^2] = E[e(t)^2] + E[(\hat{y}(t|t-1, \theta^0) - \hat{y}(t|t-1, \theta))^2] + 2E[e(t)(\hat{y}(t|t-1, \theta^0) - \hat{y}(t|t-1, \theta))]$$

The last term is =0 because e(t) cannot be correlated with $\hat{y}(t|t-1,\theta)$ or $\hat{y}(t|t-1,\theta^0)$. Remembering that $E[\epsilon(t,\theta)^2] = \bar{J}(\theta)$ and $E[e(t)^2] = var[e(t)] = \lambda^2$:

$$\bar{J}(\theta) = \lambda^2 + E[(\hat{y}(t|t-1,\theta^0) - \hat{y}(t|t-1,\theta))^2]$$

$$E[(\hat{y}(t|t-1,\theta^0) - \hat{y}(t|t-1,\theta))^2] = \begin{cases} \geq 0 & \text{if } \theta \neq \theta^0 \\ 0 & \text{if } \theta = \theta^0 \end{cases}$$

So $\bar{J}(\theta) \geq \lambda^2 = \bar{J}(\theta^0)$ which means that θ^0 is the global minimum of $\bar{J}(\theta)$

$$\bar{\theta} = \theta^0$$

The P.E.M provides the **true model** if $\mathbf{S} \in m(\theta)$