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## Invited Review

### Review of energy-efficient train control and timetabling

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## ABSTRACT

The energy consumption of trains is highly efficient due to the low friction between steel wheels and rails, although the efficiency is also influenced largely by the driving strategy applied and the scheduled running times in the timetable. Optimal energy-efficient driving strategies can reduce operating costs significantly and contribute to a further increase of the sustainability of railway transportation. The railway sector hence shows an increasing interest in efficient algorithms for energy-efficient train control, which could be used in real-time Driver Advisory Systems (DAS) or Automatic Train Operation (ATO) systems. This paper gives an extensive literature review on energy-efficient train control (EETC) and the related topic of energy-efficient train timetabling (EETT), from the first simple models from the 1960s of a train running on a level track to the advanced models and algorithms of the last decade dealing with varying gradients and speed limits, and including regenerative braking. Pontryagin's Maximum Principle (PMP) has been applied intensively to derive optimal driving regimes that make up the optimal energy-efficient driving strategy of a train under different conditions. Still, the optimal sequence and switching points of the optimal driving regimes are not trivial in general, which led to a wide range of optimization models and algorithms to compute the optimal train trajectories and more recently to use them to optimize timetables with a trade-off between energy efficiency and travel times.

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## 1. Introduction

Global warming is an increasingly important topic these days. One of the causes of global warming is the increasing amount of carbon dioxide ( $\text{CO}_2$ ) emissions which comes for a large part from transport. Therefore, the European Union (EU) set targets to decrease these  $\text{CO}_2$  emissions. One of the sectors affected by these measures is the railway sector. For the railway sector targets are set by the UIC (International Union of Railways) and CER (Community of European Railway and Infrastructure Companies). The short term target is to decrease  $\text{CO}_2$  emissions by 30% over the period 1990 to 2020, with a further decrease by 50% in 2030 (UIC, 2012). Furthermore, energy consumption of railway companies should be decreased in 2030 by 30% compared to 1990. A further incentive for railway undertakings to reduce energy consumption is the

reduced operating costs and enlarged competitive advantages involved.

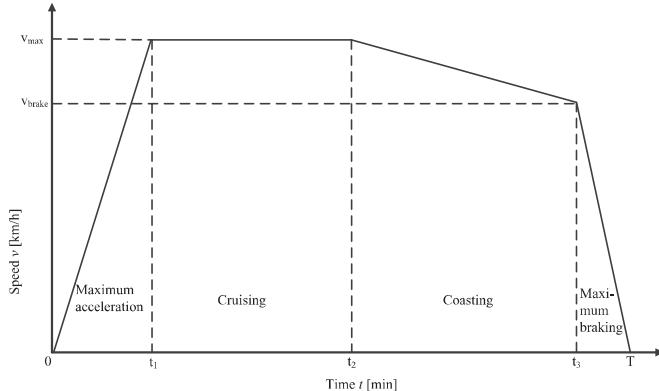
As a consequence, railway companies in Europe have started research on opportunities to decrease energy consumption in order to be sustainable and more profitable in the future. Several ways to achieve this goal are as follows:

- An operator can deploy rolling stock that is more energy-efficient (due to more efficient engines or streamlining).
- An operator may better match the capacities of the trains with the demand, so that fewer empty seats are moved around.
- An operator can deploy measures concerning heating, cooling, lighting, etc. of parked trains during nights in order to save energy.
- Energy-efficient train control (EETC) or eco-driving may be applied, in which a train is driven with the least amount of traction energy, given the timetable.
- The timetable may be constructed in such a way that it allows EETC most effectively, resulting in energy-efficient train timetabling (EETT).

This paper focuses on the last two options: energy-efficient train control (EETC) and energy-efficient train timetabling (EETT). A good

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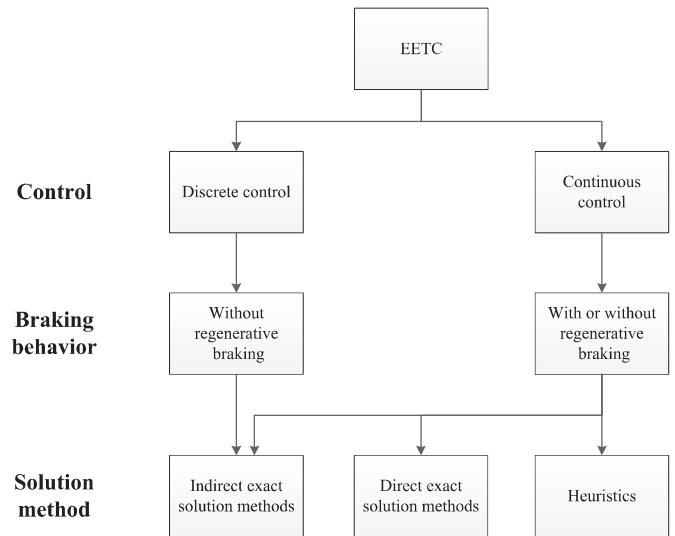


**Fig. 1.** Optimal driving regimes for energy-efficient driving on flat track as function of time with switching points at  $t_1$ ,  $t_2$  and  $t_3$ .

overview of different measures in order to decrease energy consumption for urban rail transport can be found in [González-Gil, Palacin, Batty, and Powell \(2014\)](#).

EETC has been and is a hot topic in the literature. Much research effort aims at finding the optimal driving strategies of a train that minimize energy consumption ([Khmelnitsky \(2000\)](#); [Liu and Golovitcher \(2003\)](#); [A. Albrecht, Howlett, Pudney, Vu, and Zhou \(2015b,c\)](#)). Most of this research is based on optimal control theory, and in particular on Pontryagin's Maximum Principle (PMP) ([Pontryagin, Boltyanskii, Gamkrelidze, & Mishchenko, 1962](#)), to derive the optimal control. This leads to optimal driving regimes such as maximum acceleration, cruising, coasting and maximum braking, see Fig. 1. The problem is then to find the optimal sequence of these driving regimes and the switching points between the regimes for a range of different circumstances and train types. The optimal driving strategy must then be translated into feasible and understandable advice to train drivers in real-time. This generated considerable research in developing *Driver Advisory Systems* (DAS) that provide specific speed advice to the train drivers with the main challenge to incorporate the current delays into the advice ([Kent, 2009](#); [ON-TIME, 2013](#); [Panou, Tzieropoulos, & Emery, 2013](#)). Energy savings between 20–30% have been reported when applying EETC in a DAS compared to normal train operation, for example see [Franke, Terwiesch, and Meyer \(2000\)](#) and [ON-TIME \(2014a\)](#).

The impact of train operation on energy savings depends on the timetable. More recently this led to research on the topic of optimal running time supplements ([Scheepmaker & Goverde, 2015](#)). A running time supplement is the extra running time on top of the technically minimum running time between two stations which is included in the timetable primarily to manage disturbances in operations and to recover from small delays. However, if a train is punctual then these supplements can be used for energy-efficient driving. Nevertheless, in practice energy efficiency is not yet considered in the construction of timetables which sometimes leads to allocating most running time supplements before main stations where punctuality is measured at the cost of insufficient supplements or even unrealizable running times earlier on the route. Another recent stream of research considers the synchronization of accelerating and braking trains to support *regenerative braking*, like [T. Albrecht \(2004\)](#). With regenerative braking, kinetic energy is converted into electricity that is fed back to the power supply system to be used by other (nearby) trains. A more detailed description about the working of regenerative braking and different regenerative braking technologies for urban transport can be found in the review paper of [González-Gil, Palacin, and Batty \(2013\)](#). Energy savings up to 35% have been reported after timetable optimization compared to using the normal timetable, for example see [T. Albrecht and Oettich \(2002\)](#) and [Sicre et al. \(2010\)](#).



**Fig. 2.** Framework of EETC.

This paper provides a thorough review of the literature on energy-efficient train control and timetabling, starting with the first simple models from the 1960s of a train running on a level track to the advanced models and algorithms of the last decade dealing with varying gradients and speed limits, and including regenerative braking. The focus is on the differences between the mathematical models and algorithms in terms of applicability, accuracy and computation time, and their main conclusions on the structure of the optimal driving strategy.

Our method is based on a literature study focussed on EETC and EETT. We structured the publications based on the frameworks shown in Figs. 2 and 9. Our review paper includes publications up to January 2016. The recent paper by [X. Yang, Li, Ning, and Tang \(2016\)](#) also provides a review of EETC and EETT with a focus on urban rail. In contrast to that paper, we consider general railway systems and focus on the differences in the mathematical problem formulations and solution approaches.

[Section 2](#) introduces a basic EETC problem and outlines the mathematics involved. [Section 3](#) reviews the EETC literature building on the concepts and terminology of the basic model. The application of EETC in EETT is the topic of [Section 4](#), which reviews the related literature on the optimization of running time supplements and the synchronization of accelerating and braking trains. Finally, [Section 5](#) ends this literature review with the main conclusions and an outlook to future research directions of EETC and EETT.

## 2. A basic model and solution approaches

This section considers a basic optimal train control problem to define the basic notation and illustrate the main modelling concepts which will be extended later in the paper. This problem was analysed by [Milroy \(1980\)](#) in the late 1970s as one of the first optimal train control problems. Here, we give a modern analysis. A rigorous mathematical treatment and further extensions are given in [Howlett and Pudney \(1995\)](#) and [A. Albrecht et al. \(2015b,c\)](#).

### 2.1. A basic energy-efficient train control model

Consider the problem of driving a train from one station to the next along a flat track within a given allowable time  $T$  in such a way that energy consumption is minimized. The train speed  $v(t)$  at time  $t$  is governed by a tractive or braking effort  $F(t)$  and a

resistance force  $R(v)$  according to the Newton force equilibrium

$$\rho m \dot{v}(t) = F(t) - R(v(t)), \quad (1)$$

where  $\dot{v} = dv/dt$  is the derivative of speed to time,  $m$  is the train mass and  $\rho$  the dimensionless rotating mass factor (Brünger & Dahlhaus, 2014). The force  $F$  is the tractive effort of the engine for  $F \geq 0$  and the braking effort due to the brakes for  $F < 0$ . The maximum tractive effort  $F_{\max}$  is a non-increasing function of speed, which is approximated by a piecewise linear, quadratic and/or hyperbolic function of speed depending on the engine (Brünger & Dahlhaus, 2014). The maximum braking force  $F_{\min}$  is usually approximated based on a constant braking rate (independent of speed). The resistance force is given by the Davis equation  $R(v) = R_0 + R_1 v + R_2 v^2$  with non-negative coefficients  $R_i \geq 0$ ,  $i \in \{0, 1, 2\}$ , which is a strictly increasing quadratic function in speed (Davis, 1926). The constant and linear coefficients are rolling resistances and depend on mass, while the quadratic term is the air resistance which is mass independent.

It is convenient to normalize the equations as mass-specific by the specific resistance  $r(v) = R(v)/\rho m = r_0 + r_1 v + r_2 v^2$  and the specific tractive effort  $u(t) = F(t)/\rho m$  with  $u(t) \in U = [-u_{\min}, u_{\max}(v(t))]$  for  $t \in [0, T]$ , where

$$u_{\max}(v) = \frac{F_{\max}(v)}{\rho m} > 0 \text{ and } u_{\min} = \frac{F_{\min}}{\rho m} > 0.$$

Recall that  $r(v)$  is strictly increasing and  $r_i \geq 0$ ,  $i \in \{0, 1, 2\}$ .

The energy consumption to be minimized is the work done by the traction power  $P(t) = F(t)v(t)$  over time, i.e.,  $\int u^+(t)v(t)dt$ , where the integral is only over the (positive) specific tractive effort denoted as

$$u^+(t) = \max(u(t), 0). \quad (2)$$

Note that in this example we assume that braking does not cost nor generate energy. We finally get the basic optimal train control problem

$$J = \min_u \int_0^T u^+(t)v(t)dt \quad (3)$$

subject to

$$\dot{x}(t) = v(t) \quad (4)$$

$$\dot{v}(t) = u(t) - r(v(t)) \quad (5)$$

$$x(0) = 0, x(T) = X, v(0) = 0, v(T) = 0 \quad (6)$$

$$v(t) \geq 0, u(t) \in [-u_{\min}, u_{\max}(v(t))], \quad (7)$$

where  $x(t)$  is the distance travelled over time, and  $X$  is the total distance travelled. The variables  $(x, v)$  are the state variables and  $u$  is the control variable.

## 2.2. Pontryagin's Maximum Principle

This optimal control problem has the standard form  $\min_u \int_0^T f_0(x, v, u)dt$ , subject to the ordinary differential equations  $\dot{x}(t) = f_1(x, v, u)$  and  $\dot{v}(t) = f_2(x, v, u)$  with boundary conditions for  $x$  and  $v$  and (algebraic) path constraints  $g_i(x, v, u) \geq 0$ ,  $i = 1, \dots, n$ , as given in (7). Note that the control is bounded from above by a mixed constraint that depends on the state  $v$ . Necessary conditions for these optimal control problems are given by Pontryagin's Maximum Principle (Pontryagin et al., 1962). According to the PMP the optimal control variable  $\hat{u}$  should be selected from the admissible control variables that maximize the Hamiltonian

$$H(x, v, \varphi, \lambda, u) = -f_0(x, v, u) + \varphi f_1(x, v, u) + \lambda f_2(x, v, u), \quad (8)$$

where  $(\varphi, \lambda)$  are the co-state (or adjoint) variables which satisfy the differential equations

$$\dot{\varphi}(t) = -\frac{\partial \tilde{H}}{\partial x}(x, v, \varphi, \lambda, \mu, u) \text{ and } \dot{\lambda}(t) = -\frac{\partial \tilde{H}}{\partial v}(x, v, \varphi, \lambda, \mu, u) \quad (9)$$

without boundary conditions. Here,  $\tilde{H}$  is the augmented Hamiltonian (or Lagrangian)

$$\tilde{H}(x, v, \varphi, \lambda, \mu, u) = H(x, v, u, \varphi, \lambda, u) + \sum_{i=1}^n \mu_i g_i(x, v, u), \quad (10)$$

with respect to the additional path constraints  $g_i(x, v, u) \geq 0$ , where  $\mu_i$  are Lagrange multipliers satisfying the complementary slackness conditions  $\mu_i \geq 0$  and  $\mu_i g_i(x, v, u) = 0$ . Moreover, the Karush–Kuhn–Tucker (KKT) necessary condition  $\partial \tilde{H}/\partial u = 0$  must be satisfied by the optimal solution.

Note that the differential Eqs. (4) and (5) of the state variables satisfy  $\dot{x}(t) = \partial \tilde{H}/\partial \varphi = v$  and  $\dot{v}(t) = \partial \tilde{H}/\partial \lambda = u - r(v)$ , so that we end up with a special boundary value problem of four differential equations in four variables with four boundary conditions. Unfortunately, the boundary conditions are both the initial and final conditions for the state equation, and none for the co-state equation. (If the final state is free, then the final co-states must be zero,  $\varphi(T) = \lambda(T) = 0$ , which is easier to solve.)

For the example problem we get the following Hamiltonian:

$$H(x, v, \varphi, \lambda, u) = -vu^+ + \varphi v + \lambda(u - r(v)), \quad (11)$$

and augmented Hamiltonian

$$\tilde{H}(x, v, \varphi, \lambda, \mu, u) = H(\cdot) + \mu_1(u_{\max}(v) - u) + \mu_2(u + u_{\min}), \quad (12)$$

with the additional differential equations for the co-state  $(\varphi, \lambda)$

$$\dot{\varphi}(t) = 0 \quad \text{and} \quad \dot{\lambda}(t) = \lambda r'(v) - \varphi + u^+ - \mu_1 u'_{\max}(v). \quad (13)$$

From the first equation of (13) it follows that  $\varphi = \varphi_0$  is a constant. Moreover, from the complementary slackness conditions follows that  $\mu_1 = 0$  if  $u < u_{\max}(v)$ , and  $\mu_1 \geq 0$  if  $u = u_{\max}(v)$  (maximum acceleration).

According to the PMP the optimal control is

$$\hat{u}(t) = \arg \max_{u \in U} H(\hat{x}(t), \hat{v}(t), \hat{\varphi}(t), \hat{\lambda}(t), u), \quad (14)$$

where  $(\hat{x}, \hat{v})$  and  $(\hat{\varphi}, \hat{\lambda})$  are the state and co-state trajectories associated to the control trajectory  $\hat{u}$ . Typical for an optimal train control problem is that the Hamiltonian is (piecewise) linear in the control variable  $u$ , by which the optimal control may not be uniquely defined from the necessary conditions on some nontrivial interval. For the example problem, the Hamiltonian (11) can be split around  $u = 0$ , yielding

$$H(x, v, \varphi, \lambda, u) = \begin{cases} (\lambda - v)u + \varphi v - \lambda r(v) & \text{if } u \geq 0 \\ \lambda u + \varphi v - \lambda r(v) & \text{if } u < 0, \end{cases} \quad (15)$$

which is linear both for non-negative and negative values of  $u$ . Discarding for the moment the control constraints  $u \in U$ , the optimal control must satisfy the necessary optimality condition  $\partial H/\partial u(x, v, \varphi, \lambda, u) = 0$ , giving  $\lambda - v = 0$  for  $u \geq 0$  and  $\lambda = 0$  for  $u < 0$ , which are independent of the value of the control variable  $u$  (besides its sign). Taking also the control constraints (7) into

account, the optimal control is characterized as

$$\hat{u}(t) = \begin{cases} u_{\max}(v(t)) & \text{if } \lambda(t) > v(t) \\ u \in [0, u_{\max}] & \text{if } \lambda(t) = v(t) \\ 0 & \text{if } 0 < \lambda(t) < v(t) \\ -u_{\min} & \text{if } \lambda(t) \leq 0 \end{cases} \quad (\text{MA}) \quad (\text{CR}) \quad (\text{CO}) \quad (\text{MB}). \quad (16)$$

The optimal control is illustrated in Fig. 1. Clearly, the maximum control  $\hat{u} = u_{\max}$  implies maximum acceleration (MA), zero control  $\hat{u} = 0$  implies coasting (CO), i.e., rolling with the engine turned off, and the minimum control  $\hat{u} = -u_{\min}$  implies maximum braking (MB). The singular solution defined by  $\lambda(t) = v(t)$  corresponds to speed-holding or cruising (CR), i.e., driving at a constant optimal cruising speed using partial tractive effort  $\hat{u} \in [0, u_{\max}]$ . To see this, note that the singular solution only holds over some nontrivial interval if also the derivatives are the same,  $\dot{\lambda}(t) = \dot{v}(t)$ . Moreover,  $\mu_1 = 0$  since  $u < u_{\max}(v)$  except maybe at some isolated points as otherwise we are back in regime (MA). Then from (5), (13),  $\lambda(t) = v(t)$  and  $u > 0$ , it follows that the optimal cruising speed must satisfy

$$\varphi_0 = vr'(v) + r(v). \quad (17)$$

This equation has a unique solution  $v_c$  which gives the optimal cruising speed over some interval. Recall that  $r(v)$  is a non-negative strictly-increasing quadratic function and thus convex in  $v$ . Then also  $\psi(v) = vr(v)$  is a non-negative strictly-increasing convex function for  $v \geq 0$  with  $\psi'(v) = vr'(v) + r(v)$ , and in particular  $\psi'(v) \geq \psi'(0) = r(0) = r_0$ . Hence, by (17) a unique optimal cruising speed exists if  $\varphi_0 > r_0$ . Also note that this implies  $\varphi_0 > 0$  and thus the solution  $\lambda = 0$  with  $u < 0$  cannot hold except at a single time point, since in this case we get  $\dot{\lambda}(t) = -\varphi_0 < 0$  and therefore  $\lambda$  is not constant over a nontrivial interval. So without loss of generality, we added the singular point  $\lambda = 0$  in (16) to the (MB) regime. Later, we will see that the singular solution  $\lambda = 0$  may occur when considering gradients.

However, finding the optimal cruising speed usually takes some creativity since (17) has two unknowns  $v$  and  $\varphi_0$ . An additional equation can be obtained from the PMP which also states that the Hamiltonian is constant along the optimal control and state trajectories (if the cost and dynamic equations are independent of time), i.e.,

$$H(\hat{x}(t), \hat{v}(t), \hat{\lambda}(t), \hat{\varphi}(t), \hat{u}(t)) = C \quad \text{for all } t \in [0, T]. \quad (18)$$

So the Hamiltonian is kept at its maximum value along the optimal control and state trajectories. In the example problem for the singular solution under  $v = \lambda$ , (18) gives  $\varphi_0 v - vr(v) = C$ . After substituting (17) this gives the additional equation

$$v^2 r'(v) = C \quad (19)$$

with the additional unknown  $C$ . Note that from (19) follows  $C \geq 0$ . Still we end up with two equations in three unknowns. In general, the cruising speed  $v_c$  can be parameterized in either  $\varphi_0$  or  $C$  and then solved for the optimal parameter using a numerical procedure. Nevertheless,  $v_c$  can also be considered as a parameter itself.

### 2.3. Solution approaches

The optimal control problem can be reformulated as a boundary value problem in  $(x, v, \varphi, \lambda)$  connected by the optimal control structure (16). Starting with estimates for the initial values  $\varphi(0) = \varphi_0$  and  $\lambda(0)$ , first the optimal cruising speed  $v_c$  is computed from (17) which is then used for the cruising regime in (16). Then the trajectories for  $(x, v, \lambda)$  could in principle be computed as an initial value problem forward in time  $t$  using a shooting method (Sroer & Bulirsch, 2002), with  $u(t)$  specified by (16) depending on the computations of  $v(t)$  and  $\lambda(t)$ . If the computed final values  $x(T)$

and  $v(T)$  are equal to the boundary conditions (6) then we have found the optimal trajectories. Otherwise, the initial values are adjusted and the procedure starts again. However, shooting methods are really sensitive to the initial values and this procedure does not work well in practice.

A different approach to solve the optimal train control problem is by constructive methods. These methods are based on the observation that an optimal driving trajectory must be a concatenation of the four optimal driving regimes given by (16), in the case of flat track. Then the problem is replaced by finding the optimal order of driving regimes and the switching times between regimes, along with a possible optimal cruising speed. For the example problem, the optimal order of the driving regimes is maximum acceleration, cruising, coasting and maximum braking, while the cruising regime may also be absent. The basic decision variables then become the switching time from acceleration to cruising (and thus the cruising speed  $v_c$ ) and the switching time from cruising to coasting, if both these regimes are optimal, or a direct switching from acceleration to coasting. Note that the switching time to the final braking regime is implicitly determined when the speed trajectory reaches the braking curve in time to reach the destination  $X$  at time  $T$ . This braking curve can be computed by solving  $\dot{v}(t) = -u_{\min} - r(v(t))$  backwards from  $v(T) = 0$ .

The switching times and the number of driving regimes depend on the terminal time  $T$ . For the example problem, the optimal driving trajectory may consist of maximum acceleration to some switching speed, coasting and maximum braking (the case of short distance with sufficient time) or maximum acceleration to the cruising speed, cruising, coasting and maximum braking (long distance with sufficient time). Note that coasting is always present due to the continuity of the co-state variable (in this case that the dynamic equations do not depend explicitly on time), although the coasting regime can be very short depending on the terminal time. The minimal feasible terminal time corresponds to maximum acceleration to the maximum speed, cruising at maximum speed, and maximum braking, i.e., time-optimal driving for the minimal running time  $T_{\min}$ . Note that the minimal-time train control problem is a slightly different optimal control problem with a variable terminal time that needs to be minimized. Hence, the time-optimal solution is not energy-efficient. In the energy-efficient train control problem the lengths of the coasting and cruising regimes depend on the available running time supplement  $T - T_{\min}$ . An energy-efficient solution exists only if the scheduled running time exceeds the minimal running time.

### 3. Energy-efficient train control

This section gives a literature review of EETC models and solution methods. The review is mainly chronological where the first simple models are extended and adapted to derive more complex models. We will use the concepts and terminology introduced in the description of the basic model in Section 2 to provide a consistent terminology throughout the review.

A distinction can be made between models with continuous traction control (such as in Section 2) and models with discrete traction throttle settings. Moreover, regenerative braking may be used or not. The review is clustered in these distinct classes.

Another clustering can be obtained through the solution method applied. Two main solution approaches can be distinguished which are both explicitly or implicitly based on the optimal control structure derived from the optimality conditions of Pontryagin's Maximum Principle (PMP) such as discussed in Section 2. These are exact solutions by numerical algorithms that solve the differential equations indirectly using the derived optimal control structure, and heuristics that find suboptimal solutions to the dynamic equations by artificial intelligence or search

algorithms using knowledge of the optimal control structure. A third solution approach is to solve the optimal control problem by transcribing the problem into a nonlinear optimization problem and solving this problem directly, as opposed to indirectly solving the necessary optimality conditions.

The remainder of this section considers subsequently indirect exact methods without and with regenerative braking, indirect exact methods with discrete control, direct methods, and heuristic methods. An overview of the framework that we used for the classification of EETC can be found in Fig. 2.

### 3.1. Exact methods without regenerative braking

The first study on energy-efficient train control was carried out by Ichikawa (1968) in Japan. His model is similar to the basic model discussed in Section 2, but the resistance force was simplified as  $r(v) = v$  by which (5) reduces to the linear differential equation  $\dot{v}(t) = u(t) - v(t)$ . Since now both differential equations are linear, Ichikawa could derive analytical expressions for the various regimes by applying the PMP. He gave a complete analysis of all four driving regimes on level tracks:

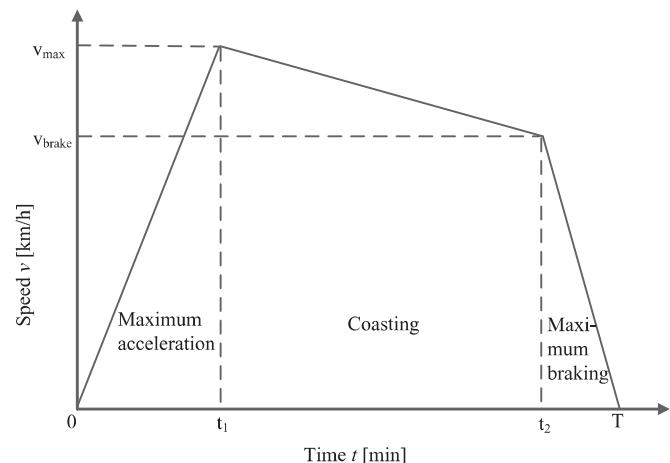
1. Maximum acceleration (MA),
2. Cruising by partial traction force (CR),
3. Coasting (CO), and
4. Maximum braking (MB),

as well as the resulting optimal control rules. In the conclusions he mentioned that "Considerable idealization has been made on the equations of motion for the train in this report, but the basic point seems to have been revealed about the optimal operation of a train. The author believes that the report will serve to make the beginning of scientific and reliable research on the economization of train operation for which huge amount of energy is consumed everyday." (Ichikawa, 1968, p. 865)

Strobel, Horn, and Kosemund (1974) continued the research for the optimal control strategy of a train with a model similar to that of Ichikawa (1968), but they modelled the resistance force as a quadratic function of speed with an additional term for gradient resistance. Nevertheless, they then linearized the resistance function and thus could derive analytical expressions for all driving regimes using the PMP as well. As a result of the possible negative slopes they found a second singular solution consisting of partial braking to maintain cruising, although they stated that this solution was "practically without significance" (Strobel et al., 1974, p. 379). Strobel et al. (1974) thus found five driving regimes for varying gradients:

1. Maximum acceleration (MA),
2. Cruising by partial traction force (CR1),
3. Coasting (CO),
4. Cruising by partial braking (CR2), and
5. Maximum braking (MB).

They mentioned that for suburban train traffic the cruising regimes could be neglected. This further simplification allowed them to derive a suboptimal algorithm for real-time computation. They implemented their algorithm and compared the resulting computer-aided train operation with manually controlled train movements in a train simulator, which revealed a substantially improved adherence to timetables and driving energy savings of approximately 15%. Note that the energy savings compared to using technical minimum running times will be higher than the energy savings that are achieved in practice, since not all drivers without DAS drive as fast as possible. The approach was translated into a driver advice about the optimal driving regime. The algorithm of Strobel et al. (1974) formed the basis for the first DAS implemented



**Fig. 3.** Optimal driving regimes without cruising (for metro and suburban railway systems) over time with switching points between driving regimes at  $t_1$  and  $t_2$ .

in board computers of the Berlin S-Bahn (suburban trains) in Germany at the beginning of the 1980s. However, the computations to determine the switching points were made offline, due to the limited computational power of the computers in those days (Oettich & Albrecht, 2001).

T. Albrecht and Oettich (2002) revisited the research of Strobel et al. (1974) to determine the optimal driving strategy for a single train using the linearized resistance equations. They used Simulink to numerically calculate switching curves that could be used to calculate the switching points in the optimal trajectory backwards from the target station. The control algorithm was applied in a DAS on the train driving simulator at Dresden University of Technology (TU Dresden), and in real-time passenger operation at the suburban railway line S1 in Dresden, see T. Albrecht (2005a). The successful real-time test showed energy savings of 15% to 20% compared with manual driving.

Since 1982 a lot of research about optimal train control has been carried out by the University of South Australia (UniSA). The research started with the Ph.D research of Milroy on continuous train control similar to the basic model of Section 2 (Howlett & Pudney, 1995; Milroy, 1980). Milroy (1980) applied the PMP and concluded based on his research on urban railway transport that there are three driving regimes in the optimal driving strategy for urban railways on level track and with a fixed speed limit (see Fig. 3):

1. Maximum acceleration (MA),
2. Coasting (CO), and
3. Maximum braking (MB).

Later, Howlett proved mathematically based on the PMP that the general optimal driving strategy for level track and a fixed speed limit consists of four driving regimes including cruising (Howlett, 1990), which had already been found by Strobel et al. (1974).

The theoretical ideas of continuous energy-efficient train control were implemented by UniSA in a commercial system named Metromiser. The system consisted of two parts: a software package for timetable planners to generate energy-efficient timetables, and a DAS for energy-efficient train operation (Howlett & Pudney, 1995). The DAS part of Metromiser advised the train driver when to coast and when to brake in order to minimize energy consumption using light and sound indications (Benjamin, Long, Milroy, Payne, & Pudney, 1987; Cheng, 1997; Howlett, 1996; Howlett, Milroy, & Pudney, 1994; Howlett & Pudney, 1995). However, Metromiser assumed a constant effective gradient during coasting and

braking phases (Pudney & Howlett, 1994). The first successful runs with the system were done on the (sub)urban trains in Adelaide (Australia) in 1984, and later in Toronto (Canada), Melbourne (Australia) and Brisbane (Australia). The achieved energy savings were more than 15% compared to the trains running without Metromiser, and also punctuality increased. Benjamin et al. (1987) and Howlett et al. (1994) showed that for suburban trains to which Metromiser was applied the coasting phase is the most important driving regime due to the short stop distances.

Around 1990, Netherlands Railways (NS, Nederlandse Spoorwegen) also investigated the EETC problem. Van Dongen and Schuit (1989a,b, 1991) investigated the optimal driving strategy and found the four optimal driving regimes by measurements and experience. Static advice about cruising and coasting was included in the timetable for the train drivers on the intercity line between Zandvoort and Maastricht/Heerlen in the Netherlands. Results with the optimal driving strategy indicated energy savings of 10% compared to the normal practice of train operation with reduced constant timetable speeds. Moreover, they found that both optimizing the cruising speed (by applying constant power) and the coasting distance led to the most energy savings. In addition, Van Dongen and Schuit (1989a,b, 1991) found that it is even better not to apply maximum acceleration for the Dutch power supply system with its low voltage, since this led to a high drop in voltage and energy losses. They therefore recommended to apply a low and constant line current in consideration of the low catenary voltage.

Liu and Golovitcher (2003) considered the EETC problem with varying gradients and speed limits. Since both gradients and speed limits are functions of distance, they reformulated the optimal control problem with distance as the independent variable instead of time. This change of independent variable had been proposed before by Howlett et al. (1994) and Pudney and Howlett (1994) to deal with varying speed limits, and by Howlett and Pudney (1995) for both varying speed limits and gradients. With distance  $x$  as independent variable, the state variables now become time  $t(x)$  and speed  $v(x)$ , and the energy consumption equals  $\int u^+(x)dx$ , where now  $u(x) = F(x)/\rho m$  with  $u(x) \in U = [-u_{\min}, u_{\max}(v(x))]$  for  $x \in [0, X]$ . Note that

$$\int_0^T u^+(t)v(t)dt = \int_0^T u^+(t)\frac{dx}{dt}dt = \int_0^X u^+(x)dx.$$

The equivalent EETC problem in terms of distance with the additional speed limit upper bounds is then:

$$J = \min_u \int_0^X u^+(x)dx \quad (20)$$

subject to the constraints

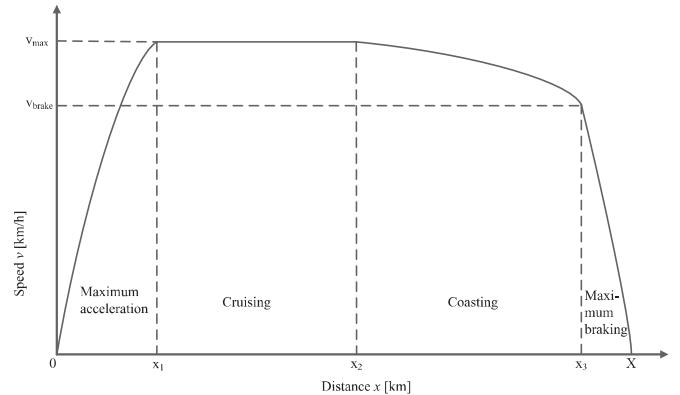
$$\dot{t}(x) = 1/v(x) \quad (21)$$

$$\dot{v}(x) = (u(x) - r(v(x)) - g(x))/v(x) \quad (22)$$

$$t(0) = 0, t(X) = T, v(0) = 0, v(X) = 0 \quad (23)$$

$$v(x) \in [0, v_{\max}(x)], u(x) \in [-u_{\min}, u_{\max}(v(x))], \quad (24)$$

where  $X$  is the total distance travelled,  $t(x)$  is the time over the distance travelled, and  $T$  is the total available running time. The variables  $(t, v)$  are the state variables and  $u$  is the control variable. Note that now  $\dot{v}(x)$  and  $\dot{t}(x)$  denote the derivatives of  $v$  and  $t$  with respect to the independent variable  $x$ . The resistance force now consists of a train resistance  $r(v)$  and a line resistance  $g(x)$ , where line resistance  $g(x)$  is defined as the specific external force due to track gradient or curvature. It is assumed that tracks have piecewise constant gradients. Note that on uphill slopes  $g(x) > 0$  and on downhill slopes  $g(x) < 0$ . Total resistance may be defined now also as  $r(v, x) = r(v) + g(x) = r_2 v^2 + r_1 v + r_0 + g(x)$ , which is thus explicitly a function of both speed and distance.



**Fig. 4.** Speed profile of a basic energy-efficient driving strategy with switching points between driving regimes at  $x_1$ ,  $x_2$  and  $x_3$ .

The optimal control  $\hat{u}(x)$  can be derived similar to Section 2 by applying Pontryagin's Maximum Principle as follows. The Hamiltonian is

$$\begin{aligned} H(t, v, \varphi, \lambda, u, x) \\ = -u^+ + \frac{\phi}{v} + \frac{\lambda(u - r(v) - g(x))}{v} \\ = \begin{cases} \left(\frac{\lambda}{v} - 1\right)u + \frac{\varphi}{v} - \frac{\lambda}{v}(r(v) + g(x)) & \text{if } u \geq 0 \\ \frac{\lambda}{v}u + \frac{\varphi}{v} - \frac{\lambda}{v}(r(v) + g(x)) & \text{if } u < 0, \end{cases} \end{aligned} \quad (25)$$

and the associated augmented Hamiltonian

$$\tilde{H}(x, v, \varphi, \lambda, \mu, u) = H + \mu_1(u_{\max}(v) - u) + \mu_2(u + u_{\min}) + \mu_3(v_{\max} - v),$$

where  $\varphi(x)$  and  $\lambda(x)$  are the co-state variables satisfying the differential equations

$$\dot{\varphi}(x) = -\frac{\partial \tilde{H}}{\partial t} = 0 \quad (26)$$

$$\begin{aligned} \dot{\lambda}(x) &= -\frac{\partial \tilde{H}}{\partial v} \\ &= \frac{\lambda u + \lambda v r'(v) - \lambda r(v) + \varphi}{v^2} - \mu_1 u'_{\max}(v) + \mu_3. \end{aligned} \quad (27)$$

Note that the Hamiltonian is now also a function of the independent variable  $x$  due to the line resistance  $g(x)$ . Similar to Section 2, the optimal control  $\hat{u}(x)$  that maximizes the Hamiltonian for varying gradients is

$$\hat{u}(x) = \begin{cases} u_{\max}((v(x))) & \text{if } \lambda(x) > v(x) \\ u \in [0, u_{\max}] & \text{if } \lambda(x) = v(x) \\ 0 & \text{if } 0 < \lambda(x) < v(x) \\ u \in [-u_{\min}, 0] & \text{if } \lambda(x) = 0 \\ -u_{\min} & \text{if } \lambda(x) < 0 \end{cases} \quad \begin{array}{l} (\text{MA}) \\ (\text{CR1}) \\ (\text{CO}) \\ (\text{CR2}) \\ (\text{MB}). \end{array} \quad (28)$$

The optimal speed-distance profile for a level track is illustrated in Fig. 4.

Liu and Golovitcher (2003) derived the above five driving regimes from the PMP where the cruising regime is split into partial power and partial braking. The latter may occur on negative gradients. They also showed that the optimal cruising speed  $v_c$  is the root of (19) or the maximum speed, whichever is lower. To determine the sequence of optimal controls they derived four control switching graphs describing the possible switchings between the five driving regimes depending on speed  $v(x)$  and speed limit  $v_{\max}(x)$  at the switching moment, which could either be

$v(x) < v_{\max}(x)$ , or  $v(x) = v_{\max}(x)$  with  $v_{\max}(x)$  remaining constant at  $x$ , dropping down or jumping up. In each of these switching graphs, conditions were derived for switching to another regime depending on the value of speed, optimal cruising speed, (changed) speed limit, and the beginning of a steep climb or end of a steep descent, see also Golovitcher (2001) for more details. For the final determination of the optimal control, Liu and Golovitcher (2003) divided the distance in intervals with constant line resistance  $g(x) = g_i$  on  $(x_{n-1}, x_n)$ . On each of these intervals the dynamic Eq. (22) is again independent of distance. Thus the Hamiltonian (25) is constant there, providing a complementary optimality condition on each interval, cf. (18). Based on the optimal driving regimes, the control switching graphs and the complementary optimality conditions, Liu and Golovitcher (2003) finally derived a numerical algorithm consisting of an outer loop that finds the cruising speed  $v_c$  on each interval of constant line resistance and an inner loop that builds the optimal trajectory for the given values of  $v_c$ . They implemented the algorithm in a simulation and optimization package, which has been applied for crew training and timetable optimization. Several case studies were reported. A simulation of a metro system with Automatic Train Operation (ATO) showed energy savings of 3% compared to using technical minimum running times. Here a simple control algorithm, which computes the required speed based on the remaining time and distance only, was compared with the energy-efficient algorithm that constantly re-calculates the optimal trajectory to the next station using the track gradient profile to find the optimal speed and locations for switching the control.

Vu (2006) also considered the optimal train control problem in speed and time as function of distance and showed that the optimal control for a specific journey on a non-steep track is unique. Based on this research, Howlett, Pudney, and Vu (2009) developed a new local energy minimization principle to calculate the critical switching points on tracks with steep gradients. A steep uphill section is a section in which the train has insufficient power to maintain a cruising speed when climbing, while a steep downhill section is a section in which the train is increasing speed when applying coasting (Vu, 2006). They showed that a maximum acceleration regime is necessary for a steep uphill section and a coasting regime for a steep downhill section. Furthermore, they showed that the necessary conditions defining the optimal switching points near steep gradients are also necessary conditions for minimization of local energy usage subject to a weighted time penalty. This minimization was adopted as a more efficient means to compute the optimal switching points in the DAS Freightmiser for freight trains, the follow-up of CruiseMiser described in Section 3.3. During trial tests in Australia and India in the period between 2002 and 2007, energy savings of about 15% were achieved for freight trains with Freightmiser compared to freight trains without this DAS (Howlett, Pudney, & Vu, 2008). Freightmiser was also tested on a passenger high speed line in the UK with energy savings of 22% compared to normal operation (Coleman, Howlett, Pudney, Vu, & Yee, 2010).

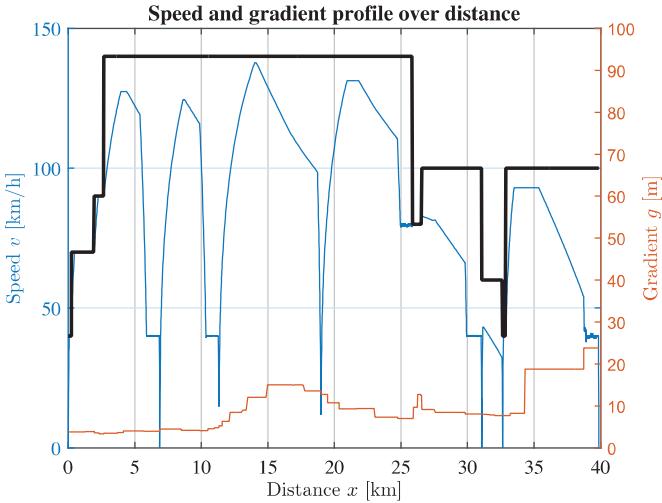
A. Albrecht, Howlett, Pudney, and Vu (2013a) proved that the switching points obtained from the local energy minimization principle are uniquely defined for each steep section of track and therefore also deduced that the global optimal strategy is unique. They now reported an implementation of the algorithm in a DAS called Energymiser, the follow up of Freightmiser. Energymiser has been used with energy savings between 7% and 20% compared to normal driving without Energymiser, see A. Albrecht, Howlett, Pudney, Vu, and Zhou (2015a). In addition, A. Albrecht et al. (2014) showed by means of numerical examples using Energymiser that the optimal train control strategy indeed consists of maximum power instead of partial power for acceleration. The power is then applied for a smaller time resulting in a lower total energy consumption. Recently, the French railway undertaking

SNCF (Société Nationale des Chemins de fer Français) applied Energymiser on their TGV high speed trains using tablets to display driving advice to the train drivers A. Albrecht, Howlett, Pudney, Vu, and Zhou (2015c).

Aradi, Becsi, and Gaspar (2013) used a predictive optimization model to calculate the energy-efficient speed profile taking into account varying gradients and speed limits. Their algorithm considers both the current location of the train and some distance further ahead to make a prediction about the speed profile. The multi-objective function of the algorithm aims at minimizing the total energy consumption and at maximizing punctuality. Sequential quadratic programming (SQP) was used to solve the model. The model was applied in a case study of a locomotive-hauled train on a 15 kilometres track on the Swiss line between Fribourg and Bern, showing energy savings of 15.3% compared to normal operation.

Scheepmaker and Goverde (2015) also considered the EETC model (20)–(24) with varying gradients and speed limits and derived the PMP optimality conditions, see also Scheepmaker (2013). To find the switching points, they developed a two-stage iterative algorithm that calculates the optimal cruising speed using Fibonacci search and the optimal coasting point for the given cruising speed using the bisection method. The algorithm was implemented in MATLAB and applied in a case study on the regional train line between Utrecht Central and Rhenen in the Netherlands. The results from the EETC model were compared with the UZI method applied by train drivers at NS. The UZI method (*Universel Zuinigrijden Idee*, Dutch for universal energy-efficient driving idea) is a simple coasting strategy for short and long distances derived empirically by an enthusiastic train driver. In the UZI method, for short distances with scheduled running time  $t \in \{2, \dots, 8\}$  minutes, the driver accelerates with maximum power to the coasting speed  $v_{Coast}(t) = 60 + 10 \cdot t$  kilometres per hour and then starts coasting. For longer distances, the time to start coasting  $t_{coast}$  before the arrival time at the next station is defined as a function of the track speed limit as  $t_{Coast}(v_{\max}) = 4 + (v_{\max} - 100)/10$  minutes for  $v_{\max} \in \{100, \dots, 140\}$  kilometres per hour (Scheepmaker, 2013; Velthuizen & Ruijsendaal, 2011). The results of Scheepmaker and Goverde (2015) showed that extra energy savings compared to the UZI method of at least 5% were possible by using the EETC model which considers both cruising and coasting, as well as the exact running time supplement and the track and train characteristics. Compared to time-optimal running times the energy savings were 15.7%. With respect to an improved timetable with uniform running time supplements the energy savings increased to 15.9% for UZI and 21.8% for the EETC model compared to time-optimal running. An example of an energy-efficient speed profile including varying gradients and speed limits can be found in Fig. 5. The reported computation time of the EETC algorithm was on average 190 seconds (laptop with 2.1 gigahertz processor speed and 8 gigabytes RAM) for a train run between two stops (including varying gradients). The method could be used for static energy-efficient speed advice with optimal cruising speed and coasting point information for punctual trains.

Su, Li, Tang, and Gao (2013) considered the EETC problem in time (3)–(7) on level track with the simplifying assumptions that the maximum traction, maximum braking and resistance forces are all constants. As a result they could derive analytical expressions for all regimes based on the PMP. Su, Tang, Li, and Gao (2014) extended the previous model with maximum traction, maximum braking and resistance forces as functions of speed although the exact functions are not given. Since their focus is on subway systems, they assumed that the maximum speed is the optimal cruising speed. The computation of the energy-efficient speed profile is based on a given amount of energy available for each section between two stops. The algorithm first discretizes the section into parts of equal distances and then starts with maximum accelera-



**Fig. 5.** Example of an energy-efficient speed profile with varying gradients and speed limits for the line Utrecht Central-Rhenen (adapted from Scheepmaker & Goverde (2015)).

tion on the first part. Then as long as energy is left the train either accelerates with maximum power if the speed is below the maximum speed or cruises with an energy consumption that counters the train and line resistances. When all available energy has been used, both speed profiles for coasting until the end of the section and for maximum braking from the end of the section are computed, with the minimum speed of these profiles giving the final coasting and braking regimes (and their intersection). They applied their algorithms on the Beijing Yizhuang metro line in China in a timetabling algorithm, see Section 4.

A. Albrecht, Howlett, Pudney, Vu, and Zhou (2013b) observed that timetabled arrival times are not always efficient when meeting the specified times requires the train to vary its pace throughout the journey and thus to waste energy. Therefore, they proposed to use time windows which define the earliest and latest arrival time at a specific location to improve the energy-efficient driving strategies. In a case study from the UK they showed that 13% extra energy could be saved with a 1-minute time window, and 18% with a 3-minute time window both compared to no time window. T. Albrecht, Binder, and Gassel (2013c) also described the use of time windows instead of target times at minor stops and junctions to decrease energy consumption. They mentioned that target windows should only be applied if slight delays of a train do not have immediate consequences to surrounding trains. They extended the algorithm described in T. Albrecht and Oettich (2002) to include time intervals without giving the details. Jaekel and Albrecht (2013) further developed the concept of time windows to so-called Train Path Envelopes (sequences of time windows) to limit the time-distance search space for energy-efficient train control with respect to adjacent trains.

### 3.2. Exact methods with regenerative braking

A different option for energy saving is to incorporate regenerative braking where the kinetic energy of the running train is fed back to the catenary system when the train brakes using the regenerative braking. This energy can then be used by other trains so that the overall energy consumption of the train decreases.

Asnis, Dmitruk, and Osmolovskii (1985) studied the energy-efficient train control problem including regenerative braking for level track. They considered the basic problem (3)–(7) but with the adjusted objective function

$$J = \min_u \int_0^T (u^+(t) - \eta u^-(t))v(t)dt, \quad (29)$$

where the second term  $\eta u^- v$  gives the energy regenerated by the braking of the train. Here,  $u^-(t) = -\min(u(t), 0)$  denotes the specific braking force (the negative part of the control) similar to (2), and  $\eta \in [0, 1]$  is the recuperation coefficient which determines the efficiency of the regenerative braking system. Note that the problem reduces to the basic problem if  $\eta = 0$  (no regenerative braking). The resistance force  $r(v)$  was modelled in an abstract way that included the usual quadratic function in speed but with  $r_0 = 0$ . Asnis et al. (1985) derived necessary conditions by applying PMP, resulting in the following optimal control strategy

$$\hat{u}(t) = \begin{cases} u_{\max}(v(t)) & \text{if } \lambda(t) > v(t) \\ u \in [0, u_{\max}] & \text{if } \lambda(t) = v(t) \\ 0 & \text{if } \eta v(t) < \lambda(t) < v(t) \\ u \in [-u_{\min}, 0] & \text{if } \lambda(t) = \eta v(t) \\ -u_{\min} & \text{if } \lambda(t) < \eta v(t) \end{cases} \quad (30)$$

Here the driving regime RB denotes regenerative braking. Hence, the possibility for regenerative braking generates an additional singular solution corresponding to partial braking. Note that we distinguish between the partial braking regime CR2 from (28) and the regenerative braking regime RB, since the latter may also contribute energy to the cost function and may thus lead to potential different strategies. However, for level track, Asnis et al. (1985) showed that this singular solution does not occur over a nontrivial interval. Hence, regenerative braking is only used with maximum braking. However, the optimal driving regime sequence may now also contain maximum braking before a coasting regime. Asnis et al. (1985) also derived analytical expressions in the special case of  $r(v) = v$ . They did not provide an algorithm to construct an optimal driving regime sequence with the associated switching times.

Khmelnitsky (2000) considered the EETC problem with variable gradient profiles and speed restrictions as well as regenerative braking. He used the same objective function (29) as Asnis et al. (1985) but using distance as independent variable, by which it transforms to

$$J = \min_u \int_0^X (u^+(x) - \eta u^-(x))dx. \quad (31)$$

This is equal to (20) with an additional term for the regenerative braking.

However, Khmelnitsky (2000) used time  $t(x)$  and total energy  $E(x) = K(x) + P(x)$  as state variables. Total energy is the sum of kinetic energy  $K(x)$  and potential energy  $P(x)$  at position  $x$ . Potential energy is the energy due to the track height  $P(x) = mgh(x)$  which other authors model using the track gradients, and kinetic energy is the energy due to motion  $K(x) = \frac{1}{2}mv^2$ . The train resistance force is now a function of kinetic energy  $w(K) = w_0 + w_1\sqrt{K} + w_2K$ , which equals the usual Davis equation  $r(v)$  using the transformation  $v = \sqrt{2K/m}$ . We use a different notation  $w(x)$  to distinguish it from the squared function  $r(v)$ . The constraints can now be described as:

$$\dot{t}(x) = 1/\sqrt{2K(x)} \quad (32)$$

$$\dot{E}(x) = u(x) - w(K(x)) \quad (33)$$

$$t(0) = 0, t(X) = T, E(0) = 0, E(X) = E_X \quad (34)$$

$$K(x) \in [0, \tilde{K}(x)], u(x) \in [-u_{\min}, u_{\max}(K(x))], \quad (35)$$

where  $\tilde{K}(x)$  is the maximum kinetic energy at position  $x$ , which can be derived from the speed profile using the transformation  $\tilde{K} = \frac{1}{2}mv_{\max}^2$ . Khmelnitsky (2000) derived the PMP necessary conditions for problem (31)–(35) and also found the optimal control structure

(30) with five regimes, but now in terms of distance:

$$\hat{u}(x) = \begin{cases} u_{\max}(K(x)) & \text{if } \lambda(x) > K(x) \\ u \in [0, u_{\max}] & \text{if } \lambda(x) = K(x) \\ 0 & \text{if } \eta K(x) < \lambda(x) < K(x) \\ u \in [-u_{\min}, 0] & \text{if } \lambda(x) = \eta K(x) \\ -u_{\min} & \text{if } \lambda(x) < \eta K(x) \end{cases} \quad (\text{MA}) \quad (\text{CR}) \quad (\text{CO}) \quad (\text{RB}) \quad (\text{MB}). \quad (36)$$

In order to keep a constant speed (and kinetic energy) along an interval, the tractive or braking force has to change according to the grade profile

$$u(x) = \dot{P}(x) + w(K(x)). \quad (37)$$

Hence for varying gradients, Khmelnitsky (2000) showed that both singular solutions could occur in a cruising regime with partial traction or partial braking equal to the track and train resistance forces on intervals with minor grades or falls where the right-hand side of (37) stays within the bounds of traction and braking forces, respectively. Note that the definition of minor grades and falls depends on both the grade profile and the speed. He also proved that the equations  $\lambda(x) = K(x)$  and  $\lambda(x) = \eta K(x)$  have no more than one root for each minor grade and each minor fall interval, respectively, so that the cruising speeds are well-defined. Moreover, he proved that the smaller the running time, the higher the optimal cruising speeds. For steep grades where the speed decreases even at full traction or for steep falls where speed increased even at full braking, he showed that the cruising phase should be interrupted in advance by maximum acceleration or maximum braking, respectively. Khmelnitsky (2000) solved the problem with a numerical algorithm that first locates the intervals of singular cruising regimes (CR, RB) and then links them together with a sequence of regular driving regimes (MA, CO, MB). For full recovery of braking energy ( $\eta = 1$ ) he remarked that the cruising and coasting regimes merge constituting a unique stabilization regime on intervals with minor grades and falls. A case study on a 40 kilometres railway line with two hills and three different speed limits showed fast computation times within 10 seconds on an IBM PC-586 computer.

Franke et al. (2000) considered the EETC problem with regenerative braking with mass-specific kinetic energy  $E(x) = \frac{1}{2}v^2$  and time  $t$  as state variable of distance as independent variable. They simplified the resistance equation into the linear equation  $w(E) = w_0 + w_2E$  and thus neglected the term in  $\sqrt{E}$ . In terms of the normal Davis resistance this discards the linear speed term but not the quadratic speed. Moreover, they approximated the traction and braking force as piecewise constant. The objective function is the integral of electric power  $P = uv$  and an additional nonlinear term  $P_{\text{loss}}(u, v)$  for the power losses of the propulsion (traction and regenerative braking) system, formulated in speed. Hence, they considered the following optimal control problem:

$$J = \min_u \int_0^X (u(x)v(x) + P_{\text{loss}}(u(x), v(x)))dx, \quad (38)$$

subject to

$$\dot{t}(x) = 1/\sqrt{2E(x)} \quad (39)$$

$$\dot{E}(x) = u(x) - w(E(x)) - g(x) \quad (40)$$

$$t(0) = 0, t(X) = T, E(0) = 0, E(X) = 0 \quad (41)$$

$$E(x) \in [0, \bar{E}(x)], u(x) \in [-u_{\min}, u_{\max}(E(x))]. \quad (42)$$

Since the simplified dynamic equation in  $E$  is linear, they could derive analytical expressions for the various driving regimes and solve the problem by a Discrete Dynamic Programming (DDP) algorithm. For this, they reformulated the optimal control problem (38)–(42) as a multistage optimization problem by discretizing the problem into  $K$  stages  $k = 0, \dots, K - 1$ , such that the resistance including the grade profile  $w^k$  and the traction/braking force  $u^k$  could

be considered constant in each stage. Stage  $k$  covers the distance interval  $[x^k, x^{k+1})$  with length  $\Delta x^k = x^{k+1} - x^k$ . This then results in the Dynamic Programming problem

$$\min \sum_k f_0(E^k, \Delta x^k, u^k, w^k), \quad (43)$$

subject to

$$E^{k+1} = f_1(E^k, \Delta x^k, u^k, w^k) \quad (44)$$

$$t^{k+1} = t^k + f_2(E^k, \Delta x^k, u^k, w^k) \quad (45)$$

$$E_{\min}^k \leq E^k \leq E_{\max}^k, t_{\min}^k \leq t^k \leq t_{\max}^k \quad (46)$$

$$-u_{\min}(w^k) \leq u^k \leq u_{\max}(w^k, E^k, E^{k+1}) \quad (47)$$

and the given initial state  $(t^0, E^0) = (0, 0)$  and scheduled finite state  $(t^K, E^K) = (T, 0)$ . Here, the function  $f_0$  is obtained using a numerical approximation of (38) over each stage with fixed resistance  $w^k$  and control  $u^k$ , and  $f_1$  and  $f_2$  are the analytical expressions to (39) and (40) depending on the values of  $w^k$  and  $u^k$ . The DPP algorithm was implemented in a Nonlinear Model Predictive Controller (NMPC) to optimize the driving strategy in real-time. The algorithm was applied in a case study on the Swiss line Zürich HB–Luzern where in two controlled runs the driver operated the train exactly according to the pre-calculated optimization results. Results from simulations and the pilot runs showed potential energy savings between 10% and 30% compared to mean manual driving strategies and fastest driving. A remarkable result from their model was that no maximum acceleration or maximum braking is applied at high speeds due to the nonlinear power losses.

Baranov, Meleshin, and Chin' (2011) considered the EETC problem with both mechanical and regenerative braking with distance as independent variable. Denote by  $u_f$ ,  $u_b$  and  $u_r$  the mass-specific force due to traction, braking and regenerative braking, respectively. Then the optimal train control problem is formulated as:

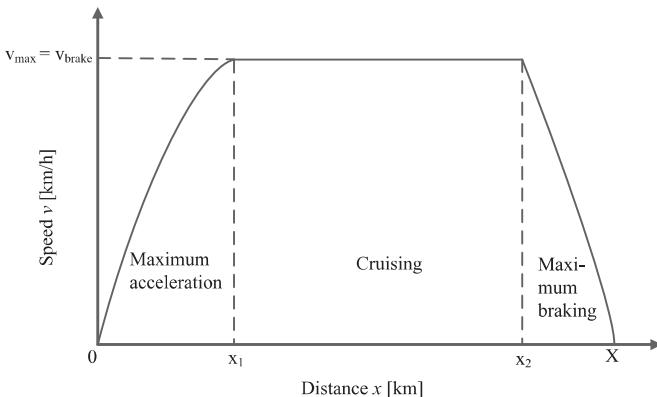
$$\min_{u_f, u_r} \int_0^X (u_f(x) - \eta u_r(x))dx,$$

subject to

$$\dot{v}(x) = (u_f(x) - u_b(x) - u_r(x) - r(v(x)) - g(x))/v(x),$$

together with (21), (23),  $v(x) \in [0, v_{\max}(x)]$ , and the control limits  $u_f(x) \in [0, \bar{u}_f(v(x))]$ ,  $u_b(x) \in [0, \bar{u}_b(v(x))]$ ,  $u_r(x) \in [0, \bar{u}_r(v(x))]$ . Here,  $\eta$  is a the efficiency factor of regenerative energy returned to the network. Using the PMP, they found seven driving regimes: the three familiar regimes MA, CO and MB (maximum braking with both mechanical and regenerative braking), three cruising regimes with either partial traction (CR), with partial regenerative braking (RB) or with full regenerative braking and partial mechanical braking, and finally a regime with full regenerative braking. An algorithm to construct the optimal control sequence of these driving regimes is mentioned as an open question.

Rodrigo, Tapia, Mera, and Soler (2013) discretized distance into  $n - 1$  intervals and transformed the optimal control problem into an optimization problem with speed  $v_i$  at the fixed points  $i = 1, \dots, n$  as  $n$  decision variables. They therefore expressed the objective function, dynamic equations and constraints as functions of the  $n$ -tuple of speed values and solved the resulting optimization problem by the Lagrange multiplier method using MATLAB. They included the option of regenerative braking with an efficiency coefficient and considered two case studies of Madrid's metro Line 8 in Spain. For regenerative braking they concluded that it is optimal to start with maximum acceleration until some average cruising speed. In the central sections, traction and regenerative braking are alternated with preferably braking on downward slopes to generate energy, and at the destination braking is applied to recover as much energy as possible. The average regenerated energy



**Fig. 6.** Speed profile of an energy-efficient driving strategy without coasting and with switching points between driving regimes at  $x_1$  and  $x_2$ .

was approximately 23%. When regenerative braking is not possible, however, they found that it is optimal to start with maximum acceleration until a speed is reached that ensures arrival on time using coasting. In central sections partial traction is used for cruising and in case of speed restrictions coasting is preferred before braking if time allows. The energy consumption of mechanical braking was higher than with regenerative braking. The two case studies showed a big increase in computation time if mechanical braking was applied instead of regenerative braking. The computation time for the first case study between the stations Nuevos Ministerios and Colombia was 34.82 seconds for regenerative braking and 290 seconds for mechanical braking. In the second case study between the stations Colombia and Mar de Cristal the computation time increased to 90.22 seconds for regenerative braking up to 1977 for mechanical braking.

Regenerative braking in the optimal control for metro trains is considered by Qu, Feng, and Wang (2014). They used the objective function (31) but assumed full recovery of regenerative braking energy ( $\eta = 1$ ) and no steep slopes. In this case, coasting is not used and the optimal driving strategy consists of a sequence of the three driving regimes maximum acceleration, cruising and maximum braking, see Fig. 6. They presented an iterative numerical algorithm to compute the optimal cruising speeds for given speed restrictions and scheduled running time. The authors applied it to a case study based on the Shenzhen Metro Line 1 in China to show that the presence of a speed restriction changes the cruising speeds.

In the European rail project ON-TIME (Optimal Networks for Train Integration Management across Europe) an iterative algorithm was developed for an on-board DAS to calculate the optimal control of a train (ON-TIME, 2014a). The algorithm is based on PMP and includes regenerative braking as well as traction efficiency with time as independent variable. They thus assumed the five optimal driving regimes as in (30), which was used in an iterative gradient-based algorithm that computes the switching times between regimes by iteratively replacing regimes on a subsection as long as the running time can be increased. The regime changes are selected as the ones that provide most energy savings with the smallest change in running times. The three options to increase the running time on a subsection with given start and end speed are (i) reducing the duration of maximum acceleration and replacing it with cruising at a lower speed or coasting; (ii) reducing the duration of cruising, and (iii) replacing part of it by coasting; and reducing the cruising speed. Results on a case study on the Dutch railway network between Utrecht Central and Eindhoven showed energy savings of 20% to 30% by the use of the algorithms compared to non-optimized train driving.

A. Albrecht et al. (2015b,c) discussed the key principles of optimal train control and extended their previous work by including regenerative braking. Their problem statement includes varying (steep) gradients. The problem formulation is the same as the dynamic constraints with respect to distance (21)–(24) with the objective equal to (31). This leads to the following optimal control strategy:

$$\hat{u}(x) = \begin{cases} u_{\max}(v(x)) & \text{if } \lambda(x) > v(x) \\ u \in [0, u_{\max}] & \text{if } \lambda(x) = v(x) \\ 0 & \text{if } \eta v(x) < \lambda(x) < v(x) \\ u \in [-u_{\min}, 0] & \text{if } \lambda(x) = \eta v(x) \\ -u_{\min} & \text{if } \lambda(x) < \eta v(x) \end{cases} \quad (48)$$

(MA)      (CR)      (CO)      (RB)      (MB).

The results indicate that regenerative braking should only be included in a cruising phase to maintain a certain cruising speed during a steep downhill section. Moreover, they derived analytical equations for the modified dimensionless co-state variable  $\eta = \lambda/v - 1$ . Phase plots of the state and modified-co-state variables were drawn for constant gradients to find the optimal switching points between control regimes for different initial conditions. They implemented their model in MATLAB and successfully tested it on two examples with steep uphill and steep downhill sections, and checked the calculations with the results from Energymiser. Moreover, they showed an example of Energymiser in a case study of the high speed TGV trains of SNCF between Lyon and Valance (France) without consideration of regenerative braking. Results indicated that the amount of running time supplement influences the optimal driving strategy. Energy savings of 22.6% can be achieved with 10% running time supplement in relation to time-optimal running.

### 3.3. Exact methods with discrete control

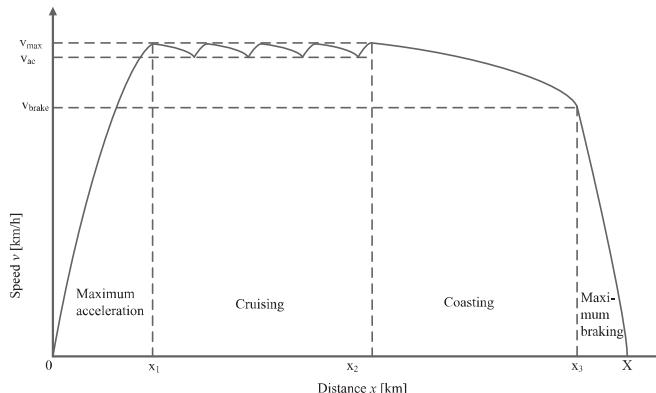
The models considered up to now assumed continuous traction control, which is applicable to most trains nowadays (Liu & Golovitcher, 2003). Nevertheless, there are also trains where traction is controlled using discrete throttle settings. For example, in Australia most freight trains used to have diesel-electric traction with discrete throttle settings (Howlett, 2000). Therefore, the literature also considered energy-efficient train control models where traction control is restricted to a finite number of discrete values. In particular, this changes the cruising regime since not all control settings are possible to maintain an optimal constant cruising speed. Still, for freight trains the distance between two stops is much longer than for suburban trains and therefore some kind of approximate cruising phase would be the dominant phase.

Cheng and Howlett (1992) first described the energy-efficient train control problem with discrete throttle settings as follows. Assume that there are  $m + 1$  distinct throttle settings  $f_j$ ,  $j = 0, \dots, m$ , with  $f_0 = 0$  the zero fuel case corresponding to coasting, and  $f_j < f_{j+1}$ ,  $j = 1, \dots, m$  a sequence of increasing fuel supply rates. Moreover, let  $t_i$ ,  $i = 0, \dots, n + 1$  be a sequence of switching times between throttle settings with  $f_{j_{k+1}}$  the rate of fuel supply maintained in the interval  $(t_k, t_{k+1})$  for a duration of  $\tau_{k+1} = t_{k+1} - t_k$ . Let  $t_0 = 0$  and  $t_{n+1} = T$ . Furthermore, it is assumed that braking is only applied at the final stage with maximum braking rate  $b$ . Then the minimum fuel consumption optimization problem is formulated as

$$\min \sum_{k=0}^{n-1} f_{j_{k+1}} \tau_{k+1} \quad (49)$$

subject to

$$\dot{v}(t) = \frac{H f_{j_{k+1}}}{v(t)} - r(v(t)), \quad t \in [t_k, t_{k+1}) \quad (50)$$



**Fig. 7.** Speed profile of an energy-efficient driving strategy with discrete control (throttle settings) with switching points between driving regimes at  $x_1$ ,  $x_2$  and  $x_3$  (the cruising phase consists of different phases of acceleration and coasting).

for  $k = 0, \dots, n - 1$ , and

$$\dot{v}(t) = b - r(v(t)), \quad t \in [t_n, t_{n+1}], \quad (51)$$

with the additional constraints (4), (6) and  $v(t) \geq 0$ . Here,  $H$  is some constant. Note that this first problem formulation assumes flat track. Cheng and Howlett (1992) solved this problem using Lagrange multiplier theory by formulating a Lagrangian function and applying the Karush–Kuhn–Tucker necessary conditions.

Cheng and Howlett (1992) showed that cruising is now approximated by alternating between maximum acceleration and coasting which leads to a sawtooth pattern between two speeds  $V$  and  $W$ , where a train repetitively accelerates to some critical speed  $W$  and then coasts until a certain critical speed  $V < W$ , where it will accelerate again to the critical speed  $W$ , et cetera, see Fig. 7. This strategy was coined a ‘strategy of optimal type’. The critical speeds are obtained from the equation

$$\lambda v - \mu = vr(v), \quad (52)$$

where  $\lambda$  and  $\mu$  are the non-negative Lagrange multipliers corresponding to the fixed distance  $X$  and fixed running time  $T$ , respectively. Since  $vr(v)$  is a convex function, there are exactly two solutions to (52), of which  $V$  denotes the lower and  $W$  the higher. Furthermore, the speed where braking begins was shown to be the solution  $U$  to  $\lambda v - \mu = 0$ . Starting with maximum acceleration, a strategy of optimal type is then characterized by the three speeds  $0 < U < V < W$ , where  $\lambda$  and  $\mu$  can be computed in terms of  $V$  and  $W$  as

$$\lambda = \frac{Wr(W) - Vr(V)}{W - V} \quad \text{and} \quad \mu = \frac{VW(r(W) - r(V))}{W - V}.$$

Then the braking speed follows by  $U = U(V, W) = \mu/\lambda$ . A numerical procedure to solve the optimal control problem for a strategy of optimal type is now obtained by finding speeds  $V$  and  $W$  such that the resulting errors in the total distance  $X$  and time  $T$  are zero. With  $n = 2p + 3$ , for any nonnegative integer  $p$ , the solution starts with maximum acceleration to  $W$ , oscillates  $p$  times with coasting–maximum acceleration between the critical speeds  $V$  and  $W$ , coasts to the braking speed  $U < V$ , and then brakes with maximum braking. Cheng and Howlett (1993) showed that the critical speeds  $V_p$  and  $W_p$  converge to an idealized strategy with speed  $V_p = W_p = Z$  as  $p \rightarrow \infty$  which minimizes the fuel consumption. The oscillation strategy can thus be interpreted as an approximate cruising regime.

Howlett (1996) extended the energy-efficient train control problem with discrete throttle settings to varying gradients using the associated formulation with distance as independent variable. For non-steep gradients again an approximate cruising regime is

obtained with oscillations between two critical speeds. The switching points now depend on the gradients and hence the oscillation interval durations  $\tau_k$  are no longer regular. Recall, that by definition for a non-steep track the speed increases in an acceleration regime and decreases during each coasting or braking regime. The critical speeds and corresponding switching times are computed for a fixed number of acceleration-coasting phases by adjusting the values of the Lagrange multipliers  $\mu$  and  $\lambda$ . For steep gradients, the approximate cruising regime is interrupted by segments of coasting and traction.

Howlett et al. (1994) extended the problem to varying speed limits. In this case, each segment  $(x_i, x_{i+1})$  of constant speed limit is associated with a separate Lagrange multiplier  $\lambda_i$  that takes a different value on the different segments, and (52) is rewritten as

$$\lambda_i = r(v) + \mu/v$$

on  $(x_i, x_{i+1})$ . Since the critical speeds for various track segments are defined by different values of the parameter  $\lambda_i$ , the critical speed intervals  $(V_i, W_i)$  are nested over the various segments. The models usually consider a train as a point mass. However, Howlett et al. (1994) also showed that a real train with a distributed mass can be treated as a point mass by constructing a modified gradient profile.

Cheng, (1997) contained most results for discrete control and also Howlett and Pudney (1995) captured all the results for discrete control with some additional information (next to their results for continuous control up to 1995). The models and algorithms for discrete throttle control setting were used in a DAS named Cruisemiser (Benjamin, Milroy, & Pudney, 1989; Cheng, 1997; Howlett et al., 1994), which extended the ideas of Metromiser (see Section 3.1) to long-haul freight trains in Australia.

### 3.4. Direct exact methods

The solution approaches considered so far first derived the optimal driving regimes from the necessary conditions for optimality using Pontryagin's Maximum Principle, and then tried to solve the resulting optimization problem of finding the optimal sequence and switching points of the optimal driving regimes by solving the differential equations of the train movements for the optimal driving regimes. This approach worked well for special cases but the general problem with varying gradients and speed limits is very difficult to solve, while the inclusion of regenerative braking made the problem even harder to solve.

A different approach for solving optimal control problems is obtained by discretizing the dynamic system into a problem with a finite set of variables and then solving the resulting static nonlinear programming problem by nonlinear programming (NLP) methods (Betts, 2010). Only recently this direct approach has been considered for solving EETC problems.

Y. Wang, Ning, Cao, De Schutter, and van den Boom (2011) considered the EETC problem with varying gradients, curves and speed restrictions. They used kinetic energy and time as state variables in the independent variable distance and assumed a linear resistance force as in Franke et al. (2000). As objective function they used a trade-off between energy consumption and driving comfort:

$$\min_u \int_0^X \left( u(x) - \alpha \left| \frac{du(x)}{dx} \right| \right) dx,$$

subject to (39)–(42), but assuming also a constant maximum traction force, i.e.,  $u(x) \in [-u_{\min}, u_{\max}]$ . Here,  $\alpha$  is a weight factor to balance between the two objectives. They then discretized the problem into a discrete-space problem by dividing distance in discrete intervals similar to Franke et al. (2000), where they approximated the nonlinear terms through piecewise affine (PWA) functions, and finally reformulated it into a MILP (mixed integer linear

programming) problem. A case study of a 10 kilometres long line was considered with a fixed speed limit, 20 fixed discretization intervals of 500 metre, and  $\alpha = 500$ . Still the computation time was about 10 minutes and, as a result of the rough discretization, the optimal control and train trajectory were not very accurate.

[Y. Wang, De Schutter, Van den Boom, and Ning \(2013\)](#) reconsidered the EETC problem of [Y. Wang et al. \(2011\)](#) but now considered maximum traction force as a usual nonlinear function of speed, which they approximated by PWA functions. The resistance equation was still assumed linear in the kinetic energy. They now also proposed a Pseudospectral method ([Rao et al., 2010; Ross & Karpenko, 2012](#)) to solve the problem and compared this with the MILP and DDP methods of [Franke et al. \(2000\)](#). For the Pseudospectral method, the optimal control problem was first reformulated into a multiple-phase optimal control problem with each phase corresponding to a constant gradient, curve and speed limit ([Betts, 2010](#)). This multiple-phase optimal control problem could then be transformed into a nonlinear programming problem using the Pseudospectral method, where the state and control functions are approximated by orthogonal polynomials based on interpolation of orthogonal collocation points. The PSOPT solver was used to transform and solve the NLP problem. The three approaches were compared in the case study of the 10 kilometres long line where now varying gradients and speed limits were added. The MILP problem used again a 500 metre interval while for the DDP method a space interval of 100 metre was chosen. The solvers were PSOPT, CPLEX for the MILP problem, and a generic MATLAB Dynamic Programming function for the DDP problem. For a scenario with a constant maximum traction force, PSOPT took 6 minutes, DDP took 2 minutes, and CPLEX required 32 seconds, but the DDP and MILP models were highly inaccurate with respect to the PSOPT solution. In a second scenario, the maximum traction was considered as a nonlinear function of speed. Again the DDP and MILP approaches were highly inaccurate compared to the smooth PSOPT solution. The computation time for PSOPT was however very high with 19 minutes, compared to 2 minutes and 1 minute for the DDP and MILP approaches.

[P. Wang, Goverde, and Ma \(2015\)](#) considered the EETC problem (20)–(24) with the state variables speed and time as function of distance and varying gradients and speed limits, plus additional timetable constraints using the Train Path Envelope (TPE) to model intermediate stops as mandatory target points and through-passing of stations as time windows. The model was reformulated as a multiple-phase optimal control problem with each phase corresponding to a constant gradient and speed limit, and then solved using the Gauss Pseudospectral Method using the GPOPS solver ([Rao et al., 2010](#)). The model was applied to a case study based on the 50 kilometres corridor Utrecht Central–'s-Hertogenbosch in the Netherlands with eight stations. In a first scenario, a regional train was considered with four scheduled stops from Utrecht Central to Houten Castellum for both the original timetable and an adapted timetable with smaller running times as would be provided by a Traffic Management System (TMS). The solutions are computed within 6 seconds for each train run between two stops, and the speed-distance and force-distance diagrams show a driving behaviour as expected from the optimal driving regimes known from applying the PMP. The faster train uses less coasting and needs more energy as a result. In a second scenario, an intercity train is considered from Utrecht Central to 's-Hertogenbosch which overtakes a regional train in a station halfway. Two strategies are computed: one with mandatory target points at all intermediate stations according to the scheduled pass-through times, and another with time windows on the five intermediate stations besides the overtaking and end stations. The optimal trajectories use coasting before all scheduled target points and speed restrictions, while for the time-window case a smoother operation was obtained with

a constant cruising speed over all intermediate stations. The case with the time windows saves 4.5% extra energy. The computation time for time-window case was less than 30 seconds for the entire trajectory.

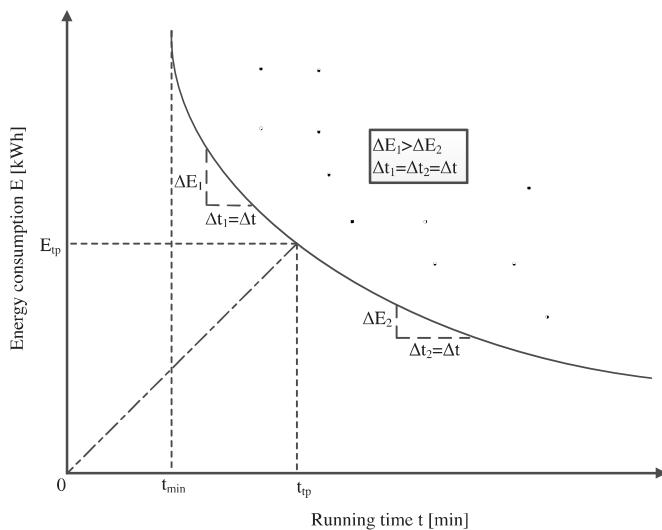
### 3.5. Heuristics

In addition to exact solution methods based on PMP to determine the optimal driving strategy, also artificial intelligence or search algorithms have been applied to directly solve the energy-efficient train control problem.

[Chang and Sim \(1997\)](#) developed a coast control driving strategy for metro systems with varying speed limits and using regenerative braking. They developed an algorithm with the objective to minimize the total traction energy consumption by taking into account punctuality and riding comfort in penalty functions. The riding comfort is described by the jerk, which is the change of acceleration over time. The successive driving regimes are translated into a coast control lookup table, which gives the distance to start each driving regime and could be incorporated in an ATO system. The authors developed a Genetic Algorithm (GA), which was implemented in C++ on an IBM 486 PC. The algorithm was applied to a case study consisting of a track of 923 metre between two stops with a speed restriction halfway of 40 kilometer per hour. Two scenarios were tested with the model: a normal schedule with 90 seconds scheduled running time, and a tight timetable in which the scheduled running time between the two stops is assumed 0 second, which forces the algorithm to consider a delay. The model results were compared with a fuzzy ATO controller. Results indicated that energy savings of 2.5% for the tight timetable to 6.8% for the normal timetable were achieved and that punctuality also increased in both cases. For the normal timetable scenario the jerk was higher than with the fuzzy ATO controller, but it remained within the boundaries of passenger comfort. The results were generated within 30 seconds.

[Lechelle and Mouneimne \(2010\)](#) developed a GA approach to find energy-efficient speed profiles. A GA generates numerous operating speed profiles according to certain rules specified by the users. In turn, a single-train simulator simulates the movement of a train for each of these speed profiles and computes the corresponding traction energies. Through an iterative process, the GA gradually converges towards an energy-optimised speed profile. This approach was implemented in a tool called OptiDrive and applied on a case study of the tramway network of Rouen in France. Results showed 7% energy saving compared to normal real-time operation.

[Domínguez, Fernández, Cucala, and Lukaszewicz \(2011\)](#) developed a simulation model for the Madrid metro system driven by ATO. The simulation model includes four independent modules (ATO, engine, train dynamics, and energy consumption). The configuration data for the ATO consists of four parameters: traction, cruising speed, coasting, and braking deceleration rate. The considered ATO system provided only certain discrete values for each parameter, resulting in a solution space of 156 alternative speed profiles per inter-station run, which enabled an exhaustive simulation of all feasible ATO speed profiles. In addition the resulting speed profiles were filtered based on comfort and operational conditions such as a minimum speed throughout the journey (20 kilometres per hour), maximum number of reaccelerations, maximum slope for coasting, and minimum speed limits along curves. The resulting running times were plotted against the associated energy consumption, after which a Pareto curve could be used to determine the most energy-efficient speed profile given the available running time, see [Fig. 8](#). The curves were used to determine a set of four alternative speed profiles per inter-station run associated with the fastest running time, the scheduled running time (at most



**Fig. 8.** Pareto frontier curve of multi-objective function between energy consumption versus running time ( $t_{\min}$  is the technical minimum running time, solid black line is the Pareto frontier (optimal solutions) and black dots are dominated solutions,  $t_p$  and  $E_{tp}$  are the turning points, i.e.  $\Delta t_1 = \Delta t_2$  and  $\Delta E_1 = \Delta E_2$ ).

20 seconds more than the time-optimal running time), and two running times uniformly distributed in between. The approach was applied on the Madrid Metro line 3 in Spain resulting on average in 13% energy savings compared to the previous ATO design without affecting the scheduled running times.

Domínguez, Fernández-Cardador, Cucala, and Pecharromán (2012) extended the simulation model of Domínguez et al. (2011) by considering the energy in the substations to include regenerative braking. Using the same method as before the model was also tested on the Madrid Metro line 3. Energy savings of 6% to 11% were reported for the optimal speed profiles including optimal use of regenerative braking energy compared to operation without using the optimal speed profiles.

Three different optimization algorithms for finding energy-efficient speed profiles were studied by Lu, Hillmansen, Ho, and Roberts (2013). The authors make use of a graphic model to simplify calculation of the optimal control by avoiding nonlinear complexity. The objective of the optimization is to minimize total traction energy considering punctuality constraints. The optimal speed profile is determined by constructing a complete weighted and directed speed graph. The authors compared the heuristic methods Ant Colony Optimization (ACO) and the Genetic Algorithm, and Dynamic Programming. Varying gradients as well as speed limits were taken into account. The algorithms were applied in different examples in which the total distance was fixed, but the available running time was variable. The results showed that the speed profiles with the lowest total energy consumption were gained with the DP algorithm, but this costs the most computation time. On the other hand, the heuristic algorithms did not guarantee finding an optimal solution.

Sicre, Cucala, and Fernández-Cardador (2014) proposed a Genetic Algorithm with Fuzzy parameters based on the accurate simulation of a train motion to determine the optimal driving strategy for delayed high-speed trains. Fuzzy cruising speeds and switching times were used to consider manual train driving. Moreover, regenerative braking was included. The objective of the GA is to find the solution that has the target running time with the minimum energy consumption. A general structure for efficient manual driving was proposed that was easy to implement for drivers. This structure replaced the cruising regime by a partial traction phase that maintains a speed as long as traction is required. However,

if braking is necessary to keep the speed constant on a downhill section braking would not be applied, but the train would coast instead and thus increasing its speed where braking is applied if a speed limit would be reached. The approach requires several runs of the Fuzzy GA with different target running times starting with the scheduled running time where in each iteration the target time is updated by the estimated delay of the former run. To allow a real-time calculation, the algorithm was restricted to two GA runs with a computation time limit of 15 seconds each. The approach was applied in a numerical case of the Spanish High Speed line between Calatayud and Zaragoza in Spain for recovery from a temporary speed restriction at the beginning of the stretch that would lead to a 1:04 minute delay if the nominal driving strategy would not be updated. The scheduled running time was 26 minutes which included 4:08 minute (15.9%) running time supplement. Energy savings of 5.5% on pantograph level and 6.7% on substation level were achieved compared to the nominal operations.

Chevrier, Pellegrini, and Rodriguez (2013) proposed a bi-objective optimization approach for computing running times as a trade-off between minimizing running time and energy consumption using a heuristic Evolutionary Algorithm (EA). They consider the time formulation (3)–(7) with varying gradients and speed limits, but with both the energy consumption  $J$  and total time  $T$  as objectives to be minimized. They decomposed the problem into  $i = 1, \dots, n$  sections of constant speed limit and in each section  $i$  they built the speed profile by splitting the section in two parts with target speeds  $v_{a,i}$  and  $v_{b,i}$  in the first and second part, respectively, which are the decision variables of the EA. Continuity is guaranteed by incorporating the speed limit in the next section and linking the exit and entry speeds of successive sections. In the first part only maximum acceleration or braking is used to reach the first target speed  $v_{a,i}$  from the entry speed. Then in the second part the second target speed  $v_{b,i}$  is reached initially by coasting and possibly braking after which cruising is used to complete the rest of the section. The algorithm is followed by a post-processing step for smoothing the speed profile with cruising regimes where a maximum braking regime is followed by either maximum acceleration or acceleration while coasting (on a steep descent). The EA algorithm finds multiple and well-spread non-dominated (Pareto) solutions in a single run that a planner can choose from. The algorithm has been applied in two case studies of a 2.2 kilometres long line with five sections and a 20.2 kilometres long line with also five sections. In both cases the algorithm runs for 60 seconds to produce a wide set of Pareto optimal speed profiles, where a planner can select a solution with given scheduled running time and associated energy consumption. The results show that the energy consumption can be reduced significantly already by slightly increasing the running times. When more running time is allowed (up to 22% more than the time-optimal running time), energy savings may be even close to 50% in comparison to the technical minimum running time.

### 3.6. Summary on EETC

**Table 1** gives an overview of the different literature on the EETC. The fourth column 'Method' indicates Approximation (A) if a heuristic method is used or the model contained simplifications. Most research is on the topic of continuous train control using PMP with algorithms for finding exact solutions. All rail modes are considered with an emphasis on metro/urban and regional/IC (intercity) railway systems. To compute realistic speed profiles it is important to take into account nonlinear train traction and resistance, line resistance with in particular varying gradient profiles, and varying speed limits. Curve resistances and tunnels are often ignored.

**Table 1**  
Summary of EETC literature.

Publication	Control	RB	Method	Real-time	Varying features		Mode of transport			
					Speed limits	Gradients	High speed	Regional, IC	Freight	Metro, urban
Ichikawa (1968)	C	E								x
Strobel et al. (1974)	C	E		x						x
T. Albrecht and Oettich (2002)	C	E		x						x
Milroy (1980)	C	E		x						x
Howlett (1990)	C	E		x				x		x
Cheng and Howlett (1992)	D	E		x					x	
Cheng and Howlett (1993)	D	E		x					x	
Howlett et al. (1994)	D	E			x	x			x	
Howlett (1996)	D	E		x		x			x	
Cheng, 1997)	D	E		x	x	x			x	
Van Dongen and Schuit (1989a,b, 1991)	C	A			x	x		x		
Howlett and Pudney (1995)	C, D	E		x	x	x		x	x	x
Howlett (2000)	C, D	E		x		x		x	x	x
Oettich and Albrecht (2001)	C	E			x	x		x		x
T. Albrecht (2014)	C	E			x	x		x		x
Liu and Golovitcher (2003)	C	E		x	x	x		x		x
Vu (2006)	C	E			x	x				
Howlett et al. (2009)	C	E		x	x	x			x	
A. Albrecht et al. (2013a)	C	E			x	x				
A. Albrecht et al. (2013b)	C	E		x	x	x			x	
A. Albrecht et al. (2014)	C	E			x	x		x	x	x
A. Albrecht et al. (2015b,c)	C	E		x	x	x	x	x	x	x
Lechelle and Mouneimne (2010)	C	A		x	x	x				x
Domínguez et al. (2011)	C	A			x	x				x
Aradi et al. (2013)	C	E			x	x		x		
Chevrier et al. (2013)	C	E			x	x		x		
Lu et al. (2013)	C	A/E			x	x				x
Rodrigo et al. (2013)	C	E			x			x		
Scheepmaker and Goverde (2015)	C	E			x	x		x		
Su et al. (2013)	C	A			x					x
Su et al. (2014)	C	A			x	x				x
T. Albrecht et al. (2013c)	C	E		x	x	x		x		x
Jaekel and Albrecht (2013)	C	E		x	x	x		x		x
ON-TIME (2014a)	C	E		x	x	x		x	x	x
Qu et al. (2014)	C	E			x	x		x		
Y. Wang et al. (2011)	C	A			x	x				
Y. Wang et al. (2013)	C	A, E			x	x				
P. Wang et al. (2015)	C	E		x	x	x		x		
Asnis et al. (1985)	C	x	E							x
Chang and Sim (1997)	C	x	A		x					x
Franke et al. (2000)	C	x	A	x	x	x		x		
Khmelnitsky (2000)	C	x	E	x	x	x		x		x
Baranov et al. (2011)	C	x	E		x					x
Domínguez et al. (2012)	C	x	A		x	x				x
Sicre et al. (2014)	C	x	A		x	x	x	x		

Legend: RB = Regenerative Braking, C = Continuous, D = Discrete, E = Exact, A = Approximation, IC = intercity.

Some algorithms are applied in a real-time Driver Advisory System. For those algorithms, fast calculation times are achieved by using efficient algorithms, simplifications of the problem, or offline computations of a set of solutions that can be chosen from online. For the analysis of more complex driver behaviour or regenerative braking over networks, simulation and/or heuristics like GA have been applied. Also regenerative braking has been considered with still the main focus on a single train (energy consumption at the pantograph).

Finally, there are differences between different modes of rail transport. Table 2 shows the main differences. The differences are mostly related to the maximum speed of the trains and the average distance between two stations. The table indicates that cruising becomes more important when the average distance between two stations increases. This can be explained by the fact that at short distances the maximum speed cannot be reached and the running time supplements of the timetable are very small. Therefore any optimal speed to start coasting can be reached, while for longer distances cruising naturally comes in play.

#### 4. Energy-efficient train timetabling

This section discusses energy-efficient train timetabling, which is the problem of finding a timetable for one or more trains on a railway corridor or network that allows as much as possible energy-efficient driving. The total running time of each train over the corridor may be pre-specified or may still have to be determined. In both cases, the aim is to determine the running time between each pair of consecutive stops for each train such that the total (planned) energy consumption of the involved trains is minimum.

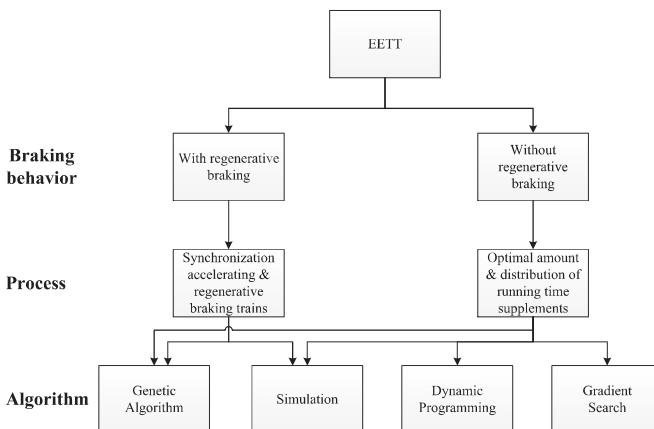
For each trip between two consecutive stations, the planned running time consists of the minimum running time between the stations plus a running time supplement. The minimum energy consumption that is needed on a single trip between two stations is decreasing in the amount of running time supplement. Indeed, if more running time supplement is available, then less energy is needed by running at a lower cruising speed or starting earlier with coasting. This effect is shown in Fig. 8, which also shows the

**Table 2**

Comparison of EETC solutions by mode of transport.

Train type	Driving regimes	Most important driving regime	Maximum speed [kilometre per hour]	Average stop spacing [kilometre]	Regenerative braking included	Algorithms
High speed	MA, CR, CO, MB	CR (& CO)	200–300	> 50	Possible	E, A
Regional and IC	MA, CR, CO, MB	CR & CO	120–160	> 25	Possible	E
Freight	MA, CR, CO, MB	CR & CO	80–100	> 30	Possible	E
(Sub)urban	MA, CR, CO, MB	CO	100–120	> 5	Possible	E, A
Metro	MA, CO, MB	CO	80	> 1	Yes	E, A

Legend: MA = Maximum Acceleration, CR = Cruising, CO = Coasting, MB = Maximum Braking, E = Exact method, A = Approximation, IC = intercity.

**Fig. 9.** Framework of EETT.

decreasing added value of more running time supplement. Increasing the amount of running time supplement is often also useful for increasing the robustness of the timetable. However, it also leads to increased (planned) journey times for the passengers.

We used a framework based on EETT with and without regenerative braking. With regenerative braking the models consider synchronization of accelerating and regenerative braking trains. Without regenerative braking the models from literature focus on finding the optimal amount and distribution of the running time supplements. An overview of our framework is given in Fig. 9.

Section 4.1 first describes a basic version of a model for EETT. Then Section 4.2 focuses on EETT without regenerative braking. Finally, Section 4.3 reviews the literature including the possibility of regenerative braking.

#### 4.1. Basic timetabling model

In this section we give a brief description of a basic timetabling model for a single train. Suppose a single train is to be scheduled along the consecutive trips in the set  $Q = \{1, 2, \dots, n\}$ . The minimum running time of trip  $q \in Q$  is denoted by  $r_q$ , and the dwell time of the stop between trips  $q$  and  $q + 1$  is denoted by  $d_q$ .

The basic problem here is to distribute a given amount of time supplement  $Z$  in such a way among the trips that the total energy consumption of the train is minimum. To that end, let the decision variables  $D_q$ ,  $A_q$  and  $S_q$  denote the departure time, the arrival time and the time supplement of trip  $q \in Q$ , respectively. Then the timetabling model for scheduling this single train can be described as follows:

$$\min f(D, A, S) \quad (53)$$

subject to

$$A_q - D_q = r_q + S_q \quad \text{for } q = 1, \dots, n \quad (54)$$

$$D_{q+1} - A_q = d_q \quad \text{for } q = 1, \dots, n - 1 \quad (55)$$

$$\sum_{q=1}^n S_q \leq Z. \quad (56)$$

The objective function (53) expresses the total energy consumption of the train in terms of the departure times, the arrival times, and the time supplements of the trips. For evaluating the objective function (53), a trade-off curve as shown in Fig. 8 is used. Constraints (54) and (55) express the running time and dwell time of trip  $q$  in terms of the trip's departure time, arrival time and time supplement. Constraint (56) indicates that the total amount of slack time should not exceed the maximum amount of slack time  $Z$ .

The above model is a basic model for scheduling just a single train, where minimizing the energy consumption is the only objective. For generating a timetable for more than one train, at least headway constraints and connection constraints must be taken into account as well. Furthermore, regenerative braking requires modelling the interaction between the energy production and the energy consumption of nearby trains. In addition, also other objectives, such as robustness, may be pursued. For different approaches to deal with an objective like robustness, we refer to Cacchiani and Toth (2012).

#### 4.2. EETT without regenerative braking

T. Albrecht and Oettich (2002) belonged to the first ones researching EETT and dynamic train operations. They used a simulation model to compute the energy utilization for each discretized running time between two consecutive stops of a train. Then they calculated the optimal timetable with Dynamic Programming, in which the total running time of each train is optimally distributed along the line. They also aimed at increasing the probability that passenger connections can be maintained in case of delays. Therefore, a multi-objective function was used: minimizing expected waiting time at transfer stations (passengers) and minimizing energy consumption (operator). The final solution was based on the minimum Euclidean distance from the *ideal point*. The results show that the algorithm inserted extra running time supplements to decrease the waiting time at a connection and to reduce energy consumption. The developed method was successfully tested at the suburban railway system of Dresden in Germany: it led to 15–20% reduction in energy consumption compared to using the normal timetable, see T. Albrecht (2005a,b). The EETT algorithm has been implemented in the driver simulator of TU Dresden, see T. Albrecht (2005b).

Ghoseiri, Szidarovszky, and Asgharpour (2004) considered an optimization model for scheduling a number of passenger trains on a railway network including single and double track sections and several stations. Their model is a large non-linear mathematical programming model that is solved with the commercial solver LINGO. They consider the multi-objective function of minimizing the total travel time of the passengers and minimizing the fuel consumption costs of the operator. The solution process for the multi-objective problem consists of two steps. In the first step the Pareto curve of the trade-off between running time and

energy consumption is determined by the  $\epsilon$ -constraint method. Then a multi-objective optimization is performed in which different distance-based methods are used to select the final timetable from the Pareto curve. The model was tested on a number of relatively small artificial instances. The results clearly show the trade-off between the two objectives: lower travel times cost more energy. Due to the nature of the model, it is not possible to explicitly describe the resulting optimal driving strategies.

**Ding, Liu, Bai, and Zhou (2011)** used a two-level iterative optimization model to determine the energy-efficient driving strategy as well as the optimal timetable for a metro line. They consider the driving regimes acceleration, coasting, and braking. At the first level, an optimization model computes the energy-efficient driving strategy by determining the switching points. At the second level, an optimization model determines the optimal running time supplements. They use a Genetic Algorithm for solving the iterative two-level optimization model. The authors apply their model on a single case study. They report that the energy consumption can be reduced by up to 19.1% compared to the timetable without optimization.

**Sicre et al. (2010)** considered optimizing the running time distribution for a high speed train. A simulation model computes the relation between the amount of running time supplement and the energy consumption per trip, which results in trade-off curves between running time and energy consumption per trip between two stops. In this simulation model the energy-efficient driving strategy is based on a ‘manual’ driving module with heuristic rules to change between the optimal driving regimes known from the EETC literature. An optimization model then distributes the available running time over the trips in order to minimize the total energy consumption. The model only redistributes the available running time supplement, so there is no focus on finding the optimum total amount of running time supplement. The authors report savings of about 33.6% compared to using the commercial timetable with technically minimum running times over a high-speed journey from Madrid to Zaragoza with two intermediate stops.

**Cucala, Fernández, Sicre, and Domínguez (2012)** further optimized the model of **Sicre et al. (2010)** for high speed trains. They included uncertain delays in the model by using fuzzy numbers and punctuality constraints. In addition, they changed the single objective function into a bi-criteria objective function aiming at minimizing total energy consumption and minimizing delays. The preferred timetable is found by distributing the available running time supplement among the trips. For this, first an EETC problem is solved per trip using a Genetic Algorithm and a simulator. As in **Sicre et al. (2010)**, these models determine per trip the trade-off curve between running time and energy consumption. Then a fuzzy mathematical programming model is used to distribute the running time supplement among the trips, where fuzzy models are used to determine uncertainty in delays. Since the problem is solved by a mathematical programming model, no analytical descriptions of the solutions are derived. On a journey from Madrid to Barcelona with four intermediate stops, the authors report a reduction in energy consumption of 5.25% in the optimized timetable without delays, and of 6.67% in the delayed optimized timetable both in comparison with using the commercial timetable. They thus conclude that it is useful to consider delays in the optimization.

**L. Yang, Li, Gao, and Li (2012)** considered the EETC problem with variable running times for multiple high-speed trains without regenerative braking and discarding cruising. Their objective was to find a trade-off between total traction energy and total travel time in the railway network, where weight factors are used to determine the balance between the two objectives. Cruising at a fixed speed was not taken into account; the authors mention that cruising is just a special sort of traction operation and could be con-

sidered easily. Hence, the optimal driving strategy is a sequence of acceleration and coasting phases until final braking. The EETC was calculated based on a simulation model for determining the driving behaviour and a Genetic Algorithm to find the optimal control strategy for multiple trains in a network by taking into account the headway between trains, the speed limits and the riding comfort. Fuzzy variables are included in the simulation part to simulate uncertainty in the performance of the train during operation. Penalty functions were included to differentiate between the priority of trains. The model was implemented in VC++ 6.0 on a PC with 2.67 gigahertz processor and was tested on two numerical examples for high speed trains. In the first example, the difference between three objective functions is compared, i.e., technical minimum running times, energy-efficient train operation (single train optimization) and a combination of both (multiple train optimization). Results indicated that a combination of minimizing total travel time and energy consumption is the most realistic. In the second example the weight factor for the objective function was varied. The best total objective function was achieved when equal weights were used. The computation time of the model varied between 400 seconds and 2000 seconds, depending on the settings of the parameters in the model.

**Su et al. (2013)** developed an optimization model that determines both an energy-efficient driving strategy and an optimal distribution of the running time supplements in the timetable of a metro line. The aim is to minimize the total energy consumption. To that end, the authors first explicitly calculate the energy-efficient switching points for the different driving regimes. A second algorithm calculates the minimum running time for a train, given the maximum speed limits. Then a third algorithm distributes the running time supplements among consecutive trips in order to minimize the total energy consumption, based on the gradients of the curves between running time and energy consumption. The running time calculations are simplified by assuming constant acceleration, braking, and running resistances. Based on experiments involving the Beijing Yizhuang metro line, the authors conclude that the energy consumption can be reduced on average by 10.3% by applying EETC in the current timetable compared to normal operation. If the timetable is modified based on the results of the model, then the energy consumption can be reduced by 14.5% compared to normal operation.

In a follow-up paper, **Su et al. (2014)** aimed at overcoming some of the shortcomings of the model of **Su et al. (2013)**. To that end, they extend the energy-efficient optimization model into an integrated energy-efficient optimization model. In this model they also consider the headway times between consecutive trains in order to incorporate passenger demand and multiple trains. Moreover, the authors now allow variable gradient profiles. In the same way as in **Su et al. (2013)**, the model first calculates the optimal train control per trip. Then the optimal distribution of the running time supplements is determined, again based on the per trip trade-off curves of running time and energy consumption. Finally the headway times between consecutive trains are determined, thereby taking into account passenger demand and fleet size. This is done by iterating over the first two algorithms for different values of the cycle time and the fleet size. In a case study of the Beijing Yizhuang metro line, the authors determine that the energy consumption could be reduced by 25.4% during peak hours and 15.9% during off-peak hours compared to normal operation. Over a whole day, the energy consumption could be reduced by 24.0% in comparison with normal operation.

**Li, Wang, Li, and Gao (2013)** described another multi-objective timetable optimization approach. They considered three objectives: minimizing energy consumption, carbon emissions, and passenger travel time. All objectives are equally weighted in the optimization. To solve the problem, a multi-objective optimization model is

proposed. This model is a deterministic model, but it is solved by fuzzy multi-objective optimization techniques available in LINGO. The model is applied to the Wuhan–Guangzhou high speed railway line in China, which includes 10 stations. In the resulting timetable, a reduction in energy consumption of about 17.6% is realized in comparison with the timetable that minimizes passenger travel times. However, this comes at a cost of an increase in total passenger travel time of 8.6%. Especially, the journey times increased for the heavier trains with high resistance coefficients and the trains with high carbon emission factors.

[Y. Wang, De Schutter, Van den Boom, and Ning \(2014\)](#) considered the optimal trajectory planning problem for two trains without regenerative braking, and incorporating varying gradients and speed limits. The authors studied the effects of using two different signalling systems, namely a fixed block system (FBS) or a moving block systems (MBS). The objective of their algorithm is to minimize total traction energy consumption for a leading and a following train. The nonlinear terms in the train model and constraints were approximated by piecewise affine functions. In order to separate the two trains, minimum headways were applied. Two different approaches were considered in the paper to solve the problem. A greedy approach where first the leading train is optimized and afterwards the following train, and a simultaneous approach, where the optimization of the two trains is done simultaneously. Two solution approaches were used: a MILP method and a Pseudospectral method, both implemented in MATLAB. The models were applied on a case study of a 1332 metre track between two stops of the Beijing Yizhuang metro line in China. The results indicated that moving block leads to more energy savings, since the headways between two trains can be shorter. Moreover, a simultaneous approach performed better in energy minimization than a greedy approach. The Pseudospectral method performed better in terms of energy consumption and punctual running than the MILP approach, however, the computation times of the MILP approach were much faster. For the instances with parameter values corresponding to the more accurate models the computations times for the MILP approach were above a minute, while the fastest Pseudospectral method took more than several minutes.

[P. Wang and Goverde \(2016\)](#) considered the train trajectory optimization problem for two successive trains without regenerative braking with consideration of general infrastructure (varying gradients and speed limits) and operational constraints, as well as signalling constraints. Operational constraints refer to time and speed restrictions from the actual timetable, while signalling constraints refer to the influences of signal aspects and automatic train protection on train operation. They extended the multiple-phase optimal control model of [P. Wang et al. \(2015\)](#) by modelling also signalling constraints in the TPE and applied a Pseudospectral method as solution method. They considered various delays of the first train and computed the resulting optimal trajectory for the second train, with the objective to minimize a trade-off between delays and energy consumption. For this purpose, two optimization policies were developed with either limited or full information of the train ahead. A regional signal response policy only assumed information on the signal ahead such as with a stand-alone DAS that is not connected to a TMS. This policy ensures that the train makes safe responses to different signalling aspects. On the other hand, a global green wave policy assumed that the signal release times by the train ahead are available corresponding to a DAS connected to a centralized Traffic Management System that monitors and predicts the movements of the trains and communicates the corresponding earliest signal approach times to the following trains. The green wave policy then aims at avoiding yellow signals and thus proceeds with all green signals. A case study considered a 50 kilometres corridor with eight stations in the Netherlands operated by regional and nonstop intercity trains in a 15 minute cyclic

timetable with the intercity trains overtaking the regional trains halfway. The optimal trajectories were computed for various delays of a regional train. The results showed the benefits on energy consumption and train delay of the following trains if accurate predictive information of the leading train is available. The more delay of the leading train, the better the performance of the green wave policy in both energy consumption and delay.

The combination of timetabling and energy-efficient train operation is also studied by [Binder and Albrecht \(2013\)](#) for the European rail project ON-TIME ([ON-TIME, 2014b](#)). They developed a Dynamic Programming algorithm for regional trains that determines the optimal arrival and departure times at intermediate stops in a corridor between two main stations with fixed departure and arrival time, and thus the optimal distribution of the running time supplements and the dwell times along a corridor. The objective includes three criteria: minimization of the expected energy consumption, minimization of the expected arrival delay at the main station at the end of the corridor, and minimization of the expected delay at the intermediate stations. First, Train Path Envelopes are determined which limit the solution space in the optimization. Within a TPE, the optimal train trajectory is computed. The model considers stochastic dwell times, but the running times are assumed to be deterministic and follow the optimal energy-efficient driving strategy for the running time obtained by fixing the arrival and departure times. [Binder and Albrecht \(2013\)](#) tested the model on a German corridor between two main stations with five intermediate stations. Depending on the weights of the three objectives, the authors report expected energy savings between 4.3% and 12.9% in comparison to the technical minimum running times. In the ON-TIME project the model was applied to fine-tune the arrival and departure times at the intermediate stops of the regional trains on corridors between main intercity stations ([Goverde et al., 2016](#)). First, the arrival and departure times of all trains at the intercity stations were optimized using a micro-macro two-level timetabling approach generating a conflict-free timetable with an optimal trade-off between travel times and robustness. Then the energy-efficient speed profiles were computed for the intercity and freight trains with respect to the fixed scheduled running times, after which the TPEs were determined for the regional trains between the intercity stations. Then the Dynamic Programming model was applied to the regional trains. [Goverde et al. \(2016\)](#) applied this method to a Dutch railway network of several interconnected railway lines and report energy savings of 35.5% for all trains over the network with respect to the energy consumption of the minimum running times.

[Mills, Perkins, and Pudney \(1991\)](#) studied a different version of EETT. They focused on solving the meet-and-pass problem for freight trains in Australia. Since the main part of the Australian railway network is single track, one of the main issues is the meet-and-pass problem for freight trains in different directions. The dynamic rescheduling system they describe aims at rescheduling train movements in such a way that train lateness and energy consumption are minimized. The described model is a non-linear optimization model for determining energy-efficient speed profiles, and a discrete heuristic for solving the meet-and-pass problem. The model was tested on a railway corridor between Port Augusta and Tarcoola (Australia). The reported savings in this experiment are about 6%. The non-linear model used about 21.5 minutes on a HP9000/340 workstation. The discrete heuristic required about 3.3 minutes.

#### 4.3. EETT with regenerative braking

Another way to save energy during train operation is by using regenerative braking, i.e., to use the released kinetic energy by regenerative braking of a train as traction energy for other

nearby trains. This regenerated energy can be transmitted over the catenary (overhead line) system to the other trains. The effective distance to transfer this regenerated energy over the catenary system depends on the voltage and the current of the power supply. Catenary systems using high voltage alternating current (AC) have less energy loss and thus can transmit the regenerated energy over a larger distance than low voltage direct current (DC) systems. Examples of the former are the German, Swiss and Austrian 15 kilovolts AC electrification systems, and of the latter are the Belgian, Italian or Spanish 3 kilovolts DC or the Dutch 1.5 kilovolts DC catenary systems. For low voltage DC systems it is thus important to have overlapping time intervals of the accelerating and regenerative braking train in the same electrified section to make efficient use of the regenerated energy. Thus the aim is to synchronize the processes between accelerating and regenerative braking of trains, where the gain of the synchronization is higher when the electrical voltage is lower.

One of the first papers studying this topic was [T. Albrecht \(2004\)](#). He considered additional running time for the synchronization of acceleration and regenerative braking instead of additional dwell time at stations for synchronization. The model aims at finding the optimal distribution of the running time supplements in order to minimize total energy consumption and power peaks. The model was solved by a Genetic Algorithm and applied on a case study of the S-Bahn of Berlin. It was shown that, in the case of constant dwell times, regenerative braking may lead to an extra 4% of energy savings in comparison to running time optimization for individual trains. In case of stochastic dwell times, energy consumption is about 6% higher than with constant dwell times.

Regenerative braking is also considered by [Peña-Alcaraz, Fernández, Cucala, Ramos, and Pecharrón \(2012\)](#). They developed a mathematical programming model to determine a timetable for metro systems that minimizes the total energy consumption by optimally using the regenerative braking energy of the trains. The synchronization of acceleration and braking is modelled by a power flow model, which considers increased running times instead of increased dwell times. Then the timetable model determines the optimally synchronized timetable. However, energy-efficient train control is not considered in the model, because the focus is on maximizing the use of regenerative braking by synchronization of acceleration and regenerative braking. Based on a simulation model of the metro of Madrid, the authors report energy savings of the optimized timetable of about 7% on average compared to using the published timetable without loss of service to the passengers.

[X. Yang, Li, Gao, Wang, and Tang \(2013\)](#) studied the topic of synchronization of accelerating and regenerative braking trains for metro systems. They did not explicitly look at regeneration of energy, but at maximizing time overlaps of nearby accelerating and braking trains. The authors first described the problem in terms of a mathematical programming model. Then a Genetic Algorithm was developed to find the optimally synchronized timetable by using headway and dwell time control. On a case study involving the Beijing Yizhuang metro line, the model increased the time overlaps of nearby accelerating and braking trains with 22.1% during peak hours and with 15.2% during off-peak hours. The authors did not consider transmission losses nor converter inefficiency of the regenerated energy.

[X. Yang, Ning, Li, and Tang \(2014\)](#) further developed the model of [X. Yang et al. \(2013\)](#). In addition to energy savings by using regenerative braking, the authors also take into account passenger waiting time. Again they first described the problem in terms of a mathematical programming model with two objective functions. The model does not take into account uncertainty in the dwell time process at stations: it treats passenger behaviour as determin-

istic. The model is solved by a Genetic Algorithm. Again in a case study involving the Beijing Yizhuang metro line, the model could reduce the energy utilization by 8.9%, while at the same time the passenger waiting time could be reduced by 3.2% compared to using the published timetable. However, the authors assumed that all regenerative energy is used by another train, while this is not always realistic.

[X. Yang, Chen, Li, Ning, and Tang \(2015\)](#) again improved the model of [X. Yang et al. \(2013\)](#). Instead of just focusing on time overlaps of nearby accelerating and braking trains, they consider all trains in the same track interval of electricity supply, and they extend the time horizon to the whole day. The authors assume that trains are operating according to the optimal speed profiles, and aim at synchronizing the arrivals and departures of trains such that as much as possible regenerated energy can be used. They first describe this scheduling problem in terms of a Mixed Integer Programming model. Again they develop a Genetic Algorithm for solving this model. They test their algorithm on the Beijing Yizhuang metro line. Their conclusion is that their algorithm leads to 7.0% reduction in energy consumption in comparison with the currently operated timetable, and to 4.3% reduction in comparison with the algorithm of [X. Yang et al. \(2013\)](#). The improved algorithm leads to an increase of 36.2% in the utilization of regenerated energy.

Next, [Li and Lo \(2014a\)](#) developed an integrated energy-efficient operation model. In their model they both optimize the timetable and the speed profiles by taking into account the headways between trains. They applied a Genetic Algorithm to solve their model. The timetabling part of the model tries to synchronize the accelerating and regenerative braking of trains in order to reuse the regenerated energy. The speed profile part calculates the optimal train control in order to minimize the net energy consumption. The model was again applied to the Beijing Yizhuang metro line. One of the results of the model is that the energy savings are about 25% if the headways between trains are minimal (i.e. 90 seconds). In that case the energy savings of the integrated approach are about 20% higher than those of a two-step approach. The energy savings are smaller when the headways increase, which is also observed by [Feng et al. \(2013\)](#). Also the difference between the integrated approach and the two-step approach gradually decreases for increasing headways. However, [Li and Lo \(2014a\)](#) assumed simplified train dynamics with a constant acceleration rate, deceleration rate, running resistance, and energy transmission loss factor.

Finally, [Li and Lo \(2014b\)](#) developed a model to determine the cycle time, the headway time, and the speed profiles for a metro line, dynamically depending on the passenger demand and such that the energy consumption is minimized. If the passenger demand is high, then a small cycle time and short headway times are required to be able to handle the passenger demand. If the demand is lower, then the cycle time and the headway times may be longer. The authors make several simplifying assumptions, e.g., they assume that there is no coasting phase, but only a cruising phase. Based on these assumptions, they develop an explicit quadratic expression for the net energy consumption of a train during one cycle, thereby also considering regenerated energy. Then they set up the corresponding KKT conditions, and they solve these in an iterative way. The model was applied to the Beijing Yizhuang metro line. The results obviously depend on the assumptions for the passenger demand. In the experiment with fluctuating passenger demand that was carried out, using a dynamic cycle time and headway time could save up to about 8% energy in comparison with the shortest possible fixed cycle time and headway time that allow satisfying all passenger demand. If the fixed cycle time and headway time are increased, then the corresponding energy consumption may be the same as that of the dynamic ones,

**Table 3**

Literature on energy-efficient timetabling.

Publication	Objective(s)	Process	Algorithm	Transport mode	Scope	RB
T. Albrecht and Oettich (2002)	Expected waiting time, energy consumption	Distribution of running time supplements	Dynamic Programming	Suburban trains with ATO	Single train on a line with 10 stations	
Ghoseiri et al. (2004)	Travel time, fuel consumption	Timetabling	Non-linear Mathematical Programming, $\epsilon$ -constraint method	Passenger trains	Several trains on lines with varying numbers of stations	
Ding et al. (2011)	Energy consumption	Distribution of running time supplements, switching points	Genetic Algorithm	Metro trains	Single train on a line with 6 stations	
Sicre et al. (2010)	Energy consumption	Distribution of running time supplements	Simulation, Optimization	High speed trains	Single train on a line with 4 stations	
Cucala et al. (2012)	Delays, energy consumption	Distribution of running time supplements	Genetic Algorithm, Simulation, Fuzzy Linear Programming,	High speed trains	Single train on a line with 6 stations	
L. Yang et al. (2012)	Travel time, energy consumption	Coasting control	Genetic Algorithm	High speed trains	Three trains in an example network	
Su et al. (2013)	Energy consumption	Distribution of running time supplements	Explicit formulas based on simplifying assumptions	Metro trains	Single metro on a line with 14 stations	
Su et al. (2014)	Energy consumption	Distribution of running time supplements and headway control	Iterative application of the methods from Su et al. (2013)	Metro trains	Multiple metros on a line with 14 stations	
Li et al. (2013)	Travel time, carbon emission, energy consumption	Distribution of running time supplements	Fuzzy multi-objective non-linear optimization	Regional trains and high speed trains	Multiple trains on a line with 10 stations	
Y. Wang et al. (2014)	Energy consumption	Headway optimization	Mixed integer linear programming and Pseudospectral method	Metro trains	Two following trains between two stops with fixed or moving blocks	
P. Wang and Goverde (2016)	Delays, energy consumption	Headway optimization	Pseudospectral method	Mixed regional and intercity trains	Multiple trains on a line with fixed blocks and 8 stations	
Binder and Albrecht (2013)	Expected delays, energy consumption	Distribution of running time supplements	Dynamic Programming	Regional trains	Single train on a line with 7 stations	
Mills et al. (1991)	Train lateness, energy consumption	Meet-and-pass problem	Non-linear optimization, discrete heuristic	Freight trains	Several trains on a single track corridor	
T. Albrecht (2004)	Power peaks, energy consumption	Running time optimization for synchronization	Genetic Algorithm	Suburban trains	One train on a line with 16 stations, taking into account also other trains	x
Peña-Alcaraz et al. (2012)	Energy consumption, regenerated energy	Running time optimization for synchronization	Mathematical Programming and DC power flow model	Metro trains	One metro on a line with 36 stations, taking into account also other metros	x
X. Yang et al. (2013)	Overlap time	Headway and dwell time optimization for synchronization	Genetic Algorithm	Metro trains	Several metros on a line with 14 stations in two directions	x
X. Yang et al. (2014)	Passenger waiting time, regenerated energy	Headway and dwell time optimization for synchronization	Genetic Algorithm	Metro trains	Several metros on a line with 14 stations in two directions	x
X. Yang et al. (2015)	Energy consumption	Headway and dwell time optimization for synchronization	Genetic Algorithm	Metro trains	Several metros on a line with 14 stations in two directions	x
Li and Lo (2014a)	Net energy consumption	Timetable optimization for synchronization	Genetic Algorithm	Metro trains	Several metros on a line with 14 stations in two directions	x
Li and Lo (2014b)	Net energy consumption	Dynamic cycle time, headway and speed profiles	Simplifying assumptions, KKT conditions	Metro trains	Several metros on a line with 14 stations in two directions	x

Legend: ATO = Automatic Train Operation, RB = Regenerative Braking, KKT = Karush-Kuhn-Tucker.

but in that case the transport capacity may not be sufficient in the cycles with a high passenger demand.

#### 4.4. Summary on EETT

The main results of the literature survey are summarized in Table 3. In this table the column “RB” indicates whether regenerative braking is taken into account. Part of the research focuses on optimizing speed profiles of trains in order to minimize the energy consumption, where the total running time is used as a variable

to influence the total energy consumption. Another part of the research focuses on optimally distributing running time supplements over the successive train runs according to different multi-criteria objectives including the minimization of total energy consumption. A third research stream focuses on regenerative braking and tries to synchronize the timetable in order to maximize the use of regenerated braking energy.

The models for finding optimal running time supplements over a train line are mainly divided into Gradient Search, Dynamic Programming, Genetic Algorithms, and Simulation. Moreover, when

regenerative braking is available, the synchronization of accelerating and braking trains is optimized mainly using Genetic Algorithms and Simulation. Different kinds of railway modes have been considered, although most focus is on regional trains and metro trains. Most research considers single-train line optimization, while surrounding trains are mainly considered with regenerative braking.

## 5. Conclusions

The general energy-efficient train control problem is characterized by nonlinear dynamics from the traction and train resistance forces as function of speed, distance-dependent state constraints from speed restrictions, bounded controls, and a fixed time horizon. Since the state constraints and the line resistance forces from varying gradients depend on distance, most models in the literature take distance as the independent variable rather than time. The objective is typically minimization of energy consumption, which is the integral of the (scaled) applied forces over distance. The states are mostly speed and time as function of distance, with some authors taking energy as an alternative to speed. To solve the resulting optimal train control problem, distance is typically partitioned into sections of constant gradient and speed limit, and the problem becomes a multiple-phase optimal control problem where each phase (section) is linked with its adjacent phases via continuity constraints in the state variables. In this case, the number and order of driving regimes become less obvious, by which solving the problem becomes numerically challenging.

The optimal train control structure can be derived by applying Pontryagin's Maximum Principle, which gives necessary conditions for the optimal train control. For level track and no or fixed speed limit the optimal train control structure consists of a sequence of the four driving regimes maximum acceleration by full traction, cruising by partial traction, coasting with no traction, and maximum braking, in this order, where cruising and coasting may be absent depending on the time horizon and speed limit. When varying speed limits are considered, additional maximum acceleration regimes may occur in the optimal control structure at each speed limit increase and additional coasting regimes before each speed limit decrease. With varying gradients the cruising regime can be realized by partial traction or partial braking depending on the gradient, while steep gradients may require maximum traction or maximum braking, even in front of a steep uphill or downhill section. When also regenerative braking is possible, the optimal train control structure is extended to seven driving regimes where also (partial or full) regenerative braking can be used for cruising or braking. In addition, recent literature considers further operational constraints such as various target points, flexible time and/or speed windows, or signalling constraints.

Pontryagin's Maximum Principle gives the optimal driving regimes but not the optimal sequence of these regimes nor the optimal switching points between regimes. The literature therefore describes many numerical algorithms to solve the optimal control problem by determining the switching points between driving regimes and associated optimal cruising speeds. Several efficient algorithms have been developed for special cases or assumptions such as level track, absence of steep gradients, assumed linear train resistance, discarding the coasting regime, or setting the cruising speeds equal to the speed limits. These special cases can be used for suboptimal train control in particular situations. A recent approach based on a direct Pseudospectral method is promising to solve the most general energy-efficient train control problem with varying gradients and speed limits.

The energy consumption is largely determined by the timetable and in particular by the amount of scheduled running time supplements. In the energy-efficient train timetabling problems the run-

ning time supplements are the decision variables and the objective is to find the optimal distribution of the running time supplements for a train on one or more legs of its journey, where the objective is mainly a trade-off between minimizing both travel time and energy consumption, and in some cases also delay. The energy-efficient train control problem is mostly used as a subproblem and the overall optimization problem is solved by Gradient Search, Dynamic Programming, Simulation or Genetic Algorithms. When regenerative braking is possible, the focus shifts to synchronizing braking and accelerating trains so that the regenerated braking energy can be used by nearby accelerating trains. Simulation and Genetic Algorithms are here the main solution methods.

The literature on energy-efficient train control is struggling between developing more accurate advanced models on the one hand and faster algorithms on the other. The algorithms in the existing DAS and ATO systems rely on some simplifications to be able to compute (sub)optimal driving advice in real-time or are based on offline computed solutions for a large set of scenarios. The main difference between the theoretical models for EETC and a DAS in practice is that the theoretical models try to find the optimal driving strategy, while a DAS often settles for suboptimal solutions using heuristics. The computation time increases for EETC models when more realistic behaviour is included, like varying gradients and speed limits. In real-time operation fast algorithms for a DAS are needed. Furthermore, additional constraints may be included in a DAS to provide the train driver with a stable driving advice (i.e. no continuously changing advice) or specific driving regimes may be excluded, such as coasting or cruising at less than the speed limit. This also relates to the drivability of the optimal driving strategy in terms of the number and time intervals between driving regime changes, which applies more to DAS than ATO, as well as how the advice is presented to the driver and the acceptability of a driver to use a DAS. This needs more research from a human factors point of view. Moreover, the option of regenerative braking increases the number of driving regimes even more. In this respect, a comparison of EETC with or without regenerative braking is also required with respect to energy consumption and the complexity of the associated driving strategy. Nevertheless, even suboptimal driving strategies can lead to significant energy savings. In due time, power of computers will increase allowing more advanced algorithms to be used in real-time in a DAS. And of course, there is active research in model and algorithm development where the increased knowledge about the optimal driving strategies under various conditions and constraints will be a guide to find more evident algorithms.

Recent research also focused on including more operational (schedule and signalling) constraints in the train control problem, which also paves the way to extend the single-train optimal control problem to multi-train optimal control problems where the energy consumption of multiple trains is optimized dynamically including their interaction. This multi-train optimization problem can be included in real-time railway Traffic Management Systems, where the aim is first to avoid real-time conflicts and second to minimize the total traction energy consumption of all trains in a network. An initial study in this area was presented by Mills et al. (1991), but there has not yet been much follow-up research. The additional operational constraints will also be useful for more realistic energy-efficient timetabling problems for multiple trains, which is yet a largely unexplored topic.

Incorporating energy-efficiency in timetable design is another area of future research. There are currently limited papers on this topic, but the attention to this field has been increasing recently. Besides theoretical results, railway undertakings also start showing interest in EETT. For example, recently NS is investigating a change in their timetable design process by redistributing the amount of running time supplements and scheduling more realistic running

times, which increases the opportunities for energy-efficient driving. Also the Swiss Federal Railways SBB (*Schweizerische Bundesbahnen*) is investigating EETT for their regional train services by uniformly redistributing the running time supplements over the trajectories and using flexible arrival times. In the future, research will investigate the optimal amount and distribution of the running time supplements as well as the balance between different objectives for timetable design, like minimizing total running time, total delay, energy consumption, and maximizing passenger comfort. Moreover, the efficiency of these models and the interaction with the applied EETC strategies and possible DAS implementations need to be tested in pilots. The awareness of the impact of EETC to exploit time allowances for energy savings will change both operational railway traffic management and timetabling. The review of models and algorithms presented in this paper may guide future research directions and lead to a further reduction of energy consumption and costs in the railways of the future.

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