Chapter 1

Martingale Approach

In this section, the results of the martingale approach are summarized. To quickly summarize: The first approach resulted in the correct optimal policy but the chosen random variable was never a martingale. The analysis just seemed to involve minizing the expected total discounted cost just like earlier. The approach similar to the notes from Glynn gave a martingale but the resulting conclusion was $\mu = \mu^*$, which isn't very useful.

1.1 Definitions and notation

We denote the time step by $\delta > 0$. We consider the martingale on time stages $x_i = i\delta$ $(i = 0, 1, ...)$. Time is then divided into intervals $\Delta_i = (x_{i-1}, x_i]$ $(i = 1, 2, ...)$. The n'th lifetime is a positive random variable Q_n. These Q_n's are i.i.d. random variables. When the machine is repaired preventively, a cost c is paid. When the machine breaks and is repaired correctively, a cost $c + a$ is paid. Costs that occur at a time t in the future, are discounted by a discount factor $e^{-\beta t}$ for some $\beta > 0$. Let v^* be the expected total discounted cost for the optimal control limit. When a machine breaks in an interval Δ_i, it is repaired at the end of the interval (i.e., at x_i). When a machine happens to break exactly at the time that preventive maintenance is scheduled, the cost for corrective maintenance is paid.

1.2 Approach from the last meeting

We consider the cost of one run using control limit μ, with terminal cost v^*. This cost is represented by the following random variable

$$V_\mu = (c + \mathbb{1}\{Q_0 \geq \mu\}a + v^*)e^{-\beta Q_0 \wedge \mu}.$$

We consider the following sequence of random variables

$$M_n^\mu = \mathbb{1}\{Q_0 \wedge \mu > x_n\}e^{\beta x_n}\mathbb{E}[V_\mu | Q_0 \wedge \mu > x_n]. \tag{1.1}$$

We are now going to minimize

$$
\begin{aligned}
g_n(\mu) &= \mathbb{E}[M_{n+1}^\mu - M_n^\mu | M_n^\mu] \\
&= \mathbb{E}[M_{n+1}^\mu | M_n^\mu] - \mathbb{E}[M_n^\mu | M_n^\mu] \\
&= \mathbb{E}[M_{n+1}^\mu | M_n^\mu] - M_n^\mu
\end{aligned}
\tag{1.2}
$$

Note that

$$
\begin{aligned}
\mathbb{E}[M_{n+1}^\mu] &= \mathbb{E}[\mathbb{1}\{Q_0 \wedge \mu > x_{n+1}\} e^{\beta x_{n+1}} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_{n+1}]] \\
&= \mathbb{P}(Q_0 \wedge \mu > x_{n+1}) e^{\beta x_{n+1}} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_{n+1}] \\
&= \mathbb{P}(Q_0 \wedge \mu > x_{n+1}) e^{\beta x_{n+1}} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_{n+1}] \\
&= \mathbb{P}(Q_0 \wedge \mu > x_n) e^{\beta x_{n+1}} \begin{pmatrix} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_n] \\ - \mathbb{P}(Q_0 \in \Delta_{n+1} | Q_0 > x_n) \mathbb{E}[V_\mu | Q_0 \in \Delta_{n+1}] \end{pmatrix} \\
&= \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) e^{\beta(x_{n+1} - x_n)} (\mathbb{E}[M_n^\mu] - e^{\beta x_n} \mathbb{P}(Q_0 \in \Delta_{n+1} | Q_0 > x_n) \mathbb{E}[V_\mu | Q_0 \in \Delta_{n+1}]) \\
&= \begin{matrix} \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) e^{\beta(x_{n+1} - x_n)} \mathbb{E}[M_n^\mu] \\ - e^{\beta x_{n+1}} \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) \mathbb{P}(Q_0 \in \Delta_{n+1} | Q_0 > x_n) \mathbb{E}[V_\mu | Q_0 \in \Delta_{n+1}] \end{matrix}
\end{aligned}
\tag{1.3}
$$

So that for $g_n(\mu)$, we get

$$
\begin{aligned}
g_n(\mu) &= \mathbb{E}[M_{n+1}^\mu | M_n^\mu] - M_n^\mu \\
&= \begin{matrix} \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) e^{\beta(x_{n+1} - x_n)} \mathbb{E}[M_n^\mu | M_n^\mu] \\ - e^{\beta x_{n+1}} \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) \mathbb{P}(Q_0 \in \Delta_{n+1} | Q_0 > x_n) \mathbb{E}[V_\mu | Q_0 \in \Delta_{n+1}] \\ - M_n^\mu \end{matrix} \\
&= \begin{matrix} (\mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) e^{\beta(x_{n+1} - x_n)} - 1) M_n^\mu \\ - e^{\beta x_{n+1}} \mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) \mathbb{P}(Q_0 \in \Delta_{n+1} | Q_0 > x_n) \mathbb{E}[V_\mu | Q_0 \in \Delta_{n+1}]. \end{matrix}
\end{aligned}
\tag{1.4}
$$

If we now take the derivative of $g_n(\mu)$ to μ, the second term disappears as it does not depend on μ. The factor $(\mathbb{P}(Q_0 > x_{n+1} | Q_0 > x_n) e^{\beta(x_{n+1} - x_n)} - 1)$ also does not depend on μ. So only the derivative of M_n^μ is of interest.

$$
\begin{aligned}
\frac{d}{d\mu} M_n^\mu &= \frac{d}{d\mu} \mathbb{1}\{Q_0 \wedge \mu > x_n\} e^{\beta x_n} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_n] \\
&= \mathbb{1}\{Q_0 \wedge \mu > x_n\} e^{\beta x_n} \frac{d}{d\mu} \mathbb{E}[V_\mu | Q_0 \wedge \mu > x_n].
\end{aligned}
\tag{1.5}
$$

We rewrite this expectation to make it easier to derive

$$
\begin{aligned}
\frac{d}{d\mu} \mathbb{E}[V_\mu | Q_0 > x_n] &= \frac{d}{d\mu} \mathbb{E}[(c + \mathbb{1}\{Q_0 \geq \mu\} a + v^*) e^{-\beta Q_0 \wedge \mu} | Q_0 > x_n] \\
&= \frac{d}{d\mu} ((c + v^*) \mathbb{E}[e^{-\beta Q_0 \wedge \mu} | Q_0 > x_n] + a \mathbb{P}(Q_0 leq \mu | Q_0 > x_n) \mathbb{E}[e^{-\beta Q} | Q \in (x_n, \mu]]) \\
&= -\beta \frac{\mathbb{P}(Q_0 > \mu)}{\mathbb{P}(Q_0 > x_n)} (c + v^*) e^{-\beta\mu} + a \frac{f(\mu)}{\mathbb{P}(Q_0 > x_n)} e^{-\beta\mu} \\
&= 0 \\
&\Rightarrow f(\mu) = \beta \frac{(c + v^*) \mathbb{P}(Q_0 > \mu)}{a}.
\end{aligned}
\tag{1.6}
$$

As you can see, this solution is exactly the same as in the earlier approaches. We also need to assure that $\frac{d^2}{d\mu^2}g_n(\mu) > 0$. Again, we only need to derive $\mathbb{E}[V_\mu|Q_0 > x_n]$.

$$\begin{aligned}\frac{d^2}{d\mu^2}\mathbb{E}[V_\mu|Q_0 > x_n] &= \frac{d}{d\mu}(-\beta\frac{\mathbb{P}(Q_0 > \mu)}{\mathbb{P}(Q_0 > x_n)}(c+v^*)e^{-\beta\mu} + a\frac{f(\mu)}{\mathbb{P}(Q_0 > x_n)}e^{-\beta\mu}) \\ &= \frac{\beta^2\mathbb{P}(Q > \mu) + \beta f(\mu)}{\mathbb{P}(Q_0 > x_n)}(c+v^*)e^{-\beta\mu} + \frac{-\beta f(\mu) + f'(\mu)}{\mathbb{P}(Q_0 > x_n)}ae^{-\beta\mu}.\end{aligned} \tag{1.7}$$

We now multiply by $\frac{1}{a}\mathbb{P}(Q_0 > x_n)e^{\beta\mu}$ to get

$$(\beta^2\mathbb{P}(Q > \mu)+\beta f(\mu))\frac{c+v^*}{a}+(-\beta f(\mu)+f'(\mu)) = \beta^2\mathbb{P}(Q > \mu)\frac{c+v^*}{a}+\beta f(\mu)(\frac{c+v^*}{a}-1)+f'(\mu).$$

Now we substitute $f(\mu) = \beta\frac{(c+v^*)\mathbb{P}(Q_0>\mu)}{a}$ and get

$$\begin{aligned}&\beta^2\mathbb{P}(Q > \mu)\frac{c+v^*}{a} + \beta^2\mathbb{P}(Q_0 > \mu)\frac{(c+v^*)}{a}(\frac{c+v^*}{a} - 1) + f'(\mu) \\ &= \beta^2\mathbb{P}(Q_0 > \mu)(\frac{c+v^*}{a})^2 + f'(\mu)\end{aligned} \tag{1.8}$$

For the final steps, we need the following simple lemma:

Lemma 1. Let Q be a random variable with increasing failure rate. Then

$$f'(x) > \frac{f(x)^2}{\mathbb{P}(Q_0 > \mu)}$$

Proof. The failure rate is increasing, so its derivative is positive. Hence

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{\mathbb{P}(Q > x)} &= \frac{f'(x)\mathbb{P}(Q > x) + f(x)^2}{\mathbb{P}(Q > x)^2} > 0 \\ &\Rightarrow f'(x) > -\frac{f(x)^2}{\mathbb{P}(Q > x)}\end{aligned} \tag{1.9}$$

□

We now apply this lemma

$$\beta^2\mathbb{P}(Q_0 > \mu)(\frac{c+v^*}{a})^2 + f'(\mu) > \beta^2\mathbb{P}(Q_0 > \mu)(\frac{c+v^*}{a})^2 - \frac{f(\mu)^2}{\mathbb{P}(Q > \mu)}.$$

We multiply by $\mathbb{P}(Q > \mu)$ and substitute $f(\mu) = \beta\frac{(c+v^*)\mathbb{P}(Q_0>\mu)}{a}$ again

$$\left(\beta\mathbb{P}(Q_0 > \mu)\frac{c+v^*}{a}\right)^2 - f(\mu)^2 = \left(\beta\mathbb{P}(Q_0 > \mu)\frac{c+v^*}{a}\right)^2 - \left(\beta\mathbb{P}(Q_0 > \mu)\frac{c+v^*}{a}\right)^2 = 0. \tag{1.10}$$

Since we only multiplied by positive values, we conclude that $\frac{d^2}{d\mu^2}g_n(\mu) > 0$ and the found solution is indeed optimal.

However, the sequence of random variables is not a martingale for any policy. This can easily be seen by the fact that $\mathbb{E}[M_0^\mu] > 0$ while for any n such that $x_n > \mu$, we have $\mathbb{E}[M_n^\mu] = 0$. Hence we proceed with a definition of a martingale similar to the notes of Glynn.

1.3 Approach similar to Glynn chapter 11

In the notes from Glynn, the martingale is taken to be

$$\sum_{j=0}^{T\wedge n-1} r(X_j, A_j) + \mathbb{1}\{T > n-1\}V^*(X_n), \tag{1.11}$$

i.e. the cost of using controls $(A_j : j \geq 0)$ up until stage $n-1$ and using the expected value of the rest of the cost using the optimal policy. Which is of course a supermartingale for every policy and a martingale for every optimal policy.

In our approach, we define the martingale M_n^μ to be the total discounted cost of having used control limit μ up until time x_n and taking the expected discounted cost of using optimal control limit μ^* for the rest of time. Let $R_0 = 0$ and $R_{n+1} = R_n + Q_n \wedge \mu$ be the time of the n'th repair. For convenience, we define the following random variables

- $R^-(x) = \max\{R_i | R_i \leq x\}$ to be the time of the last repair at time x.
- $R^+(x) = \min\{R_i | R_i > x\}$ to be the time of the next repair at time x.
- $K(x) = \max\{i | R_i \leq x\}$ to be the number of repairs that have occurred before time x.
- $Q(x) = Q_{K(x)}$ to be the (total) lifetime of the current machine.

We denote expectations and probabilities conditioned to the observations up to time x by a subscript x. For example

$$\mathbb{E}_x[X] = \mathbb{E}[X | R_0, ..., R_{K(x)}].$$

Furthermore, let

$$V^*(x) = \mathbb{E}_x[(c + a\mathbb{1}\{Q(x) \geq \mu^* a + v^*)e^{-\beta(R^-(x)+Q(x)\wedge\mu^*}\})]$$

be the expected discounted cost of all costs after x, using the optimal control limit. We then arrive at the following definition of the supermartingale

$$M_n^\mu = \sum_{k=0}^{K(x_n)-1} (c + a\mathbb{1}\{Q_k \geq \mu\})e^{-\beta R_{k+1}} + V^*(x_n). \tag{1.12}$$

This is a martingale for $\mu = \mu^*$.

When we try to minimize

$$\begin{aligned} g_n(\mu) &= e^{\beta x_{n+1}}\mathbb{E}_{x_n}[M_{n+1}^\mu - M_n^\mu] \\ &= \mathbb{E}_{x_n}[\sum_{k=K(x_n)}^{K(x_{n+1})-1} (c + a\mathbb{1}\{Q_k \geq \mu\})e^{-\beta R_{k+1}} - (V^*(x_n) - V^*(x_{n+1})]. \end{aligned} \tag{1.13}$$

We neglect the possibility of two repairs within an interval of time δ, i.e. we assume that

$$\mathbb{P}(K(x_{n+1}) - K(x_n) > 1) = o(\delta^*). \tag{1.14}$$

Note that $V^*(x_n)-V^*(x_{n+1})$ equals the expected discounted cost in the interval $(x_n, x_{n+1}]$ so that

$$\begin{aligned}
e^{\beta x_{n+1}}(V^*(x_n) - V^*(x_{n+1})) &= \mathbb{E}_{x_n}[\mathbb{1}\{R^-(x_n) + Q(x) \wedge \mu^* \in \Delta_{n+1}\}c \\
&+ \mathbb{1}\{R^-(x_n) + Q(x) \in \Delta_{n+1}\}a] + o(\delta^2) \\
&= \mathbb{P}_{x_n}(Q(x) > x_{n+1} - R^-(x_n))\mathbb{1}\{\mu^* = x_{n+1} - R^-(x_n)\}c \\
&+ \mathbb{P}_{x_n}(Q(x) \le x_{n+1} - R^-(x_n))(c + a) + o(\delta^2).
\end{aligned} \tag{1.15}$$

Similarly, we can rewrite the other part of (1.13) to

$$\begin{aligned}
&e^{\beta x_{n+1}}\mathbb{E}_{x_n}[\sum_{k=K(x_n)}^{K(x_{n+1})-1} (c + a\mathbb{1}\{Q_k \ge \mu\})e^{-\beta R_{k+1}}] \\
&= \mathbb{P}_{x_n}(Q(x) > x_{n+1} - R^-(x_n))\mathbb{1}\{\mu = x_{n+1} - R^-(x_n)\}c \\
&+ \mathbb{P}_{x_n}(Q(x) \le x_{n+1} - R^-(x_n))(c + a) + o(\delta^2).
\end{aligned} \tag{1.16}$$

Combining these, results in

$$\begin{aligned}
g_n(\mu) &= \mathbb{P}_{x_n}(Q(x) > x_{n+1} - R^-(x_n))\mathbb{1}\{\mu = x_{n+1} - R^-(x_n)\}c \\
&+ \mathbb{P}_{x_n}(Q(x) \le x_{n+1} - R^-(x_n))(c + a) \\
&- \mathbb{P}_{x_n}(Q(x) > x_{n+1} - R^-(x_n))\mathbb{1}\{\mu^* = x_{n+1} - R^-(x_n)\}c \\
&- \mathbb{P}_{x_n}(Q(x) \le x_{n+1} - R^-(x_n))(c + a) \\
&= \mathbb{P}_{x_n}(Q(x) > x_{n+1} - R^-(x_n))(\mathbb{1}\{\mu = x_{n+1} - R^-(x_n)\} - \mathbb{1}\{\mu^* = x_{n+1} - R^-(x_n)\})c.
\end{aligned} \tag{1.17}$$

M_n^μ is a martingale if $g_n(\mu) = 0$ for all n. Hence, we conclude that $\mu = \mu^*$, which isn't very helpful.