1 Solving the simple discounted problem

We are going to solve problem with the following Bellman equations:

$$J(l,1) = \min\{c + e^{-\alpha dt} J(0,1), e^{-\alpha dt}\mathbb{E}[J(l+dt, I(t_{k+1}))]\}.$$

$$J(l,0) = c + a + e^{-\alpha h} J(0,1).$$

Where $I(t_{k+1})$ is a random variable and it is 0 if $l < Q_{n(t_k)} < l + dt$ and 1 else. Let the Q_i's have cdf $F(.)$, pdf $f(.)$, hazard rate $h(.)$ and cumulative hazard rate $H(.)$. Then

$$\mathbb{P}(l < Q_{n(t_k)} < l + dt | l < Q_{n(t_k)}) = \mathbb{P}(l < Q_0 < l + dt | l < Q_0)$$

$$= \frac{F(l+dt) - F(l)}{1 - F(l)} = \frac{dt\, f(l) + o(dt^2)}{1 - F(l)} = dt\, h(l) + o(dt^2).$$

So that $I(t_{k+1}) = 0$ with probability $dth(l) + o(h^2)$ and 1 with probability $1 - dt\, h(l) + o(dt^2)$. Resulting in

$$J(l,1) = \min\{c + e^{-\alpha dt} J(0,1), e^{-\alpha dt}[dt\, h(l) J(l+dt,0) + (1 - dt\, h(l)) J(l+dt,1) + o(dt^2)]\}.$$

When we assume that $J(l+dt,1) < c + e^{-\alpha dt} J(0,1)$, we can write

$$J(l,1) = (1 - \alpha dt + o(dt^2))[dt\, h(l) J(l+dt,0) + (1 - dt\, h(l)) J(l+dt,1) + o(dt^2)]$$

$$= dt\, h(l) J(l+dt,0) + (1 - dt\, h(l)) J(l+dt,1) - \alpha dt J(l+dt,1) + o(dt^2).$$

Or equivalently

$$J(l,1) - J(l+dt,1) = dt[h(l) J(l+h,0) - (\alpha + h(l)) J(l+dt,1)] + o(dt^2).$$

Dividing by $-dt$ results in

$$\frac{J(l+dt,1) - J(l,1)}{dt} = -h(l) J(l+dt,0) + (\alpha + h(l)) J(l+dt,1) + o(dt).$$

Since we know that $J(l+dt,0) = c + a + e^{-\alpha dt} J(0,1) = c + a + (1 + o(dt)) J(0,1)$, we can write

$$\frac{J(l+dt,1) - J(l,1)}{dt} = -h(l)(c + a + J(0,1)) + (\alpha + h(l)) J(l+dt,1) + o(dt).$$

If we let dt approach 0, this results in

$$\frac{d}{dl} J(l,1) = -h(l)(c + a + J(0,1)) + (\alpha + h(l)) J(l,1).$$

Which seems counterintuitive since for small α, $J(l,1)$ would be decreasing as $J(l,1) < c + e^{-\alpha dt} J(0,1) < c + a + J(0,1)$. We will try to solve this O.D.E. anyway. We use the method of the integrating factor. Our integrating factor will be

$$e^{\int_0^l (-\alpha - h(x)) dx} = e^{-\alpha l - H(l)}.$$

So that we get

$$J(l,1) = e^{\alpha l + H(l)}[C + \int_0^l e^{-\alpha x - H(x)}(-h(x)(c + a + J(0,1)))dx]$$

$$= \frac{e^{\alpha l}}{1 - F(l)}[C - (c + a + J(0,1)) \int_0^l e^{-\alpha x} f(x)dx].$$

Using the identities $e^{H(l)} = (e^{-H(l)})^{-1} = \frac{1}{1-F(l)}$ and $h(l)e^{-H(l)} = f(l)$. The C is an integrating constant and since $J(l) = J(0,1)$ should hold, we find $C = J(0,1)$. We can rewrite the expression to

$$J(l,1) = \frac{e^{\alpha l}}{1 - F(l)}[J(0,1) - (c + a + J(0,1))\mathbb{P}(L < l)\mathbb{E}[e^{-\alpha L}|L < l]].$$

Concluding

$$J(l,1) = \min\{c + J(0,1), \frac{e^{\alpha l}}{1 - F(l)}[J(0,1) - (c + a + J(0,1))\mathbb{P}(L < l)\mathbb{E}[e^{-\alpha L}|L < l]]\}$$

and preventive maintenance is chosen if and only if $J(l,1) = c + J(0,1)$. However, the value of $J(0,1)$ depends on the policy that is chosen and it seems difficult to solve $J(l,1) = c + J(0,1)$ analytically for l.