Chapter 1

Martingale Approach

In this section, we will attempt to find a solution to the simple discounted problem using a martingale approach.

1.1 Discrete-time martingale

We consider a policy of repairing after a control limit μ, and denote the optimal control limit by μ^*. The fluid levels at the beginning of the n'th run are given by Q_n.

Let T_n denote the time at which the n'th run starts. Let $T_0 = 0$ and $T_{n+1} = T_n + \mu \wedge Q_n$. By V_n^μ we denote the random variable of the discounted cost of the n'th run when using control limit μ. We construct the following submartingale

$$M_N = \sum_{n=0}^{N-1} V_n^\mu + \mathbb{E} \sum_{n=N}^{\infty} V_n^{\mu^*}$$

This is a submartingale since

$$\mathbb{E}[M_{N+1}|M_N] = M_N + \mathbb{E}[V_N^\mu - V_N^{\mu^*}] \geq M_N,$$

because $\mathbb{E}V_N^\mu \geq \mathbb{E}V_N^{\mu^*}$ with equality iff μ is an optimal control limit. Hence M_N is a martingale when μ is an optimal control limit. To find an optimal control limit, we solve the equality $\mathbb{E}[M_1|M_0] = M_0$ for μ:

$$
\begin{aligned}
\mathbb{E}[V_0^{\mu} + \mathbb{E}\sum_{n=1}^{\infty} V_n^{\mu^*}] &= \mathbb{E}[V_0^{\mu} + e^{-\beta(\mu\wedge Q_0)}\mathbb{E}\sum_{n=0}^{\infty} V_n^{\mu^*}] \\
&= \mathbb{E}V_0^{\mu} + \mathbb{E}[e^{-\beta(\mu\wedge Q_0)}]\mathbb{E}\sum_{n=0}^{\infty} V_n^{\mu^*} \\
&= \mathbb{E}\sum_{n=0}^{\infty} V_n^{\mu^*} = M_0 \qquad (1.1) \\
\Rightarrow \mathbb{E}V_0^{\mu} &= (1 - \mathbb{E}e^{-\beta(\mu\wedge Q_0)})\mathbb{E}\sum_{n=0}^{\infty} V_n^{\mu^*} \\
\Rightarrow \mathbb{E}\sum_{n=0}^{\infty} V_n^{\mu^*} &= \frac{\mathbb{E}V_0^{\mu}}{1 - \mathbb{E}e^{-\beta(\mu\wedge Q_0)}}
\end{aligned}
$$