Chapter 1

Maximum Likelihood estimators for MMFM fluid rates and constant jump quantities

In these notes, a maximum likelihood estimator will be presented to estimate the fluid rates and jump quantities of a Markov Modulated fluid model using the lifetimes. We assume that all transition rates λ_{ij} and the initial distribution $f(q)$ are already known.

Suppose we have observed a run of the machine and have seen that it started in state s_{i_1}, stayed there for a period time of length τ_1. Suppose also that after this time, a transition occurred to s_{i_2} and the machine stayed there for a time τ_2 and so forth. Hence we have observations in the following form

$$\sigma = [[i_1, \tau_1], ..., [i_L, \tau_L]] \, .$$

We assume that no preventive maintenance is done so that after the last observation in the trace, it fails.

For a given MMFM model M with rates r_i and jump quantities J_{ij}, this would mean that initially the fluid level was

$$q_0(M, \sigma) = \tau_1 r_{i_1} + \sum_{k=2}^{L} \tau_k r_{i_k} - J_{i_{k-1} i_k}$$

Such that the likelihood of this trace would be

$$L(\sigma) = f(q_0(M, \sigma)) \left[\prod_{k=1}^{L-1} \lambda_{i_k i_{k+1}} e^{-\lambda_{i_k} \tau_k} \right] e^{-\lambda_{i_L} \tau_L}$$

using $\lambda_i = \sum_j \lambda_{ij}$.

To determine the ratein s_i, we can derive this to r_i. For simplicity, we take the log likelihood.

$$\frac{d}{dr_i} \log L(\sigma) = \frac{d}{dr_i} \left(\log f(q_0(M, \sigma)) + \left[\sum_{k=1}^{L-1} \log(\lambda_{i_k i_{k+1}} e^{-\lambda_{i_k} \tau_k} \right] + \log(e^{-\lambda_{i_L} \tau_L}) \right) .$$

Note that only $\log f(q_0(M,\sigma))$ depends on r_i such that all other terms vanish and we get

$$\frac{d}{dr_i}\log L(\sigma) = \frac{d}{dr_i}\log f(q_0(M,\sigma)) = \frac{f'(q_0(M,\sigma))}{f(q_0(M,\sigma))}\tau(i,\sigma).$$

Where

$$\tau(i,\sigma) = \sum_{k|i_k=i} \tau_k.$$

Similarly, for the jump quantities J_{ij}, we'll get

$$\frac{d}{dJ_{ij}}\log L(\sigma) = \frac{d}{dr_i}\log f(q_0(M,\sigma)) = \frac{f'(q_0(M,\sigma))}{f(q_0(M,\sigma))}\#(i,\sigma)$$

where $\#(i,\sigma)$ is the number of times a jump to s_i was observed in σ.

For multiple measurements $\sigma_1, ..., \sigma_K$, where

$$\sigma_n = \left[[i_1^{(n)}, \tau_1^{(n)}], ..., [i_{L^{(n)}}^{(n)}, \tau_{L^{(n)}}^{(n)}]\right],$$

the MLE will be the solution of the following equations

$$\frac{d}{dr_i}\log L(\sigma_1, ..., \sigma_K) = \sum_{n=1}^{K} \frac{f'(q_0(M,\sigma_n))}{f(q_0(M,\sigma_n))}\tau(i,\sigma_n) = 0$$

for all i and

$$\frac{d}{dJ_{ij}}\log L(\sigma_1, ..., \sigma_K) = \sum_{n=1}^{K} \frac{f'(q_0(M,\sigma_n))}{f(q_0(M,\sigma_n))}\#(i,\sigma_n) = 0$$

for all J_{ij}.