1 Simplified models

Instead of tackling the complete problem at once, we start with a simplified version of the problem, solve this and then make the model incrementally more realistic.

1.1 First model

In our first and simplest model, the machine has a lifetime L that follows some distribution $f(l)$ (cumulative function $F(l)$). When the machine breaks, it needs to be repaired and a certain cost for corrective maintenance needs to be paid (c_c). We start with an open loop problem where we have to choose at the beginning at what time the machine will be repaired. When the machine is repaired, the problem starts again with a new lifetime according to the same distribution.
We want to minimize the average cost. Since the machine is renewed after each repair, we can do this by minimizing the expected average cost until the first repair.

If we decide to repair the machine at some time $u > 0$, then the expected cost will be

$$J(u) = \frac{(1 - F(u))c_p}{u} + \int_0^u \frac{f(l)c_c}{l} dl$$

We want to minimize this. To find this minimum, we first try the extreme values. For $u = 0$, there would be an infinite cost per time unit so this is clearly not minimal. $u = \inf$ is equivalent to not doing any preventive maintenance ($F(\inf) = 1$ so the first term vanishes). As you can see, the expected cost then equals c_c times the expected value of one divided by the lifetime. It can be computed by solving the integral or, if the moment generating function is known, by integrating the moment generating function. To find the minimum, we also need to check the function at points where it is not differentiable. The minimum can also be at a zero of the derivative of the cost function:

$$\frac{d}{du}J(x) = -\frac{c_p}{u^2} - \frac{f(u)c_p}{u} + \frac{F(u)c_p}{u^2} + \frac{f(u)c_c}{u} = 0$$

We multiply by u^2 ($u > 0$ since preventive maintenance at time 0 would result in infinite average cost)

$$c_pF(u) - c_p + uf(u)(c_c - c_p) = 0$$

Which can be solved either numerically or algebraic for some specific distribution. The minimum can then be found by comparing the costs at the extreme values, undifferentiable points and at the zeroes of the derivative. For example, for lifetimes uniformly distributed on the interval $[0, B]$, this would result in solving

$$c_p\frac{u}{B} - c_p + \frac{u(c_c - c_p)}{B} = 0 \Rightarrow u = L\frac{c_p}{c_c}$$

1.2 Closed loop

We now make the problem a little more difficult by discretizing the time and having a limited set of options at each stage. There are two states, one where the machine is broken (s_b), and one where it is not (s_0). If the machine is broken, the only available action is u_c (corrective maintenance) with cost c_c. After u_c, the machine is renewed. When the machine is not broken, there are two actions:

- Preventive maintenance (u_p) with cost $c_p < c_c$, renewing the machine.
- Do nothing (u_w) with cost 0. This action will lead to s_0 if the machine does not break in the next time interval, or to s_b if it does break.

But if the machine breaks between two stages, it moves to the broken state and spends one stage there. In the continuous case, the machine does indeed spend zero time there but in this descretized problem, it does spend Δ time there. For simplicity we introduce a discount α and are interested in the discounted cost instead of the average cost. The value function would then be

$$J_k(s_0) = \min\{c_p + \alpha J_0(s_0), \alpha p_k J_{k+1}(s_0) + (1-p_k)J_{k+1}(s_b)\}$$

$$J_k(s_b) = c_c + \alpha J_0(s_0)$$

Where p_k denotes the probability that the machine does not break until the next stage. If we discretize time as $t_k = k\Delta$, this probability would be $\mathbf{P}(L > t_{k+1}|L > t_k) = \frac{1-F(t_{k+1})}{1-F(t_k)}$.

$$J_k(s_0) = \min\{c_p + J_0(s_0), p_k J_{k+1}(s_0) + (1-p_k)(c_c + J_0(s_0))\}$$

1.3 Fluid models

The last problem could be seen as a fluid model with one state with rate -1 and a random initial fluid level (ignoring the broken state for now). We could extend this to fluid models of more states. We introduce a fluid model with two states:

- s_0: With fluid rate $r_0 < 0$
- s_1: With fluid level $r_1 < 0$.

The system transitions in the fluid model occur as a CTMC when u_w is chosen and the machine does not break, when u_p is chosen, the system is renewed and transitions to s_0. When the machine breaks, it transitions to s_b from which the only available action is u_c which transitions to s_0.

To calculate the probability that the machine breaks between two stages, we need to have the distribution of the fluid decrease in a period of length Δ. Let $\overline{r} = \max r_0, r_1$ and $\underline{r} = \min r_0, r_1$. Let $\Delta\underline{r} < q < \Delta\overline{r}$, then the probability that the fluid decreases less than q in the next period, equals the probability that the machine spends less than $\frac{q-\underline{r}\Delta}{\overline{r}-\underline{r}}$ time in the state with the lowest rate. Let $f_i^j(t^*, t)$ denote the density of spending t^* out of t time in state j given that it starts in state i. We have that for small h:

$$f_0^1(t^*, t) = \lambda_0 h f_1^1(t^*, t-h) + (1-\lambda_0 h) f_0^1(t^*, t-h)$$

$$\Rightarrow \frac{f_0^1(t^*,t)-f_0^1(t^*,t-h)}{h} = \lambda_0(f_1^1(t^*,t-h)-f_0^1(t^*,t-h))$$

And in the limit this results in

$$\frac{d}{dt}f_0^1(t^*,t) = \lambda_0(f_1^1(t^*,t)-f_0^1(t^*,t))$$

Similarly, for $f_1^1(t^*,t)$ we have

$$(\frac{d}{dt}+\frac{d}{dt^*})f_1^1(t^*,t) = \lambda_1(f_0^1(t^*,t)-f_1^1(t^*,t))$$

It is difficult to solve these equations, therefore we apply a probabilistic approach and check whether the results adhere to the differential equations afterwards. If we condition to the number of transitions that occur from s_0 to s_1, we can define $g_0^{(k)}(t_0,t_1)$ as the density of spending t_0 out of t_0+t_1 time in s_0 given that k transitions from s_0 to s_1 occurred. We assume that $s_1 > 0$. Now we can split this into two cases: One where the system is in s_0 at time t_0+t_1 and one where the system is in s_1 at that time.
If the system ends in s_0, this means that the k-th transition from s_1 to s_0 occurred exactly after it has spent t_1 (total) time in s_1, while exactly k transitions occurred from s_0 to s_1 in the t_0 time it has spent in s_0. This corresponds to the probability of k arrivals of a Poisson process with rate $\lambda_0 t_0$ multiplied with the density of an Erlang distributed variable with rate λ_1 and shape k at time t_1.
If the system ends in s_1, this means that exactly $k-1$ transitions from s_1 to s_0 occurred in the time t_1 it has spent in s_1 while the k-th transition from s_0 to s_1 occurred after spending t_0 time in s_0. This corresponds to the distribution of an Erlang distributed variable with rate λ_0 and shape k at time t_0 multiplied by the probability of $k-1$ arrivals of a Poisson process with rate $\lambda_1 t_1$.
Since these events are mutually exclusive, they can be summed. We introduce some notation: Let P_λ be a Poisson distributed variable with rate λ, let $E_{\lambda,n}$ be an Erlang distributed random variable with rate λ and shape n and let $f_x(t)$ be the density of some random variable X. Then we can write:

$$\begin{aligned}
g_0^{(k)}(t_0,t_1) &= \mathbf{P}(P_{\lambda_0 t_0}=k)f_{E_{\lambda_1,k}}(t_1) + f_{E_{\lambda_0,k}}(t_0)\mathbf{P}(P_{\lambda_1 t_1}=k-1) \\
&= \frac{(\lambda_0 t_0)^k e^{-\lambda_0 t_0}}{k!}\frac{\lambda_1^k t_1^{k-1} e^{-\lambda_1 t_1}}{(k-1)!} + \frac{\lambda_0^k t_0^{k-1} e^{-\lambda_0 t_0}}{(k-1)!}\frac{(\lambda_1 t_1)^{k-1} e^{-\lambda_1 t_1}}{(k-1)!} \\
&= e^{-\lambda_0 t_0 - \lambda_1 t_1}(\frac{(\lambda_0\lambda_1 t_0)^k t_1^{k-1}}{k!(k-1)!} + \frac{\lambda_0^k(\lambda_1 t_0 t_1)^{k-1}}{(k-1)!^2})
\end{aligned}$$

The density $f_0^0(t^*,t)$ can then be obtained by summing over k from 1 to infinity (note that since $t_1 > 0$ we have $t^* < t$):

$$f_0^0(t^*,t) = \sum_{k=1}^{\infty} g_0^{(k)}(t^*,t-t^*)$$

And we have that $f_0^1(t^*,t) = f_0^0(t-t^*,t)$. Moreover, we can obtain $f_1^1(t_0,t_0+t_1)$ by interchanging λ_0 with λ_1 and interchanging t_0 with t_1 in the expression of $f_0^0(t_0,t_0+t_1)$.

It can be seen that these expressions adhere to the differential equations mentioned earlier by filling them in.

So now the next difficulty will be integrating these expressions to obtain the cumulative distribution function.

References