1 Simplified models

Instead of tackling the complete problem at once, we start with a simplified version of the problem, solve this and then make the model incrementally more realistic.

1.1 First model

In our first and simplest model, the machine has a lifetime that follows some distribution $f(l)$ (cumulative function $F(l)$). When the machine breaks, it needs to be repaired and a certain cost for corrective maintenance needs to be paid (c_c). We start with an open loop problem where we have to choose at the beginning at what time the machine will be repaired. When the machine is repaired, the problem starts again with a new lifetime according to the same distribution.
We want to minimize the average cost. Since the machine is renewed after each repair, we can do this by minimizing the expected average cost until the first repair.

If we decide to repair the machine at some time $u > 0$, then the expected cost will be

$$J(u) = \frac{(1 - F(u))c_p}{u} + \int_0^u \frac{f(l)c_c}{l} dl$$

We want to minimize this, so we search for a zero of its derivative

$$\frac{d}{du}J(x) = -\frac{c_p}{u^2} - \frac{f(u)c_p}{u} + \frac{F(u)c_p}{u^2} + \frac{f(u)c_c}{u} = 0$$

We multiply by u^2 ($u > 0$ since preventive maintenance at time 0 would result in infinite average cost)

$$c_pF(u) - c_p + uf(u)(c_c - c_p) = 0$$

Which can be solved either numerically or algebraic for some specific distribution. For example, for lifetimes uniformly distributed on the interval $[0, L]$, this would result in solving

$$c_p\frac{u}{L} - c_p + \frac{u(c_c - c_p)}{L} = 0 \Rightarrow u = L\frac{c_p}{c_c}$$

1.2 Closed loop

We now make the problem a little more difficult by discretizing the time and having a limited set of options at each stage. There are two states, one where the machine is broken (s_b), and one where it is not (s_0). If the machine is broken, the only available action is u_c (corrective maintenance) with cost c_c. After u_c, the machine is renewed. When the machine is not broken, there are two actions:

- Preventive maintenance (u_p) with cost $c_p < c_c$, renewing the machine.

- Do nothing (u_w) with cost 0.

For simplicity we introduce a discount α and are interested in the discounted cost instead of the average cost. The value function would then be

$$J_k(s_0) = \min\{c_p + \alpha J_0(s_0), \alpha p_k J_{k+1}(s_0) + (1 - p_k) J_{k+1}(s_b)\}$$

$$J_k(s_b) = c_c + \alpha J_0(s_0)$$

Where p_k denotes the probability that the machine does not break until the next stage. If we discretize time as $t_k = k\Delta$, this probability would be $\mathbf{P}(L > t_{k+1} | L > t_k) = \frac{1-F(t_{k+1})}{1-F(t_k)}$.

1.3 Fluid models

The last problem could be seen as a fluid model with one state with rate -1 and a random initial fluid level (ignoring the broken state for now). We could extend this to fluid models of more states. We introduce a fluid model with two states:

- s_0: With fluid rate $r_0 < 0$
- s_1: With fluid level $r_1 < 0$.

The system transitions in the fluid model occur as a CTMC when u_w is chosen and the machine does not break, when u_p is chosen, the system is renewed and transitions to s_0. When the machine breaks, it transitions to s_b from which the only available action is u_c which transitions to s_0.

To calculate the probability that the machine breaks between two stages, we need to have the distribution of the fluid decrease in a period of length Δ. Let $\overline{r} = \max r_0, r_1$ and $\underline{r} = \min r_0, r_1$. Let $\Delta\underline{r} < q < \Delta\overline{r}$, then the probability that the fluid decreases less than q in the next period, equals the probability that the machine spends less than $\frac{q - r\Delta}{\overline{r} - \underline{r}}$ time in the state with the lowest rate. Let $f_i^j(t^*, t)$ denote the density of spending t^* out of t time in state j given that it starts in state i. We have that for small h:

$$f_0^1(t^*, t) = \lambda_0 h f_1^1(t^*, t - h) + (1 - \lambda_0 h) f_0^1(t^*, t - h)$$

$$\Rightarrow \frac{f_0^1(t^*, t) - f_0^1(t^*, t - h)}{h} = \lambda_0 (f_1^1(t^*, t - h) - f_0^1(t^*, t - h))$$

And in the limit this results in

$$\frac{d}{dt} f_0^1(t^*, t) = \lambda_0 (f_1^1(t^*, t) - f_0^1(t^*, t))$$

Similarly, for $f_1^1(t^*, t)$ we have

$$(\frac{d}{dt} + \frac{d}{dt^*}) f_1^1(t^*, t) = \lambda_1 (f_0^1(t^*, t) - f_1^1(t^*, t))$$

The first equation is an ordinary differential equation and can be solved using the method of the integrating factor, this leads to

$$f_0^1(t^*, t) = \int \lambda_0 f_1^1(t^*, t) e^{\lambda_0 t} dt e^{-\lambda_0 t}$$

(constants omitted for readability). The second equation is a partial differential equation and could be approached using the method of characteristics, which leads to

$$f_1^1(t_0, t_0 + x) = \int^x \lambda_1 f_0^1(t_0, t_0 + y) e^{\lambda_1 y} dy d^{-\lambda_1 x}$$

(Lower limit of the integral must be set such that $f_1^1(t_0, t_0) = e^{-\lambda_1 t_0}$) I currently have not found a way to finish these differential equations.

References