Chapter 1

MMFM Optimal Policy

In this section, the structure of the optimal policy for a machine which deteriorates according to the Markov Modulated Fluid Model will be determined. We will start with the simple model from earlier and will incrementally add complexity.

1.1 One CTMC-state, no jumps

The initial fluid level is given by $Q_0 \sim F(q)$ and the fluid rate in this single state equals r. The hazard rate then equals

$$\begin{aligned} \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(rx < Q_0 < r(x+\delta) | rx < Q_0) &= \lim_{\delta \to 0} \frac{\mathbb{P}(rx < Q_0 < r(x+\delta))}{\delta \mathbb{P}(rx < Q_0)} \\ &= rh(rx). \end{aligned} \tag{1.1}$$

The rest of the problem can be solved in the same way as the simple problem. Hence, the optimal control limit μ^* is the first μ such that

$$rh(r\mu) = \beta \frac{c + V_\mu}{a} \tag{1.2}$$

(where V_μ is the expected total discounted cost when using control limit μ), or $\mu^* = \infty$ else.

1.2 Two CTMC-states with equal rate, constant jumps at exponentially distributed time intervals

We extend the previous model by adding constant fluid jumps to the model at exponentially distributed intervals. The time intervals between jumps have rate λ and fluid quantity J. As earlier, we denote the state of the machine at time t by

$$X(t) = (S(t), L_0(t), L_c(t)),$$

but in this simple one-state problem $S(t)$ isn't relevant since they have equal fluid rate and outward jump size. The current state is completely observable, so at time t, $L_0(t)$ and $L_c(t)$ are known. The hazard rate at time t is given by

$$h(t) = \begin{cases} 0 & \text{if } L_c(t) > 0 \\ rh(rL_0(t)) & \text{else.} \end{cases} \tag{1.3}$$

This suggests that preventive maintenance should be chosen at time t when $L_c(t) = 0$ and

$$rh(rL_0(t)) = \beta \frac{c + V_{L_0(t)}}{a}, \tag{1.4}$$

where V_l equals the expected total discounted cost when the machine is repaired whenever $L_0(t)$, the observed lower bound of Q_0, equals l. Note that when there are no jumps ($J = 0$), this corresponds to the optimal policy (and cost) of the previous problem. Hence, we will refer to this limit for the lower bound as the control limit μ and use the notation V_μ for the corresponding cost.

In the simple problem, we could easily calculate at each time t, how long it takes until preventive repair should take place ($\mu - t$, assuming $\mu < Q_0$). But in this case, the presence of fluid jumps complicates this. A lower bound for the time at which preventive maintenance should be scheduled is given by $L_c(t) + \mu - L_0(t)$ and this is also the optimal time to do preventive maintenance if no jumps occur in the meantime.

1.2.1 Calculation of the expected total discounted cost

For a given μ, V_μ is still difficult to compute. The presence of jumps complicates the calculation of the time between repairs. Let $T_t(q)$ be the random variable denoting the time until the fluid level is q lower than it was at time t, i.e.

$$T_t(q) = \min\{\tau \geq 0 | Q(t+\tau) \leq Q(t) - q\}.$$

Note that, using this definition, $T_0(\mu)$ equals the time until the control limit is reached ($L_0(t) = \mu$) and $T(Q_0)$ equals the time until the machine fails.

Lemma 1. $Q_0 \leq \mu \Leftrightarrow T_t(Q_0) \leq T_t(\mu)$

Proof. $\Rightarrow$:

$$\begin{aligned} Q_0 \leq \mu &\Rightarrow Q(t) - \mu \leq Q(t) - Q_0 \\ &\Rightarrow (Q(t+\tau) \leq Q(t) - \mu \Rightarrow Q(t+\tau) \leq Q(t) - Q_0) \\ &\Rightarrow T_t(Q_0) \leq T_t(\mu) \end{aligned} \tag{1.5}$$

$\Leftarrow$: We will prove that $Q_0 > \mu \Rightarrow T_t(Q_0) > T_t(\mu)$:
We know that

$$Q_0 > \mu \Rightarrow Q(t) - \mu > Q(t) - Q_0.$$

Since $Q(t)$ is piecewise continuous and does not decrease at the discontinuities, we know that

$$Q(t + T_t(\mu)) = Q(t) - \mu > Q(t) - Q_0 \Rightarrow T_t(Q_0) > T_t(\mu).$$

□

Also, let $N_t(q)$ be the random variable denoting the number of jumps that occur in the interval $(t, t + T_t(q)]$. We will now compute its distribution. For simplicity, we will assume that $r = 1$. The probability that zero jumps occur equals the probability that the exponentially distributed time interval is larger than q:

$$\mathbb{P}(N_t(q) = 0) = e^{-\lambda q}.$$

The probability that exactly one jump occurs equals the probability that exactly one Poisson event happens in the interval $(t, t + q]$ while none happen in $(t + q, t + q + J]$. Resulting in

$$\mathbb{P}(N_t(q) = 1) = \lambda q e^{-\lambda q} e^{-\lambda J} = \lambda q e^{-\lambda(q+J)}.$$

For each $k \geq 0$, by conditioning on the time until the first jump, we get the following recursion

$$\begin{aligned} \mathbb{P}(N_t(q) = k + 1) &= \int_0^q \lambda e^{-\lambda x} \mathbb{P}(N_t(q - x + J) = k) dx \\ &= \int_0^q \lambda e^{-\lambda(q-y)} \mathbb{P}(N_t(y + J) = k) dy, \end{aligned} \tag{1.6}$$

since after this first jump, the fluid level equals $q - x + J$ and k jumps should occur. The second equality follows after the substition $y = q - x$. The solution of this recursion, is given by

$$\mathbb{P}(N_t(q) = k) = \lambda q \frac{(\lambda(q + kJ))^{k-1}}{k!} e^{-\lambda(q+kJ)}.$$

Which can be seen by substituting it into (1.6) and setting $k = 1$ to see that it also satisfies the calculated expression for $\mathbb{P}(N_t(q) = 1)$.

Now we know the distribution of the amount of jumps that occur in a run where the initial fluid level equals q, we can derive the distribution of the time length of a run with initial fluid level Q_0 random with density function $f_{Q_0}(q)$. This distribution $f(x)$, is given by

$$\begin{aligned} f(x) &= \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(x < T_0(Q_0) < x + \delta) \\ &= \sum_{k=0}^{\lfloor \frac{x}{J} \rfloor} \mathbb{P}(N_0(q - kJ) = k) \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(x - kJ < Q_0 < x - kJ + \delta) \\ &= \sum_{k=0}^{\lfloor \frac{x}{J} \rfloor} \mathbb{P}(N_0(q - kJ) = k) f_{Q_0}(x - kJ) \end{aligned} \tag{1.7}$$

The total discounted cost V_μ when using control limit μ can, similarly as with the simple problem, be calculated in the following way

$$V_\mu = \frac{(c + a)\mathbb{P}(Q_0 < \mu)\mathbb{E}[e^{-\beta Q_0} | Q_0 < \mu] + c\mathbb{P}(Q_0 \geq \mu)e^{-\beta\mu}}{1 - \mathbb{E}e^{-\beta T(\mu \wedge Q_0)}}. \tag{1.8}$$

The same iteration approaches can also be applied.

1.3 Multiple CTMC-states with equal fluid rates

TODO

1.4 Multiple CTMC-states with different fluid rates

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