Chapter 1

The evolution of the distribution of the fluid level

In this section, we will derive the distribution of the remaining fluid level in a Markov Modulated Fluid Model with constant jumps. In this model, the initial fluid level is a random variable $Q_0 \sim F(q)$ with corresponding hazard rate $h(q)$ which is assumed to be nondecreasing. A CTMC with n states $s_1, ..., s_n$ and initial state s_1 is considered. Each state has a corresponding fluid rate $r_i > 0$ so that when the machine is in state i for a time period of length τ, then the fluid level will decrease by $r_i\tau$. When a transition occurs between state i and j, the fluid level instantaneously increases by J_{ij}. We denote the random variable representing the fluid level at time t by $Q(t)$.

1.1 State description

As time passes and jumps occur, the distribution of the remaining fluid changes. If the first jump (with quantity J) happens at time t after the process has started (in initial state s_1), we know at time t that $Q_0 > r_1 t$ and that $Q(t) \geq J$. This suggests that we can determine the distribution of the remaining fluid level from the distribution of Q_0, a lower bound l_0 on Q_0 and a lower bound of the current fluid level l_c. Hence, we describe the state of the process at time t in the following way:

$$X(t) = (S(t), L_0(t), L_c(t)).$$

Where $S(t)$ denotes the CTMC state the process is in at time t, $L_0(t)$ is the lower bound of Q_0 at time t and $L_c(t)$ is the lower bound of $Q(t)$ at time t. Initially $S(0) = s_1$ and $L_0(0) = L_c(0) = 0$, when the process is in state (s_i, l_0, l_c) and a CTMC jump to s_j occurs, the state evolves in the following way

$$(s_i, l_0, l_c) \mapsto (s_j, l_0, l_c + J_{ij}).$$

And when in state (s_i, l_0, l_c), a time period of length τ passes, the fluid level decreases by τr_i. The lower bound of the current level then decreases maximally

by τr_i and it increases the lower bound of Q_0 maximally by the same quantity. The exact evolution happens in the following way

$$(s_i, l_0, l_c) \mapsto (s_i, l_0 + \max\{0, \tau r_i - l_c\}, l_c - \min\{l_c, \tau r_i\}).$$

Theorem 1. At time t, the fluid level is given by

$$Q(t) = L_c(t) + Q_0 - L_0(t) \tag{1.1}$$

Proof. At time $t = 0$, $L_0(0) = L_c(0) = 0$ and $Q(0) = Q_0$ so that (1.1) holds. When (1.1) holds at time t and a time period τ passes while the machine stays in the same state s_i, then the left side of (1.1) decreases by $r_i\tau$ while the right side decreases by

$$\begin{aligned} \min\{r_i\tau, l_c\} + \max\{0, r_i\tau - l_c\} &= \min\{r_i\tau, l_c\} - \min\{0, l_c - r_i\tau\} \\ &= \min\{r_i\tau, l_c\} - \min\{r_i\tau, l_c\} + r_i\tau \\ &= r_i\tau. \end{aligned} \tag{1.2}$$

□

Hence, passage of time preserves (1.1). When (1.1) holds at time t and a jump from s_i to s_j occurs, the left side of (1.1) increases by J_{ij} and L_c also increases by this quantity such that fluid jumps also preserve (1.1).

1.2 Evolution of the distribution

Given that we are in state $X(t) = (S, L_0, L_c)$, we can calculate the distribution of the current fluid level in the following way

$$\begin{aligned} \mathbb{P}_X(Q < q) &= \mathbb{P}(L_c + Q_0 - L_0 < q | Q_0 > L_0) \\ &= \mathbb{P}(Q_0 < q + L_0 - L_c | Q_0 > L_0) \\ &= \frac{F(q + L_0 - L_c) - F(L_0)}{1 - F(L_0)}. \end{aligned} \tag{1.3}$$

The hazard rate is given by

$$\begin{aligned} h_X(q) &= \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}_X(q < Q < q + \delta | Q > q, Q > L_c) \\ &= \begin{cases} 0 & \text{if } q < L_c \\ \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(Q_0 < q + L_0 - L_c + \delta | q + L_0 - L_c < Q_0) & \text{else.} \\ \quad = h(q + L_0 - L_c) \end{cases} \end{aligned} \tag{1.4}$$

The next theorem follows from the assumption that h is nondecreasing.

Theorem 2. $h_X(q)$ is nondecreasing in q for all states X in the state space.