1 Solving the (reformulated) simple discounted problem

We are going to solve

$$V(x) = \begin{cases} \min\{c + \alpha_\delta V(1), \alpha_\delta \mathbb{E}[V(S(x))]\}, & \text{if } x > 0 \\ c + a + \alpha_\delta V(1), & \text{else.} \end{cases} \tag{1}$$

Where $\mathbb{P}(S(x) = 0) = \mathbb{P}(Q \leq x + \delta | Q \geq x) = \delta h(x) + o(\delta^2)$ (for lifetime $Q \sim F(.)$) and $\mathbb{P}(S(x) = x+1) = 1 - \delta h(x) + o(\delta^2)$ and $\alpha_\delta = e^{-\beta\delta} = 1 - \beta\delta + o(\delta^2)$ for $\beta > 0$. We define $V'_\delta(n\delta) := V(n)$ and for convenience, we define $V'_\delta(0^+) := V'_\delta(\delta)$. If we assume that

$$c + \alpha_\delta V'_\delta(0^+) > \alpha_\delta \mathbb{E}[V'_\delta(S(x)\delta)],$$

we can write

$$V'_\delta(x) = \alpha_\delta \mathbb{P}(Q \leq x+\delta | Q \geq x)(c+a+\alpha_\delta V'(0^+)) + \alpha_\delta \mathbb{P}(Q > x+\delta | Q \geq x) V'_\delta(x+\delta) \tag{2}$$

We are now going to let δ approach zero.

$$\begin{aligned} \lim_{\delta \to 0} V'_\delta(x) = \lim_{\delta \to 0} &(1 - \beta\delta + o(\delta^2))(\delta h(x) + o(\delta^2))(c + a + (1 - \beta\delta + o(\delta^2))V'(0^+)) \\ &+ (1 - \beta\delta + o(\delta^2))(1 - \delta h(x) + o(\delta^2)) V'_\delta(x + \delta). \end{aligned} \tag{3}$$

Gathering the terms of $o(\delta^2)$, we get

$$\lim_{\delta \to 0} V'_\delta(x) = \lim_{\delta \to 0} \delta h(x)(c + a + V'_\delta(0^+)) + (1 - \delta\beta - \delta h(x)) V'_\delta(x + \delta) + o(\delta^2). \tag{4}$$

And by moving one $V'_\delta(x + \delta)$ to the left and dividing by $-\delta$, we get

$$\begin{aligned} \frac{d}{dx} V'_0(x) &= \lim_{\delta \to 0} \frac{V'_\delta(x + \delta) - V'_\delta(x)}{\delta} \\ &= \lim_{\delta \to 0} -h(x)(c + a + V'_\delta(0^+)) + (\beta + h(x)) V'_\delta(x + \delta) + o(\delta) \\ &= -h(x)(c + a + V'_0(0^+)) + (\beta + h(x)) V'_0(x). \end{aligned} \tag{5}$$

Where

$$V'_0(x) := \lim_{\delta \to 0} V'_\delta(x).$$

(Note that $V'_0(0^+) = V'_0(0) - c - a$). This differential equation seems counterintuitive since for small β, $V'_0(x)$ would be decreasing as $V'_0(x) < c + e^{-\beta\delta} V'_0(0^+) < c + a + V'_0(x)$. We will try to solve this O.D.E. anyway. We use the method of the integrating factor. Our integrating factor will be

$$e^{\int_0^x (-\beta - h(q)) dq} = e^{-\beta x - H(x)}.$$

Where $H(x)$ is the cumulative hazard function. We get

$$V'_0(x) = e^{\beta x + H(x)} [C + \int_0^x e^{-\beta q - H(q)} (-h(q)(c + a + V'_0(0^+))) dq]$$

$$= \frac{e^{\beta x}}{1-F(x)}[C - (c + a + V_0'(0^+)) \int_0^x e^{-\beta q} f(q) dq].$$

Using the identities $e^{H(x)} = (e^{-H(x)})^{-1} = \frac{1}{1-F(x)}$ and $h(x)e^{-H(x)} = f(x)$. The C is an integrating constant and since $\lim_{x \to 0} V_0'(x) = V_0'(0^+)$ should hold, we find $C = V_0(0^+)$. We can rewrite the expression to

$$V_0'(x) = \frac{e^{\beta x}}{1-F(x)}[V_0'(0^+) - (c + a + V_0'(0^+))\mathbb{P}(Q < x)\mathbb{E}[e^{-\beta Q} | Q < x]].$$

Concluding

$$\begin{aligned} V_0'(x) = \min\{&c + V_0'(0^+), \\ &\frac{e^{\beta x}}{1-F(x)}[V_0'(0^+) - (c + a + V_0'(0^+))\mathbb{P}(Q < x)\mathbb{E}[e^{-\beta Q} | Q < x]]\} \end{aligned} \tag{6}$$

and preventive maintenance is chosen if and only if $V_0'(x) = c + V_0'(0^+)$. However, the value of $V_0'(0^+)$ depends on the policy that is chosen and it seems difficult to solve $V_0'(x) = c + V_0'(0^+)$ analytically for x. In the rest of this text we will write $V(x)$ instead of V_0' to not clutter the notation too much. Let x^* be the smallest positive x that satisfies $V(x) = c + V(0^+)$ if such x exist and $x^* = \infty$ else. The policy that we just derived, schedules preventive maintenance at time x^* if the machine has not already failed by then. Distinguising these two cases (machine survives until x^* and machine breaks before x^*), we get the following expression for $V(0^+)$

$$V(0^+) = \mathbb{P}(Q > x^*)e^{-\beta x^*}(c + V(0^+)) + \mathbb{P}(Q \le x^*)\mathbb{E}[e^{-\beta Q} | Q \le x^*](c + a + V(0^+)). \tag{7}$$

While for any $0 < x < x^*$ we get a similar expression for $V(x)$

$$\begin{aligned} V(x) = &\mathbb{P}(Q > x^* | Q > x)e^{-\beta(x^* - x)}(c + V(0^+)) \\ &+ \mathbb{P}(Q \le x^* | x < Q)\mathbb{E}[e^{-\beta(Q-x)} | x < Q \le x^*](c + a + V(0^+)), \end{aligned} \tag{8}$$

which also adheres to (6).

Returning to (5), we can rewrite

$$\frac{d}{dx}V(x) = -h(x)(c + a + V(0^+)) + (\beta + h(x))V(x) = h(x)(V(x) - V(0^+) - c - a) + \beta V(x). \tag{9}$$

And since this differential equation only holds when $V(x) < V(0^+) + c$, we have

$$\frac{d}{dx}V(x) < -ah(x) + \beta V(x). \tag{10}$$

Which is negative for $h(x) > \frac{\beta}{a}V(x)$. This seems even more counterintuitive: a high hazard results in a decreasing expected total discounted cost when not repairing. This could be because on the other hand, this high hazard also represents risks that have been overcome. It is important to mention that this $V(x)$ is also very dependant on the hazard rate.