1 Simple discounted problem

We consider a machine that is subject to deterioration over time. We denote the calendar time by t and we denote the fitness of the machine at time t by $Q(t)$. This fitness is not directly observed. At each time t, we can observe for each $0 < t' \leq t$ the machine was broken or not, i.e. $I(t) = \mathbb{1}\{Q(t') > 0\}$. We define i.i.d. positive random variables Q_i's ($i \geq 0$). When the machine is renewed for the k-th time ($k > 0$) at time t, $Q(t) = Q_k$ will hold. At time $t = 0$ the machine will have fitness Q_0. We introduce a continuous discount $\alpha > 0$ so that costs incurred at time t are discounted by a continuous discount factor $e^{-\alpha t}$. When $Q(t) > 0$, there are two actions available:

- a_R: Renew (i.e. repair or replace) the machine. This has cost $c > 0$.
- a_W: Wait and do not renew the machine. This has cost 0.

When $Q(t) \leq 0$, the only available action is a_R. Also, next to~~t~~he cost c for renewing the machine, an additional cost of $a > 0$~~i~~s paid.

At time t, we define $n(t)$ as the number of times the machine has been renewed in the interval $(0, t)$.~~W~~e denote the age of the machine at time t by $L(t) = t - \arg\min_{t'} n(t') = n(t)$~~S~~ince $Q(t) = Q_{n(t)} - L(t)$, the only observable information relevant for the future costs is $L(t)$ and $I(t)$. Time is discretized in steps of size h such that $t_k = kh$.

We want to minimize costs, so if at time t_k with $I(t_k) = 1$ and $L(t_k) = l$ hold, we choose the action minimizing

$$J(l, 1) = \min\{c + e^{-\alpha h} J(0, 1), e^{-\alpha h} \mathbb{E}[J(l + h, I(t_{k+1}))]\}.$$

Where $I(t_{k+1})$ is a random variable and it is 0 if $l < Q_{n(t_k)} < l + h$ and 1 else. If $I(t_k) = 0$ (or equivalently: $L(t_k) \geq Q_{n(t_k)}$), we have

$$J(l, 0) = c + a + e^{-\alpha h} J(0, 1).$$

2 Constant fitness jumps at Exponentially distributed intervals

The previous problem can be extended by allowing the occurrence of instantanious increases (jumps) in the fitness quantity. These jumps all have the same known size Δ. We denote the time at which the i-th jump occurred by T_i~~.~~ The time interval between each two consequtive jumps is exponentially distributed with rate λ. Hence, T_i is Erlang distributed with rate λ and shape i. The number of jumps that occurred before time t is denoted by $K(t)$ so that $K(t)$ is Poisson distributed random variable with rate λt. The number of jumps that occurred since the last lifetime is denoted by $K_L(t) = K(t) - K(t - L(t))$, this is Poisson distributed with rate $\lambda L(t)$. The times at which jumps have occurred are known i.e. at time t $T_1, ..., T_{K(t)}$ are known.

The costs, actions and discount remain the same as in the previous problem. Again we want to minimize the costs. Since in this case

$$Q(t) = Q_{n(t)} - L(t) + K_L(t)\Delta$$

(the same is in the previous problem but then with the addition of the jumps that occurred after the last renewal), the fluid quantity (and hence the cost) only depends on $L(t)$, $I(t)$ and $K_L(t)$. The Bellman Equations for $I(t_k) = 1, L(t_k) = l$ and $K_L(t_k) = k$ then read

$$J(l,k,1) = \min\{c + e^{-\alpha h}J(0,0,1), e^{-\alpha h}\mathbb{E}[J(l+h, K_L(t_{k+1}), I(t_{k+1}))]\}.$$

Where $I(t_{k+1})$ is 0 if for all $h' \in (0,h]$ $l < Q_{n(t_k)} + K_L(t_k + h')\Delta < l + h'$ and 1 else and $K_L(t_k + h')$ is a random variable and equal to k if no jump occurred and equal to $k+1$ if a jump has occurred in $(t_k, t_k + h']$.

And for $I(t_k) = 0$ the Bellman Equations read

$$J(l,k,0) = c + a + e^{-\alpha h}J(0,0,1).$$