# 水平集变分推导

## SA20218108 聂文尚：

$$
\begin{gathered}
\text{能量函数}: \epsilon_{g,\lambda,\nu} \lambda \mathcal{L}_g(\phi) + \nu \mathcal{A}_g(\phi) \\
\mathcal{L}_g(\phi) = \int_\Omega g\delta(\phi)|\nabla\phi|dxdy, \\
\mathcal{A}_g(\phi) = \int_\Omega gH(-\phi)dxdy, \\
g = \frac{1}{1+|\nabla G_\sigma * I|^2} \\
\delta \text{ is the univariate Dirac function, and it is a non-derivable function} \\
H(t) = \begin{cases} 0, & t<0 \\ \frac{1}{2}, & t=0 \\ 1, & t>0 \end{cases}
\end{gathered}
$$

Solution:

先解$\mathcal{L}_g(\phi) = \int_\Omega g\delta(\phi)|\nabla\phi|dxdy$:

$$
\begin{gathered}
\text{记} E(\phi) = \mathcal{L}_g(\phi) = \int_\Omega g\delta(\phi)|\nabla\phi|dxdy \\
\text{记} F(\phi) = g\delta(\phi)|\nabla\phi| = g\delta(\phi)\sqrt{\phi_x^2+\phi_y^2}
\end{gathered}
$$

引入极小变量$t$和任意函数$h$，其中函数$h$满足：$h|_{\partial\Omega} = 0$

$$
\begin{aligned}
\therefore F(\phi+th) &= g\delta(\phi)\sqrt{(\phi+th)_x^2+(\phi+th)_y^2} \\
\therefore \frac{\partial F(\phi+th)}{\partial t} &= g\delta(\phi)\frac{h_x(\phi+th)_x + h_y(\phi+th)_y}{2\sqrt{(\phi+th)_x^2+(\phi+th)_y^2}} \\
&= \frac{1}{2}g\delta(\phi)\frac{\nabla h\cdot\nabla(\phi+th)}{\sqrt{(\phi_x^2+\phi_y^2)+t^2(h_x^2+h_y^2)+2t\nabla h\cdot\nabla\phi}} \\
\therefore \frac{\partial F(\phi+th)}{\partial t}|_{t->0} &= \frac{1}{2}g\delta(\phi)\frac{\nabla h\cdot\nabla\phi}{\sqrt{\phi_x^2+\phi_y^2}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{\partial E(\phi+th)}{\partial t}|_{t->0} &= \int_\Omega \frac{1}{2}g\delta(\phi)\frac{\nabla h\cdot\nabla\phi}{\sqrt{\phi_x^2+\phi_y^2}}dxdy \\
&= \frac{1}{2}\delta(\phi)\int_\Omega g\frac{\nabla h\cdot\nabla\phi}{\sqrt{\phi_x^2+\phi_y^2}}dxdy \\
&= \frac{1}{2}\delta(\phi)\int_\Omega g\frac{h_x\phi_x+h_y\phi_y}{\sqrt{\phi_x^2+\phi_y^2}}dxdy
\end{aligned}
$$

$$
\begin{aligned}
\because \frac{\partial}{\partial x}[g\phi_x h] &= gh_x\phi_x + gh\phi_{xx} \\
\frac{\partial}{\partial y}[g\phi_y h] &= gh_y\phi_y + gh\phi_{yy}
\end{aligned}
$$

$$\therefore \frac{\partial E(\phi+th)}{\partial t}|_{t->0} = \frac{1}{2}\delta(\phi)[\int_\Omega (\frac{\partial}{\partial x}[g\frac{\phi_x}{|\nabla\phi|}h] + \frac{\partial}{\partial y}[g\frac{\phi_y}{|\nabla\phi|}h])dxdy - \int_\Omega (\frac{\partial}{\partial x}[g\frac{\phi_x}{|\nabla\phi|}]h + h\frac{\partial}{\partial y}[g\frac{\phi_y}{|\nabla\phi|}])dxdy]$$

$$\text{According to Green Equation: } \oint_{\partial\Omega} Rdy + Sdx = \iint_\Omega (\frac{dS}{dy} - \frac{dR}{dx})dxdy$$

$$\therefore \int_\Omega (\frac{\partial}{\partial x}[g\frac{\phi_x}{|\nabla\phi|}h] + \frac{\partial}{\partial y}[g\frac{\phi_y}{|\nabla\phi|}h])dxdy = \oint_{\partial\Omega} hg\Big(\frac{\phi_y}{|\nabla\phi|} - \frac{\phi_x}{|\nabla\phi|}\Big)dxdy = 0$$

$$\therefore \begin{aligned} \frac{\partial E(\phi+th)}{\partial \phi}|_{t->0} &= -\frac{1}{2}\delta(\phi)\int_\Omega h\cdot(\frac{\partial}{\partial x}[g\frac{\phi_x}{|\nabla\phi|}] + \frac{\partial}{\partial y}[g\frac{\phi_y}{|\nabla\phi|}])dxdy \\ &= -\frac{1}{2}\delta(\phi)\int_\Omega h\cdot\nabla[g\frac{\nabla\phi}{|\nabla\phi|}]dxdy \end{aligned}$$

$$\text{When } E(\phi) \text{ reach the minimal,} \frac{E(\phi+th)}{\phi t}|_{t->0} = -\frac{1}{2}\delta(\phi)\int_\Omega h\cdot\nabla[g\frac{\nabla\phi}{|\nabla\phi|}]dxdy = 0$$

$$\text{Since function h is arbitrary, we obtain: } \delta(\phi)\nabla[g\frac{\nabla\phi}{|\nabla\phi|}] = 0$$

再解$\mathcal{A}_g(\phi) = \int_\Omega gH(-\phi)dxdy$：

$$\text{同样记 } E(\phi) = \mathcal{A}_g(\phi) = \int_\Omega gH(-\phi)dxdy$$

$$\text{记 } F(\phi) = gH(-\phi)$$

引入极小变量$t$和任意函数$h$，其中函数$h$满足：$h|_{\partial\Omega} = 0$

$$F(\phi+th) = gH(-\phi-th)$$

$$\therefore \frac{\partial F(\phi+th)}{\partial t} = g\delta(-\phi-th)(-h) = -h\cdot g\delta(-\phi) = -h\cdot g\delta(\phi)$$

$$\therefore \frac{\partial E(\phi+th)}{\partial t} = -\int_\Omega hg\delta(\phi)dxdy$$

$$\text{When } E(\phi) \text{ reach the minimal,} \frac{E(\phi+th)}{\partial t}|_{t->0} = -\int_\Omega h\cdot g\delta(\phi)dxdy = 0$$

$$\text{Since function h is arbitrary, we obtain: } g\delta(\phi) = 0$$

综上所述演化方程：

$$\frac{\partial\phi}{\partial t} = \lambda\delta(\phi)\nabla[g\frac{\nabla\phi}{|\nabla\phi|}] + \nu g\delta(\phi)$$