Exercise 34.1

1. Imagine a NEATO moving along a circle given by:

$$\boldsymbol{r}(t) = R\cos(\alpha t)\hat{\boldsymbol{i}} + R\sin(\alpha t)\hat{\boldsymbol{j}}$$

How does the sign of α affect the path that the Neato takes around the circle?

Solution. The sign of α determines the direction that the Neato takes. If positive, it will travel counterclockwise. If negative, it will travel clockwise. ■

2. What are the linear velocity vector and linear speed?

Solution.

```
syms R alpha t
assume(R, {'real', 'positive'})
assume(t, {'real', 'positive'})
assume(alpha, {'real'})

ri = R*cos(alpha*t);
rj = R*sin(alpha*t);
r = [ri, rj, 0]

vel = diff(r)
speed = simplify(norm(vel))
```

$$v = -R\alpha\sin(\alpha t)\hat{\boldsymbol{i}} + R\alpha\cos(\alpha t)\hat{\boldsymbol{j}}$$
$$s = R\,|\alpha|$$

■

3. What is the unit tangent vector for the circle?

Solution.

```
T = simplify(vel ./ norm(vel))
```

$$\hat{\boldsymbol{T}} = -\frac{\alpha\sin(\alpha t)}{|\alpha|}\hat{\boldsymbol{i}} + \frac{\alpha\cos(\alpha t)}{|\alpha|}\hat{\boldsymbol{j}}$$

■

4. What is the unit normal vector?

Solution.

```
dT = diff(T);
N = simplify(dT ./ norm(dT))
```

$$\hat{\boldsymbol{N}} = -\cos(\alpha t)\hat{\boldsymbol{\imath}} - \sin(\alpha t)\hat{\boldsymbol{\jmath}}$$

■

5. What is the angular velocity vector?

Solution.

```
w = simplify(cross(T, dT))
```

$$\boldsymbol{\omega} = 0\hat{\boldsymbol{k}}$$

■

6. For the uniform circular motion we have been investigating so far, what does the parameter we have labeled α represent? How is it related to the time it takes to complete one traverse of the circular trajectory?

Solution. α represents the angular frequency of motion. ■

7. How would you modify this equation for a circle of radius 1m?

Solution. Set $R = 1$. ■

8. What value would you choose for α if you want your robot to complete a counterclockwise path around the circle in 30 seconds?

Solution.

$$T = \frac{2\pi}{|\alpha|}$$
$$30 = \frac{2\pi}{|\alpha|}$$
$$\alpha \approx 0.2094$$

■

9. What are the equations for the left and right wheel velocities for the uniform circle?

Solution.

$$V_L = V - \omega\frac{d}{2}$$
$$V_R = V + \omega\frac{d}{2}$$

```
syms d
assume(d, {'real', 'positive'})

V_L = simplify(V_T - w(3) * d / 2)
V_R = simplify(V_T + w(3) * d / 2)
```

$$V_L = R\alpha - \frac{\alpha\, d}{2}$$
$$V_R = R\alpha + \frac{\alpha\, d}{2}$$

■

10. What are the left and right wheel velocities needed for a 1m radius counterclockwise circle to be completed in 30 seconds?

Solution.

$$V_L = 0.1844$$
$$V_R = 0.2336$$

■

Exercise 34.2

1. The vector for a counterclockwise path around an ellipse is given by:

$$\boldsymbol{r}(t) = a\cos(\alpha t)\hat{\boldsymbol{\imath}} + b\sin(\alpha t)\hat{\boldsymbol{\jmath}}$$

What is the tangent vector for the ellipse?

Solution.

```
syms a b alpha t

assume(a, {'real', 'positive'})
assume(b, {'real', 'positive'})
assume(alpha, {'real', 'positive'})
assume(t, {'real', 'positive'})

ri = a * cos(alpha*t);
rj = b * sin(alpha*t);
r = [ri, rj, 0]

vel = diff(r)
```

$$\boldsymbol{V} = -a\alpha\sin(\alpha t)\hat{\boldsymbol{i}} + b\alpha\cos(\alpha t)\hat{\boldsymbol{j}}$$

■

2. What is the unit tangent vector for the ellipse?

Solution.

```
T = simplify(vel ./ norm(vel))
```

$$\hat{\boldsymbol{T}} = -\frac{a\sin(\alpha t)}{\sqrt{a^2\sin(\alpha)^2 + b^2\cos(\alpha t)^2}}\hat{\boldsymbol{i}} + \frac{b\cos(\alpha t)}{\sqrt{a^2\sin(\alpha)^2 + b^2\cos(\alpha t)^2}}\hat{\boldsymbol{j}}$$

■

3. What is the linear velocity vector?

Solution.

$$\boldsymbol{V} = -a\alpha\sin(\alpha t)\hat{\boldsymbol{i}} + b\alpha\cos(\alpha t)\hat{\boldsymbol{j}}$$

■

4. What is the unit normal vector?

Solution.

```
dT = diff(T);
N = simplify(dT ./ norm(dT))
```

$$\hat{\boldsymbol{N}} = -\frac{b\cos(\alpha t)}{\sqrt{a^2\sin(\alpha)^2 + b^2\cos(\alpha t)^2}}\hat{\boldsymbol{i}} - \frac{a\sin(\alpha t)}{\sqrt{a^2\sin(\alpha)^2 + b^2\cos(\alpha t)^2}}\hat{\boldsymbol{j}}$$

■

5. What is the angular velocity vector?

Solution.

```
w = simplify(cross(T, dT))
```

$$\boldsymbol{\omega} = \frac{ab\alpha}{a^2\sin^2(\alpha t) + b^2\cos^2(\alpha t)}\hat{\boldsymbol{k}}$$

■

6. What are the equations for the left and right wheel velocities?

Solution.

$$V_L = V - \omega\frac{d}{2}$$
$$V_R = V + \omega\frac{d}{2}$$

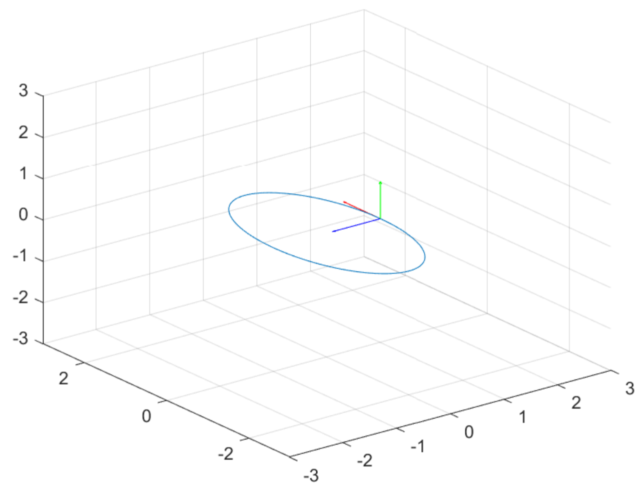
```
syms d
assume(d, {'real', 'positive'})

V_L = simplify(V_T - w(3) * d / 2)
V_R = simplify(V_T + w(3) * d / 2)
```

$$V_L = \alpha\sqrt{a^2\sin(\alpha t)^2 + b^2\cos(\alpha t)^2} - \frac{ab\alpha}{a^2\sin^2(\alpha t) + b^2\cos^2(\alpha t)}\frac{d}{2}$$
$$V_R = \alpha\sqrt{a^2\sin(\alpha t)^2 + b^2\cos(\alpha t)^2} + \frac{ab\alpha}{a^2\sin^2(\alpha t) + b^2\cos^2(\alpha t)}\frac{d}{2}$$

■

7. Plot the linear velocity vector and angular velocity vectors as a function of time for various combinations of the parameters a, b, and *alpha*.

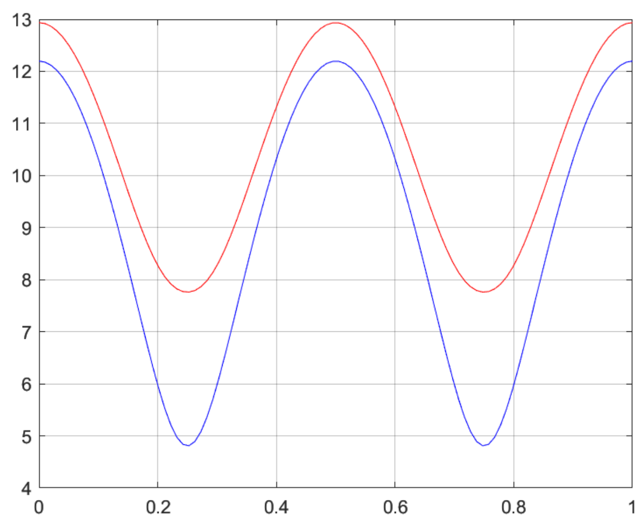
Solution.



■

8. Plot the left and right wheel velocities as a function of time for various combinations of the parameters a, b, and α.

   Solution.



■

Exercise 34.4

1. In line 2 the 'rossubscriber' command is introduced. From the code, what sensor output are we monitoring?

   Solution. We are monitoring the bump sensor. ■

2. The variable 'bumpmessage' is a structure. What is the size of 'bumpmessage.Data'? What do the values contained in that variable mean?

Solution. bumpmessage.Data is a 4x1 structure. It contains which sensors are tripped. ∎

3. What is the 'driveUntilBump' code commanding the robot to do?

   Solution. The driveUntilBump code drives until one of the sensors is tripped. ∎

4. Modify the 'driveUntilBump' code to make it a function where the robot velocity is an input.

   Solution.

```
function [] = driveUntilBump(speed)
    pub = rospublisher('/raw_vel');
    sub_bump = rossubscriber('/bump');
    msg = rosmessage(pub);

    % get the robot moving
    msg.Data = [speed, speed];
    send(pub, msg);

    while 1
        % wait for the next bump message
        bumpMessage = receive(sub_bump);
        % check if any of the bump sensors are set to 1 (meaning
            ↪ triggered)
        if any(bumpMessage.Data)
            msg.Data = [0.0, 0.0];
            send(pub, msg);
            break;
        end
    end
end
```

∎

5. Using what you have learned from the examples above, write a program that meets the following requirements:

   - The program commands the robot to drive a designated distance at a chosen speed, and stops when that distance is reached.
   - If the bump sensor is triggered, the robot reverses direction and backs up for 5 seconds then stops.

   Solution.

```
function [] = driveUntilBumpThenRunAway(speed, distance)
    pub = rospublisher('/raw_vel');
    sub_bump = rossubscriber('/bump');
    msg = rosmessage(pub);

    % get the robot moving
    msg.Data = [speed, speed];
    send(pub, msg);

    start = rostime('now');

    reverseFlag = 0;
    reverseStart = 0;

    while 1
        current = rostime('now');
        elapsed = current - start;
        if (elapsed.seconds > distance/speed) && (reverseFlag == 0) %
           ↪ Here we are saying the if the elapsed time is greater
           ↪ than
            %distance/speed, we have reached our desired distance and we
               ↪  should stop

            message.Data = [0,0]; % set wheel velocities to zero if we
               ↪ have reached the desire distance
            send(pubvel, message); % send new wheel velocities
            break %leave this loop once we have reached the stopping
               ↪ time
        end

        % wait for the next bump message
        bumpMessage = receive(sub_bump);
        % check if any of the bump sensors are set to 1 (meaning
           ↪ triggered)
        if any(bumpMessage.Data)
            reverseFlag = 1;
            reverseStart = rostime('now');
        end

        reverseElapsed = current - reverseStart;
        if (reverseElapsed.seconds > 5) && (reverseFlag == 1)
            message.Data = [0, 0];
            send(pubvel, message);
            break
```

```
        end
    end
end
```

■

Exercise 34.5

1. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions.

$$f(x, y) = x^2 \sin(xy^2)$$

Solution.

$$\frac{\partial f}{\partial x} = 2x \sin(xy^2) + x^2 y^2 \cos(xy^2)$$
$$\frac{\partial f}{\partial y} = 2x^3 y \cos(xy^2)$$

■

2.

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

Solution.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y$$
$$\frac{\partial f}{\partial y} = 3y^2 - 3x$$

■

Exercise 34.6

1. Evaluate all four second-order derivatives of the following functions.

$$f(x, y) = x^2 \sin(xy^2)$$

Solution.

$$\frac{\partial f}{\partial x} = 2x\sin(xy^2) + x^2y^2\cos(xy^2)$$
$$\frac{\partial f}{\partial y} = x^3y\cos(xy^2)$$

$$\frac{\partial^2 f}{\partial x^2} = \left(2 - x^2y^4\right)\sin(xy^2)$$
$$\frac{\partial^2 f}{\partial y^2} = 2x^2\cos(xy^2) - 2x^4y^2\sin(xy^2)$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2y\cos(xy^2) - 2x^3y^3\sin(xy^2)$$
$$\frac{\partial^2 f}{\partial y \partial x} = 6x^2y\cos(xy^2) - 2x^3y^3\sin(xy^2)$$

■

2. Evaluate all four second-order derivatives of the following functions.

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

Solution.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y$$
$$\frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$
$$\frac{\partial^2 f}{\partial y^2} = 6y$$
$$\frac{\partial^2 f}{\partial x \partial y} = -3$$
$$\frac{\partial^2 f}{\partial y \partial x} = -3$$

■

Exercise 34.7

1. Under what conditions are the mixed partial derivatives of f equal?

   Solution. The mixed partial derivatives of f are equal when the derivatives exist and are continuous. ■

Exercise 34.8

1. Find the gradient vector of the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$ and evaluate it at point $(1, 2)$.

   Solution.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 6xy - 6x \\ 6x^2 + 3y^2 - 6y \end{bmatrix}$$

$$\nabla f(1, 2) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

■

Exercise 34.9

1. Find the Hessian matrix of the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$ and evaluate it at point $(1, 2)$.

   Solution.

$$\text{Hess} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{bmatrix}$$

$$\text{Hess} f(1, 2) = \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}$$

■