Exercise 38.1

1. Find $\frac{dz}{dt}$.

$$z = x^2 + y^2 + xy, x = \sin(t), y = e^t$$

Solution.

$$\frac{dz}{dt} = (2x+y)(-\cos(t)) + (2y+x)\left(e^t\right)$$
$$\frac{dz}{dt} = -2x\cos(t) - y\cos(t) + 2ye^t + xe^t$$
$$\frac{dz}{dt} = x\left(e^t - 2\cos(t)\right) + y\left(2e^t - \cos(t)\right)$$

■

2. Find $\frac{dw}{dt}$.

$$w = xe^{\frac{y}{z}}, x = t^2, y = 1-t, z = 1+2t$$

Solution.

$$\frac{dw}{dt} = \left(e^{\frac{y}{z}}\right)(2t) + \left(\frac{x}{z}e^{\frac{y}{z}}\right)(-1) + xe^{\frac{y}{z}}(-1)\left(\frac{y}{z^2}\right)(2)$$
$$\frac{dw}{dt} = e^{\frac{y}{z}}\left(2t - \frac{x}{z} - 2x\left(\frac{y}{z^2}\right)\right)$$
$$\frac{dw}{dt} = e^{\frac{y}{z}}\left(2t - \frac{x}{z}\left(1 + \frac{2y}{z}\right)\right)$$

■

3. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$z = e^r\cos(\theta), r = st, \theta = \sqrt{s^2+t^2}$$

Solution.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$
$$\frac{\partial z}{\partial s} = (e^r \cos\theta)(t) + (-e^r \sin\theta)\left(s\left(s^2+t^2\right)^{-}\frac{1}{2}\right)$$
$$\frac{\partial z}{\partial s} = e^r\left(t\cos\theta - \frac{s+\sin\theta}{\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$
$$\frac{\partial z}{\partial t} = (e^r \cos\theta)(s) + (-e^r \sin\theta)\left(t\left(s^2+t^2\right)^{-}\frac{1}{2}\right)$$
$$\frac{\partial z}{\partial t} = e^r\left(s\cos\theta - \frac{t+\sin\theta}{\sqrt{s^2+t^2}}\right)$$

■

Exercise 38.2

1. Find the critical points of $f(x)$.

$$f(x) = x^r - x^2 + 1$$

Solution.

$$\frac{d}{dx} = 4x^3 - 2x = 0$$
$$2x(2x^2-1) = 0$$
$$x = 0, \pm\sqrt{\frac{1}{2}}$$
$$x = 0, \pm\frac{\sqrt{2}}{2}$$

■

Exercise 38.3

1. Classify the critical points of $f(x)$

Solution.

$$f''(x) = 12x^2 - 2$$
$$f''(0) = -2$$
$$f''\left(\pm\frac{\sqrt{2}}{2}\right) = 4$$

The critical point $x = 0$ is a local maximum and the critical points at $x = \pm\frac{\sqrt{2}}{2}$ are local minimums. ■

Exercise 38.4

1. Determine the critical points of the following functions.

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

Solution.

$$\nabla f = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix} = \mathbf{0} 3x^2 - 3y = 0$$
$$3y^2 - 3x = 0$$
$$y = x^2$$
$$x^4 - x = 0$$
$$x = 0, 1$$
$$y = 0, 1$$

The critical values of $f(x, y)$ is $(0, 0)$ and $(1, 1)$. ■

2.

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Solution.

$$\nabla f = \begin{bmatrix} 6xy - 6x \\ 3x^2 - 3y^2 \end{bmatrix} = \mathbf{0} 6xy - 6x = 0$$
$$3x^2 - 3y^2 = 0$$
$$\pm x = \pm y$$
$$\pm x^2 - x = 0$$
$$x = 0, 1, -1$$
$$y = 0, 1, -1$$

The critical values of $f(x, y)$ is $(0, 0)$, $(1, 1)$, and $(-1, -1)$. ■

Exercise 38.5

1. Classify the critical points of the following functions.

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

Solution.

$$\text{Hess} f = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

At $(0, 0)$

$$\text{Hess} f = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$
$$\lambda = \pm 3$$

The critical point $(0, 0)$ is a saddle point.
At $(1, 1)$

$$\text{Hess} f = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$
$$\lambda = 3, 9$$

The critical point $(1, 1)$ is a local minimum. ■

2. Classify the critical points of the following functions.

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Solution.

$$\text{Hess} f = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{bmatrix}$$

At $(0, 0)$

$$\text{Hess} f = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$$
$$\lambda = -6$$

The critical point $(0, 0)$ is a local maximum

At $(1,1)$

$$\text{Hess} f = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$$
$$\lambda = 6$$

The critical point $(1,1)$ is a local minimum.

At $(-1,-1)$

$$\text{Hess} f = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix}$$
$$\lambda = -6, -18$$

The critical point $(1,1)$ is a local maximum. ■