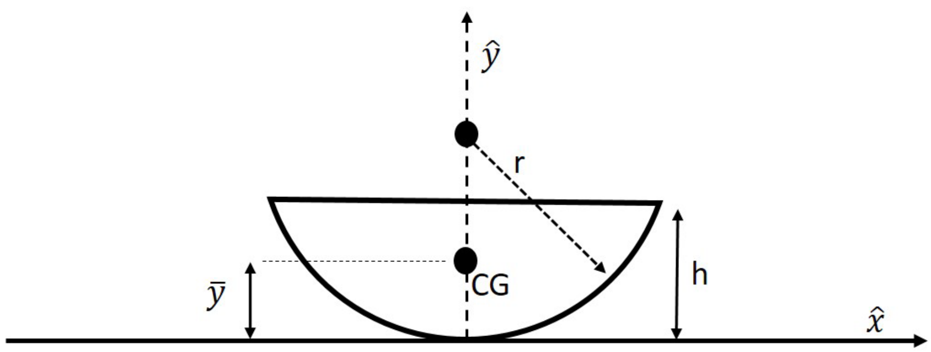
Exercise 29.1

1. When solving for the centroid of the circular segment, do you need to setup and solve the multiple integrals for both $\bar{x}$ and $\bar{y}$? Explain

   Solution. No. Because we know the circular segment is symmetric, $\bar{x}$ is at 0 in the $\hat{x}$ direction. ■

2. For the y-centroid, $\bar{y}$, write the double integral for area as a single integral, including an appropriate representation of x.



   Solution.

$$A = \int_0^h 2\sqrt{r^2 - (y-r)^2}dy$$

■

3. What are the bounds of the integrals in the equation for $\bar{y}$.

   Solution. 0, h ■

4. Write, but do not solve, the full equation for $\bar{y}$

   Solution.

$$\bar{y} = \frac{1}{M}\int\int m(y)ydy$$

$$\bar{y} = \frac{2\int_0^h y\sqrt{r^2 - (y-r)^2}dy}{2\int_0^h \sqrt{r^2 - (y-r)^2}dy}$$

■

Exercise 29.2

1. Use symbolic math in Matlab to find $\bar{y}$ and the total area of the segment.

   Solution.

```
%define circular segment parameters
r=6;
h=2;

%specify symbolic values using the matlab 'syms' function
syms r_sym h_sym y

%first define x using the definition of a circle with center at [0,R],
    ↪ ...
%x^2+(y-R)^2=R^2
x=sqrt(r_sym^2-(y-r_sym)^2);

%Setup the integral, symbolically, for the area of the circular segment
    ↪ . ...
%This integral is A=2*int(x). We will do this using the 'int' function
    ↪ ...
%in Matlab which takes inputs as int(symbolic_equation, ...
%symbolic_variable_of_integration, [lower bound, upper bound])
A=2*int(x,y,[0 h_sym]);

%Do the integral for the centroid symbolically
ybar=int(y*x,y,[0 h_sym])/int(x,y,[0 h_sym]);

%substitute numerical values for r and h and convert the output to a
    ↪ ...
%double. The 'subs' function takes inputs in the form subs(equation,
    ↪ ...
%old, new)

A=double(subs(A,[r_sym, h_sym],[r, h]));
ybar=double(subs(ybar,[r_sym, h_sym],[r, h]));
```

   ■

2. For the case of $r = 6$ and $h = 2$, what are A and $\bar{y}$?

   Solution.

$$A = 12.3899$$
$$\bar{y} = 1.1873$$

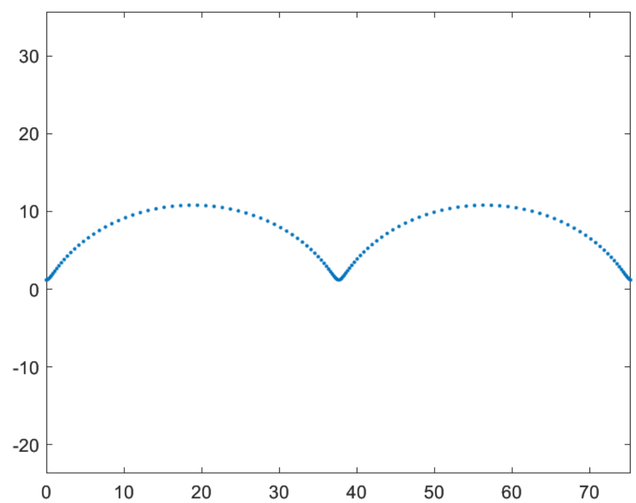
   ■

Exercise 29.3

1. Plot the position of the center of mass for the range $0 \le \phi \le 4\pi$.

   Solution.

```
phi = [0:0.1:4*pi];
x_cg = r * phi - (r - ybar)*sin(phi);
y_cg = r - (r - ybar)*cos(phi);
plot(x_cg, y_cg, '.')
axis equal
```



■

2. What are the equations for the position of the contact point, as a function of ϕ in the reference frame of the ramp?

   Solution.

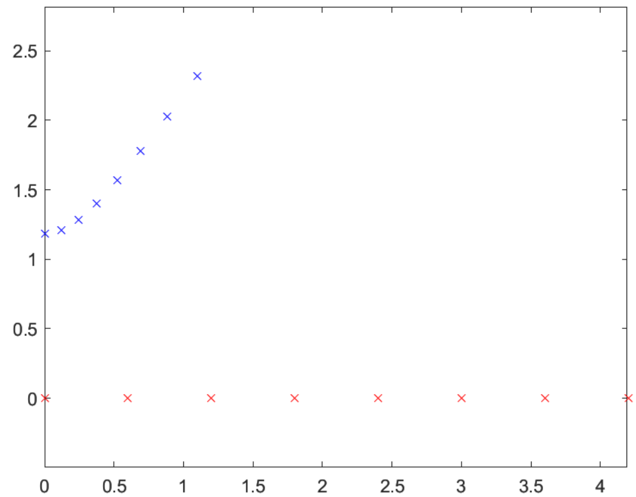
$$x_{cp} = \phi r$$
$$y_{cp} = 0$$

■

3. Plot the position of the CG and the contact point on the same plot for the range $0 \le \phi \le \frac{\pi}{4}$. What do you notice?

   Solution.

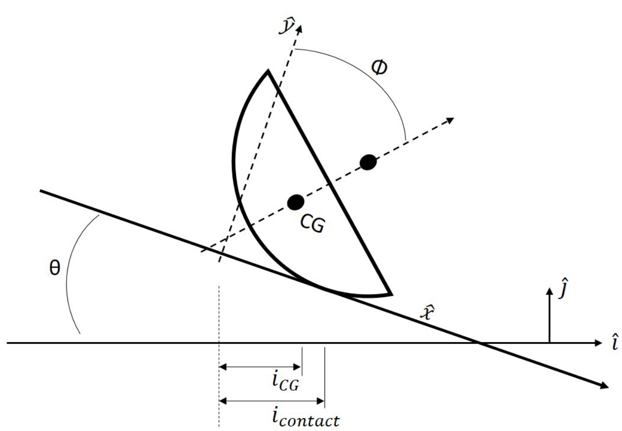
```
phi = [0:0.1:pi/4];
x_cg = r * phi - (r - ybar)*sin(phi);
y_cg = r - (r - ybar)*cos(phi);
x_cp = r * phi;
y_cp = zeros(size(phi));
plot(x_cg, y_cg, 'bx', x_cp, y_cp, 'rx')
axis equal
```



The center of gravity is always closer to 0 in the $\hat{x}$ direction. ■

Exercise 29.5

1. Write expressions for the unit vectors $[\hat{x}, \hat{y}]$ that define the *ramp frame* in terms of the vectors $\left[\hat{i}, \hat{j}\right]$ which define the *global frame*.



Solution.

$$\hat{x} = \cos\theta\hat{\imath} - \sin\theta\hat{\jmath}$$
$$\hat{y} = \sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

■

2. Represent the positions of the CG and contact point in the global frame by substituting the expressions for the $[\hat{x}, \hat{y}]$ in terms of $\left[\hat{\imath}, \hat{\jmath}\right]$.

Solution.

$$\begin{aligned} x_{cg} &= (r\phi - (r - \bar{y})\sin(\phi))(\cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath}) \\ y_{cg} &= (r - (r - \bar{y})\cos(\phi))(\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}) \\ x_{\text{contact}} &= (r\phi)(\cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath}) \\ y_{\text{contact}} &= 0 \end{aligned}$$

■

3. Collect terms and find the vertical and horizontal positions of the CG and contact point in the global frame.

Solution.

$$\begin{aligned} i_{cg} &= ((r\phi - (r - \bar{y})\sin(\phi))\cos(\theta) + (r\phi - (r - \bar{y})\sin(\phi))\sin(\theta)\hat{\jmath} \\ j_{cg} &= -((r\phi - (r - \bar{y})\sin(\phi))\sin(\theta) + (r\phi - (r - \bar{y})\sin(\phi))\sin(\theta)\hat{\jmath} \\ i_{\text{contact}} &= r\phi\cos(\theta)\hat{\jmath} \\ j_{\text{contact}} &= -r\phi\sin(\theta)\hat{\jmath} \end{aligned}$$

■

4. Based on your FBD of the circular segment on an inclined ramp, what condition must the horizontal positions of the CG and contact point satisfy for the "ducky" to be in static equilibrium (assuming no slip)?

Solution. To put the ducky in static equilibrium, the CG and contact point must be aligned in the $\hat{\jmath}$ direction. ■

Exercise 29.6

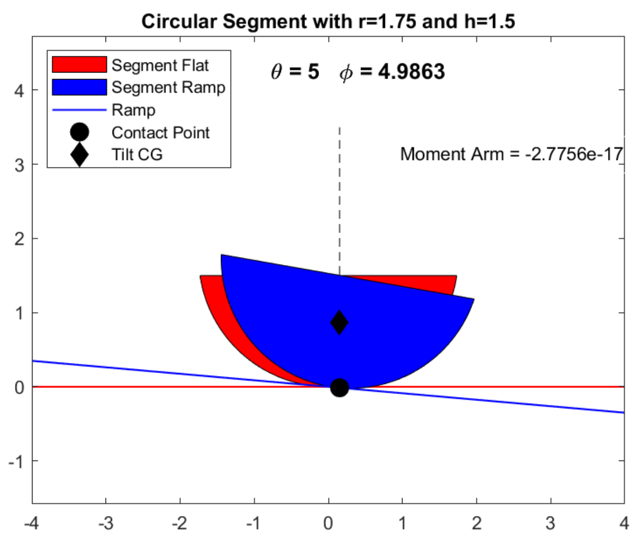
1. Find the static equilibrium angle for a circular segment with $r = 1.75$ and $h = 1.5$ for $\theta = 5$, 15, 25, 35 degrees. Is equilibrium possible for all of these ramp angles?

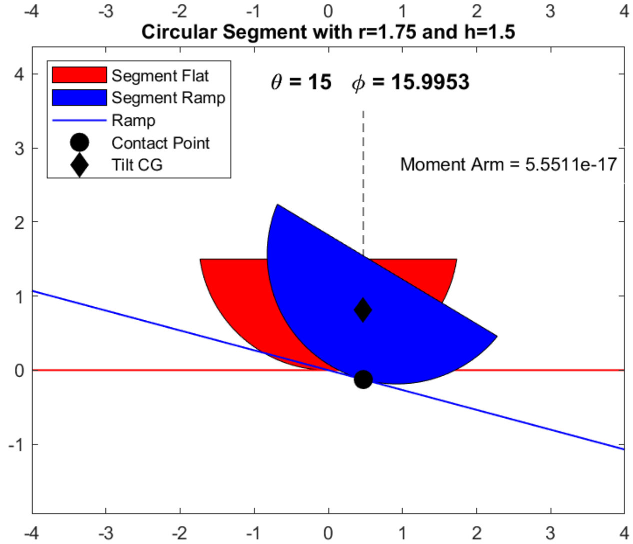
Solution.

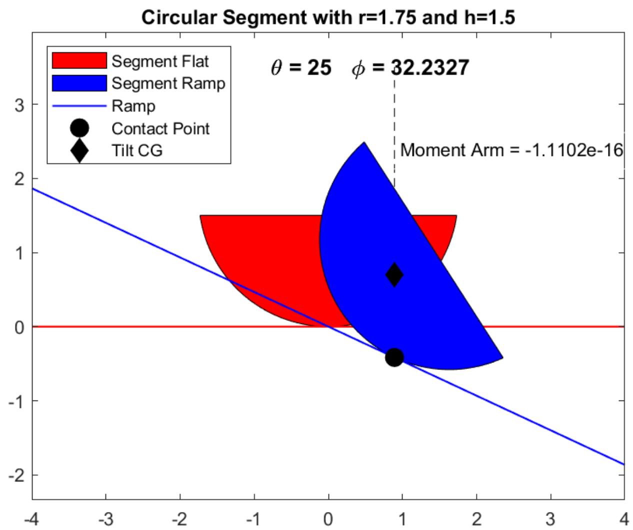
θ	ϕ
5	5
15	16
25	32
35	NaN

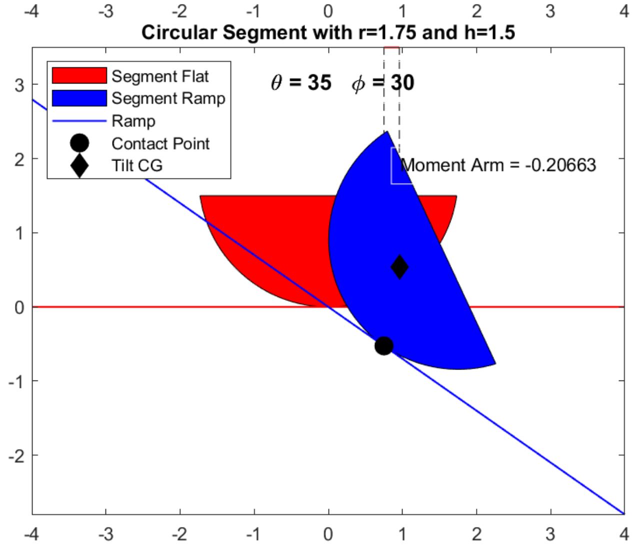
2. Visualize the circular segment on the ramp. For the 35 degree case, why is the solution not solvable?

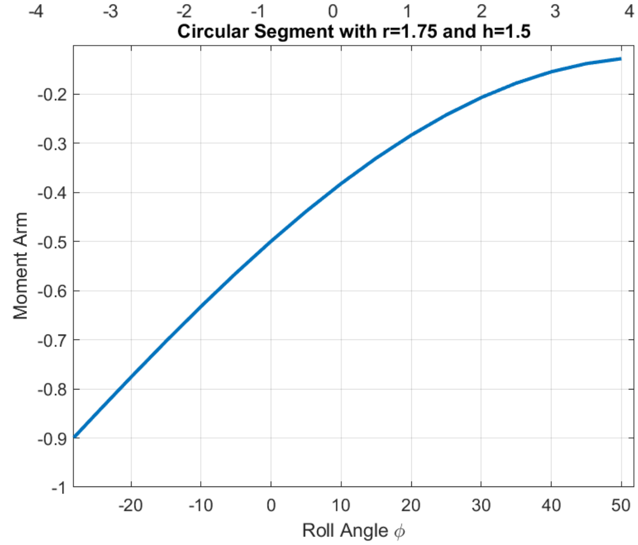
Solution.











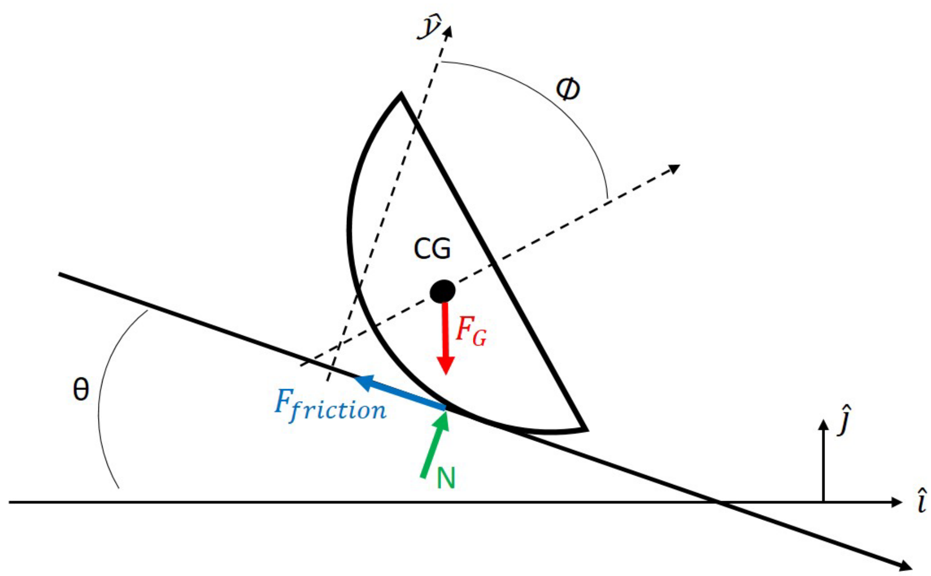
With the 35 degree case, there is no angle of the ducky where the moment arm equals 0. ■

3. The `solve` function returns two ϕ angles for a given ramp angle θ. Are both results physically valid? Why or why not?

   Solution. No. One of the solutions would return a ducky that is upside down, so the contact point would be part of the circle that doesn't exist. ■

Exercise 29.7

1. Write the force equations in the ramp $[\hat{x}, \hat{y}]$ frame for the "ducky" in static equilibrium.



   Solution.

$$F_{\hat{x}} = -F_{fx} + F_G \sin\theta = 0$$
$$F_{\hat{y}} = N - F_g \cos\theta = 0$$

■

2. When will the ducky slip?

   Solution.

$$F_{friction} < F_{G\hat{x}}$$
$$\mu N < F_G \sin\theta$$
$$\mu F_G \cos\theta < F_G \sin\theta$$
$$\mu < \tan\theta$$

■

3. What can be done to increase the slip-angle for the ducky? What are the tradeoffs for the modifications?

   Solution. To increase the slip angle for the ducky, the coefficient of friction between the ducky and the ramp must be increased. This most likely means changing the material of the ducky or the ramp. ■

Exercise 29.8

1. For the composite ducky shape, find the center of mass.

   Solution.

$$\bar{x} = 0$$
$$\bar{y} = 4.25$$

■

2. Predict the equilibrium angle $\phi_{eq}(\theta)$ for $\theta = [2, 4]$.

   Solution. Matlab script to solve this can be found here.

θ	ϕ
2	17.5
4	37.8

■