Exercise 32.1

1. Define a vector function $\boldsymbol{r}(u)$ in the xy plane whose trace is a circle centered at (x_0, y_0) with radius R.

   Solution.

$$\boldsymbol{r}(u) = (R\cos u + x_0)\,\hat{\boldsymbol{\imath}} + (R\sin u + y_0)\,\hat{\boldsymbol{\jmath}}$$

■

2. Determine (by hand) the unit tangent vector.

   Solution.

$$\begin{aligned}\boldsymbol{r'}(u) &= -R\sin u\hat{\boldsymbol{\imath}} + R\cos u\hat{\boldsymbol{\jmath}}\\ \hat{\boldsymbol{T}} &= \frac{r'}{|r'|}\\ &= -\sin u\hat{\boldsymbol{\imath}} + \cos u\hat{\boldsymbol{\jmath}}\end{aligned}$$

■

3. Determine (by hand) the unit normal vector.

   Solution.

$$\begin{aligned}\hat{\boldsymbol{T}}' &= -\cos u\hat{\boldsymbol{\imath}} - \sin u\hat{\boldsymbol{\jmath}}\\ \hat{\boldsymbol{N}}' &= \frac{\hat{\boldsymbol{T}}'}{|\hat{\boldsymbol{T}}'|}\\ &= -\cos u\hat{\boldsymbol{\imath}} - \sin u\hat{\boldsymbol{\jmath}}\end{aligned}$$

■

4. Determine (by hand) the unit binormal vector.

   Solution.

$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{T}} \times \hat{\boldsymbol{N}} = 0$$

■

5. Determine (by hand) the curvature and torsion.

Solution.

$$\kappa = \frac{\hat{\boldsymbol{T}}'}{\boldsymbol{r}'} = \frac{1}{R}$$
$$\tau = -\hat{\boldsymbol{N}} \cdot \frac{\hat{\boldsymbol{B}}'}{|\boldsymbol{r}'|} = 0$$

■

6. Set up the integral to compute the perimeter of the circle, and evaluate it by hand.

Solution.

$$\begin{aligned} L &= \int_0^{2\pi} |\boldsymbol{r}'(u)| du \\ &= \int_0^{2\pi} R du \\ &= 2\pi R \end{aligned}$$

■

Exercise 32.2

1. A helix in 3D can be defined by the vector function

$$\boldsymbol{r}(u) = a \cos u \hat{\boldsymbol{i}} + a \sin u \hat{\boldsymbol{j}} + bu\hat{\boldsymbol{k}}, a > 0, b > 0, u \geq 0$$

2. Determine (by hand) the unit tangent vector.

Solution.

$$\begin{aligned} \boldsymbol{r}'(u) &= -a \sin u \hat{\boldsymbol{i}} + a \cos u \hat{\boldsymbol{j}} + b\hat{\boldsymbol{k}} \\ |\boldsymbol{r}'(u)| &= \sqrt{a^2 \sin^2 u + a^2 \cos^2 u + b^2} \\ &= \sqrt{a^2 + b^2} \\ \hat{\boldsymbol{T}} &= \frac{r'}{|r'|} \\ &= \frac{-a \sin u}{\sqrt{a^2+b^2}} \hat{\boldsymbol{i}} + \frac{a \cos u}{\sqrt{a^2+b^2}} \hat{\boldsymbol{j}} + \frac{b}{\sqrt{a^2+b^2}} \hat{\boldsymbol{k}} \end{aligned}$$

■

3. Determine (by hand) the unit normal vector.

Solution.

$$\begin{aligned}\boldsymbol{\hat{T}'} &= \frac{-a\cos u}{\sqrt{a^2+b^2}}\boldsymbol{\hat{i}} - \frac{a\sin u}{\sqrt{a^2+b^2}}\boldsymbol{\hat{j}} \\ \boldsymbol{\hat{N}'} &= \frac{\boldsymbol{\hat{T}'}}{|\boldsymbol{\hat{T}'}|} \\ &= -\cos u\boldsymbol{\hat{i}} - \sin u\boldsymbol{\hat{j}}\end{aligned}$$

■

4. Determine (by hand) the unit binormal vector.

Solution.

$$\begin{aligned}\boldsymbol{\hat{B}} &= \boldsymbol{\hat{T}} \times \boldsymbol{\hat{N}} \\ &= \frac{b\sin u}{\sqrt{a^2+b^2}}\boldsymbol{\hat{i}} + \frac{b\sin u}{\sqrt{a^2+b^2}}\boldsymbol{\hat{j}} + \frac{a}{\sqrt{a^2+b^2}}\boldsymbol{\hat{k}}\end{aligned}$$

■

5. Determine (by hand) the curvature and torsion.

Solution.

$$\begin{aligned}\kappa &= \frac{\boldsymbol{\hat{T}'}}{\boldsymbol{r'}} \\ &= \frac{a}{a^2+b^2} \\ \tau &= -\boldsymbol{\hat{N}} \cdot \frac{\boldsymbol{\hat{B}'}}{|\boldsymbol{r'}|} \\ &= \frac{b\cos^2 u}{a^2+b^2}\boldsymbol{\hat{i}} + \frac{b\sin^2 u}{a^2+b^2}\boldsymbol{\hat{j}}\end{aligned}$$

■

6. Set up the integral to compute the length of the helix corresponding to 5 complete turns, and evaluate it by hand.

Solution.

$$\begin{aligned}L &= \int_0^{10\pi} |\boldsymbol{r}'(u)|du \\ &= \int_0^{10\pi} \sqrt{a^2+b^2}du \\ &= 10\pi\sqrt{a^2+b^2}\end{aligned}$$

■

Exercise 32.3

1. Define a vector function $\boldsymbol{r}(u)$ in the xy plane whose trace is an ellipse centered at (x_0, y_0) with semi major axis a and semi-minor axis b, $b < a$.

   Solution.

$$\boldsymbol{r}(u) = (a\cos u + x_0)\hat{\boldsymbol{\imath}} + (b\sin u + y_0)\hat{\boldsymbol{\jmath}}$$

■

2. Determine the unit tangent vector using the Symbolic Toolbox in MATLAB.

   Solution.

```
syms a b x0 y0 u
assume(a, {'real', 'positive'})
assume(b, {'real', 'positive'})
assume(x0, {'real'})
assume(y0, {'real'})
assume(u, {'real', 'positive'})

ri = a*cos(u)+x0;
rj = b*sin(u)+y0;
rk = 0;
r = [ri, rj, rk];
dr = diff(r, u);

T_hat = simplify(dr ./ norm(dr))
```

$$\hat{\boldsymbol{T}} = -\frac{a\sin u}{\sqrt{a^2\sin^2 u + b^2\cos^2 u}}\hat{\boldsymbol{\imath}} + \frac{b\cos u}{\sqrt{a^2\sin^2 u + b^2\cos^2 u}}\hat{\boldsymbol{\jmath}}$$

■

3. Determine the unit normal vector using the Symbolic Toolbox in MATLAB.

   Solution.

```
dT = diff(T_hat, u);
N_hat = simplify(dT ./ norm(dT))
```

$$\hat{\boldsymbol{N}} = -\frac{b\cos(u)}{\sqrt{a^2\sin(u)^2 + b^2\cos(u)^2}}\hat{\boldsymbol{\imath}} + \frac{a\sin(u)}{\sqrt{a^2\sin(u)^2 + b^2\cos(u)^2}}\hat{\boldsymbol{\jmath}}$$

■

4. Determine the unit binormal vector using the Symbolic Toolbox in MATLAB.

Solution.

```
B_hat = simplify(cross(T_hat, N_hat))
```

$$\hat{\boldsymbol{B}} = 1\hat{\boldsymbol{k}}$$

■

5. Determine the curvature and torsion using the Symbolic Toolbox in MATLAB.

Solution.

```
kappa = simplify(norm(dT) ./ norm(dr))
tau = dot(-N_hat, B_hat ./ norm(dr))
```

$$\kappa = \frac{ab}{\left(a^2 \sin(u)^2 - b^2 \sin(u)^2 + b^2\right)^{3/2}}$$
$$\tau = 0$$

■

6. Set up the integral to compute the perimeter of the ellipse using the Symbolic Toolbox in MATLAB.

Solution.

```
L = simplify(int(norm(dr), u, [0 2*pi]))
```

$$L = \int_0^{2\pi} \sqrt{a^2 \sin(u)^2 + b^2 \cos(u)^2} du$$

■

Exercise 32.4

1. Consider a body undergoing non-uniform circular motion with radius R so that it's angular velocity is linearly increasing with time, $\omega = \alpha t$.

   Define a position vector for this moving body.

Solution.

$$\boldsymbol{r}(t) = R\cos\left(\alpha t^2\right)\hat{\boldsymbol{\imath}} + R\sin\left(\alpha t^2\right)\hat{\boldsymbol{\jmath}}$$

■

2. Determine the velocity of this moving body.

Solution.

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = -2R\alpha t\sin\left(\alpha t^2\right) + 2R\alpha t\cos\left(\alpha t^2\right)$$

■

3. Determine the acceleration of this moving body, and decompose the acceleration into the unit tangent and unit normal directions.

Solution.

$$\begin{aligned}
\boldsymbol{a}(t) &= \boldsymbol{v}'(t) \\
&= \left(-2R\alpha\sin\left(\alpha t^2\right) - 4R\alpha^2t^2\cos\left(\alpha t^2\right)\right)\hat{\boldsymbol{\imath}} + \left(2R\alpha\cos\left(\alpha t^2\right) - 4R\alpha^2t^2\sin\left(\alpha t^2\right)\right)\hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{T}}(t) &= \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \\
&= -\sin\left(\alpha t^2\right) + \cos\left(\alpha t^2\right) \\
\hat{\boldsymbol{N}}(t) &= \frac{\hat{\boldsymbol{T}}'(t)}{|\hat{\boldsymbol{T}}'(t)|} \\
&= -\cos\left(\alpha t^2\right) - \sin\left(\alpha t^2\right) \\
\boldsymbol{a}(t) &= 2R\alpha\hat{\boldsymbol{T}} + 4R\alpha^2t^2\hat{\boldsymbol{N}}
\end{aligned}$$

■

Exercise 32.5

1. Consider a body moving in 3D with position vector

$$\boldsymbol{r}(t) = a\cos(ct)\hat{\boldsymbol{\imath}} + a\sin(ct)\hat{\boldsymbol{\jmath}} + bct\hat{\boldsymbol{k}}$$

Describe the path that the body should take.

Solution. The body is moving in a helix of radius a at rate c, rising at rate b. ■

2. Determine the velocity of this moving body.

Solution.

$$\boldsymbol{v}(t) = -ac\sin(ct)\hat{\boldsymbol{\imath}} + ac\cos(ct)\hat{\boldsymbol{\jmath}} + bc\hat{\boldsymbol{k}}$$

■

3. Determine the acceleration of this moving body, and decompose the acceleration into the unit tangent and unit normal directions.

   Solution.

$$\begin{aligned}
\boldsymbol{a}(t) &= -ac^2\cos(ct)\hat{\boldsymbol{\imath}} - ac^2\sin(ct)\hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{T}}(t) &= \frac{a\sin(ct)}{\sqrt{a^2+b^2}}\hat{\boldsymbol{\imath}} + \frac{a\cos(ct)}{\sqrt{a^2+b^2}}\hat{\boldsymbol{\jmath}} + \frac{b}{\sqrt{a^2+b^2}}\hat{\boldsymbol{k}} \\
\hat{\boldsymbol{N}}(t) &= -\cos(ct)\hat{\boldsymbol{\imath}} - \sin(ct)\hat{\boldsymbol{\jmath}} \\
\boldsymbol{a}(t) &= 2R\alpha\hat{\boldsymbol{T}} + 4R\alpha^2t^2\hat{\boldsymbol{N}}
\end{aligned}$$

■

Exercise 32.6

1. Consider a body moving in 3D with position vector

$$\boldsymbol{r}(t) = a\cos(ct)\hat{\boldsymbol{\imath}} + b\sin(ct)\hat{\boldsymbol{\jmath}}$$

   Describe the path that the body should take.

   Solution. The body is moving in an ellipse with major and minor axes a and b at rate c. ■

2. Determine the velocity of this moving body.

   Solution.

$$\boldsymbol{v}(t) = -ac\sin(ct)\hat{\boldsymbol{\imath}} + bc\cos(ct)\hat{\boldsymbol{\jmath}}$$

■

3. Determine the acceleration of this moving body, and decompose the acceleration into the unit tangent and unit normal directions.

Solution.

$$\begin{aligned}
\boldsymbol{a}(t) &= -ac^2\cos(ct)\hat{\boldsymbol{i}} - bc^2\sin(ct)\hat{\boldsymbol{j}} \\
\hat{\boldsymbol{T}}(t) &= \frac{a\sin(ct)}{\sqrt{a^2\sin(ct)^2 + b^2\cos(ct)^2}}\hat{\boldsymbol{i}} + \frac{b\cos(ct)}{\sqrt{a^2\sin(ct)^2 + b^2\cos(ct)^2}}\hat{\boldsymbol{j}} \\
\hat{\boldsymbol{N}}(t) &= -\frac{b\cos(ct)}{\sqrt{a^2\sin(ct)^2 + b^2\cos(ct)^2}}\hat{\boldsymbol{i}} - \frac{a\sin(ct)}{\sqrt{a^2\sin(ct)^2 + b^2\cos(ct)^2}}\hat{\boldsymbol{j}} \\
\boldsymbol{a}(t) &= \frac{c^2\sin(2ct)\left(a^2 - b^2\right)}{2\sqrt{-a^2\cos(ct)^2 + a^2 + b^2\cos(ct)^2}}\hat{\boldsymbol{T}} + \frac{abc^2}{\sqrt{a^2\sin(ct)^2 - b^2\sin(ct)^2 + b^2}}\hat{\boldsymbol{N}}
\end{aligned}$$

$\blacksquare$