Exercise 22.1

1. Consider the exponential function

$$y = Ae^{kt}$$

where A and k are parameters.

What is the value of y when $t = 0$?

Solution. $y = A$ ■

2. What happens to y as $t \to \pm\infty$? How does this limiting behavior depend on the sign of A and k?

Solution.
If $A > 0$ and $k > 0$:

$$\lim_{t \to \infty} y = \infty$$
$$\lim_{t \to -\infty} y = 0$$

If $A < 0$ and $k > 0$:

$$\lim_{t \to \infty} y = -\infty$$
$$\lim_{t \to -\infty} y = 0$$

If $A > 0$ and $k < 0$:

$$\lim_{t \to \infty} y = 0$$
$$\lim_{t \to -\infty} y = \infty$$

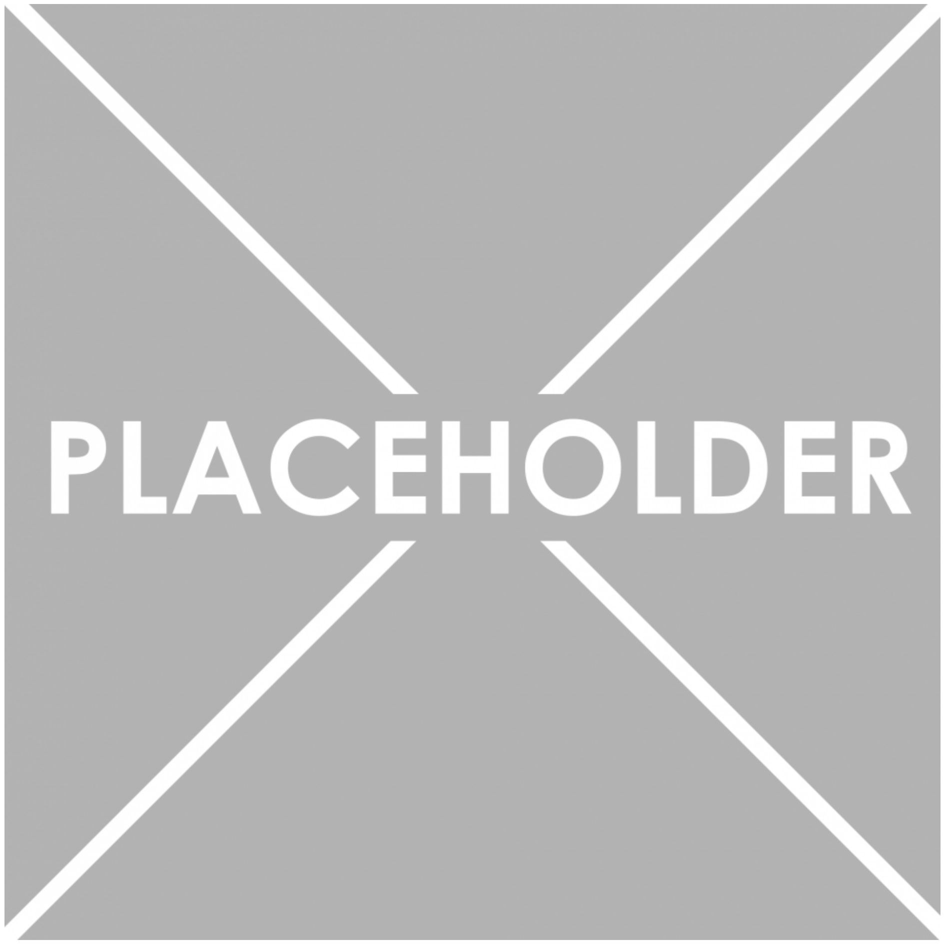
If $A < 0$ and $k < 0$:

$$\lim_{t \to \infty} y = 0$$
$$\lim_{t \to -\infty} y = -\infty$$

■

3. Sketch examples of the curves in the four quadrants of the $A - k$ space.

   Solution.



$\blacksquare$

4. What is the effect of parameters A and k on the curve?

   Solution. A determines if the curve has positive or negative values. k determines if the curve exponentially grows or exponentially shrinks. $\blacksquare$

5. Assuming that t is time and y is population, how would you interpret the parameters $A > 0$ and $k > 0$?

   Solution. A is the initial population at $t = 0$. k is the exponential growth rate. $\blacksquare$

6. How long does it take an exponentially increasing population to double if $k = 0.1$?

Solution.

$$
\begin{aligned}
y &= Ae^{0.1t} \\
2A &= e^{0.1T} \\
0.1T &= \ln 2 \\
T &\approx 6.93
\end{aligned}
$$

■

Exercise 22.2

1. Consider the logistic function

$$y = \frac{A}{1 + e^{-kt}}$$

   where A and k are parameters.

   What happens to y when $t = 0$?

   Solution. $y = \frac{A}{2}$ ■

2. What happens to y as $t \to \pm\infty$? How does this limiting behavior depend on the sign of k?

   Solution.
   If $k > 0$:

$$
\begin{aligned}
\lim_{t\to\infty} y &= A \\
\lim_{t\to-\infty} y &= 0
\end{aligned}
$$

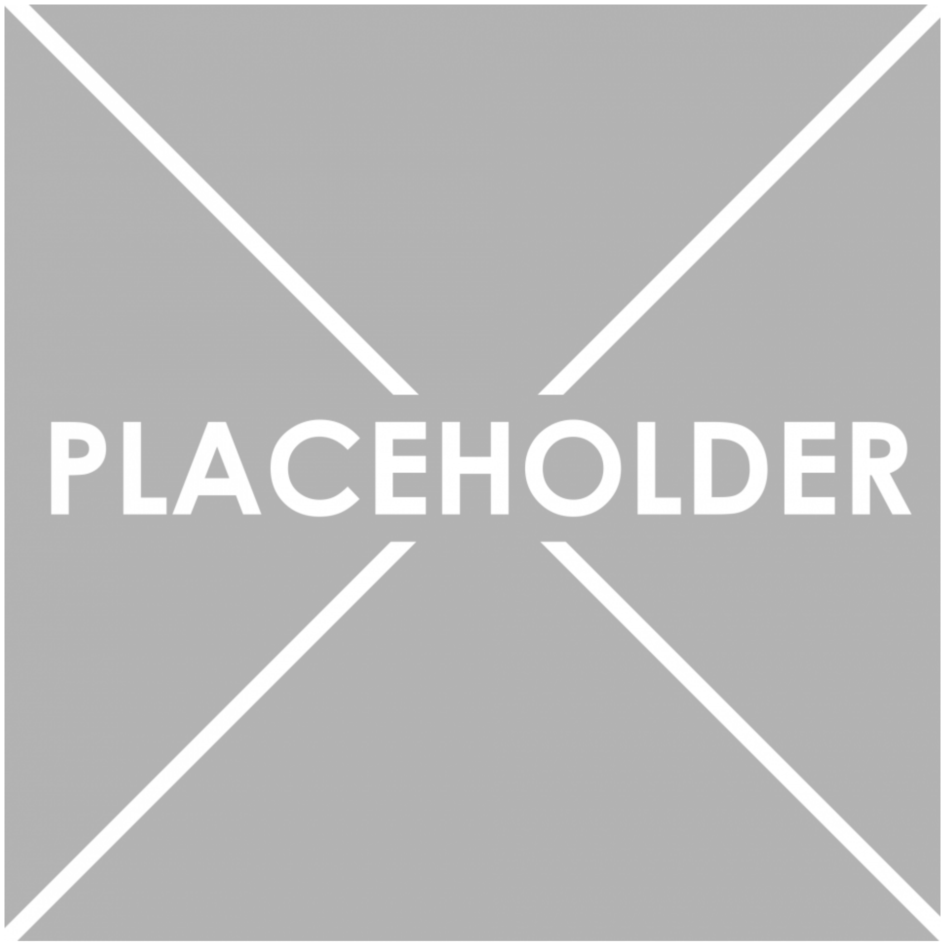
   If $k < 0$:

$$
\begin{aligned}
\lim_{t\to\infty} y &= 0 \\
\lim_{t\to-\infty} y &= A
\end{aligned}
$$

■

3. Sketch examples of the curves in the four quadrants of the $A - k$ space.

   Solution.



∎

4. What is the effect of parameters A and k on the curve?

   Solution. The curve approaches A at rate k. ∎

5. Assuming that t is time and y is population, how would you interpret the parameters $A > 0$ and $k > 0$?

   Solution. A is the carrying capacity of the environment and k is the natural growth rate of the population. ∎

6. How long does it take an initial population of $\frac{A}{2}$ to reach 99% of the carrying capacity if $k = 0.1$

Solution.

$$
\begin{aligned}
0.99A &= \frac{A}{1+e^{-0.1t}} \\
e^{-0.1T} &= \frac{0.01}{0.99} \\
-0.01T &= \ln\frac{0.01}{0.99} \\
T &\approx 46
\end{aligned}
$$

■

Exercise 22.3

1. Consider the trigonometric function

$$y = A\sin(\omega t + \phi)$$

with parameters A, ω, and ϕ.

Sketch some representative examples of these curves for different values of the parameters.

Solution.

$\blacksquare$

2. What features do A, ω, and ϕ control?

   Solution. A represents amplitude, ω represents frequency, and ϕ represents phase shift. $\blacksquare$

3. Find the value of t corresponding to $y = 2$ for $A = 3$, $\omega = 4$, $\phi = 5$.

   Solution.

$$
\begin{aligned}
y &= A \sin(\omega t + \phi) \\
2 &= 3 \sin(4t + 5) \\
4t + 5 &= \arcsin \frac{2}{3} \\
t &\approx -1.06
\end{aligned}
$$

$\blacksquare$

Exercise 22.4

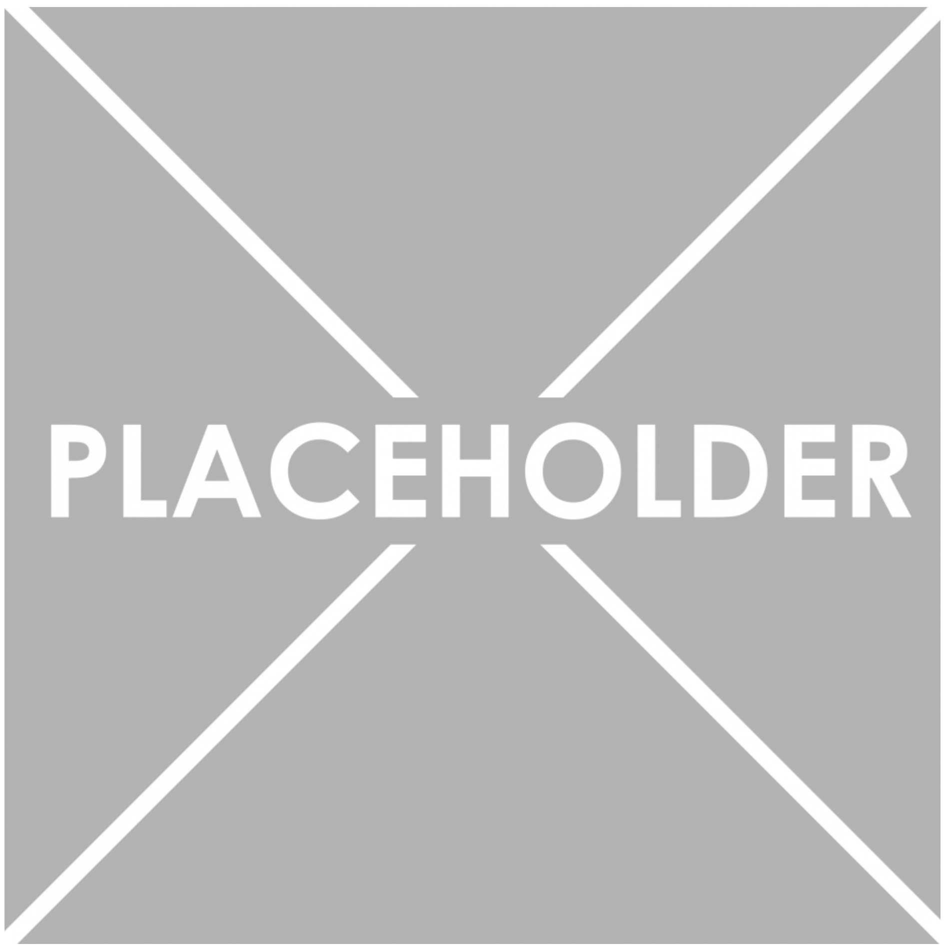
1. Consider the quadratic polynomial in vertex form

$$y = g(x - h)^2 + k$$

with parameters g, h, and k.

Sketch some representative curves for different parameter values.

Solution.



■

2. What features of the curve do g, h, and k control?

Solution. g controls the direction that the parabola opens in, with vertex at (h, k). ■

3. What is the relationship between g, h, and k in the vertex form and a, b, and c in the standard form $y = c + bx + ax^2$?

Solution. $a = g$, $b = -2gh$, $c = gh^2 + k$ ∎

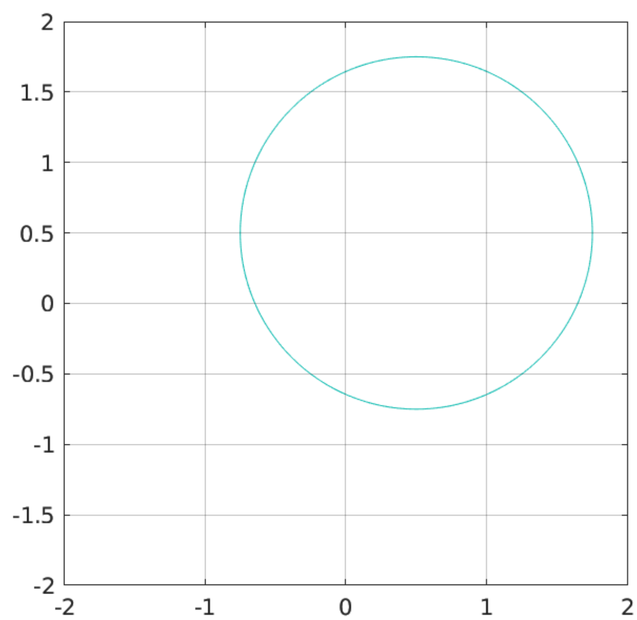
Exercise 22.5

1. Visualize a circle in MATLAB for different values of a, b, and R.

   Solution.

$$0 = (x - a)^2 + (y - b)^2 - R$$

```
r = 1.25;
a = 0.5;
b = 0.5;
[x, y] = meshgrid(linspace(-2, 2, 200), linspace(-2, 2, 200));
f = (x - a).^2 + (y - b).^2 - r^2;
contour(x, y, f, [0, 0])
axis equal
grid on
```



∎

Exercise 22.6

1. Visualize an ellipse using the implicit definition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

   for different values of a and b. What features of the ellipse do a and b control?

Solution. a and b control the amount the ellipse is stretched in the x and y directions, respectively. ■

Exercise 22.7

1. Visualize an hyperbola using the implicit definition

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

for different values of a and b. What features of the hyperbola do a and b control?

Solution. The hyperbola is asymptotic to the lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ and crosses the x axis at $x = \pm a$. ■

Exercise 22.8

1. Use the internet to find the conditions under which the solutions of

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

define an ellipse, a parabola, a hyperbola, and a circle.

Solution. The conic section is defined by $b^2 - 4ac$.
If $b^2 - 4ac < 0$, it represents an eclipse. If $a = c$ and $b = 0$, it represents a circle.
If $b^2 - 4ac = 0$, it represents a parabola.
If $b^2 - 4ac > 0$, it represents a hyperbola. ■

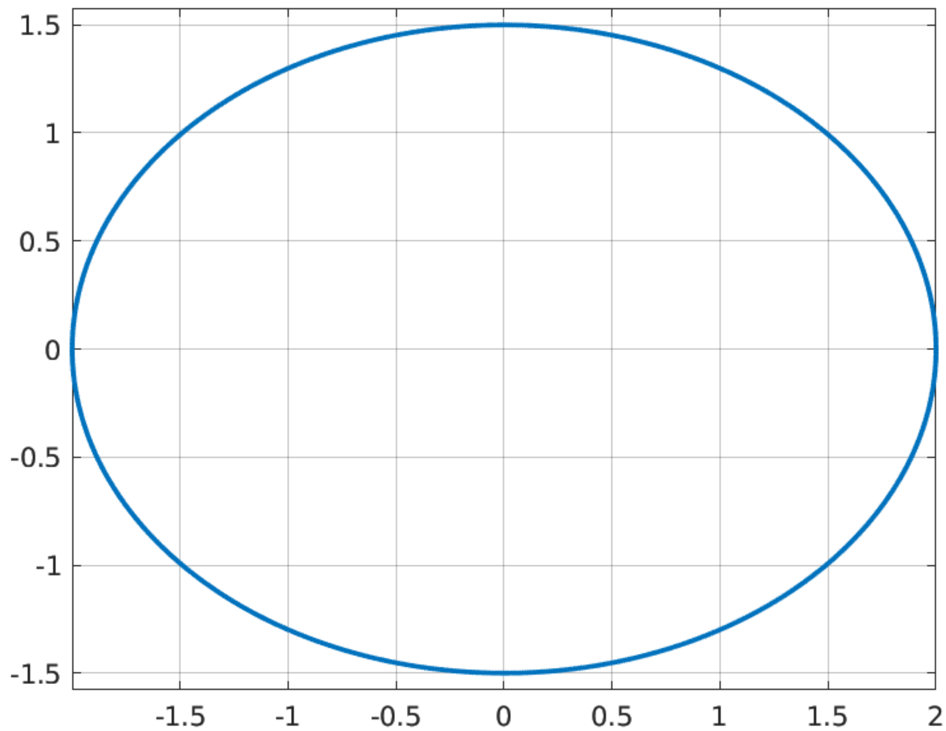
Exercise 22.9

1. Find a set of parametric equations that define an ellipse, and use MATLAB to verify them visually. Show that the parametric equations satisfy the implicit equation for an ellipse.

Solution.

$$x = a\cos u, y = b\sin u, u \in [0, 2\pi]$$

```
u = linspace(0, 2*pi, 1000);
a = 2;
b = 1.5;
x = a*cos(u);
y = b*sin(u);
plot(x, y, '.')
axis equal
grid on
```



■

Exercise 22.10

1. A logarithmic spiral can be defined by the parametric equations

$$x = ae^{-bu}\cos(u), y = ae^{-bu}\sin(u), a > 0, b > 0, u \in [0, \infty)$$

   How does a and b change the curve?

   Solution. If $b = 0$, this is a circle with radius a. If not, it is a spiral that starts at $(a, 0)$ and approaches 0. The rate that it spirals increases as $b \to 0$ ■

Exercise 22.11

1. A helix in 3D can be defined by the parametric equations

$$x = a\cos(u), y = a\sin(u), z = bu, a > 0, b > 0, u > 0$$

   How do a and b change the curve?

   Solution. b controls the pitch of the helix, and a controls the radius of the circle. ■

Exercise 22.12

1. Find the best fit parabola for these 4 data points. Recall that a parabola can be defined using the explicit function $y = ax^2 + bx + c$.

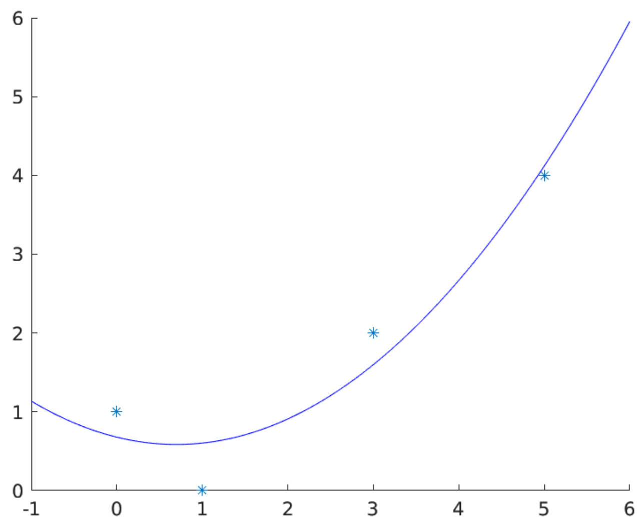
$$(0, 1), (1, 0), (3, 2), (5, 4)$$

Solution.

```
hold on
x_data = [0; 1; 3; 5];
y_data = [1; 0; 2; 4];
plot(x_data, y_data, '*')
A_data = [x_data.^2 x_data ones(size(x_data))];
p = A_data\y_data
```

$$p = \begin{bmatrix} 0.1910 \\ -0.2663 \\ 0.6784 \end{bmatrix}$$
$$y = 0.1910x^2 - 0.2663x + 0.6784$$

```
x = linspace(-1, 6, 1000)';
A = [x.^2 x ones(size(x))];
y = A * p;
plot(x, y)
```



■

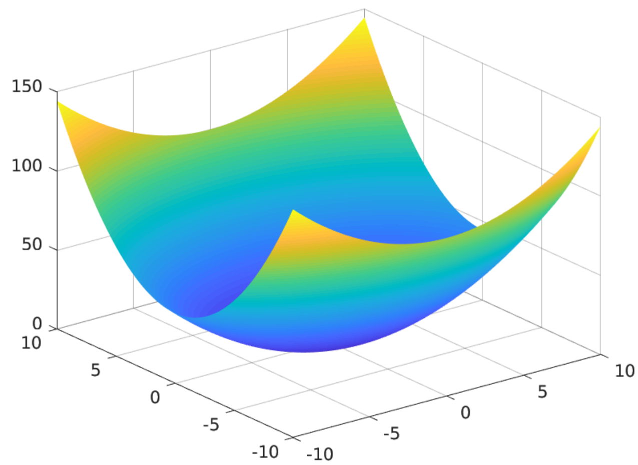
Exercise 22.13

1. Visualize the elliptic paraboloid $z = x^2/a^2 + y^2/b^2$ for different values of $a > 0$ and $b > 0$.

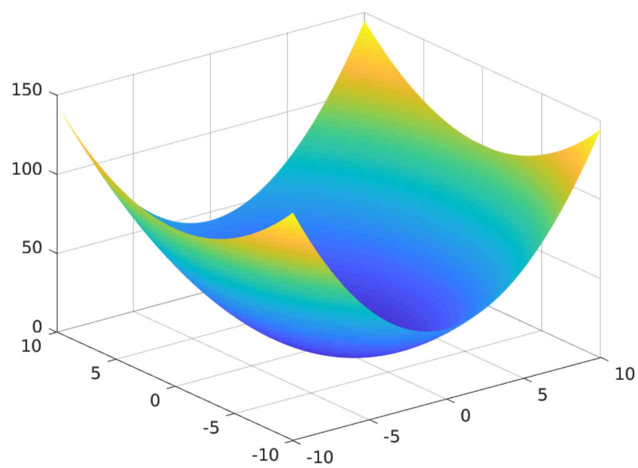
   Solution.

```
[x, y] = meshgrid(linspace(-10, 10, 100), linspace(-10, 10, 100));
z = x.^2/a^2 + y.^2/b^2;
surf(x, y, z);
shading interp
```

   With $a > b$:



   With $a < b$:



■

   Describe the contours in the yz-plane defined by $x = c$.

   Solution. This is a parabola that crosses the z-axis at c^2/a^2 ■

2. Describe the contours in the xz-plane defined by $y = c$.

Solution. This is a parabola that crosses the z-axis at c^2/b^2 ■

3. Describe the contours in the xy-plane defined by $z = c$.

Solution. This is an ellipse. ■

Exercise 22.14

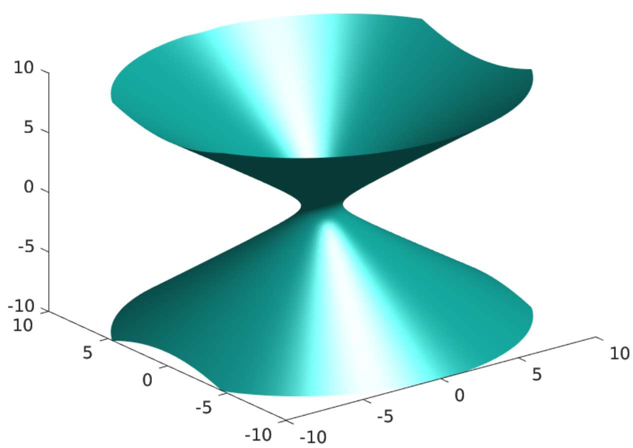
1. Visualize the hyperboloid of one sheet defined by

$$x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 = 0$$

for different values a, b, c. What features do a, b, and c control?

Solution.

```
a = 1.5;
b = 1;
c = 1;
[x,y,z] = meshgrid(linspace(-10,10,100),linspace(-10,10,100),linspace
    ↪ (-10,10,100));
f = x.^2/a^2 + y.^2/b^2 - z.^2/c^2 - 1;
isosurface(x, y, z, f, 0)
```



a, b, and c control the stretching in the x, y, and z directions, respectively. ■

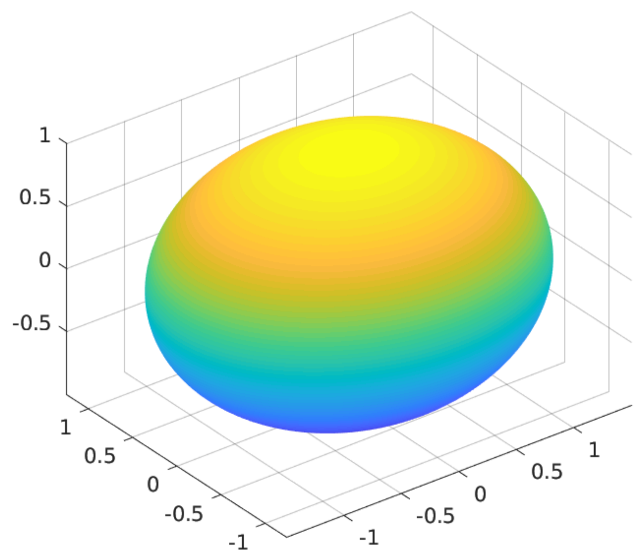
Exercise 22.15

1. Lookup the parametric equations that define an ellipsoid, and use MATLAB to visualize.

Solution.

$$x = a\cos\theta\sin\phi$$
$$y = b\sin\theta\sin\phi$$
$$z = c\cos\phi$$

```
a = 1.5;
b = 1;
c = 1;
[u, v] = meshgrid(linspace(0,2*pi,200),linspace(0,2*pi,200));
x = a*cos(u).*sin(v);
y = b*sin(u).*sin(v);
z = c*cos(v);
surf(x, y, z)
shading interp
axis equal
```



■

Exercise 22.16

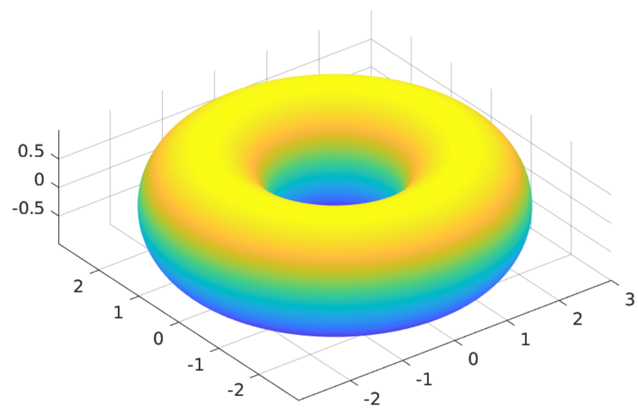
1. Visualize the following parametric surface

$$x = (a + r\cos(u))\cos(v), y = (a + r\cos(u))\sin(v), z = r\sin(u)$$

with $r < a$ and $u \in [0, 2\pi], v \in [0, 2\pi]$. Describe the surface and interpret the parameters a and r.

Solution.

```
a = 2;
r = 1;
[u, v] = meshgrid(linspace(0,2*pi,200),linspace(0,2*pi,200));
x = (a+r*cos(u)).*cos(v);
y = (a+r*cos(u)).*sin(v);
z = r*sin(u);
surf(x, y, z)
shading interp
axis equal
```



∎

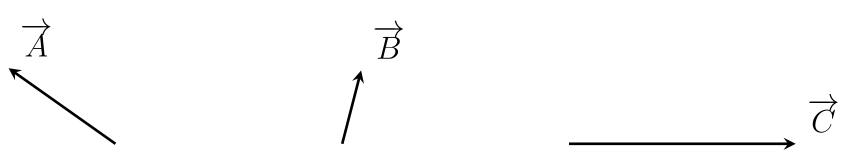
Exercise 22.19

1. What does the dot product of two vectors tell you about the two vectors? What about the cross product?

   Solution. The dot product measures how parallel two vectors are, and the cross product measures how perpendicular the two vectors are, and is in the direction orthogonal to the plane the two vectors define. ∎

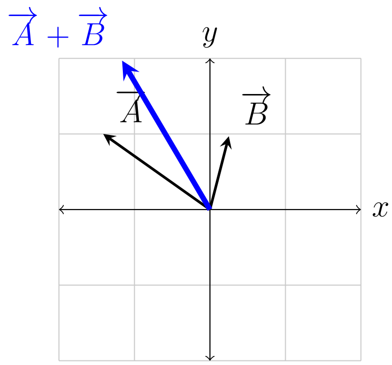
Exercise 22.20

1. The diagram shows three vectors, $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{C}$. All three are in plane with the page; their magnitudes are (respectively) 2, 1, and 3. For each operation below, either draw the results of the identified operations, or (if appropriate) give a best guess as to the value, or (if appropriate) identify the operation as nonsense.



$\overrightarrow{A} + \overrightarrow{B}$

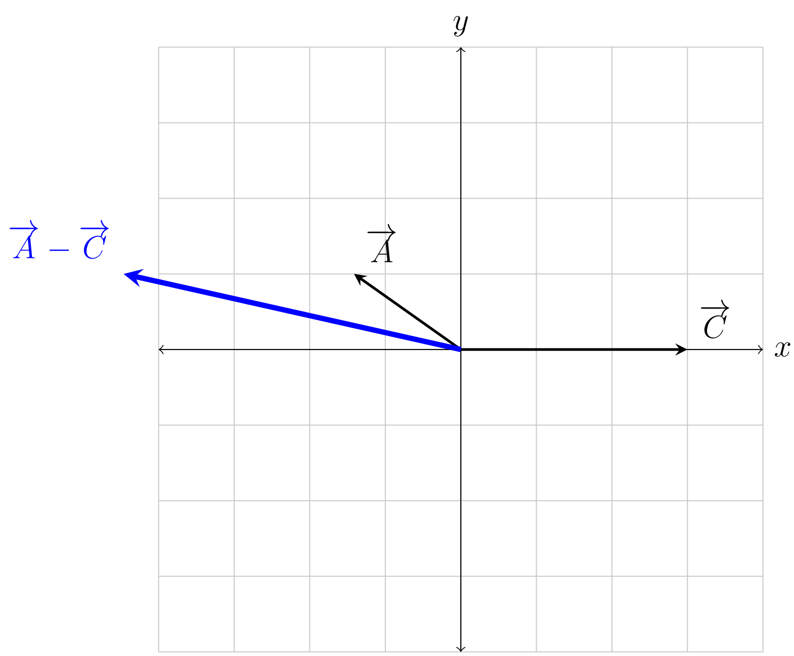
Solution.



■

2. $\overrightarrow{A} - \overrightarrow{C}$

Solution.



■

3. $\overrightarrow{A} \cdot \overrightarrow{B}$

Solution.

$$\overrightarrow{A} \cdot \overrightarrow{B} = 0.6165$$

■

4. $\left(\overrightarrow{A} \times \overrightarrow{B}\right) \times \overrightarrow{C}$

Solution.

$$\left(\overrightarrow{A} \times \overrightarrow{B}\right) \times \overrightarrow{C} = \begin{bmatrix} 0 \\ -4.8647 \\ 0 \end{bmatrix}$$

■

5. $\left(\overrightarrow{A} \cdot \overrightarrow{B}\right) \times \overrightarrow{C}$

Solution. This does not make sense. You cannot cross a scalar with a vector. ■

Exercise 22.21

1. Let $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}, \vec{B} = \hat{\imath} - \hat{\jmath}$, and $\vec{C} = -5\hat{\jmath}$. Find the results of identified operations, or (if appropriate) identify the operation as nonsense.
$\left|\overrightarrow{A} + \overrightarrow{B}\right|$

Solution.

$$\left|\overrightarrow{A} + \overrightarrow{B}\right| = 5$$

■

2. $\overrightarrow{A} \times \overrightarrow{C}$

Solution.

$$\overrightarrow{A} \times \overrightarrow{C} = -15\hat{k}$$

■

3. $\overrightarrow{A} \cdot \overrightarrow{B}$

Solution.

$$\overrightarrow{A} \cdot \overrightarrow{B} = -1$$

■

Exercise 22.22

1. $\overrightarrow{A}$ and $\overrightarrow{B}$ are two arbitrary, non-parallel vectors in three-dimensional space. Using them, construct the following vectors.

   The vector $\hat{A}$, which has length of 1, and points in the direction of $\overrightarrow{A}$.

Solution. $\hat{A} = \overrightarrow{A} / \left|\overrightarrow{A}\right|$ ■

2. The vector $\hat{n}$, which has length of 1, and is perpendicular to both $\overrightarrow{A}$ and $\overrightarrow{B}$.

   Solution. $\hat{n} = \overrightarrow{A} \times \overrightarrow{B} / \left|\overrightarrow{A} \times \overrightarrow{B}\right|$ ■

Exercise 22.24

1. "I am pushing a coffee cup across my desk. Since it is in motion, I know that the magnitude of the frictional force acting on the cup is given by $\mu_k N$, where μ_k is the coefficient of kinetic friction and N is the normal force the table exerts on the cup." True or false?

   Solution. True ■

2. "I am pushing on a coffee cup that is sitting on my desk. Since it is not moving, I know that the magnitude of the frictional force acting on the cup is given by $\mu_s N$, where μ_s is the coefficient of static friction and N is the normal force the table exerts on the cup." True or false? Why?

   Solution. False. Because the cup is not moving, the frictional force is less than $\mu_s N$. ■

3. If an object of mass m is sitting alone and stationary on a level surface, what is the magnitude of the normal force that the surface exerts on the object?

   Solution. $N = mg$ ■

4. Now consider that we drop this object onto the surface. During the time that the object is decelerating to a stop when it comes into contact with the surface is the normal force exerted by the surface on the object greater than, less than, or equal to the force from part 3?

   Solution. Greater than. The force from part 3 gives the force when the object is at rest, but during this time, the object is also experiencing a force from the deceleration. ■