Exercise 12.1

1.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

Confirm the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 2$ and $\lambda_2 = -3$.

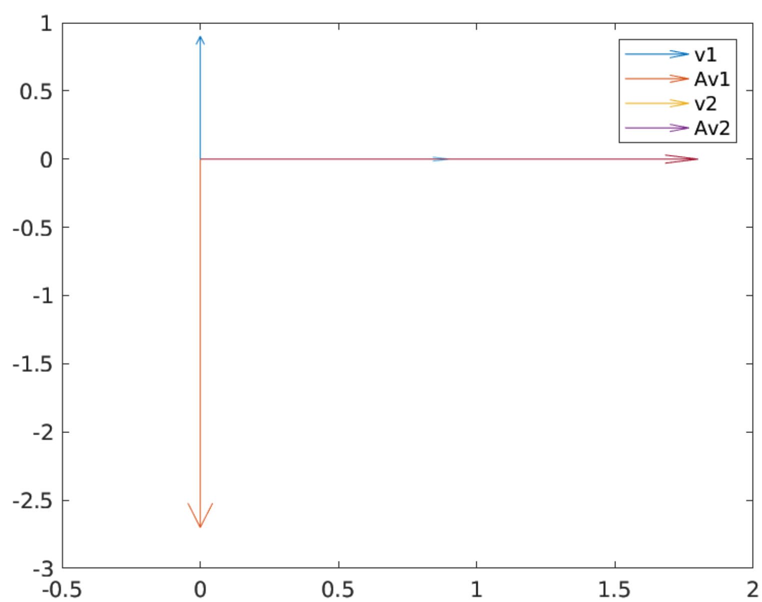
Solution.

$$\begin{aligned} \boldsymbol{Av} &= \lambda \boldsymbol{v} \\ (\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{v} &= 0 \\ \boldsymbol{A} - \lambda \boldsymbol{I} &= \begin{bmatrix} 2-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} \\ \det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= 0 \\ (2-\lambda)(-3-\lambda) &= 0 \\ \lambda &= 2, -3 \end{aligned}$$

■

2. The eigenvector associated with $\lambda_1 = 2$ is $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the eigenvector associated with $\lambda_2 = -3$ is $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Plot $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{Av}_1$, and $\boldsymbol{Av}_2$. What is the difference between a negative and positive eigenvalue?

Solution.



A negative eigenvalue inverts the direction of the transformed vector. ■

Exercise 12.2

1. Determine the trace and the determinant. Solve the characteristic equation to find the eigenvalues.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned} \operatorname{tr}(\boldsymbol{A}) &= 2 \\ \det(\boldsymbol{A}) &= -3 \\ \lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0 \\ \lambda &= 3, -1 \end{aligned}$$

■

2.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Solution.

$$\begin{aligned} \operatorname{tr}(\boldsymbol{A}) &= 3 \\ \det(\boldsymbol{A}) &= -4 \\ \lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) &= 0 \\ \lambda^2 - 3\lambda - 4 &= 0 \\ (\lambda - 4)(\lambda + 1) &= 0 \\ \lambda &= 4, -1 \end{aligned}$$

■

Exercise 12.3

1. Use the solutions of the characteristic equation to prove that $\lambda_1 + \lambda_2 = \operatorname{tr}(A)$.

Solution.

$$\lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0$$

$$\begin{aligned} \lambda &= \frac{\operatorname{tr}(\boldsymbol{A}) \pm \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}}{2} \\ \lambda_1 + \lambda_2 &= \operatorname{tr}(\boldsymbol{A}) + \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})} - \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})} \\ \lambda_1 + \lambda_2 &= \operatorname{tr}(\boldsymbol{A}) \end{aligned}$$

■

2. Use the solutions of the characteristic equation to prove that $\lambda_1\lambda_2 = \det(A)$.

Solution.

$$\begin{aligned}
\lambda &= \frac{\operatorname{tr}(\boldsymbol{A}) \pm \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}}{2} \\
\lambda_1\lambda_2 &= \frac{\left(\operatorname{tr}(\boldsymbol{A}) + \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}\right) \times \left(\operatorname{tr}(\boldsymbol{A}) - \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}\right)}{4} \\
\lambda_1\lambda_2 &= \frac{\operatorname{tr}(\boldsymbol{A})^2 - (\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A}))}{4} \\
\lambda_1\lambda_2 &= \det(\boldsymbol{A})
\end{aligned}$$

■

3. Use the solutions of the characteristic equation to prove that the eigenvalues of a symmetric 2×2 matrix are real.

Solution.

$$\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} a & b \\ b & c \end{bmatrix} \\
\lambda^2 - \operatorname{tr}(\boldsymbol{A})\,\lambda + \det(\boldsymbol{A}) &= 0 \\
\frac{\operatorname{tr}(\boldsymbol{A}) \pm \sqrt{\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A})}}{2} &= \lambda \\
\operatorname{tr}(\boldsymbol{A})^2 - 4\det(\boldsymbol{A}) &\geq 0 \\
(a+c)^2 - 4(ac - b^2) &\geq 0 \\
a^2 + 2ac + c^2 - 4ac + 4b^2 &\geq 0 \\
(a-c)^2 + 4b^2 &\geq 0
\end{aligned}$$

■

Exercise 12.4

1. Show that if $\boldsymbol{v}$ is an eigenvector for λ, then $c\boldsymbol{v}$ is also an eigenvector for λ, where c is any constant.

Solution.

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{v} &= \lambda\boldsymbol{v} \\
\boldsymbol{A}(c\boldsymbol{v}) &= \lambda(c\boldsymbol{v}) \\
c\boldsymbol{A}\boldsymbol{v} &= c\lambda\boldsymbol{v}
\end{aligned}$$

■

Exercise 12.5

1. Check that $\lambda_1 = 10$ and $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are the corresponding eigenvalue and eigenvector for $\boldsymbol{A} = \begin{bmatrix} 18 & -2 \\ 12 & 7 \end{bmatrix}$.

   Solution.

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v} &= \lambda \boldsymbol{v} \\ \begin{bmatrix} 18 & -2 \\ 12 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} &= 10 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 10 \\ 40 \end{bmatrix} &= \begin{bmatrix} 10 \\ 40 \end{bmatrix} \end{aligned}$$

■

Exercise 12.6

1. Findn the eigenvector for $\boldsymbol{A} = \begin{bmatrix} 18 & -2 \\ 12 & 7 \end{bmatrix}$ that corresponds with $\lambda_2 = 15$.

   Solution.

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v}_2 &= \lambda \boldsymbol{v}_2 \\ \begin{bmatrix} 18 & -2 \\ 12 & 7 \end{bmatrix} \boldsymbol{v}_2 &= 15\boldsymbol{v}_2 \\ \begin{bmatrix} 18 - 15 & -2 \\ 12 & 7 - 15 \end{bmatrix} \boldsymbol{v}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 3 & -2 \\ 12 & -8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 3a - 2b &= 0 \\ 12a - 8b &= 0 \\ \boldsymbol{v}_2 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

■

Exercise 12.7

1. Determine the eigenvalues and eigenvectors.

$$\boldsymbol{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution.

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{A}) &= 7 \\
\det(\boldsymbol{A}) &= 10 \\
\lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) &= 0 \\
\lambda^2 - 7\lambda + 10 &= 0 \\
(\lambda - 5)(\lambda - 2) &= 0 \\
\lambda_1 &= 2 \\
\lambda_2 &= 5
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{v} &= \lambda\boldsymbol{v} \\
\boldsymbol{A}\boldsymbol{v}_1 &= \lambda_1\boldsymbol{v}_1 \\
\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\boldsymbol{v}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{v}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\boldsymbol{v}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{v}_2 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}$$

■

2.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Solution.

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{A}) &= 3 \\
\det(\boldsymbol{A}) &= -4 \\
\lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) &= 0 \\
\lambda^2 - 3\lambda - 4 &= 0 \\
(\lambda - 4)(\lambda + 1) &= 0 \\
\lambda_1 &= -1 \\
\lambda_2 &= 4
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{v} &= \lambda\boldsymbol{v} \\
\boldsymbol{A}\boldsymbol{v}_1 &= \lambda_1\boldsymbol{v}_1 \\
\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\boldsymbol{v}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{v}_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\boldsymbol{v}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{v}_2 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\end{aligned}$$

■

Exercise 12.8

1.

$$\begin{aligned}
\boldsymbol{n} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\boldsymbol{z} &= \begin{bmatrix} -1 \\ 1.01 \end{bmatrix} \\
\boldsymbol{S} &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\end{aligned}$$

On the same axes, plot the vectors $\boldsymbol{n}$ and $\boldsymbol{z}$ using MATLAB.

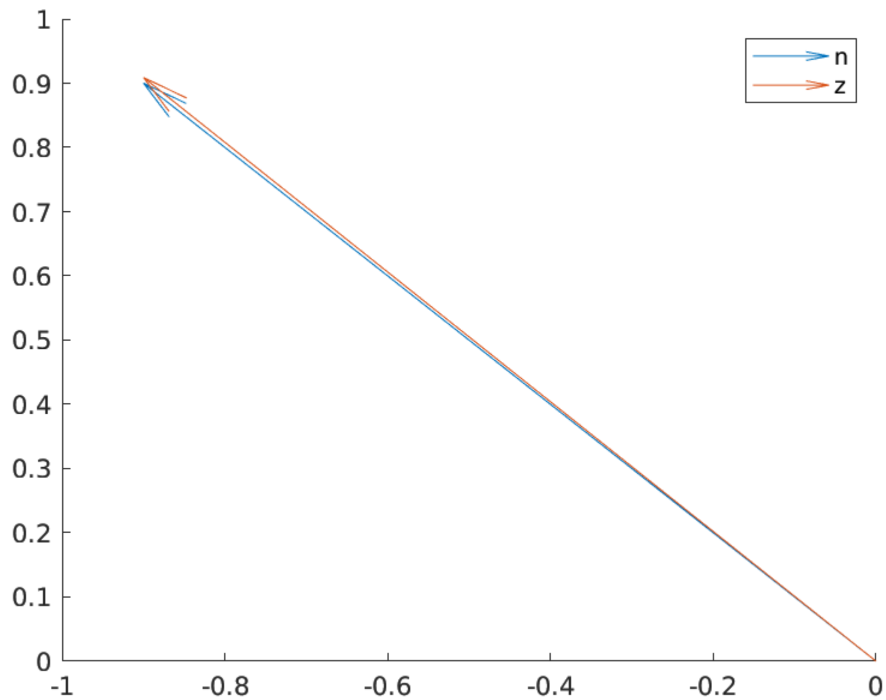
Solution.

```
n = [-1; 1];
z = [-1; 1.01];

hold on
quiver(0, 0, n(1), n(2));
```

```
quiver(0, 0, z(1), z(2));

legend('n', 'z')
```



■

2. Suppose that $\boldsymbol{n}$ and $\boldsymbol{z}$ are transformed by $\boldsymbol{S}$. On the same axes as in the previous part, plot the vectors $\boldsymbol{Sn}$ and $\boldsymbol{Sz}$ using MATLAB.

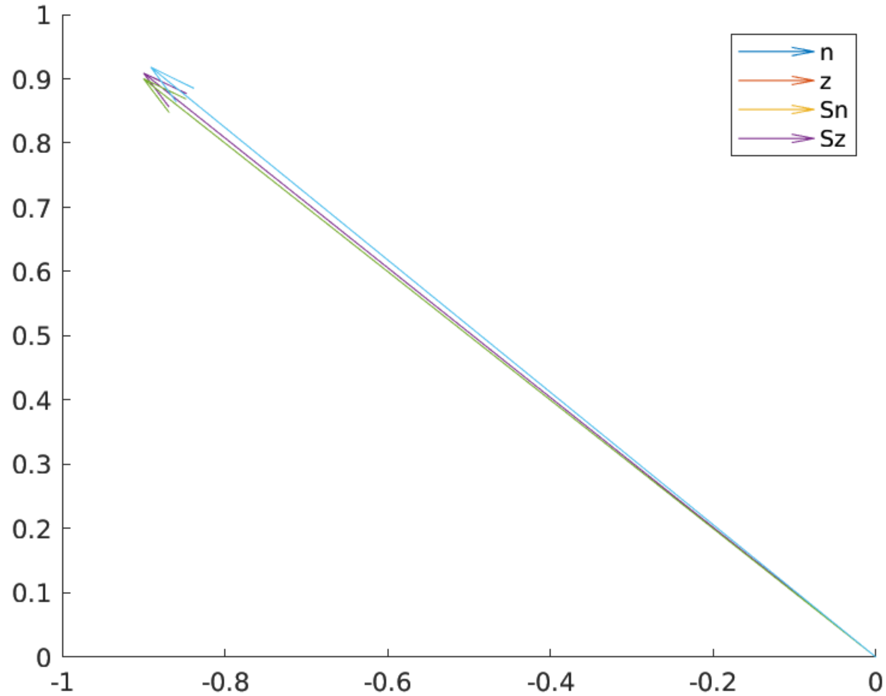
Solution.

```
n = [-1; 1];
z = [-1; 1.01];
S = [2 1; 1 2];

Sn = S*n;
Sz = S*z;

hold on
quiver(0, 0, n(1), n(2));
quiver(0, 0, z(1), z(2));
quiver(0, 0, Sn(1), Sn(2));
quiver(0, 0, Sz(1), Sz(2));

legend('n', 'z', 'Sn', 'Sz')
```



■

3. Now, we shall see what happens to these vectors under repeated transformations by $\boldsymbol{S}$. On the same axes as in the previous part, plot the vectors $\boldsymbol{SSn}$ and $\boldsymbol{SSz}$ using MATLAB.

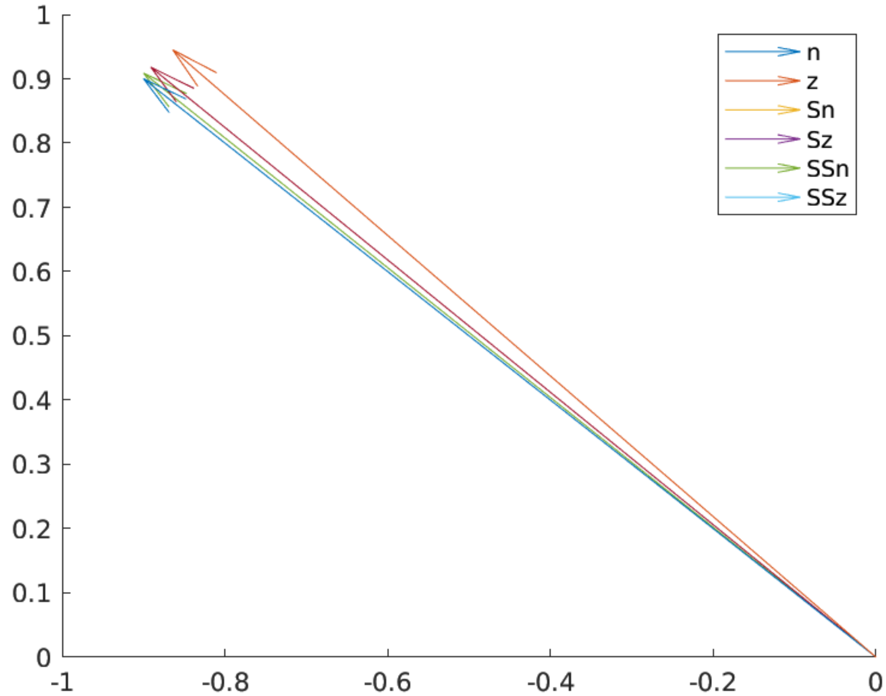
   Solution.

```
n = [-1; 1];
z = [-1; 1.01];
S = [2 1; 1 2];

Sn = S*n;
Sz = S*z;
SSn = S*S*n;
SSz = S*S*z;

hold on
quiver(0, 0, n(1), n(2));
quiver(0, 0, z(1), z(2));
quiver(0, 0, Sn(1), Sn(2));
quiver(0, 0, Sz(1), Sz(2));
quiver(0, 0, SSn(1), SSn(2));
quiver(0, 0, SSz(1), SSz(2));

legend('n', 'z', 'Sn', 'Sz', 'SSn', 'SSz')
```



■

4. On the same axes as in the previous part, plot the vectors $\boldsymbol{SSSn}$ and $\boldsymbol{SSSz}$ using MATLAB.

Solution.

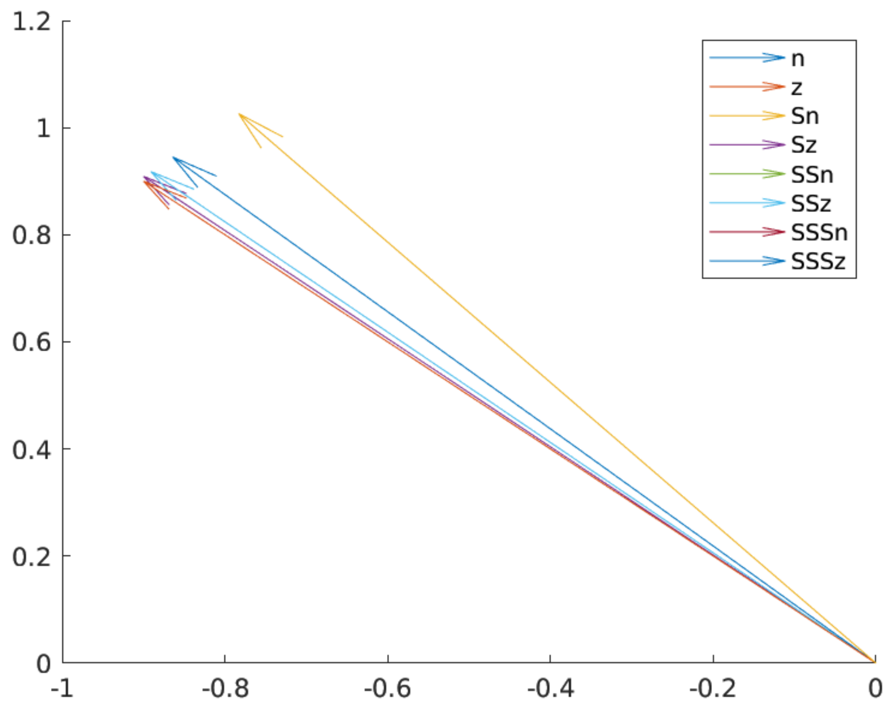
```
n = [-1; 1];
z = [-1; 1.01];
S = [2 1; 1 2];

Sn = S*n;
Sz = S*z;
SSn = S*S*n;
SSz = S*S*z;
SSSn = S*S*S*n;
SSSz = S*S*S*z;

hold on
quiver(0, 0, n(1), n(2));
quiver(0, 0, z(1), z(2));
quiver(0, 0, Sn(1), Sn(2));
quiver(0, 0, Sz(1), Sz(2));
quiver(0, 0, SSn(1), SSn(2));
quiver(0, 0, SSz(1), SSz(2));
quiver(0, 0, SSSn(1), SSSn(2));
```

```
quiver(0, 0, SSSz(1), SSSz(2));

legend('n', 'z', 'Sn', 'Sz', 'SSn', 'SSz', 'SSSn', 'SSSz')
```



■

5. On the same axes as in the previous part, plot the vectors $\boldsymbol{SSSSn}$ and $\boldsymbol{SSSSz}$ using MATLAB.

Solution.

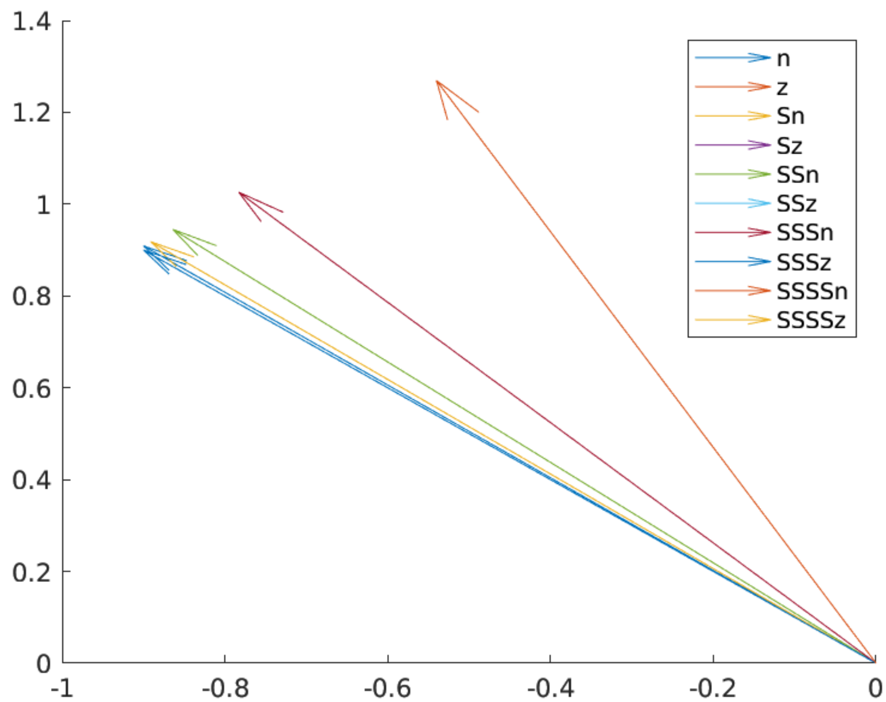
```
n = [-1; 1];
z = [-1; 1.01];
S = [2 1; 1 2];

Sn = S*n;
Sz = S*z;
SSn = S*S*n;
SSz = S*S*z;
SSSn = S*S*S*n;
SSSz = S*S*S*z;
SSSSn = S*S*S*S*n;
SSSSz = S*S*S*S*z;

hold on
quiver(0, 0, n(1), n(2));
```

```
quiver(0, 0, z(1), z(2));
quiver(0, 0, Sn(1), Sn(2));
quiver(0, 0, Sz(1), Sz(2));
quiver(0, 0, SSn(1), SSn(2));
quiver(0, 0, SSz(1), SSz(2));
quiver(0, 0, SSSn(1), SSSn(2));
quiver(0, 0, SSSz(1), SSSz(2));
quiver(0, 0, SSSSn(1), SSSSn(2));
quiver(0, 0, SSSSz(1), SSSSz(2));

legend('n', 'z', 'Sn', 'Sz', 'SSn', 'SSz', 'SSSn', 'SSSz')
hold off
```



■

6. Explain what you see in terms of eigenvalues and eigenvectors.

   Solution. The translation $\boldsymbol{Ax}$ where $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ is an eigenvector of $\boldsymbol{A}$, and therefore, will always be in the same direction, no matter how many transformations $\boldsymbol{A}$ are applied. ■

Exercise 12.9

1. Generate a 3×3 matrix with random entries using `A = randn(3, 3);`.

   Use MATLAB's `eig` function to get the eigenvalues and eigenvectors of the matrix.

Solution.

$$\boldsymbol{A} = \begin{bmatrix} 0.5377 & 0.8622 & -0.4336 \\ 1.8339 & 0.3188 & 0.3426 \\ -2.2588 & -1.3077 & 3.5784 \end{bmatrix}$$
$$\lambda = \begin{bmatrix} 3.8142 \\ -0.8472 \\ 1.4678 \end{bmatrix}$$

■

2. Using MATLAB's `trace` function, confirm that the trace equals the sum of the eigenvalues.

   Solution.

   $$\mathrm{tr}(\boldsymbol{A}) = 4.4348$$

   ■

3. Using MATLAB's `det` function, confirm that the determinant equals the product of the eigenvalues, and explain why a square matrix is invertible if and only if all its eigenvalues are nonzero.

   Solution.

   $$\det(\boldsymbol{A}) = -4.7434 = 3.8142 \times -0.8472 \times 1.4678$$

   If any of the eigenvalues are zero, the determinant would be zero, making it non-invertible. Therefore, if none of the eigenvalues are zero, the matrix is invertible. ■

4. Generate a new matrix $\boldsymbol{B} = \boldsymbol{A}^{\intercal}\boldsymbol{A}$ which must be symmetric. Find its eigenvalues and eigenvectors using `eig`, and verify that the eigenvectors are orthogonal.

   Solution.

   $$\boldsymbol{B} = \begin{bmatrix} 8.7546 & 4.0020 & -7.6878 \\ 4.0020 & 2.5550 & -4.9440 \\ -7.6878 & -4.9440 & 13.1103 \end{bmatrix}$$
   $$\boldsymbol{v} = \begin{bmatrix} 0.2562 & 0.7781 & -0.5735 \\ -0.9432 & 0.0716 & -0.3243 \\ -0.2113 & 0.6240 & 0.7523 \end{bmatrix}$$
   $$\lambda = \begin{bmatrix} 0.3605 \\ 2.9572 \\ 21.1022 \end{bmatrix}$$

   ■

Exercise 12.10

1. Subtract out the mean temperature of each city from the daily temperature data.

   Solution.

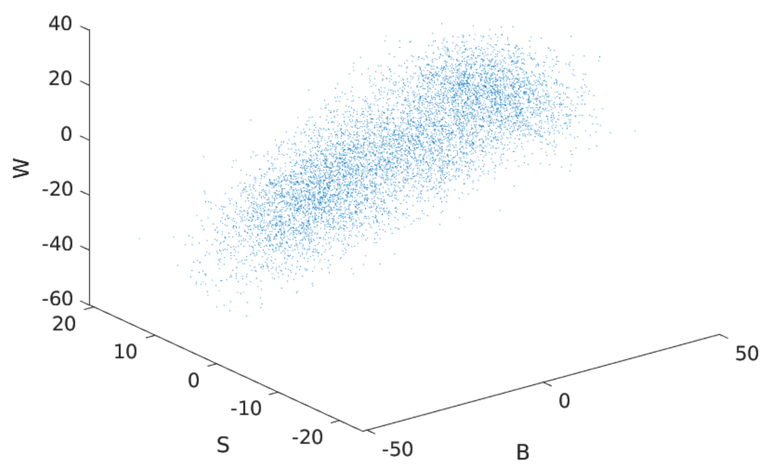
```
load('temps_bos_sp_dc.mat')
b_adj = b - mean(b);
s_adj = s - mean(s);
w_adj = w - mean(w);
```

■

2. Make a 3D scatter plot of the data points with the means subtracted out. You will find MATLAB's `plot3` function useful. You may wish to use the `MarkerSize` argument for `plot3` with a marker size of 0.1 or less to make the plots clearer.

   Solution.

```
plot3(b_adj, s_adj, w_adj, '.', 'MarkerSize', 0.1);
xlabel('B')
ylabel('S')
zlabel('W')
```



■

3. Construct a covariance matrix for the data and compute its eigenvectors.

   Solution.

```
A = 1/sqrt(length(b)-1) * [b_adj s_adj w_adj];
C = A' * A;
[v, lambda] = eigs(C)
```

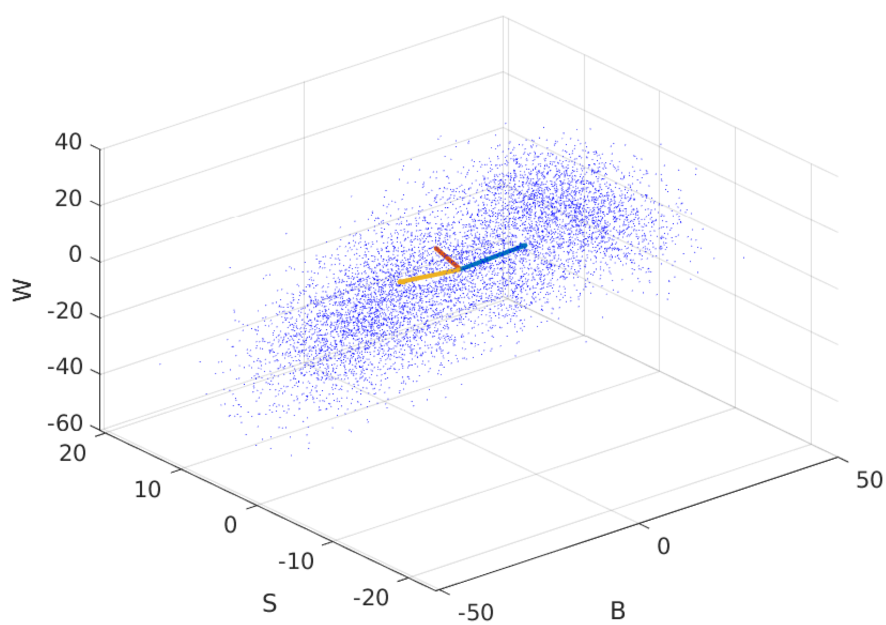
$$\boldsymbol{v} = \begin{bmatrix} 0.6994 & 0.1233 & -0.7041 \\ -0.1516 & 0.9882 & 0.0225 \\ 0.6985 & 0.0910 & 0.7098 \end{bmatrix}$$
$$\boldsymbol{\lambda} = \begin{bmatrix} 566.3086 & 0 & 0 \\ 0 & 27.0811 & 0 \\ 0 & 0 & 15.1269 \end{bmatrix}$$

■

4. On the same axes, plot the eigenvectors scaled by the square-root of their corresponding eigenvalues.

Solution.

```
clf
hold on
grid on
v_adj = v.*sqrt(diag(lambda));
quiver3(0, 0, 0, v_adj(1, 1), v_adj(2, 1), v_adj(3, 1), 'LineWidth', 2)
quiver3(0, 0, 0, v_adj(1, 2), v_adj(2, 2), v_adj(3, 2), 'LineWidth', 2)
quiver3(0, 0, 0, v_adj(1, 3), v_adj(2, 3), v_adj(3, 3), 'LineWidth', 2)
plot3(b_adj, s_adj, w_adj, 'b.', 'MarkerSize', 0.1);
xlabel('B')
ylabel('S')
zlabel('W')
```



■