Exercise 4.1

1. Calculate the determinant of the generic 2×2 rotation matrix.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Solution.

$$\det\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = \cos^2(\theta) + \sin^2(\theta) = 1$$

■

2. Calculate the determinant of the matrix that reflects over the y axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\det\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = -1$$

■

3. Calculate the determinant of the matrix that shears in the horizontal direction.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 1$$

■

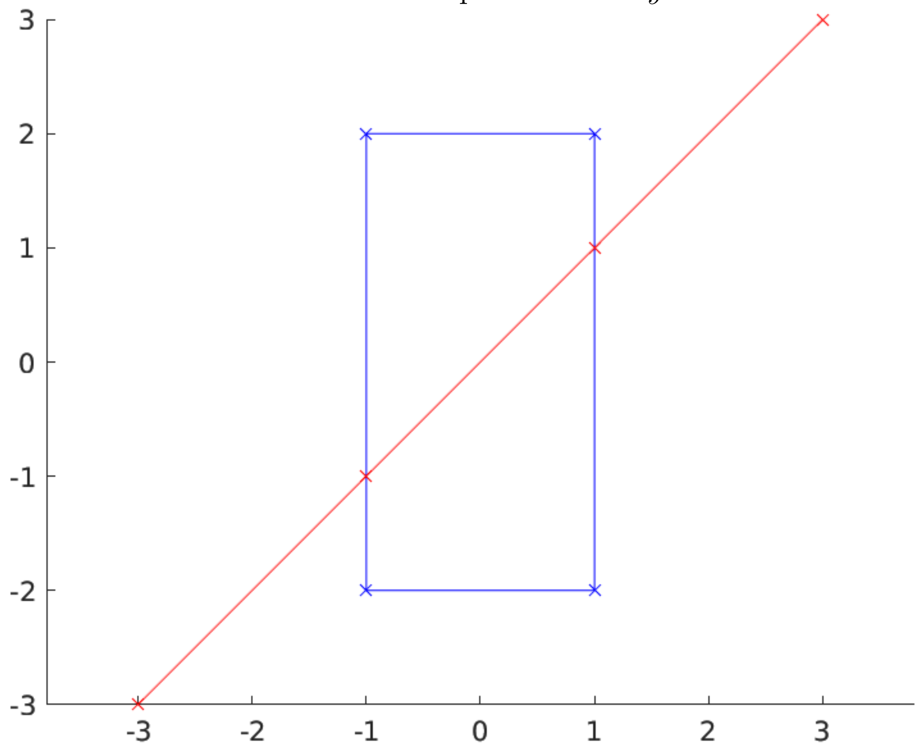
Exercise 4.2

1. What do the following matrices do? How much does the area change?

(a)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution. This matrix stretches space into the $y = x$ line.



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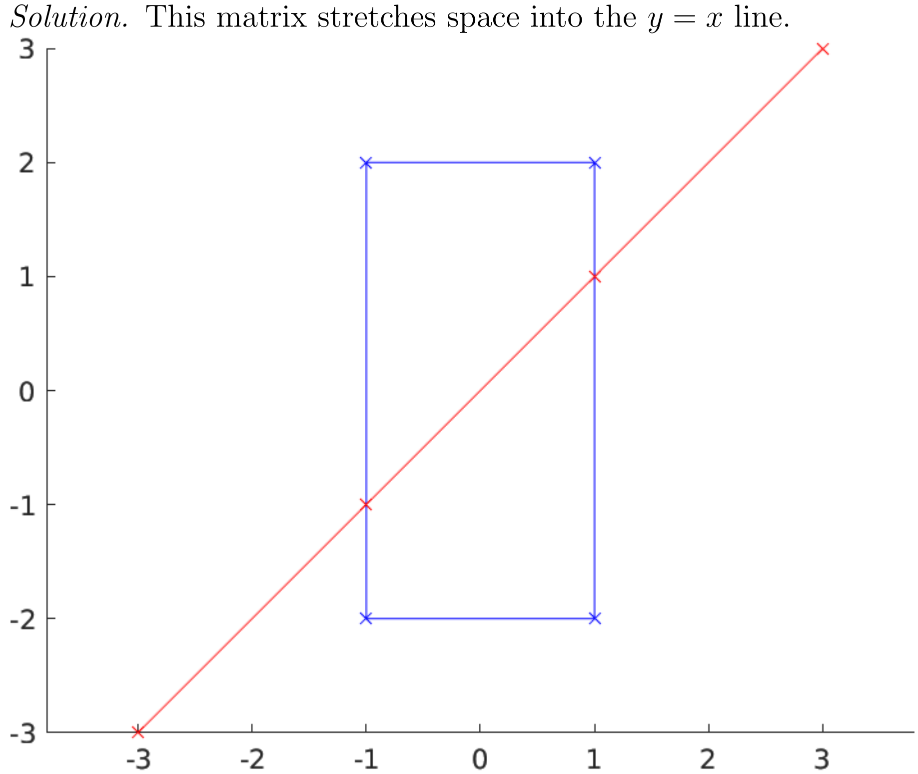
(b)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Solution. This matrix stretches space into the $y = x$ line.



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2. Is it possible to "undo" any of the matrices above? Why or why not?

   Solution. No. Because space is compressed into a straight line, the data that differentiates between the x and y components is lost. ■

Exercise 4.3

1. What are the determinants of the two matrices from the previous exercise?

   Solution. 0 ■

2. Generalizing from Exercise 4.1 and Exercise 4.2, what's the relationship between the determinant of a matrix and the result of transforming a rectangle by that matrix?

   Solution. The determinant is the scale factor of the area of a polygon from the original space to the transformed space. ■

Exercise 4.4

1.

$$\boldsymbol{P} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$
$$\boldsymbol{Q} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$
$$\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

   Find $\boldsymbol{w} = \boldsymbol{P}\boldsymbol{u}$

   Solution.

$$\boldsymbol{u} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

■

2. Find $\boldsymbol{Q}\boldsymbol{w}$. How is it related to $\boldsymbol{u}$?

   Solution.

$$\boldsymbol{Q}\boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$\boldsymbol{Q}\boldsymbol{w} = \boldsymbol{u}$$

■

3. Find $\boldsymbol{QP}$. Does this answer look familiar?

   Solution.

$$\boldsymbol{QP} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   This is the identity matrix. ■

4. Find $\boldsymbol{PQ}$.

   Solution.

$$\boldsymbol{QP} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■

5. Find the determinant of $\boldsymbol{P}$.

   Solution.

$$\det(\boldsymbol{P}) = 2$$

■

6. Find the determinant of $\boldsymbol{Q}$.

   Solution.

$$\det(\boldsymbol{Q}) = \frac{1}{2}$$

■

Exercise 4.5

1. (a) If

$$\boldsymbol{P} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$
$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

   compute $(\boldsymbol{PB})^{-}1$.

Solution.

$$\det(\boldsymbol{B}) = -1$$
$$\boldsymbol{B}^{-}1 = -1\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$
$$\boldsymbol{B}^{-}1 = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$(\boldsymbol{PB})^{-}1 = \boldsymbol{P}^{-}1\boldsymbol{B}^{-}1$$
$$(\boldsymbol{PB})^{-}1 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} (\boldsymbol{PB})^{-}1 = \begin{bmatrix} -\frac{17}{2} & \frac{7}{2} \\ 5 & -2 \end{bmatrix}$$

■

(b) For $\boldsymbol{P}$ defined above, find

$$(\boldsymbol{P}^{\intercal})^{-}1$$

Solution.

$$(\boldsymbol{P}^{\intercal})^{-}1 = \left(\boldsymbol{P}^{-}1\right)^{\intercal} = \begin{bmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

■

2. Use the inverse formula to calculate the inverses for the first three matrices in Exercise 4.1.

(a)

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Solution.

$$\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right)^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(theta) \end{bmatrix}$$

■

(b)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

■

(c)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

■

3. Suppose you know that you have apples and oranges in the fridge and the weights of all apples are 3 oz and all oranges are 4 oz. The price of each apple is \$1 and the price of each orange is \$2. Suppose you know you paid \$13 total for your fruit and the total weight of the fruit is 33 oz.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 13 \\ 33 \end{bmatrix}$$

(a) Write an equation relating $\boldsymbol{n}$ and $\boldsymbol{d}$, using a matrix-value product.

Solution.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$
$$\boldsymbol{An} = \boldsymbol{d}$$

■

(b) Calculate how many apples and oranges you have.

Solution.

$$\boldsymbol{n} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

■

(c) Why do you think this problem is often called an inverse problem?

Solution. The inverse has to be used to solve for the $\boldsymbol{n}$ vector. ■

Exercise 4.6

1. In addition to the apples and oranges, imagine there are pears that weigh 3 oz and cost \$3. Additionally, suppose the total weight of the fruit is 45 oz and you paid a total of \$21 per fruit.

   (a) If possible, find the number of oranges, apples, and pears. If not, explain why.

   Solution. This is not possible. From a matrix standpoint, the inverse of a 3×2 matrix can not be found. From an algebraic standpoint, there are 3 unknowns and only 2 equations. ■

   (b) Suppose that you additionally know that you have a total of 14 fruits. Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?

   Solution.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \\ n_p \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}$$
$$\boldsymbol{A}\boldsymbol{n} = \boldsymbol{d}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

■

   (c) What is the determinant of the matrix you have set up to solve this?

   Solution.

$$\det(\boldsymbol{A}) = 2$$

■

2. The fruit vendors bought the pricing algorithm from Uber. Oranges are still \$2, pears are now only \$1.50, and (due to an influx of teachers) apples are now surging at \$1.50 each. Their weights stay the same. You return to the market, and again purchase 14 fruits, which have the same total weight and total cost.

(a) Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?

Solution.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \\ n_p \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}$$
$$\boldsymbol{A}\boldsymbol{n} = \boldsymbol{d}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

■

(b) What is the determinant of the matrix you have set up for this?

Solution.

$$\det(\boldsymbol{A}) = 0$$

■

(c) What does this mean?

Solution. The determinant is 0. This means this cannot be solved because there are an infinite amount of solutions to this. This is because there is no perceivable difference between apples and pears with the information known. ■

Exercise 4.7

1.

$$\boldsymbol{S} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Work with a rectangle defined with the points $(1, 2), (1, -2), (-1, -2), (1, -2)$.

Predict what would happen if you operate on the rectangle with $\boldsymbol{S}$.

Solution. The rectangle will be stretched in the x direction by a factor of 2 and shrunk in the y direction by a factor of $\frac{1}{3}$. ■

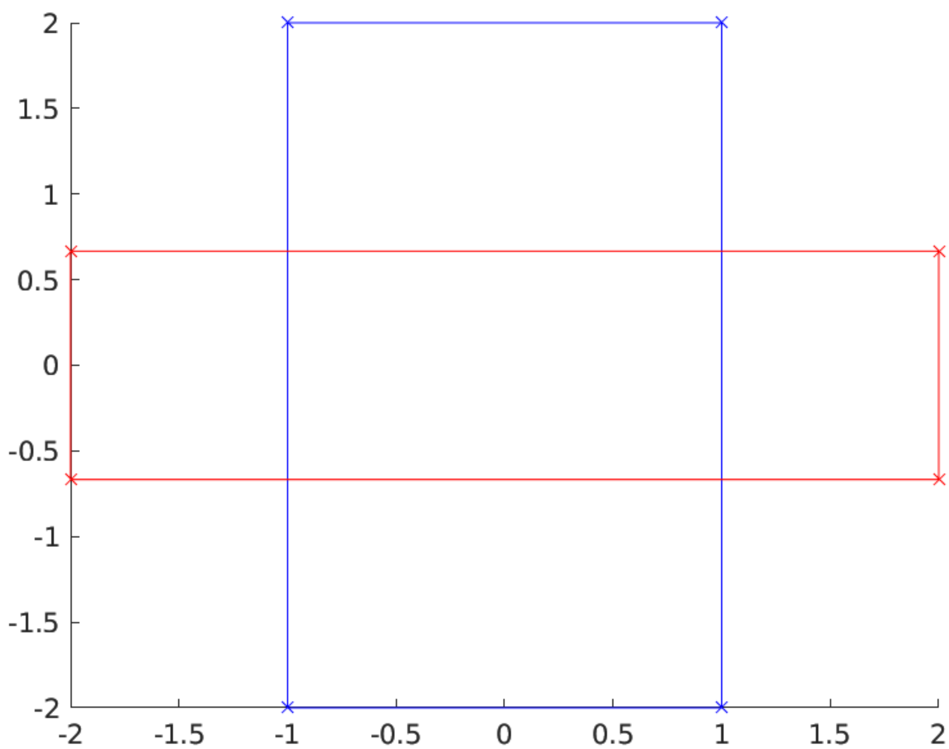
2. Write a MATLAB script to carry out this operation and check your prediction.

Solution.

```
hold on
axis equal

rect = [1 -1 -1 1 1; 2 2 -2 -2 2];
stretch = [2 0; 0 1/3];
rect_transform = stretch * rect;

% plot rectangle
plot(rect(1, :), rect(2, :), 'bx-')
plot(rect_transform(1, :), rect_transform(2, :), 'rx-')
```



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3. How does the area of the rectangle change?

Solution.

$$\det(\boldsymbol{S}) = \frac{2}{3}$$

■

4. What matrix should you use to undo this scaling? Show that the product of this matrix with the original scaling matrix is the identity matrix.

Solution.

$$\boldsymbol{S}^{-1} = \frac{3}{2}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}$$
$$\boldsymbol{S} \times \boldsymbol{S}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■

5. Define it in MATLAB and check.

Solution.

```
S = [2 0; 0 1/3];
Sn = [1/2 0; 0 3];
S * Sn
% ans = 2x2
% 1 0
% 0 1
```

■

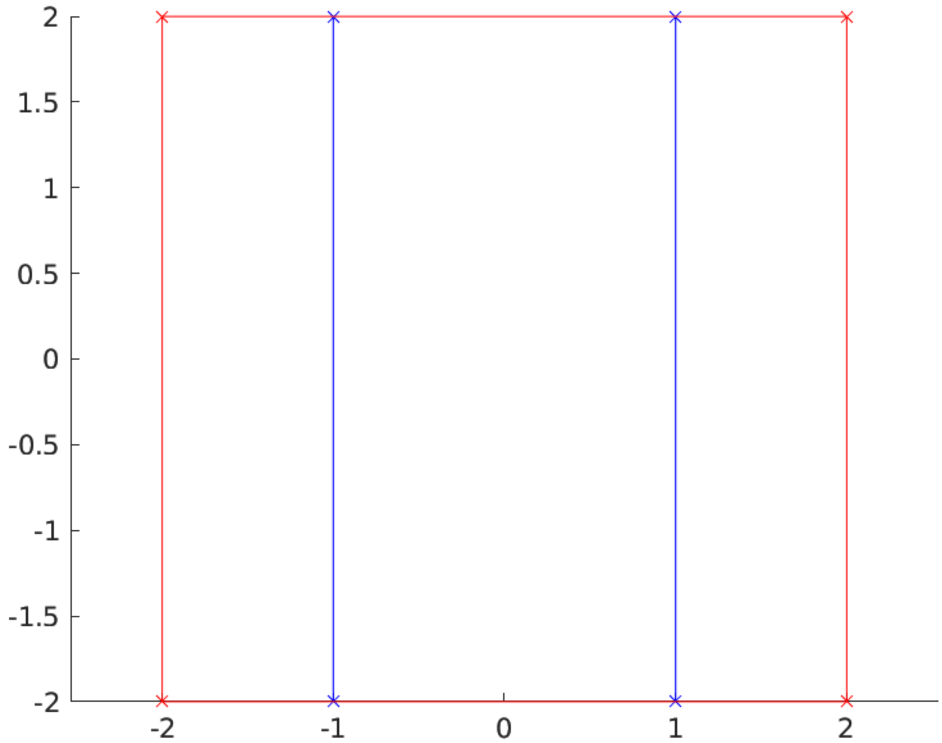
6. In MATLAB, change the value of s_2 to 1 and find the product of the new S and your rectangle. How does the area of the rectangle change?

Solution.

```
hold on
axis equal

rect = [1 -1 -1 1 1; 2 2 -2 -2 2];
stretch = [2 0; 0 1];
rect_transform = stretch * rect;

% plot rectangle
plot(rect(1, :), rect(2, :), 'bx-')
plot(rect_transform(1, :), rect_transform(2, :), 'rx-')
```



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7. Predict what would happen if you operate on the original rectangle with $\boldsymbol{SR}$, where $\boldsymbol{R}$ is the rotation matrix. How about $\boldsymbol{RS}$?

   Solution. Multiplying by $\boldsymbol{SR}$ would rotate the rectangle, then stretch it unevenly in the x and y directions. This results in a parallelogram. Multiplying by $\boldsymbol{RS}$ stretches the rectangle, then rotates it. This results in a rotated rectangle. ■

8. How would you *undo* each of these operations ($\boldsymbol{SR}$ and $\boldsymbol{RS}$)? How is the inverse of the product related to the individual inverses, i.e. what is the relationship between $(\boldsymbol{SR})^{-1}$ and $\boldsymbol{S}^{-1}$ and $\boldsymbol{R}^{-1}$? What about $(\boldsymbol{RS})^{-1}$?

   Solution.

$$(\boldsymbol{SR})^{-1} = \boldsymbol{R}^{-1}\boldsymbol{S}^{-1}$$
$$(\boldsymbol{RS})^{-1} = \boldsymbol{S}^{-1}\boldsymbol{R}^{-1}$$

■

Exercise 4.8

1. Show that $\boldsymbol{Tv}$ accomplishes the process of translation. What is the final vector?

Solution.

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}\boldsymbol{v} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

■

2. Predict what would happen if you operate on the rectangle with the translation matrix defined by $t_x = 2$ and $t_y = 3$.

   Solution. The rectangle will be translated 2 units right and 3 units up. ■

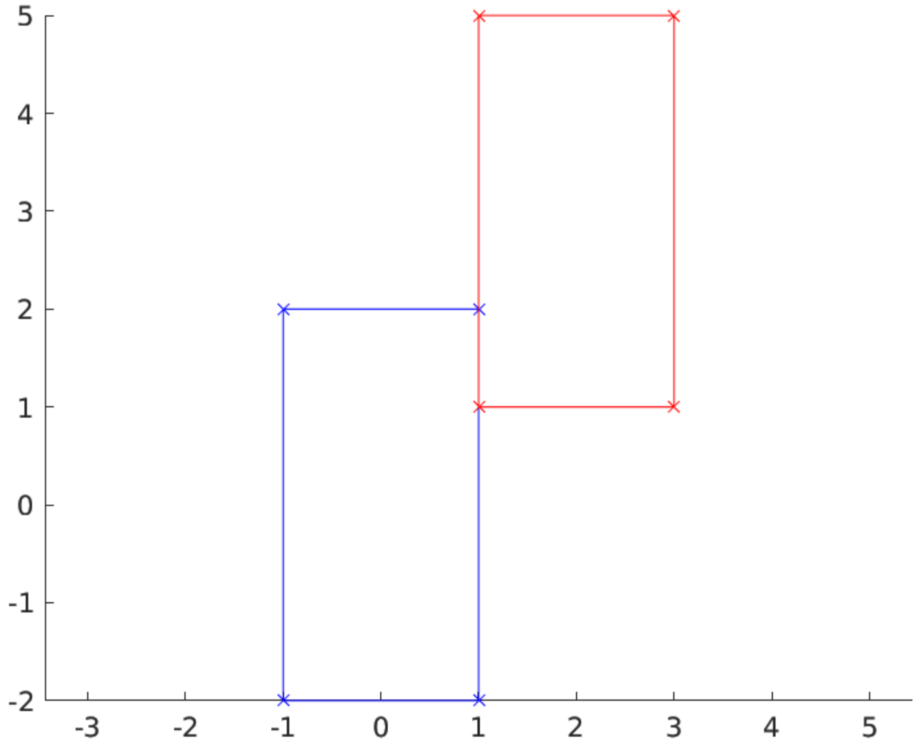
3. Write a MATLAB script to carry out this operation and check your prediction. How has the area of your rectangle changed?

   Solution.

```
4.7
hold off
clf()
hold on
axis equal
clear

rect = [1 -1 -1 1 1; 2 2 -2 -2 2; 1 1 1 1 1];
translate = [1 0 2; 0 1 3; 0 0 1];
rect_transform = translate * rect;

% plot rectangle
plot(rect(1, :), rect(2, :), 'bx-')
plot(rect_transform(1, :), rect_transform(2, :), 'rx-')
```



The area of the rectangle did not change. ■

4. What matrix should you use to *undo* this translation? Show on paper that the product of this matrix with the original translation matrix is the *identity* matrix.

   Solution.

$$\boldsymbol{T}^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

5. Choose a rotation matrix $\boldsymbol{R}$. Predict what would happen if you operate on the original rectangle with $\boldsymbol{TR}$. How about $\boldsymbol{RT}$? How would you undo each of these operations?

   Solution. Applying the $\boldsymbol{TR}$ matrix rotates the rectangle, then translates it. Applying the $\boldsymbol{RT}$ matrix translates the rectangle first, then rotates around the origin. This results in the translation also being rotated about the origin. To undo $\boldsymbol{TR}$, apply $\boldsymbol{R}^{-1}\boldsymbol{T}^{-1}$. To undo $\boldsymbol{RT}$, apply $\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}$ ■

6. Predict what would happen if you operate on the original rectangle with $\boldsymbol{STR}$. How about $\boldsymbol{TRS}$? How would you undo each of these operations?

   Solution.
   The $\boldsymbol{STR}$ matrix rotates the rectangle by θ, translates it by (t_x, t_y), and scales it by $\boldsymbol{S}$. The $\boldsymbol{TRS}$ matrix scales the rectangle by $\boldsymbol{S}$, rotates it by θ, and translates it by (t_x, t_y).

   To undo $\boldsymbol{STR}$, the resulted figure should be multiplied by $\boldsymbol{R}^{-1}\boldsymbol{T}^{-1}\boldsymbol{S}^{-1}$. To undo $\boldsymbol{TRS}$, the resulted figure should be multiplied by $\boldsymbol{S}^{-1}\boldsymbol{R}^{-1}\boldsymbol{T}^{-1}$. ■

7. How would you generalize translation to 3D?

   Solution. The 3D vector matrix should have an extra row, making it:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

   The 3D transformation matrix is:

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■

Exercise 4.9

1.