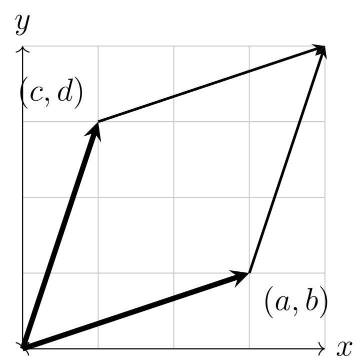
Exercise 6.1

1. Let $\boldsymbol{A}$ be a 2×2 matrix

$$\boldsymbol{A} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

Show that the magnitude of $\det(\boldsymbol{A})$ is equal to the area of the parallelogram formed by the column vectors of the matrix $\boldsymbol{A}$.

Solution.



$$area = (a + c)(b + d) - \left(2bc + 2\left(\frac{1}{2}ab\right) + 2\left(\frac{1}{2}cd\right)\right)$$
$$area = ab + ad + cb + cd - 2bc - ab - cd$$
$$area = ad - bc$$

∎

2. What is the determinant of $\boldsymbol{A}$ if its column vectors are on the same line? Graphically, what happens to the parallelogram?

Solution. If the column vectors are on the same line, $\det(\boldsymbol{A}) = 0$. Graphically, the parallelogram becomes a straight line. ∎

Exercise 6.2

1. Consider the following matrix whose columns lie on the same line: the second column is simply twice the first column.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

What is $\det(\boldsymbol{A})$?

Solution.

$$\det(\boldsymbol{A}) = 0 \tag{1}$$

■

2. Find all solutions to

$$\boldsymbol{Ax} = 0$$

Solution.

$$\boldsymbol{x} = \begin{bmatrix} a \\ -\frac{1}{2}a \end{bmatrix}$$

■

3. For which vectors $\boldsymbol{b}$ does $\boldsymbol{Ax} = \boldsymbol{b}$ have a solution? Why are there only certain $\boldsymbol{b}$ vectors that lead to solutions to $\boldsymbol{Ax} = \boldsymbol{b}$?

Solution. $\boldsymbol{b}$ has a solution where it is on the line $y = -0.5x$. This is because the basis vectors are linearly dependant. ■

Exercise 6.3

1. Suppose that the following table describes the stock holdings of three of the QEA instructors. Also suppose that on a given day the value of the Apple, IBM and General Mill's stock are \$100, \$50, and \$20, respectively.

	Apple	IBM	General Mills
Jeff	100	100	100
Emily	100	200	0
John	50	50	200

What is the total value of the holdings for each professor on the day in question? Can you formulate this as a matrix expression? If so, what is it? If not, why not?

Solution.

$$\boldsymbol{T} = \boldsymbol{Hv}$$

$$\boldsymbol{v} = \begin{bmatrix} 100 \\ 50 \\ 20 \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} 100 & 100 & 100 \\ 100 & 200 & 0 \\ 50 & 50 & 200 \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} \$17000 \\ \$20000 \\ \$11500 \end{bmatrix}$$

■

2. Now, suppose that you do not know how many shares of each stock are owned by the instructors. However, you know that the total value of the stocks for each instructor for three consecutive days is as given in the following table

	Jeff	Emily	John
Day 1	\$1500	\$2600	\$950
Day 2	\$1600	\$2810	\$1020
Day 3	\$1400	\$2550	\$1000

You also know that the price of each stock on each of the three days was as follows:

	Apple	IBM	General Mills
Day 1	\$100	\$50	\$20
Day 2	\$110	\$50	\$22
Day 3	\$100	\$40	\$30

How many stocks of each company does each professor own? Can you formulate this as a matrix equation? If so, what are the matrices/vectors? If not, why not?

Solution.

$$\boldsymbol{T} = \boldsymbol{V}\boldsymbol{H}$$

$$\boldsymbol{T} = \begin{bmatrix} \$1500 & \$2600 & \$950 \\ \$1600 & \$2810 & \$1020 \\ \$1400 & \$2550 & \$1000 \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} \$100 & \$50 & \$20 \\ \$110 & \$50 & \$22 \\ \$100 & \$40 & \$30 \end{bmatrix}$$

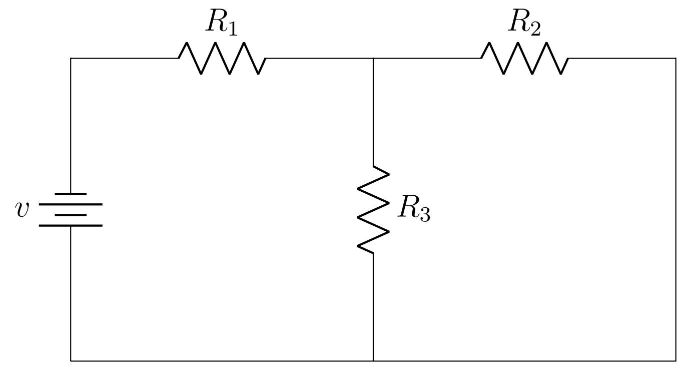
$$\boldsymbol{H} = \boldsymbol{V}^{-1}\boldsymbol{T}$$

$$\boldsymbol{H} = \begin{bmatrix} 10 & 20 & 5 \\ 10 & 10 & 5 \\ 0 & 5 & 10 \end{bmatrix}$$

■

Exercise 6.4

1. In the following circuit, consider that there is a current I_1 going through resistor R_1, a current I_2 going through resistor R_2 and a current I_3 going through resistor R_3. Find a linear algebra expression for the vector of our three unknown currents.



Solution.

$$\begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & 0 & R_3 \\ 0 & -R_2 & R_3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

■

Exercise 6.5

1. Using the technique of elimination of variables described above, determine which values of h and k result in the following system of linear algebraic equations having (a) no solution, (b) a unique solution, and (c) infinitely many solutions?

$$\begin{aligned} x_1 + hx_2 &= 1 \\ 2x_1 + 3x_2 &= k \end{aligned}$$

Solution.

(a) $h = \frac{3}{2}$, $k \neq 2$
(b) $h \neq \frac{3}{2}$
(c) $h = \frac{3}{2}$, $k = 2$

■

2. (a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_2 + x_3 &= 2 \\ x_1 - 2x_3 &= 4 \end{aligned}$$

Solution.

$$\begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

■

(b)

$$\begin{aligned} x_1 + x_2 + x_3 &= -6 \\ 2x_1 + x_2 - x_3 &= 10 \\ x_1 - 2x_3 &= 4 \end{aligned}$$

Solution.

$$\begin{bmatrix} -6 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

No solutions to $\boldsymbol{x}$ ■

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 - x_3 &= 10 \\ x_1 + 2x_3 &= 4 \end{aligned}$$

Solution.

$$\begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Inifnite solutions to $\boldsymbol{x}$ ■

Exercise 6.6

1. Suppose that you know that you have apples and oranges in the fridge and that in the genetically engineered future, the weights of all apples are 3 oz and all oranges are 4 oz. Because of inflation in this genetically engineered future, the price of each apple is \$1 and the price of each orange is \$2. Suppose that you also know that you paid

\$13 total for your fruit and the total weight of the fruit is 33 oz. Let n_o and n_a be the numbers of oranges and apples in your fridge respectively, and that you don't know what these numbers are.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 13 \\ 33 \end{bmatrix}$$

Write an equation relating $\boldsymbol{n}$ and $\boldsymbol{d}$, using a matrix-vector product.

Solution.

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \boldsymbol{n} = \boldsymbol{d}$$

■

2. Calculate how many oranges and apples you have.

   Solution.

$$\boldsymbol{n} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}^{-1} \boldsymbol{d}$$
$$\boldsymbol{n} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

   There are 3 oranges and 7 apples. ■

3. Why this kind of problem is often called an inverse problem?

   Solution. This problem is an inverse problem because the inverse of the matrix needs to be taken to solve for $\boldsymbol{n}$. ■

Exercise 6.7

1. In addition to the apples and oranges, imagine there are pears that weigh 3 oz and cost \$3. Additionally, suppose the total weight of the fruit is 45 oz and you paid a total of \$21 per fruit.

   (a) If possible, find the number of oranges, apples, and pears. If not, explain why.

   Solution. This is not possible. From a matrix standpoint, the inverse of a 3×2 matrix can not be found. From an algebraic standpoint, there are 3 unknowns and only 2 equations. ■

(b) Suppose that you additionally know that you have a total of 14 fruits. Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?

Solution.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \\ n_p \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}$$
$$\boldsymbol{A}\boldsymbol{n} = \boldsymbol{d}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

■

(c) What is the determinant of the matrix you have set up to solve this?

Solution.

$$\det(\boldsymbol{A}) = 2$$

■

2. The fruit vendors bought the pricing algorithm from Uber. Oranges are still \$2, pears are now only \$1.50, and (due to an influx of teachers) apples are now surging at \$1.50 each. Their weights stay the same. You return to the market, and again purchase 14 fruits, which have the same total weight and total cost.

   (a) Can you formulate and solve a matrix-vector equation to find out the numbers of oranges, apples and pears you have?

Solution.

$$\boldsymbol{n} = \begin{bmatrix} n_o \\ n_a \\ n_p \end{bmatrix}$$
$$\boldsymbol{d} = \begin{bmatrix} 21 \\ 45 \\ 14 \end{bmatrix}$$
$$\boldsymbol{A}\boldsymbol{n} = \boldsymbol{d}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{3}{2} \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

■

(b) What is the determinant of the matrix you have set up for this?

Solution.

$$\det(\boldsymbol{A}) = 0$$

■

3. Recall the example with fruits from class: Suppose that you have a total number of 14 apples, oranges and pears in your fridge. Suppose that each apple costs \$1, each orange costs \$2 and each pear costs \$3. Assume also that the weights of every apple is 3 oz, every orange is 4 oz and every pear is 3 oz. Additionally, suppose that the total weight of the fruits is 45 oz, and you paid a total of \$21 for the fruit.

   (a) Formulate a matrix-vector equation to find out the numbers of oranges, apples and pears you have.

   Solution.

$$\begin{bmatrix} 14 \\ 45 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & 3 \\ 2 & 1 & 3 \end{bmatrix} * \begin{bmatrix} n_o \\ n_a \\ n_p \end{bmatrix}$$

■

   (b) Solve this equation to find the numbers of apples, oranges and pears using the following approaches:

      i. Using MATLAB, compute the inverse of the matrix in part a and use it to find the numbers of apples, oranges and pears.

      Solution.

```
A = [
    1 1 1;
    4 3 3;
    2 1 3
];

d = [14; 45; 21];

n = inv(A) * d
```

$$\boldsymbol{n} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}$$

■

ii. Use MATLAB's `linsolve` function to find the numbers of apples, oranges and pears.

Solution.

```
A = [
    1 1 1;
    4 3 3;
    2 1 3
];

d = [14; 45; 21];

n = linsolve(A, d)
```

$$\boldsymbol{n} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}$$

■

iii. Use MATLAB's \ operator to find the numbers of apples, oranges and pears.

Solution.

```
A = [
    1 1 1;
    4 3 3;
    2 1 3
```

```
];

d = [14; 45; 21];

n = A \ d
```

$$\boldsymbol{n} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}$$

■