Exercise 44.1

1. Which of the following quantities could be considered a scalar field or vector field (or neither).

   The temperature at all points in your room right now.

   Solution. Scalar field ■

2. The temperature gradient at all points in your room right now.

   Solution. Vector field ■

3. The velocity of a bird over the course of a day

   Solution. Neither ■

4. The population density at all points in the USA right now.

   Solution. Scalar field ■

5. The wind velocity at all points in the USA right now.

   Solution. Vector field ■

Exercise 44.2

1. Generate a list of at least 3 quantities and classify them as a scalar field or a vector field.

   Solution.

   - The salinity at every point in the ocean (scalar field)
   - The gravitational pull around Earth (vector field)
   - Velocity of water particles in a river (vector field)

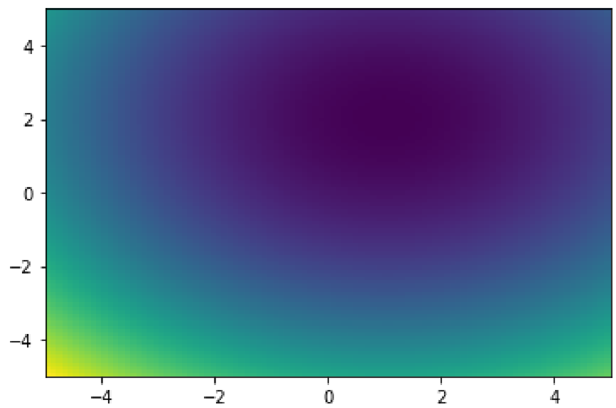
   ■

Exercise 44.3

1. Visualize the following vector or scalar fields

$$f(x, y) = (x - 1)^2 + (y - 2)^2$$

   Solution.

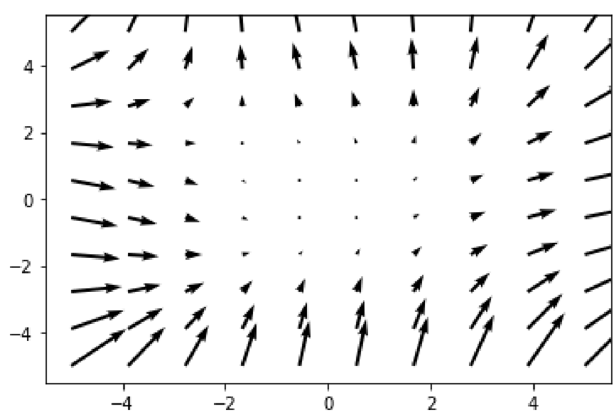


■

2.

$$\boldsymbol{F}(x, y) = (x^2 - y)\hat{\boldsymbol{\imath}} + (x + y^2)\hat{\boldsymbol{\jmath}}$$

Solution.

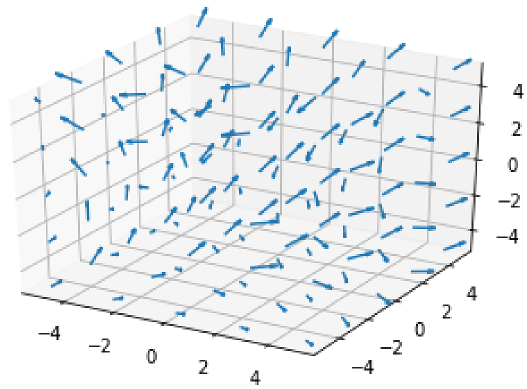


■

3.

$$\boldsymbol{F}(x, y, z) = (2y - z)\hat{\boldsymbol{\imath}} + (x + y^2 - z)\hat{\boldsymbol{\jmath}} + (4y - 3x)\hat{\boldsymbol{k}}$$

Solution.



∎

Exercise 44.4

1. Find the Jacobian for the following vector fields

$$\boldsymbol{F}(x,y,z) = (3x+4y)\hat{\boldsymbol{\imath}} + (4y-5z)\hat{\boldsymbol{\jmath}} + (x+y-z)\hat{\boldsymbol{k}}$$

Solution.

$$\begin{aligned} J\boldsymbol{F} &= \begin{bmatrix} \frac{\partial X}{x} & \frac{\partial X}{y} & \frac{\partial X}{z} \\ \frac{\partial Y}{x} & \frac{\partial Y}{y} & \frac{\partial Y}{z} \\ \frac{\partial Z}{x} & \frac{\partial Z}{y} & \frac{\partial Z}{z} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 & 0 \\ 0 & 4 & -5 \\ 1 & 1 & -1 \end{bmatrix} \end{aligned}$$

∎

2. Find the Jacobian for the following vector fields

$$\boldsymbol{F}(x,y,z) = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}$$

Solution.

$$\begin{aligned} J\boldsymbol{F} &= \begin{bmatrix} \frac{\partial X}{x} & \frac{\partial X}{y} & \frac{\partial X}{z} \\ \frac{\partial Y}{x} & \frac{\partial Y}{y} & \frac{\partial Y}{z} \\ \frac{\partial Z}{x} & \frac{\partial Z}{y} & \frac{\partial Z}{z} \end{bmatrix} \\ &= \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} \end{aligned}$$

∎

Exercise 44.5

1. Find the divergence for the following vector fields

$$\boldsymbol{F}(x, y) = \cos x \sin y \hat{\boldsymbol{\imath}} - \sin x \cos y \hat{\boldsymbol{\jmath}}$$

Solution.

$$\begin{aligned} \operatorname{div}\boldsymbol{F} &= \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \\ &= -\sin x \sin y + \sin x \sin y \\ &= 0 \end{aligned}$$

■

2.

$$\boldsymbol{F}(x, y, z) = y \sin z \hat{\boldsymbol{\imath}} - x \sin z \hat{\boldsymbol{\jmath}} + \cos z \hat{\boldsymbol{k}}$$

Solution.

$$\begin{aligned} \operatorname{div}\boldsymbol{F} &= \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \\ &= -\sin z \end{aligned}$$

■

Exercise 44.6

1. Show that the divergence of a vector field $\boldsymbol{F}$ is the dot product of the gradient operator with the vector field.

Solution.

$$\begin{aligned} \nabla \cdot \boldsymbol{F} &= \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} \cdot \begin{bmatrix} X & Y & Z \end{bmatrix} \\ &= \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \end{aligned}$$

■

Exercise 44.7

1. Find the curl for the following vector fields

$$\boldsymbol{F}(x, y) = \cos x \sin y \hat{\boldsymbol{\imath}} - \sin x \cos y \hat{\boldsymbol{\jmath}}$$

Solution.

$$\begin{aligned}\text{curl}\boldsymbol{F} &= \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\hat{\boldsymbol{k}} \\ &= (-\sin x \sin y - \sin x \sin y)\,\hat{\boldsymbol{k}} \\ &= -2\sin x \sin y \hat{\boldsymbol{k}}\end{aligned}$$

■

2. Find the curl for the following vector fields

$$\boldsymbol{F}(x,y,z) = y\sin z\hat{\boldsymbol{\imath}} - x\sin z\hat{\boldsymbol{\jmath}} + \cos z\hat{\boldsymbol{k}}$$

Solution.

$$\begin{aligned}\text{curl}\boldsymbol{F} &= \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\hat{\boldsymbol{\imath}} - \left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\hat{\boldsymbol{k}} \\ &= x\cos z\hat{\boldsymbol{\imath}} + y\cos z\hat{\boldsymbol{\jmath}} - 2\sin z\hat{\boldsymbol{k}}\end{aligned}$$

■

Exercise 44.8

1. Show that the curl of a vector field $\boldsymbol{F}$ is the dot product of the gradient operator with the vector field.

Solution.

$$\begin{aligned}\nabla\cdot\boldsymbol{F} &= \begin{bmatrix}\partial_x & \partial_y & \partial_z\end{bmatrix} \times \begin{bmatrix}X & Y & Z\end{bmatrix} \\ &= \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\hat{\boldsymbol{\imath}} - \left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\hat{\boldsymbol{k}}\end{aligned}$$

■

Exercise 44.9

1.

$$V(x,y) = \ln\sqrt{x^2+y^2}$$

Define V in polar coordinates.

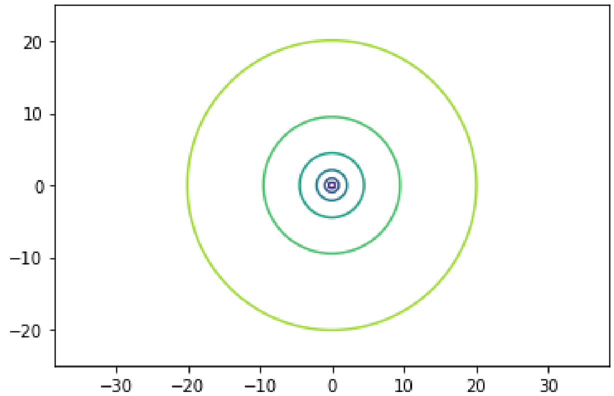
Solution.

$$V(r, \theta) = \ln r$$

■

2. Create a contour plot of V

Solution.

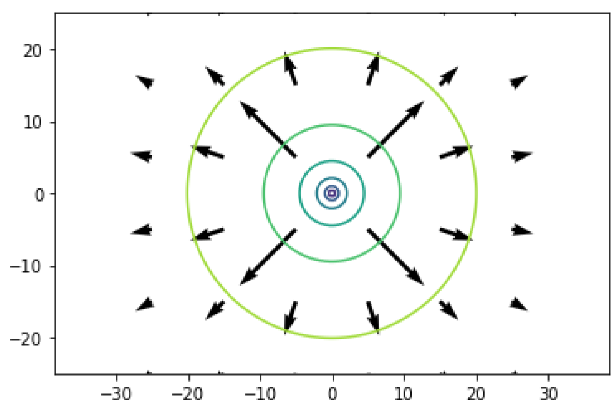


■

3. Determine the gradient field defined by ∇V , and visualize it.

Solution.

$$\nabla V = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{bmatrix}$$



■

4. How would a NEATO performing gradient descent behave if you put it on a Flatland defined by V and started it at $(3, 3)$?

   Solution. The NEATO would accelerate towards from $(0, 0)$ and stop. ■

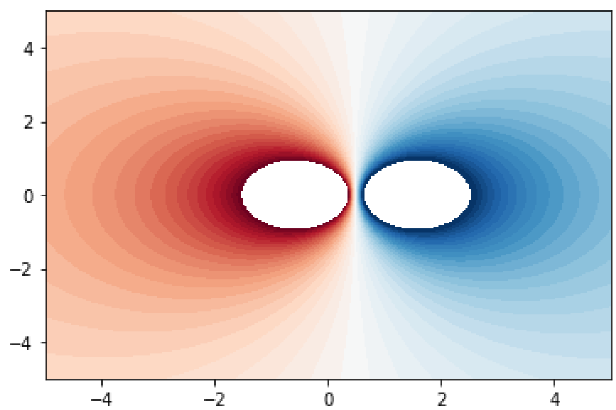
Exercise 44.10

1.

$$V(x, y) = \ln\sqrt{x^2 + y^2} - \ln\sqrt{(x-1)^2 + y^2}$$

   Create a contour plot of V.

   Solution.

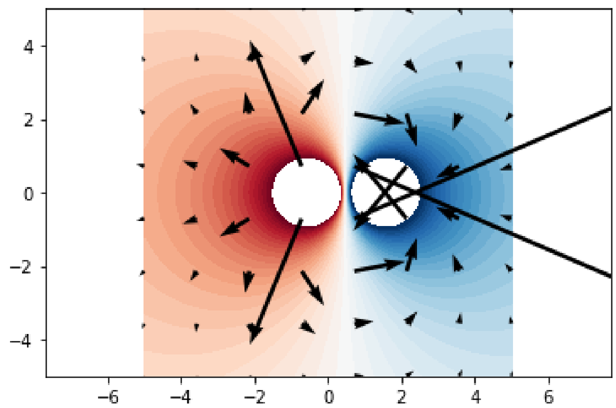


■

2. Determine the gradient field defined by ∇V , and visualize it.

   Solution.

$$\begin{aligned}\nabla V &= \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2x}{x^2+y^2} - \frac{(2x-2}{(x-1)^2+y^2} \\ \frac{2y}{x^2+y^2} - \frac{(2y}{(x-1)^2+y^2} \end{bmatrix}\end{aligned}$$



∎

3. How would a NEATO performing gradient descent behave if you put it on a Flatland defined by V and started it at $(3,3)$?

   Solution. The NEATO would drive away from $(1,0)$ and eventually find its way towards $(0,0)$. ∎

Exercise 44.11

1. Define a scalar field $V(x,y)$ that has a sink at $(0,0)$, and sources at $(1,0)$ and $(2,3)$.

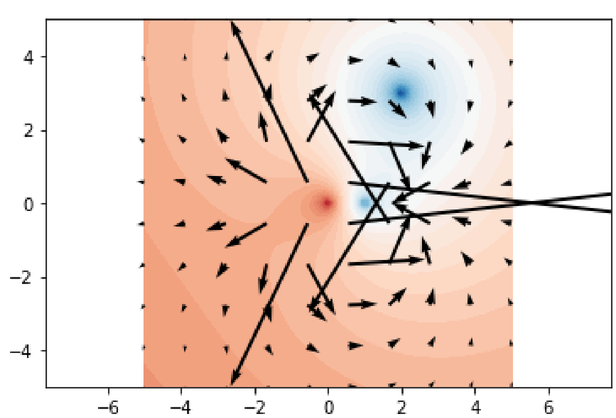
   Solution.

$$V(x,y) = \ln\sqrt{x^2+y^2} - \ln\sqrt{(x-1)^2+y^2} - \ln\sqrt{(x-2)^2+(y-3)^2}$$

∎

2. Visualize the scalar field and the gradient field.

   Solution.



∎

3. How would a NEATO performing gradient descent behave if you put it on a Flatland defined by V and started it at $(1, 2)$?

   Solution. The NEATO would drive away from $(1, 2)$ and eventually find its way towards $(0, 0)$. ■