Numerical optimization technique for constrained optimization:

Penalty function combined with Newton's method

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1. A constrained optimization problem

The general form of a non-linear constrained optimization is given as:

$$\begin{aligned}&(NLP)\quad \text{Min } f(X)\\ &\qquad\qquad s.t.\ \ g_i(X)\le 0;\ \ (i=1,2,\dots,m)\end{aligned}$$

$where\ f: R^n \to R,\ g_i: R^n \to R;\ i=1,2,\dots,m$ are differentiable convex functions.

A numerical optimization technique or constrained optimization problem aims at converting the NPL to an unconstrained optimization problem, which can be solved using numerical techniques for unconstrained problems. One of the methods to convert the constrained NLP into an unconstrained problem is the introduction of a penalty function.

Consider the following function:

$$p(X)=\begin{cases}0; & \mathrm{X}\in S\\ +\infty; & \mathrm{X}\notin S\end{cases}$$

$where\ S=\left\{X\in R^n \,\middle|\, g_i(X)\le 0\ (i=1,2,\dots,m)\right\}$ is the feasible set.

The smooth function is defined as:

$$P(X)=\sum_{i=1}^{m}\left[\max(g_i(X),0)\right]^2$$

Introduction of $P(X)$ in the objective function converts the NLP into equivalent unconstrained minimization problem as: $(UMP)_\mu\ \ \underset{X\in R^n}{Min}\ f(X)+\mu P(X)$

2. Algorithm for penalty function method

(1) Choose a termination tolerance $\varepsilon = 10^{-5}$.

(2) Choose a suitable penalty function: $P(X)$.

(3) Choose an increasing sequence of positive real numbers which tends to $+\infty$, i.e. a sequence $\{\mu_k\}_{k=1}^{\infty}$ such that for each k, $\mu_k > 0, \mu_{k+1} > \mu_k, \{\mu_k\} \to \infty$. In general, we take $\mu_1 = 1, \mu_2 = 10, \mu_3 = 100, \mu_4 = 1000$ and so on. $(scalar\ \Delta = 10)$

(4) Choose an arbitrary point that violate constraints: X_0, $X_0 \in R^n$. Construct the following unconstrained minimization problem: $\begin{array}{l} (UMP)_{\mu_1} \quad \underset{X \in R^n}{Min} f(X) + \mu_1 P(X) \\ e.g. \mu_1 = 1 \end{array}$ and solve it using Newton's method, starting with X_0. Let X_1 be the optimal solution $(UMP)_{\mu_1}$, set $k = 1$.

(5) Construct $(UMP)_{\mu_{k+1}}$ as $(UMP)_{\mu_{k+1}} : Min\ z(X, \mu_{k+1}) = f(X) + \mu_k P(X)$ and solve it using Newton's method, starting with X_{k+1}, where X_{k+1} is the optimal solution of $(UMP)_{\mu_{k+1}}$.

(6) Stopping criteria: Continue iterations of step (5) till $\mu_k P(X_k) < \varepsilon\ or\ z(X_k, \mu_k) - f(X_k) < \varepsilon$ for some tolerance level $\varepsilon > 0$.[1]

3. Compare results between Python and Maple

Example 1: Problem Set 3 Question 5 (using minimize problem instead)

$$\begin{aligned}
&Minimize\ f = (x_1 - 2)^2 + (x_2 - 2)^2 \\
&\quad s.t. \qquad x_1 + 2x_2 \le 3 \\
&\qquad\qquad\quad 8x_1 + 5x_2 \ge 10 \\
&\qquad\qquad\quad x_1, x_2 \ge 0 \\
&\Rightarrow z = (x_1 - 2)^2 + (x_2 - 2)^2 + \mu \max\{0, (x_1 + 2x_2 - 3)^2\} + \mu \max\{0, (-8x_1 - 5x_2 + 10)^2\} \\
&set\ X = (x_1, x_2),\ tolerance\ \varepsilon = 10^{-5}
\end{aligned}$$

Python result:

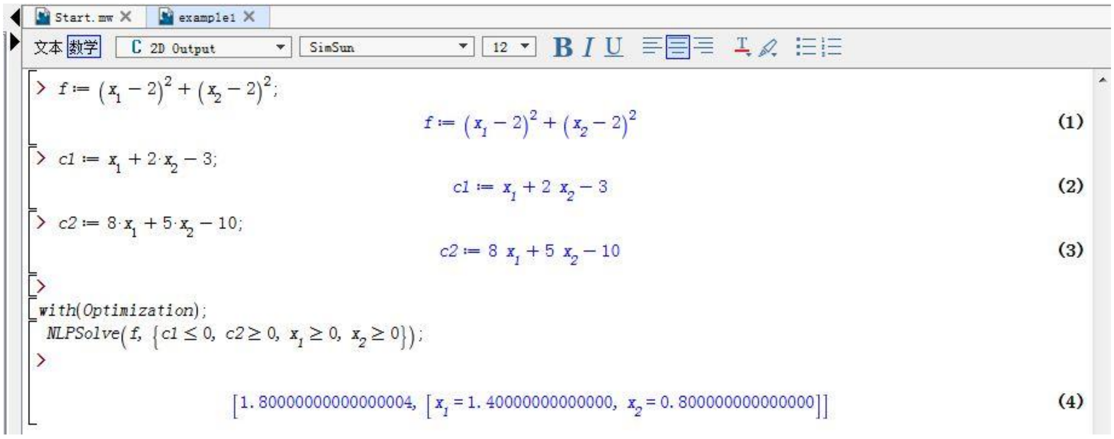
Starting point X_0 is (0, 0).

Iteration k	μ_k	X_k	$f(X_k)$	$z(X_k, \mu_k)$	$\mu_k P(X_k)$
1	1	(1.4900,1.0000)	1.2499	1.5000	0.2500
2	10	(1.4117,0.8235)	1.7301	1.7647	0.0346
3	100	(1.4012,0.8023)	1.7928	1.7964	0.0035
4	1000	(1.4001,0.8000)	1.7999	1.7999	0.00035
5	10000	(1.4000,0.8000)	1.7999	1.7999	3.5986e-06

Optimal solution is x1=1.4, x2=0.8; Optimal value of f is 1.7999.

1. Lecture 30: Newton's and Penalty Function Methods https://www.youtube.com/watch?v=z_-iUPg4Iwk

Maple result:



Example 2: Problem Set 3 Question 3

$$Minimize\ f = 4 + 3(1 - x_1)^2 + (1 - x_2)^2$$

$$s.t. \quad 3x_1 + x_2 = 5$$

$$\Rightarrow z = 4 + 3(1 - x_1)^2 + (1 - x_2)^2 + \mu(3x_1 + x_2 - 5)^2$$

$$set\ X = (x_1, x_2),\ tolerance\ \varepsilon = 10^{-5}$$

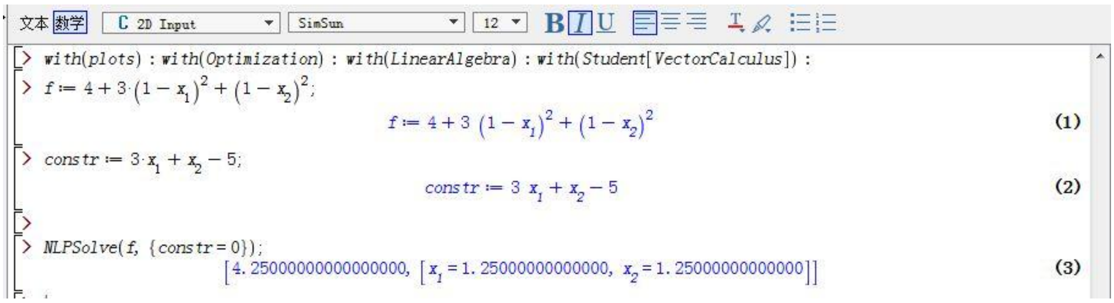
Python result:

Starting point X_0 is (0, 0).

Iteration k	μ_k	X_k	$f(X_k)$	$z(X_k, \mu_k)$	$\mu_k P(X_k)$
1	1	(1.1999,1.2000)	4.1599	4.2000	0.0400
2	10	(1.2439,1.2339)	4.2379	4.2439	00059
3	100	(1.2493,1.2493)	4.2487	4.2493	0.0006
4	1000	(1.2499,1.2499)	4.2498	4.2499	6.2479e-05
5	10000	(1.2499,1.2500)	4.2499	4.2499	6.2598e-06

Optimal solution is x1=1.2499, x2=1.2500; Optimal value of f is 4.2499.

Maple result:



4. Appendix: Python code