Homework 2 (Related to LN3 and LN4)

Try to be precise and to the point. Your answers should be short.

RUNNING TIMES:

1. Consider Euclidean algorithm. The input integers n and m are given in their decimal representations.
   (a) What is the size of the input for the algorithm?
   (b) Explain why the running time of the algorithm is linear on the size of the input. For the analysis, assume that the division process takes a constant amount of time.

2. Let p be a polynomial with positive coefficients. Show that p is $O(n^{\log(n)})$.
3. Show that $\sum_{i=1}^{n} i^2 = \Theta(n^3)$.

4. Show that $n \log(n) = \Omega(\log(n!))$.
5. Show that $a^n = O(n!)$ for any positive integer $a > 1$.
6. Show that
$$\sum_{i=1}^{n} \log(n/i) = \Theta(n).$$
   Show this directly. For instance, do not use some advanced results from calculus such as Stirling formula.

7. For the codes below analyse the running times in terms of Θ notation.

Algorithm 1

For $i = 1$ to n do
$j = n - i$
while $j \geq 0$ do
$j = j - 3$.

Algorithm 2

Set $s = 0$
for $i = 1$ to n do
for $j = 3 \cdot i$ to n do
$s = s + 1$.

Algorithm 3

for $i = 1$ to n do
$j = i$
while $j < n$ do
$j = 2 \cdot j$.

GRAPHS:

1. Let G be any graph. Explain why the sum of all degrees of the vertices of any graph G equals twice the number of edges of G.
2. How many edges does a complete bipartite graph $K_{n,m}$ have?
3. Prove that if there is a path in a graph from vertex x to vertex y and $x \neq y$ then there is a simple path from x to y. Recall that simple path is a path in which every vertex appears at most once.
4. Show that every finite connected graph G with more than 1 vertex has two vertices of the same degree.
5. We call a graph 3-regular if every vertex of the graph has degree 3. Draw 3-regular connected graphs consisting of 4 vertices, 6 vertices, and 8 vertices.
6. Let G be a connected graph. For any two vertices u, v let $d(u, v)$ be the distance from u to v. Recall that the distance from u to v is the length of the shortest path from u to v. Prove the following triangle inequality. For all vertices x, y and z of the graph we have

$$d(x, z) \leq d(x, y) + d(y, z).$$