Homework 3

Try to be precise and to the point. Your answers should be short.

1. Solve Exercise 1-4 and Exercise 6 from LN4.pdf.

   Solution to Exercise 1: The connected vertices are listed row by row. Those in different rows are not connected:
   0, 1, 3, 5.
   6,9.
   2, 4, 7, 8.

   Solution to Exercise 2: Not connected. It is very possible that there is a user, call it X, that has not friends. Hence, the graph is not connected.

   Solution to Exercise 3: For property 1, each vertex x is a path on its own; it starts with x and ends with x.
   For property 2. Assume that $x = v_1, v_1, \ldots, v_{n-1}, v_n = y$ is a path from x to y. Then $v_n, v_{n-1}, \ldots, v_1, v_0$ is a path from y to x. For property 3, assume that P is a path from x to y and Q is a path from y to z, then PQ (this is the path that follows P and then follows Q) is a path from x to z.

   Solution to Exercise 4: We already proved this in previous exercises. The solution has been provided.

   Solution to Exercise 6: Let T be a tree with n nodes and m edges. Let X_1 be the set of all the nodes of T of degree 1 (these are all leaves of T). Let k_1 be the cardinality of X_1. Remove all these nodes from T. The remaining tree T_1 has $n - k_1$ nodes $m - k_1$ edges. T_1 is still a tree. Apply this process to T_1. Repeat this process until exactly one node is left in a tree T_s. Thus we have:

   $$n - k_1 - k_2 - k_3 - \ldots - k_s = 1 \quad \text{and} \quad m - k_1 - k_2 - \ldots - k_s = 0.$$

   This proves what is needed.

2. Draw three graphs G_1, G_2 and G_3 such that (1) each graph has exactly 5 vertices; (2) G_1 has exactly 1 component, G_2 has exactly 2 components, and G_3 has exactly 3 components.

   Solution: This is very easy; can easily be drawn in a tutorial.

3. Solve Exercises 1-5 in LN5.pdf

   Solution to Exercise 1: For W_5 one BFS-tree has height 1 (if the starting vertex is the middle vertex), and the other has height 2 (if the start vertex is not the middle node). For $Cube_3$

all BFS-trees have height 3 and look as follows. There is the root, the root has 3 children. One of the children has 2 children, the other has 1, and third has none, and one node at level 2 has one child.

Solution to Exercise 2: Suppose that there is an edge between x and y; let the level of x be i and the level of y be j and $i \leq j$. The only reason that y does not appear at level $i+1$ is if y has already appeared in some of the previous levels. So either $j = i + 1$ or $j = i$.

Solution to Exercise 3: The heights are always either 1 or 2.

Solution to Exercise 4: Let n be the max of all distances $d(x, y)$ in the graph G. The height of any BFS-tree is n. If there is a a DFS-tree of height smaller that n then for each $y \in G$ there is a path from x to y of length less then n. This constradicts the choice of n.

Solution to Exercise 5: For K_n all entries of the adjacency matrix, apart from diagonal, contain 1. For $K_{n,m}$ all the positions (i, j) in the adjacency matrix such that $i \neq j$ and $i \leq n$ and $m \leq m$ contain 1, and other entries contain 0. The rest is easy.

4. Fix $n > 0$ and $k > 0$. Draw a directed graph with $k \cdot n$ vertices such that the graph has exactly n strongly connected components.

   Solution: For instance, consider the graph G that is built by drawing cycles of length k n times one after the other.

5. Assume that you have an acyclic digraph G with exactly n vertices. What is the maximum number of topological orders can G have? What is the minimum number of topological orders can G have? For each give an example that meets your bound. (You can either draw the graphs or formally write down their descriptions).

   Solution: The bounds are 1 and $n!$, respectively. For bound 1, the chain graph, whose vertices are 1, 2, ..., n and edges are from 1 to 2, from 2 to 3, ..., from $n-1$ to n, is an example. For $n!$ the graph consisting of n isolated vertices is an example.

6. What is the maximal number of edges a directed graph with n vertices can have? What is the maximal number of edges an undirected graph with n vertices can have? Explain your answer.

   Solution: The number is $n(n-1)/2$ and n^2. A set with n vertices can has exactly $n(n-1)/2$ unordered pairs. These all can be edges of a graph (e.g. complete graph). The set with n vertices has exactly n^2 number of ordered pairs. These all can be edges of a graph.
7. Give examples of $3n$ requests such that the solution based on "select the smallest interval" gives n requests, while an optimal solution consists of $2n$ intervals. You can explain your answer by drawing your solution or by explicitly writing the intervals down.

Solution: Intervals could be of the form $I_1, J_1, K_1, \ldots, I_n, J_n, K_n$ such that all intervals $J_1, \ldots, J_n$ have the same length, and these lengths are the shortest among all intervals; these intervals are not overlapping with each other. Tese intervals bind intervals I and K: For instance, J_1 overlaps with both I_1 and K_1, J_2 overlaps with both I_2 and K_2. Now the rule select the J intervals; there are n of them. But the optimal solutions has $2n$ intervals consisting of I and K intervals.

8. Suppose the selection rule for the interval scheduling problem is the following. Select a request that has the fewest possible requests overlapping it. Give an example where this rule does not provide an optimal solution.

   Solution: Not too hard.

9. For the interval scheduling problem consider the following rule to solve it. Always select the request, among available ones, with the latest starting time. Does this give an optimal solution to the problem? Explain your answer.

   Solution: The answer is "yes". This is essentially the same algorithm that selects an interval with the earliest finishing time (but in reverse order). The proof arguments are the same with the adjustment that finishing times are replaced with starting times.