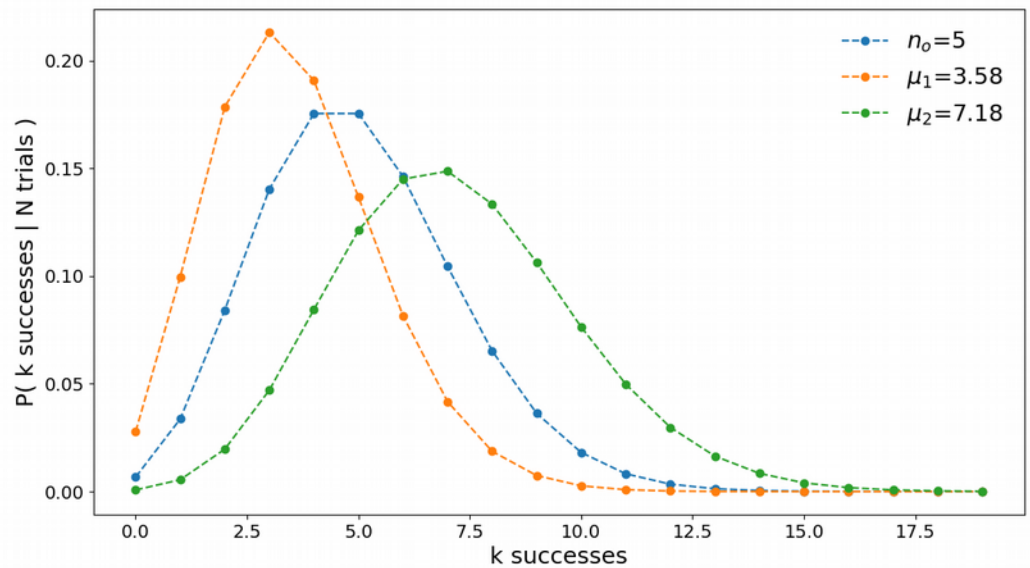
Problem 1:

Part (a)

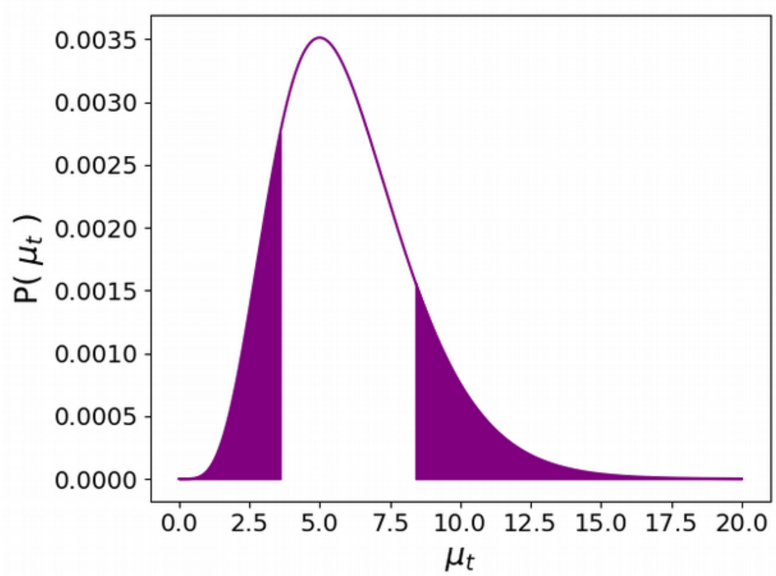
Method	Interval	Length	Coverage
RMS Dev.	(2.76, 7.24)	4.47	Yes
Classical Central	(3.58, 7.18)	3.60	Yes
Bayesian Central (uniform prior)	(3.62, 8.39)	4.76	No
Bayesian Central (inverse prior)	(2.84, 7.17)	4.32	No

Part (b)

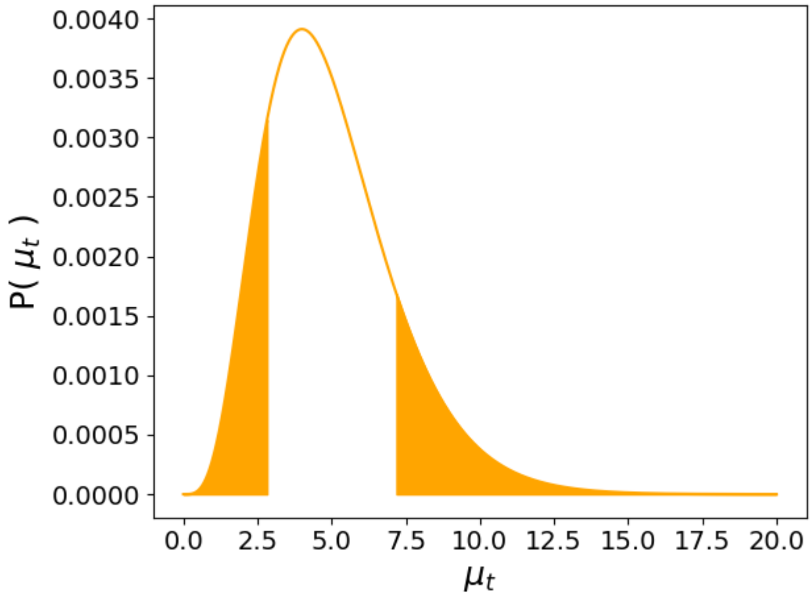
Frequentist Central:



Bayesian Central (uniform prior):



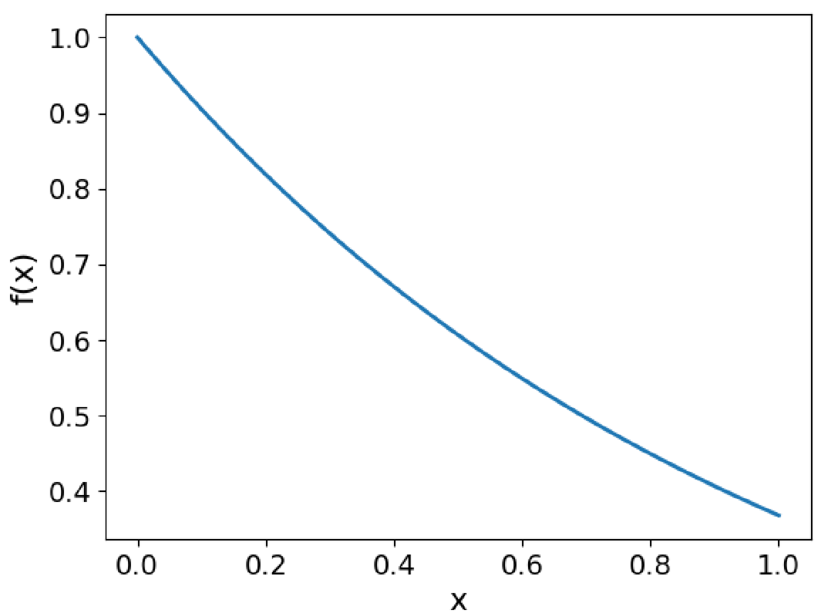
Bayesian Central (inverse prior):



Problem 2:

Part (a)

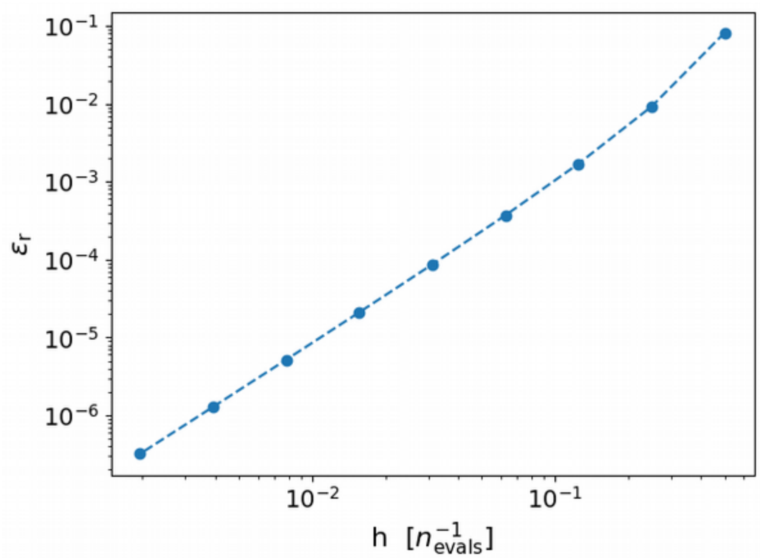
The integral of $\int_0^1 e^{-x}\,dx = 1 - \frac{1}{e}$ is approximately equal to 0.632, so let's compare with that:



Part (b)

Trapezoidal:

N	step size	I_{num}	ε_r
2	0.5	0.683939720586	0.0819767068693
4	0.25	0.637962716731	0.00924215771904
8	0.125	0.633195228312	0.00170010209037
16	0.062	0.632354660214	0.000370342938431
32	0.031	0.632175372356	8.67137233251e-05
64	0.016	0.632133830809	2.09959645756e-05
128	0.008	0.632123824788	5.16667166113e-06
256	0.004	0.632121368928	1.28155804665e-06
512	0.002	0.632120760562	3.19136829331e-07



Part (c)

Romberg:

N	step size	I_{num}	ε_r
2	0.5	0.68393972	8.19767069e-02
4	0.25	0.62263705	1.50026920e-02
8	0.125	0.63220400	1.32001917e-04
16	0.062	0.63210414	2.59781277e-05
32	0.031	0.63211861	3.08343556e-06
64	0.016	0.63212031	3.86871022e-07
128	0.008	0.63212053	4.83977921e-08
256	0.004	0.63212056	6.05094935e-09
512	0.002	0.63212056	7.56407001e-10

I interpreted this question as asking for the “final” (n,m) values in each row of the Romberg matrix. Full Romberg value and error matrices are reproduced below for reference:

Romberg Values:

0.68393972							
0.63796272	0.62263705						
0.63319523	0.63160607	0.632204					
0.63235466	0.63207447	0.6321057	0.63210414				
0.63217537	0.63211561	0.63211835	0.63211855	0.63211861			
0.63213383	0.63211998	0.63212028	0.63212031	0.63212031	0.63212031		
0.63212382	0.63212049	0.63212052	0.63212053	0.63212053	0.63212053	0.63212053	
0.63212137	0.63212055	0.63212055	0.63212055	0.63212055	0.63212055	0.63212056	0.63212056
0.63212076	….	0.63212056	0.63212056	0.63212056	0.63212056	0.63212056	0.63212056

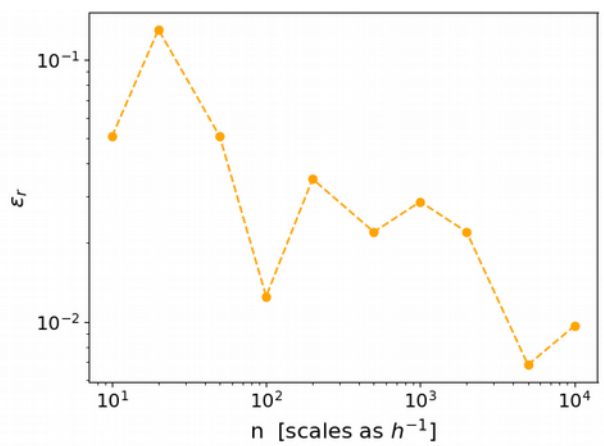
Romberg Errors:

8.1976e-02							
9.2421e-03	1.5002e-02						
1.7001e-03	8.1391e-04	1.3200e-04					
3.7034e-04	7.2910e-05	2.3509e-05	2.5978e-05				
8.6713e-05	7.8293e-06	3.4906e-06	3.1728e-06	3.0834e-06			
2.0995e-05	9.0995e-07	4.4866e-07	4.0037e-07	3.8950e-07	3.8687e-07		
5.1666e-06	1.0975e-07	5.6412e-08	5.0186e-08	4.8813e-08	4.8480e-08	4.8397e-08	
1.2815e-06	1.3479e-08	7.0611e-09	6.2778e-09	6.1056e-09	6.0638e-09	6.0535e-09	6.050e-09
3.1913e-07	….	7.8487e-10	7.6332e-10	7.5810e-10	7.5681e-10	7.564e-10	7.5640e-10

Part (d)

Hit-or-Miss:

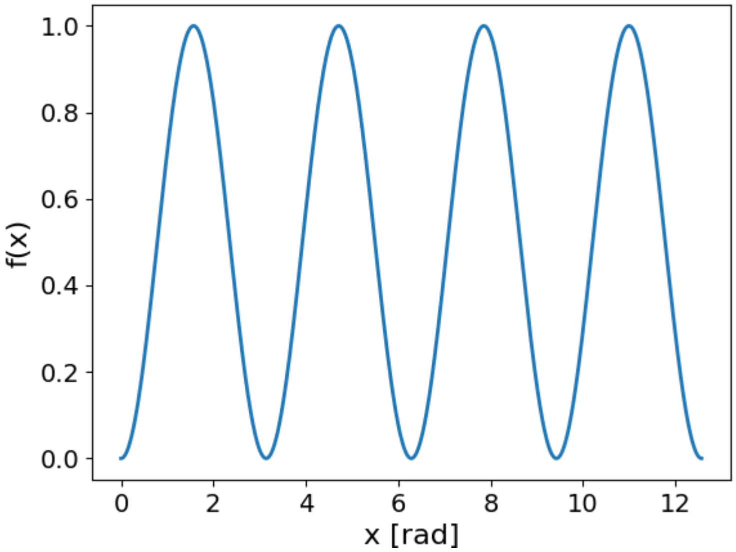
N	I_{num}	ε_r
10	0.8	0.265581365495
20	0.7	0.107383694809
50	0.64	0.0124650923964
100	0.62	0.019174441741
200	0.6	0.0508139758784
500	0.606	0.0413221156372
1000	0.635	0.00455520886202
2000	0.638	0.00930113898263
5000	0.6222	0.0156940929859
10000	0.6306	0.0024054886482



Problem 3:

Part (a)

The analytical expression is given by $\int_0^{4\pi} \sin^2 x\, dx = 2\pi$ which is going to cause problems for Romberg integration based on the trapezoid method, as any number of evaluations less than the Nyquist sampling will alias a lower-frequency harmonic and give zero or ~100% accuracy:



Part (b)

Romberg:

N	step size	I_{num}	ε_r
2	0.5	1.50e-30	1.00e+00
3	0.333	1.13e-30 1.00e-30	1.00e+00 1.00e+00
4	0.25	6.28e+00 8.37e+00 8.93e+00	2.82e-16 3.33e-01 4.22e-01
5	0.2	1.03e-30 -2.09e+00 -2.79e+00 -2.97e+00	1.00e+00 -2.09e+00 …. -2.79e+00

Ope! It reaches the absolute converge criterion of 10^-5 on the first iteration, but it doesn't stay "converged" now does it! Maybe we should've tried N = 2,4,8,16, etc…

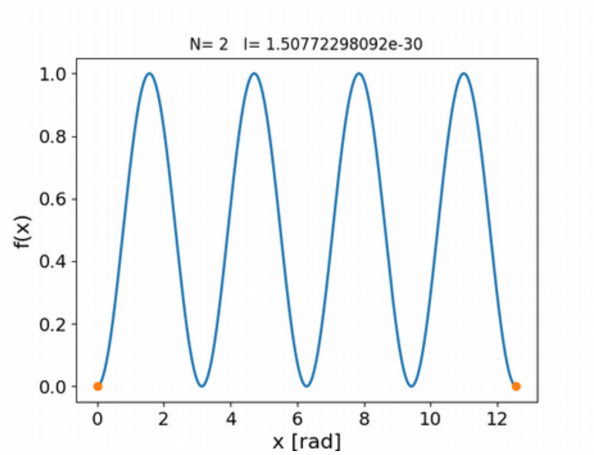
N	step size	I_{num}	ε_r
2	0.5	1.50e-30	1.00e+00
4	0.25	6.28e+00 8.37e+00	2.82e-16 3.333e-01
8	0.125	6.28e+00 6.28e+00 6.14e+00	0.00e+00 1.41e-16 2.22e-02
16	0.0625	6.28e+00 6.28e+00 …. 6.28e+00	1.41e-16 1.41e-16 …. 3.52e-04

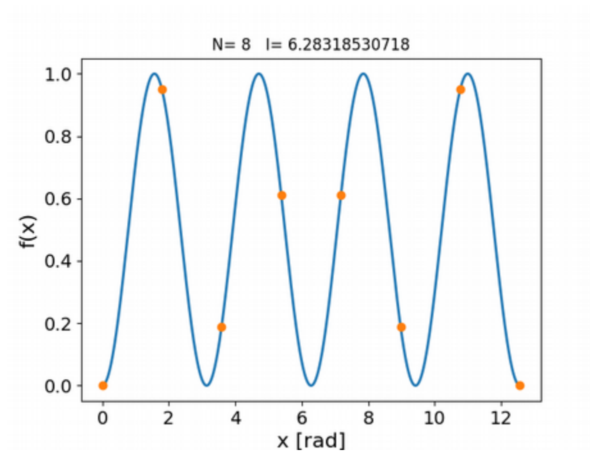
This one **almost** converges to 10^-5 by N=16 -Let's try one more N and see if we converge to 10^-6:

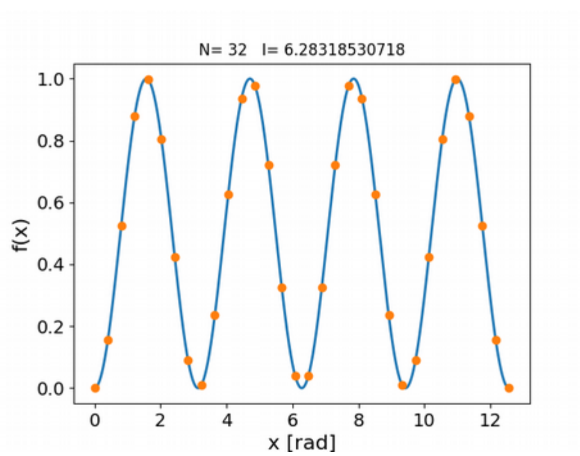
32 0.03125 6.28e+00 …. 6.28e+00 1.41e-16 2.82e-16 2.82e-16 2.82e-16 1.38e-06

We do! So here's why our iteration appeared to converge prematurely.

$\int_0^{4\pi} \sin^2 x\,dx = 2\pi \rightarrow \int_0^{4\pi} \sin^2 \frac{x}{2m}\,dx = 2\pi$ for all integer m. Therefor, when we use numbers of evaluation less than the Nyquist sampling for our function, we "accurately" integrate it's lower-frequency harmonics. Here's a few plots showing this for several step-sizes/numbers of evaluation:







Problem 4:

Part (a)

See the following table for the sample mean MC (next page):

Sample mean MC:

N	I_{num}	ε_r
2	27.7989623063	0.0760888634685
4	28.9340326942	0.120027072034
8	29.316978843	0.134850793924
16	25.3857021225	0.0173276597729
32	24.8853030085	0.0366979480569
64	28.0186763284	0.0845939223904
128	24.9114233791	0.03568683694
256	25.1552519728	0.0262483107309
512	25.9617761814	0.00497198121408
1024	26.1891555389	0.0137737627955
2048	25.4594763978	0.0144718813748
4096	25.5166393617	0.0122591214829
8192	25.8077636852	0.000989792830937
16384	25.8524320228	0.000739304108822
32768	25.7741464065	0.00229110684612
65536	25.8409619059	0.000295299582353
131072	25.8020402156	0.00121134649441

Part (b)