Ward Howard
page #1

n=0 / 1
n=1 / 1 1
n=2 / 1 2 1
n=3 / 1 3 3 1
n=4 / 1 4 6 4 1
n=5 / 1 5 10 10 5 1 → k successes
0 1 2 3 4 5

1. a. For N people with fair coins, the probability of an "odd-man-out" on the first flip is

$$P = B(1; N, p_{flip} = \tfrac{1}{2}) + B(4; N, p_{flip} = \tfrac{1}{2}) = 2\frac{N!}{(N-1)!}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{N-1}$$

$$= \boxed{\frac{N}{2^{N-1}}} \quad \left(\text{for } N=5,\ P(\text{success} \mid 1 \text{ flip}) = \frac{5}{16}\right)$$

b. The mean number of flips required to achieve an "odd-man-out" is

$$E[n_{flips}] = n_{flips} P_{single} \quad \text{where } B\left(1; n_{flips}, P_{single} = \frac{N}{2^{N-1}}\right)$$

$$E[B(1; n_{flips}, P_{single})]\Big|_{E=1} = n_{flips} P_{single} = 1 \quad \therefore \quad \boxed{n_{flips} = \frac{2^{N-1}}{N}}$$

$$\left(\text{for } N=5,\ n_{flips} = \frac{16}{5} = 3.2\right)$$

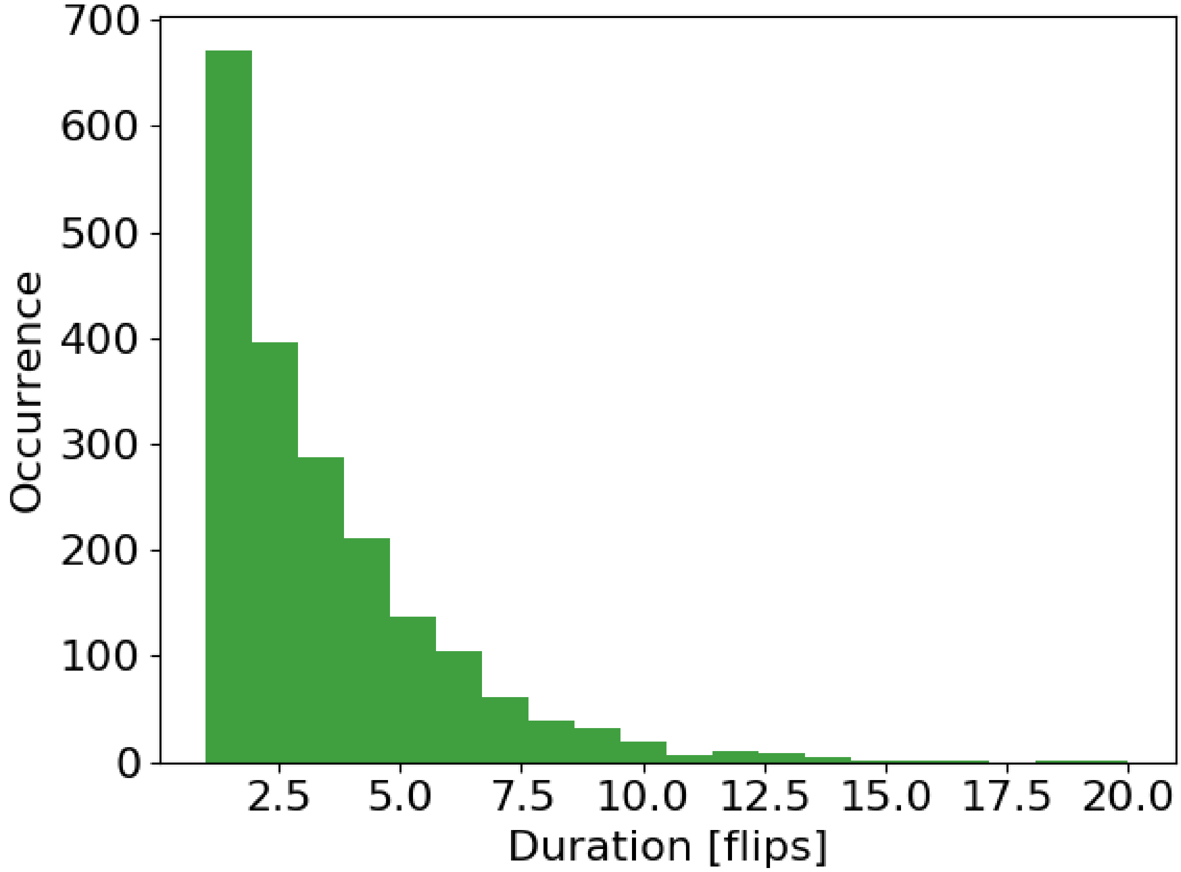
c. See my Github code; the mean of all 2000 games was $\boxed{\mu = 3.124}$, very close to the theoretical averge, 3.2!

d. Both theoretically and in Python game simulations, the biased coin does not affect the mean # of n_{flips}. This is because, with 1 coin fixed, 4 free coins may achieve the "odd man out" in the following ways:

$$B(0; 4, p=\tfrac{1}{2}) + B(3; 4, p=\tfrac{1}{2}) = \frac{1}{16} + \frac{4}{16} = \boxed{\frac{5}{16}}, \text{ the same}$$

as before.

Problem 1c:



700
600
500
400
300
200
100
0
Occurrence
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Duration [flips]

2a) Set of $n=2$ random variables $X = (X_1, \ldots, X_n) \rightarrow (m, v)$ with mean values of x_i, $\mu = (\mu_1, \ldots, \mu_n) \rightarrow (\hat{m}, \hat{v})$, and covariance matrix

$$V_{ij} = \begin{bmatrix} cov[m,m] & cov[v,m] \\ cov[m,v] & cov[v,v] \end{bmatrix} = \begin{bmatrix} \sigma_m^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}$$

Then for a set of functions $y = (y_1, \ldots, y_n) \rightarrow (p, E)$ with mean values $y_i, \mu = (\hat{p}, \hat{E})$ such that $y_i(X_1, X_2)$.

Then the variance $\sigma_y^2 \rightarrow \sigma_E^2$ or σ_p^2 is approximately

$$\sigma_p^2 \approx \sum_{i,j=1}^{n=2} \frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j}\bigg|_\mu V_{ij} = \frac{\partial p}{\partial m}\frac{\partial p}{\partial m}\bigg|_\mu V_{mm} + \frac{\partial p}{\partial m}\frac{\partial p}{\partial v}\bigg|_\mu \overset{\nearrow V_{mv}=0}{V_{mv}} +$$

$$\frac{\partial p}{\partial v}\frac{\partial p}{\partial m}\bigg|_\mu \overset{\nearrow V_{vm}=0}{V_{vm}} + \frac{\partial p}{\partial v}\frac{\partial p}{\partial v}\bigg|_\mu V_{vv} = \sigma_m^2\left[\frac{\partial p}{\partial m}\right]^2 + \sigma_v^2\left[\frac{\partial p}{\partial v}\right]^2$$

so that

$$\boxed{\sigma_p^2 \approx \sigma_m^2 v^2 + \sigma_v^2 m^2} = a^2 m^2 v^2 + b^2 v^2 m^2 = \boxed{p^2(a^2+b^2)}$$

(where $a^2 = \frac{\sigma_m^2}{m^2}$ and $b^2 = \frac{\sigma_v^2}{v^2}$ are the fractional uncertainties)

and $$\sigma_E^2 \approx \frac{\partial E}{\partial m}\frac{\partial E}{\partial m}\bigg|_{\mu=\hat{m}} V_{mm} + \frac{\partial E}{\partial v}\frac{\partial E}{\partial v}\bigg|_{\mu=\hat{v}} V_{vv} = \frac{1}{4}v^4\sigma_m^2 + \frac{1}{4}m^2(2v)^2\sigma_v^2$$

$$\sigma_E^2 \approx \boxed{\frac{1}{4}v^4\sigma_m^2 + m^2v^2\sigma_v^2} = \frac{1}{4}v^4a^2m^2 + m^2v^2b^2v^2$$

$$= \boxed{E^2(a^2+4b^2)}$$

Likewise, the covariance is

$$\mathrm{cov}[p,E] \approx \sum_{i,j=1}^{n=2} \frac{\partial p}{\partial x_i}\frac{\partial E}{\partial x_j}\Big|_{\hat{\mu}} V_{ij}$$

$$\approx \frac{\partial p}{\partial m}\frac{\partial E}{\partial m}\sigma_m^2 + \frac{\partial p}{\partial v}\frac{\partial E}{\partial v}\sigma_v^2 = v\tfrac{1}{2}v^2\sigma_m^2 + m\tfrac{1}{2}m2v\sigma_v^2$$

$$\approx \tfrac{1}{2}v^3a^2m^2 + \tfrac{1}{2}2m^2vb^2v^2 = Epa^2 + 2Epb^2$$

$$= \boxed{Ep(a^2+2b^2)}$$

The correlation coefficient $\rho(p,E)$ is $\dfrac{\mathrm{cov}(p,E)}{\sigma(p)\sigma(E)} =$

$$= \frac{Ep\,(a^2+2b^2)}{p^2(a^2+b^2)E^2(a^2+4b^2)} = \boxed{\frac{(a^2+2b^2)}{(a^2+b^2)(a^2+4b^2)}\frac{1}{Ep}}$$

For the special cases $a=0$ or $b=0$,

$$\rho(p,E)\Big|_{a=0} = \frac{2\cancel{b^2}}{\cancel{b^2}4b^2}\frac{1}{Ep} = \boxed{\frac{1}{2b^2}\frac{1}{Ep}}$$

$$\rho(p,E)\Big|_{b=0} = \frac{a^2}{a^2a^2}\frac{1}{Ep} = \boxed{\frac{1}{a^2}\frac{1}{Ep}}$$

When there is no uncertainty in a, the uncertainty in b cannot be zero, and vice versa. Furthermore, to keep $|\rho<1|$, the maximum/min values of $|a|,|b|$ are $\sqrt{\frac{1}{2Ep}} = |b_{max/min}|$, $\sqrt{\frac{1}{Ep}} = |a|$.

b) E, p and covariance matrix $cov[p,E]_{\ell k}$ are known.

$$U = AVA^T$$

$$UA = AVA^TA$$

$$A^TUA = A^TAVA^TA = V$$

$$A = \begin{bmatrix} \frac{\partial p}{\partial m} & \frac{\partial p}{\partial v} \\ \frac{\partial E}{\partial m} & \frac{\partial E}{\partial v} \end{bmatrix}, \quad A^T = \begin{bmatrix} \frac{\partial p}{\partial m} & \frac{\partial E}{\partial m} \\ \frac{\partial p}{\partial v} & \frac{\partial E}{\partial v} \end{bmatrix}, \quad U = \begin{bmatrix} \sigma_p^2 & cov[p,E] \\ cov[p,E] & \sigma_E^2 \end{bmatrix}$$

where $cov[p,E] = \frac{a^2+2b^2}{(a^2+b^2)(a^2+4b^2)} \frac{1}{Ep}$

$$V = \begin{bmatrix} \frac{\partial p}{\partial m} & \frac{\partial E}{\partial m} \\ \frac{\partial p}{\partial v} & \frac{\partial E}{\partial v} \end{bmatrix} \begin{bmatrix} \sigma_p^2 & cov[p,E] \\ cov[p,E] & \sigma_E^2 \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial m} & \frac{\partial p}{\partial v} \\ \frac{\partial E}{\partial m} & \frac{\partial E}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\partial p}{\partial m}\sigma_p^2 + \frac{\partial E}{\partial m} cov[p,E]\right) & \left(\frac{\partial p}{\partial m} cov[p,E] + \frac{\partial E}{\partial m}\sigma_E^2\right) \\ \left(\frac{\partial p}{\partial v}\sigma_p^2 + \frac{\partial E}{\partial v} cov[p,E]\right) & \left(\frac{\partial p}{\partial v} cov[p,E] + \frac{\partial E}{\partial v}\sigma_E^2\right) \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial m} & \frac{\partial p}{\partial v} \\ \frac{\partial E}{\partial m} & \frac{\partial E}{\partial v} \end{bmatrix}$$

so the uncertainty on the mass is $\sqrt{V_{mm}}$, or

$$\boxed{\sigma_m = \left[\frac{\partial p}{\partial m}\left(\frac{\partial p}{\partial m}\sigma_p^2 + \frac{\partial E}{\partial m} cov[p,E]\right) + \frac{\partial E}{\partial m}\left(\frac{\partial p}{\partial m} cov[p,E] + \frac{\partial E}{\partial m}\sigma_E^2\right)\right]^{1/2}}$$

$\sigma_m^2 = \left(\frac{\partial p}{\partial m}\sigma_p\right)^2 + 2\frac{\partial p \partial E}{\partial m \partial m} cov[p,E] + \left(\frac{\partial E}{\partial m}\sigma_E\right)^2$, so errors add in quadrature.

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$E = \frac{1}{2}vp$ $\qquad$ $p = mv$

$E = \frac{1}{2}mv \cdot v$ $\qquad$ $\frac{p}{m} = v$

$mE = \frac{1}{2}m^2v^2 \to 2mE = p^2$

$\boxed{\hat{m} = \frac{p^2}{2E}}$ $\quad v = \frac{p}{\left(\frac{p^2}{2E}\right)} = \frac{2E}{p}$

$E = \frac{1}{2}mv^2, \; p = mv$

$\frac{\partial E}{\partial m} = \frac{1}{2}v^2 = \frac{1}{2}\left(\frac{4E^2}{p^2}\right) = \frac{2E^2}{p^2}$

estimate of the mass in terms of $\bar{y} = (p, E)$

$\frac{\partial p}{\partial m} = v = \frac{2E}{p}$ $\qquad$ then

$$\sigma_m^2 = \frac{4E^2}{p^2}\sigma_p^2 + 2\left(\frac{2E}{p}\right)\left(\frac{2E^2}{p^2}\right)\mathrm{cov}[p, E] + \frac{4E^4}{p^4}\sigma_E^2$$

$$= \frac{4E^2}{\cancel{p^2}}\left(\cancel{p^2}(a^2+b^2)\right) + \frac{8E^3}{p^3}Ep(a^2+2b^2) + \frac{4E^4}{p^4}E^2(a^2+4b^2)$$

$$= 4E^2(a^2+b^2) + \frac{8E^4}{p^2}(a^2+2b^2) + \frac{4E^6}{p^4}(a^2+4b^2)$$

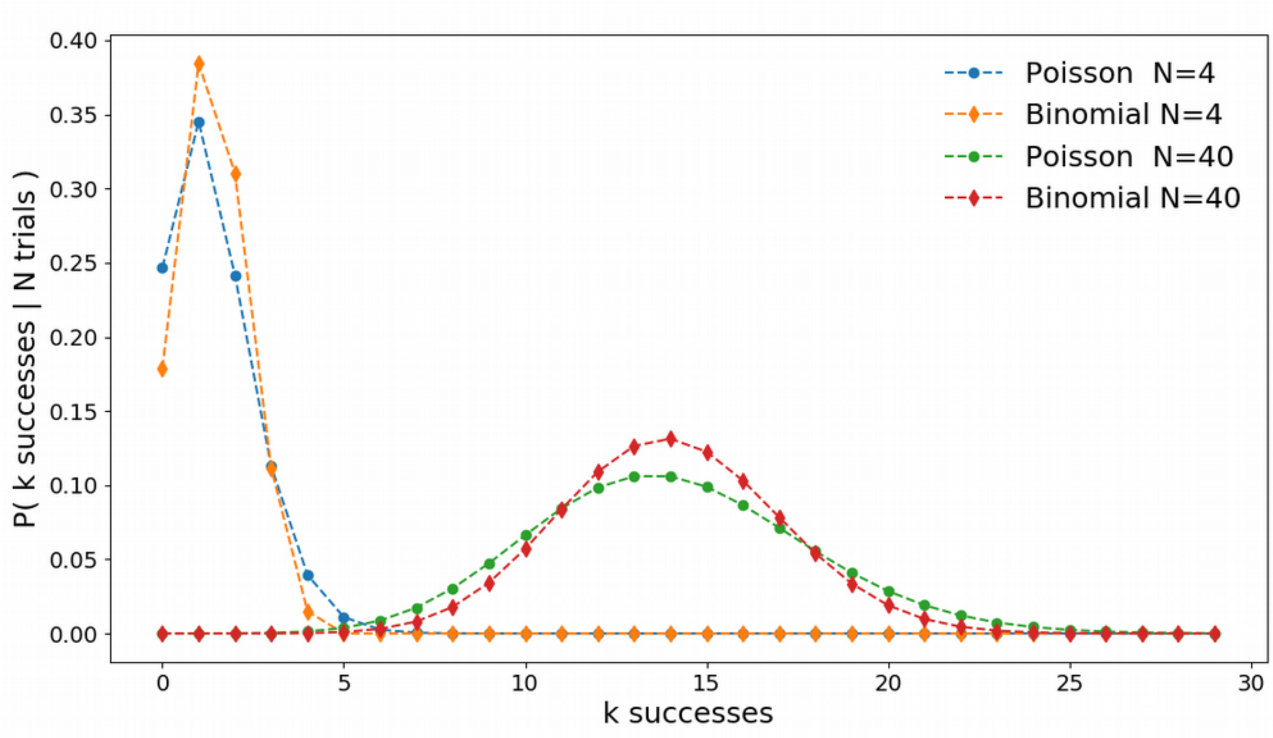
which is some kind of series expansion that doesn't seem to simplify properly... hmm... I give up for now.

- As an alternate "check", I can use Cowen's $\sigma_y^2 \approx \frac{\partial y}{\partial x_i}\sigma_i^2$ for σ_p^2, σ_E^2 in terms of σ_m^2, σ_v^2, then re-arrange and subtract off σ_v^2 for something like:

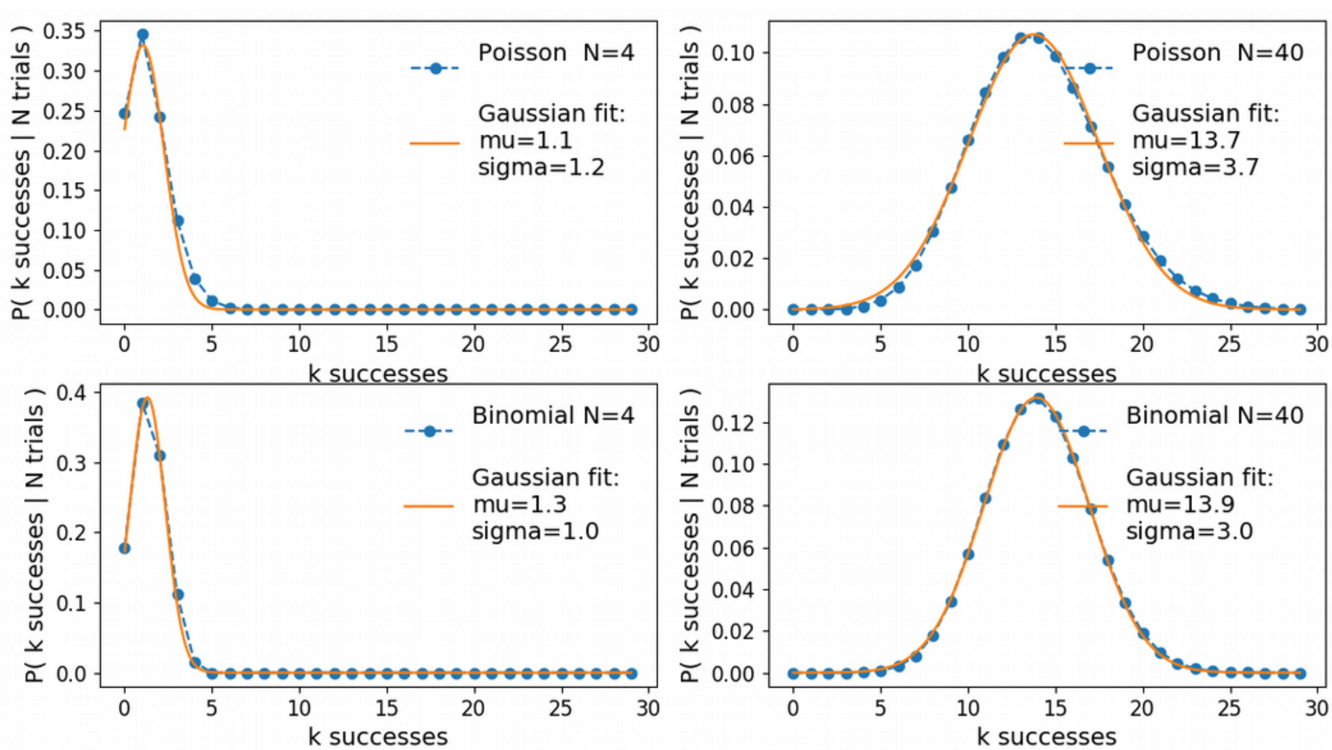
$$\sigma_m^2 \doteq -2\left(\frac{p}{2E}\right)^2\left[\sigma_E^2 - \left(\frac{2E}{p}\right)\sigma_p^2\right]$$

Problem 3:

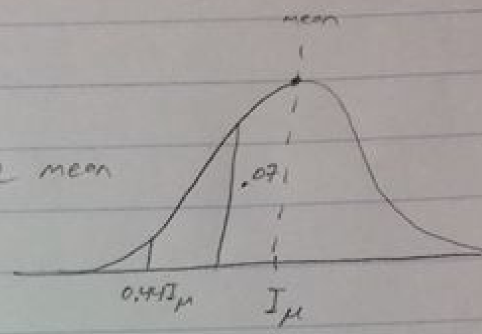
a)


Poisson N=4
Binomial N=4
Poisson N=40
Binomial N=40
P(k successes | N trials)
k successes

b)


Poisson N=4
Gaussian fit:
mu=1.1
sigma=1.2
Poisson N=40
Gaussian fit:
mu=13.7
sigma=3.7
Binomial N=4
Gaussian fit:
mu=1.3
sigma=1.0
Binomial N=40
Gaussian fit:
mu=13.9
sigma=3.0
P(k successes | N trials)
k successes

4. a. $P(0.44 I_\mu \mid 0.07 I_\mu) =$



$\frac{0.56 I_\mu}{0.07 I_\mu} = 8\sigma$ away from the mean

$$P = 1 - \int_{-8\sigma}^{8\sigma} N(I_\mu, \sigma)\, dx'$$

$$1 - 2\int_0^{8\sigma} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x')^2}{\sigma^2}}\, dx' \rightarrow 1 - \frac{2}{\sqrt{2\pi\sigma^2}} \sqrt{2}\sigma \int_0^{\frac{8}{\sqrt{2}}} e^{-\chi^2} d\chi$$

where $\chi = \frac{x'}{\sqrt{2}\sigma}$, $d\chi = \frac{1}{\sqrt{2}\sigma} dx'$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{8/\sqrt{2}} e^{-\chi^2} d\chi = \boxed{1 - \mathrm{erf}(8/\sqrt{2})} = 1 - \mathrm{erf}(5.6577)$$

$$\boxed{\approx 10^{-15}}$$

$P(\text{total}, 10^5 \text{ events}) \doteq 10^5 \cdot 10^{-15} = \boxed{10^{-10} \text{ for 1 such "fluke"}}$

b. $\frac{0.56 I_\mu}{0.14 I_\mu} = 4\sigma$ away from the mean, $P_{\text{single}} = \frac{1}{15787}$

$P_{\text{total}} = P_{\text{single}} \times 10^5 \text{ events} \cdot 10^{-2} \text{ fraction} + (\text{other distribution} = 0)$

$= 10^3 \cdot \frac{1}{15787} = \boxed{0.063}$, so <u>much</u> more likely if 1% of the 10^5 events have $2\times$ the σ. <u>Very non-linear</u> dependence on sigma!