Ladybird lost

Introduction

We investigate a system in which a ladybird starts at the centre of a circular room of radius R containing N coins of diameter d. The ladybird moves at constant speed v along straight lines, choosing a random direction after every collision with a coin (which is then removed) or the wall.

In this report, we derive approximate formulas for average times and distances in the system, including the mean times and paths between collisions, the mean time to reach the wall, and the decay law for the number of coins. We model the motion as a random walk and use results from stochastic processes and first-passage theory [1, 2]. These approximations rely on diffusive (macroscopic) properties and can fail at low coin density, so we compare results with Monte Carlo (MC) simulations [3] in Python.

The simulation disperses N coins uniformly in the circular room, then records the ladybird's trajectory in straight lines from a random initial direction $\theta \in [0, 2\pi)$ under different scenarios. We sample a large number of such trajectories to estimate means (full description in [6]).

Random walk and diffusion problems arise in many areas (e.g. diffusion in the Lorentz gas [4] and the motion of bacteria [5]). Our simple system shows approaches for related problems concerning mean free times/paths, exit times, or decay laws. Further, we offer approaches to test the validity of such models using MC simulations.

Coin Collisions

We begin our report by considering the average time between collisions (the mean free time τ), the average distance between collisions (the mean free path ℓ), and the average number of collisions per unit time (the collision rate λ).

These values tell us, on average, how often the system changes in time and space, as each collision corresponds to a change in motion. They also help us approximate more complex quantities such as Proposition 2.

Proposition 1 assumes that the coin centres are i.i.d. uniformly distributed. Further, the coin density $\rho = \frac{N}{\pi R^2}$ is low and $d \ll R$. Under this and further assumptions [6] collisions along a trajectory are approximately independent(ignoring the boundary). We can then model collision occurrences by a Poisson process.

Proposition 1. *The collision rate λ, mean free time τ, and mean free path ℓ are approximated by*

$$\lambda = \frac{vdN}{\pi R^2}, \qquad \tau = \frac{\pi R^2}{vdN}, \qquad \ell = \frac{\pi R^2}{dN}.$$

Proof(sketch). Full proof here [6]. Assume $d \ll R$ and coin centres are i.i.d. uniform, so (with small $\rho = \frac{N}{\pi R^2}$ and away from the boundary) the ladybird encounters coins as a Poisson process. In time dt the ladybird sweeps area $dA \approx vd\,dt$, so the expected number of collisions in $(t, t+dt)$ is $\rho\, dA = \rho\, vd\, dt$. Hence the collision rate is $\lambda \approx vd\rho = \frac{vdN}{\pi R^2}$, and $\tau = 1/\lambda = \frac{\pi R^2}{vdN}$ $\quad \ell = v\tau = \frac{\pi R^2}{dN}$. □

In Proposition 1 the core value is $d\rho = \frac{dN}{\pi R^2}$. We notice $\lambda \propto v\, d\rho$, $\tau \propto (v\, d\rho)^{-1}$, and $\ell \propto (d\rho)^{-1}$, so ℓ depends only on ρ and d, not on v. In particular, two rooms with the same d and ρ have the same average distance between collisions, and if v is the same then they have the same average time between collisions.

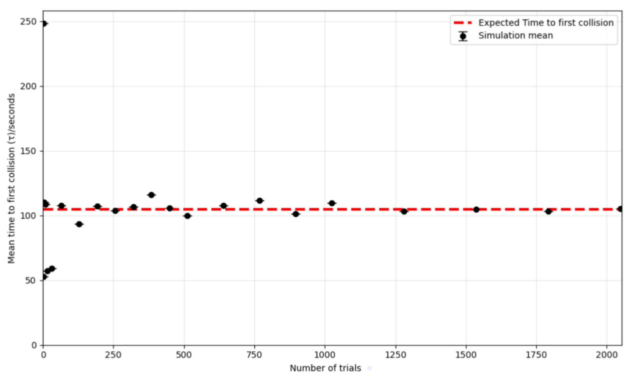

Figure 1: Plot showing the running mean of the first collision time τ against the number of trials in this MC simulation ($R = 100\,\mathrm{cm}$, $d = 2\,\mathrm{cm}$, $N = 150$, $v = 1\,\mathrm{cm\,s^{-1}}$). Black dots show the running mean after a given number of trials, and the red dotted line shows the predicted value $\tau = \frac{\pi R^2}{vdN}$.

Figure 1 shows an estimate of the mean free time τ for $R = 100\,\mathrm{cm}$, $d = 2\,\mathrm{cm}$, $N = 150$, and $v = 1\,\mathrm{cm\,s^{-1}}$ using a MC simulation. From 2000 sampled trajectories we plotted the running mean of the first-collision time (black dots). Proposition 1 predicts $\tau = \pi R^2/(vdN) = 100\pi/3$ (red dotted line). The running mean converges to this value as the number of trials increase, supporting the approximation under these parameters. Further simulations for other parameter values, or by involving inter-collision times beyond the first collision, may offer additional insights.

Wall Collisions

In this section, we consider extending our results to find the average time taken for the ladybird to reach the wall (mean first passage time (MFPT)).

First-passage times measure the time to a specified event. Often they represent a stopping time for the system, as in gambler's ruin [7] when a player is ruined. In other systems, they might give the times to trigger a transition.

Proposition 2 uses Proposition 1 and further approximates the motion by two-dimensional Brownian motion. We assume the trajectory is diffusive on macroscopic scales (i.e. $\ell \ll R$). This approximation can fail when the coin density ρ is low, as collisions are rare and the motion becomes effectively directed (see [6] for further assumptions).

Proposition 2. *The mean first passage time (MFPT) T_{wall} is approximated by*

$$T_{wall} = \frac{dN}{\pi v}.$$

Proof(sketch). Full proof here [6]. We treat the macroscopic motion as diffusive ($\ell \ll R$). Approximating the trajectory by a random walk with step length $\ell = v\tau$ and step time τ gives $\mathbb{E}[r(t)^2] \approx \frac{t}{\tau}\ell^2 = v^2\tau\, t$ (were $r(t)$ is the displacement at time t). Since in 2D $\mathbb{E}[r(t)^2] \approx 4Dt$ (see [2]), we have $D = \frac{v^2\tau}{4} = \frac{v\ell}{4}$. The mean first passage time for a wall collision satisfies $D\nabla^2 T = -1$, $T(R) = 0$ (see [2]), giving $T(0) = \frac{R^2}{4D}$. Substituting $\tau = \frac{\pi R^2}{vdN}$ yields $\mathbb{E}[T_{\text{wall}}] = \frac{dN}{\pi v}$. $\square$

Proposition 2 implies that the MFPT depends on d, N, and v, but doesn't depend on R. Intuitively, a larger R means we must go further to reach the wall, increasing T_{wall}, but it also decreases the coin density ρ and hence the collision rate λ. With fewer collisions, the mean free path(ℓ) increases, so these effects cancel in this approximation.

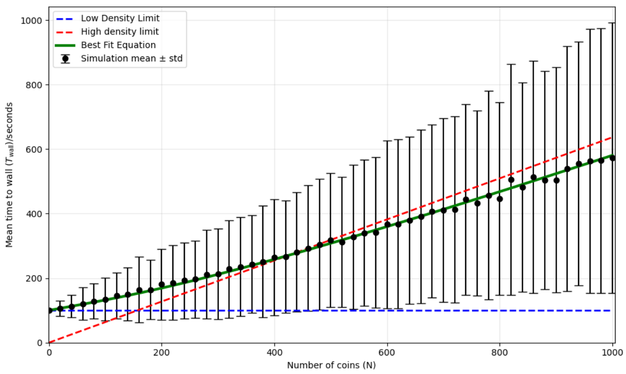

Figure 2: Plot showing the MFPT(T_{wall}) versus the number of coins in the system(black dots). This MC uses 1000 trajectories for each mean and $R = 100\,\mathrm{cm}$, $d = 2\,\mathrm{cm}$, and $v = 1\,\mathrm{cm\,s^{-1}}$. The error bars show the mean calculated plus $\pm$ one standard deviation. The red dotted line shows the predicted $T_{\text{wall}} = \frac{dN}{\pi v}$, the blue dotted line shows the MFPT with no coins $T_{\text{wall}} = \frac{R}{v}$, and the green line shows the fit from Equation 1.

Figure 2 shows Monte Carlo estimates of T_{wall} versus N (black dots), averaged over 1000 trajectories per N with $R = 100\,\mathrm{cm}$, $d = 2\,\mathrm{cm}$, and $v = 1\,\mathrm{cm\,s^{-1}}$. Error bars show mean $\pm$ one standard deviation. The red dotted line is $T_{\text{wall}} = \frac{dN}{\pi v} = \frac{2N}{\pi}\mathrm{s}$, the blue dotted line is $T_{\text{wall}} = \frac{R}{v} = 100\mathrm{s}$, and the green line is the fit from Equation 1.

We notice that for small N (low ρ) the ballistic limit works well, as collisions are rare and trajectories are straight, so $T_{\text{wall}} \approx R/v$. For larger N (high ρ), Proposition 2 approximates well, since the diffusion assumption becomes valid.

The plot shows both the ballistic and the diffusive approximations for the MFPT fail across the full range of N. To improve this, we fitted the form in Equation 1 to our Monte Carlo data using the `curve_fit` package, yielding $j = 0.798$ and $k = 0.929$ (green curve). This form was chosen because it approaches the ballistic limit for small N, and the diffusive limit for large N found by trial and error with different forms. Note that more justified forms may improve the function's fit.

$$T_{\text{wall}} = \frac{R}{v}\left(1 - \exp\left(-\frac{2\pi R}{N^j d}\right)\right) + \frac{dN}{2\pi v}\left(1 + \exp\left(-\frac{2\pi R}{N^k d}\right)\right) \quad (1)$$

Coin Decay Law

We now approximate a decay law for the number of coins in the room $N(t)$ at time using our previous results and further approximations.

Similar decay processes occur in real systems [8], so it is useful to have approaches for approximating these laws. Further, the decay law tells us when our previous approximations in Propositions 1 and 2 start to fail. Previous results rely on high coin density ρ (and so large N), but as $N(t)$ becomes small, our Poisson/diffusion approximations no longer hold.

In Proposition 3, the main assumption is that the Poisson approximation used in Proposition 1 still holds as coins disappear. Clearly, this becomes less accurate for large t when $N(t)$ is small, but for shorter times it gives a useful indication of when our earlier approximations start to fail (see [6] for further assumptions).

Proposition 3. *Suppose there are $N(t)$ coins in the room at time t and there are N_0 coins in the room at $t = 0$. Then $N(t)$ is approximated by*

$$N(t) = N_0 \exp(-\frac{vd}{\pi R^2}t)$$

Proof(sketch). Full proof here [6]. At time t the coin density is $\rho(t) = \frac{N(t)}{\pi R^2}$. In time dt the ladybird sweeps area $dA \approx vd\,dt$, so the expected number of hits in $(t, t + dt)$ is $\mathbb{E}[-\Delta N(t)] \approx \rho(t)\,dA = \frac{vd}{\pi R^2}N(t)\,dt$. Since each hit removes one coin, this gives the ODE $\frac{dN}{dt} = -\frac{vd}{\pi R^2}N(t)$, and solving with $N(0) = N_0$ yields $N(t) = N_0 \exp\left(-\frac{vd}{\pi R^2}t\right)$. □

In this proposition, we find an exponential decay with rate $\frac{vd}{\pi R^2}$. Intuitively, the decrease in $N(t)$ should be proportional to $N(t)$ itself, as with fewer coins, the density is lower, so collisions are less frequent, and collisions occur at a slower pace.

In Figure 3 we show the mean time to remove N_r coins T_r versus N_r (black dots). The Monte Carlo simulation uses 100 trajectories for each mean with $R = 100\,\text{cm}$, $d = 2\,\text{cm}$, $N_0 = 150$,

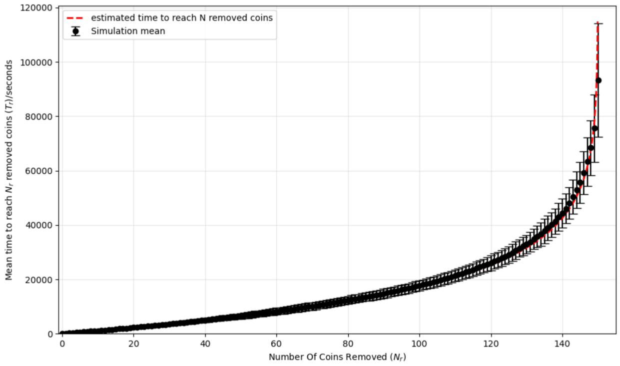

Figure 3: Plot showing the mean time to remove N_r coins(T_r) versus the number of removed coins N_r (black dots). The Monte Carlo simulation uses 100 trajectories for each mean and $R = 100\,\text{cm}$, $d = 2\,\text{cm}$, $N_0 = 150$, and $v = 1\,\text{cm}\,\text{s}^{-1}$. Error bars display the sample mean $\pm$ one standard deviation. The red dotted line shows the predicted relationship given by Equation 2.

and $v = 1\,\text{cm}\,\text{s}^{-1}$. Error bars display the sample mean $\pm$ one standard deviation. The red dotted line shows the result as predicted by Proposition 3 given by Equation 2.

$$T_r = \frac{\pi R^2}{vd} ln\left(\frac{N_0}{N_0 - N_r}\right) \quad (2)$$

Intuitively, when N_r is small the number of remaining coins N (and the density ρ) is large, so collisions are frequent and T_r is small. As N_r increases, N and ρ decrease, collisions become rarer, and the time T_r grows. The plot shows that the decay is faster when N (and ρ) are large than when N (and ρ) are small. This indicates that the rate of loss of the validity of our approximations about ρ is highest during the start of the trajectory, but becomes slower later when collisions occur less often.

Conclusion

Within this report we used Poisson and diffusion approximations to estimate the collision rate, mean free time/path, MFPT, and the decay law for the number of coins, and compared these results with Monte Carlo simulations. The approximations work well when coin density is high but fail when low so we offered curve fitting to improve on our results.

Our results provided general approaches to approximate microscopic properties(e.g. mean free path) of similar systems, and how to use these microscopic properties to approximate macroscopic properties such as the MFPT or decay laws, and means for testing these approximations through simulations. Further, as previously discussed these approaches can be used to approximate properties and evaluate models of real phenomena [4],[5].

In this report we found that the fitted curve in Figure 2 lacks justification. A different form may therefore provide a better fit to the Monte Carlo data. The model could also be extended to include multiple ladybirds or the system could be brought to three dimensions (a spherical room with spherical coins). This would bring the system closer to physical systems such as the Lorentz gas [4].

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