AMA3020 - Full derivations

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This document provides more complete proofs for the results that were presented in our submitted report.

A ladybird starts at the centre of a circular room of radius R containing N coins of diameter d. The ladybird moves at constant speed v along straight lines, choosing a random direction after every collision with a coin (which is then removed) or the wall.

1 Assumptions

1. The room is modelled by a circle of radius R, and the ladybird is treated as a point-like particle.
2. Collisions occur when the ladybird falls within a distance of $\frac{d}{2}$ of a coin, at which point the coin is removed.
3. After every collision, the ladybird reorientates it's directions uniformly at random in $[0, 2\pi)$.
4. At relevant timescales, coin centres are treated as independent and uniformly distributed. After each coin is removed, we assume this counties to hold.
5. We assume coin overlaps have a negligible effect on collision properties.
6. Over a reasonable timescale, the area swept by a trajectory can be approximated by $A(t) \approx vdt$.
7. For large coin density ρ and small d on microscopic scales, the number of collisions on a straight trajectory is approximated by a Poisson Process.
8. When average distances travelled between collisions ℓ is much smaller than R, we approximate macroscopic motion as a diffusive process.
9. In the diffusive process, we replace step lengths and step times by their means τ and ℓ.
10. We approximate the number of coins $N(t)$ at time t by its expectation.

2 Proposition 1

We consider the average time between collisions (the mean free time τ), the average distance between collisions (the mean free path ℓ), and the average number of collisions per unit time (the collision rate λ).

Proposition 1. *The collision rate λ, mean free time τ, and mean free path ℓ are approximated by*

$$\lambda = \frac{vdN}{\pi R^2}, \qquad \tau = \frac{\pi R^2}{vdN}, \qquad \ell = \frac{\pi R^2}{dN}.$$

Proof. Figure 1, shows the part of a typical trajectory we will be considering for Proposition 1.

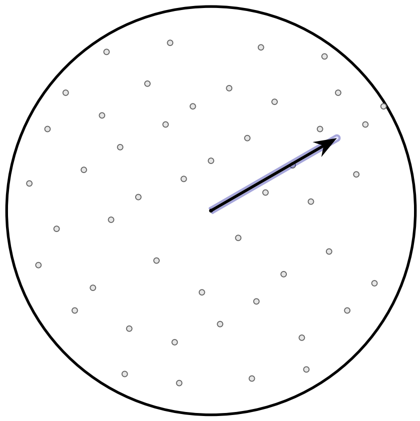

Figure 1

Proof. Let τ_1 denote the time to the first encounter with a coin. Model the ladybird as a point particle moving at constant speed v in a fixed direction until the first hit. A hit occurs whenever the ladybird comes within distance $d/2$ of the centre of a coin.

Equivalently, a hit occurs if a coin centre lies within the loci of radius $d/2$ around the trajectory. After time t, the tube has length vt and width d, with semicircles of radius $d/2$ on the top and bottom of the rectangle, hence swept area

$$A(t) = vtd + \frac{\pi d^2}{4}. \tag{1}$$

For typical times we have $vt \gg d$, so we neglect the end semicircles and use

$$A(t) \approx vtd. \tag{2}$$

Assume coins are independently and uniformly distributed over the circle of area πR^2, with the number of coins per unit area given

$$\rho = \frac{N}{\pi R^2}. \tag{3}$$

For large N and small d (and on scales $\ll R$), the number of coin centres falling in a region of area A is approximated by a Poisson random variable with mean ρA. Therefore,

$$\mathbb{P}(\tau_1 > t) = \mathbb{P}(\text{no coin centers in } A(t)) = \exp\big(-\rho A(t)\big) \approx \exp\left(-\frac{N}{\pi R^2} vdt\right) \tag{4}$$

Hence τ_1 is approximately exponential with rate

$$\lambda = \frac{vdN}{\pi R^2}, \tag{5}$$

and the mean time to the first encounter is

$$\mathbb{E}[\tau_1] = \frac{1}{\lambda} = \frac{\pi R^2}{vdN} \tag{6}$$

From this, we can find the mean distance travelled before the first encounter, given as

$$\ell_1 = v\,\mathbb{E}[\tau_1] = \frac{\pi R^2}{dN} \tag{7}$$

We note that due to our assumptions, we can extend our results to the mean time and distance between consecutive collisions. Which we call the mean free time(τ) and the mean free path ℓ, respectively. Given by

$$\tau = \frac{1}{\lambda} = \frac{\pi R^2}{vdN} \qquad \ell = v\,\tau = \frac{\pi R^2}{dN}$$

$\square$

3 Proposition 2

We consider extending our results to find the average time taken for the ladybird to reach the wall (mean first passage time (MFPT)).

Proposition 2. *The mean first passage time (MFPT) T_{wall} is approximated by*

$$T_{wall} = \frac{dN}{\pi v}.$$

Figure 2, shows the part of a typical trajectory we will be considering for Proposition 2.

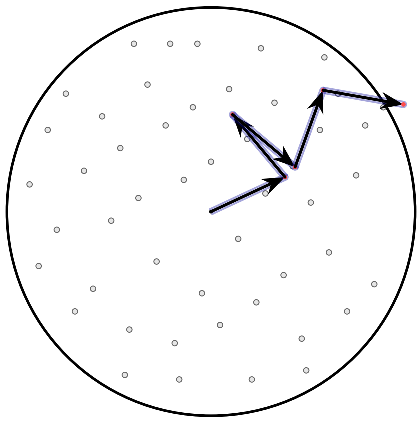

Figure 2

We assume the mean free path is much smaller than the room size,

$$\ell \ll R,$$

so that on macroscopic scales (i.e. distances comparable to R) the motion can be approximated by a 2D diffusion.

From Problem 1, coin centres have density

$$\rho = \frac{N}{\pi R^2},$$

and the ladybird sweeps area per unit time $\approx vd$. Hence, collisions occur as a Poisson process of rate

$$\lambda \approx vdn = vd\,\frac{N}{\pi R^2}.$$

Therefore the mean free time is

$$\tau = \frac{1}{\lambda} = \frac{\pi R^2}{vdN},$$

and the corresponding mean free path is

$$\ell = v\tau = \frac{\pi R^2}{dN}.$$

We model the motion as a random walk made of straight lines between reorientations. On the diffusive scale, we replace the random inter-collision time by its mean τ (and hence the step length by its mean $\ell = v\tau$), and take the direction to be isotropic. Thus, we write the step vector as

$$\mathbb{E}[\mathbf{s}_i] = \ell\,\mathbb{E}[\hat{\mathbf{e}}_i],$$

where $\hat{\mathbf{e}}_i$ are i.i.d. unit vectors with directions uniform on $[0, 2\pi)$.

After k steps,

$$\mathbf{r}_k = \sum_{i=1}^{k} \mathbf{s}_i, \qquad |\mathbf{r}_k|^2 = \sum_{i=1}^{k} |\mathbf{s}_i|^2 + 2 \sum_{1\le i<j\le k} \mathbf{s}_i \cdot \mathbf{s}_j.$$

By independence and isotropy, for $i \neq j$ we have

$$\mathbb{E}[\mathbf{s}_i \cdot \mathbf{s}_j] = \ell^2\,\mathbb{E}[\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j] = 0,$$

hence

$$\mathbb{E}[|\mathbf{r}_k|^2] = k\,\mathbb{E}[|\mathbf{s}|^2] \approx k\,\ell^2.$$

Let $k(t)$ be the number of steps by time t. On average,

$$\mathbb{E}[k(t)] \approx \frac{t}{\tau}.$$

Therefore the mean-square displacement satisfies

$$\mathbb{E}[|\mathbf{r}(t)|^2] \approx \mathbb{E}[k(t)]\,\ell^2 \approx \frac{t}{\tau}\,\ell^2 = \frac{t}{\tau}(v\tau)^2 = v^2\tau\,t.$$

For a diffusion process in 2D we define the diffusion constant D by (see [1])

$$\mathbb{E}[|\mathbf{r}(t)|^2] = 4Dt \qquad (t \to \infty),$$

so matching coefficients gives

$$D = \frac{v^2\tau}{4} = \frac{v\ell}{4}.$$

Let $T(r)$ be the mean time for a 2D Brownian motion with diffusion constant D to exit the circle of radius R, starting at distance r from the centre. As stated in [1], T satisfies

$$D\nabla^2 T(r) = -1, \qquad T(R) = 0,$$

and radial symmetry implies $\nabla^2 T = T''(r) + \frac{1}{r}T'(r)$. Solving with regularity at $r = 0$ yields

$$T(r) = \frac{R^2 - r^2}{4D}.$$

Starting at the centre ($r = 0$),

$$\mathbb{E}[T_{\text{wall}}] = T(0) = \frac{R^2}{4D}.$$

Substituting $D = \frac{v^2\tau}{4}$ and $\tau = \frac{\pi R^2}{vdN}$ gives

$$\mathbb{E}[T_{\text{wall}}] = \frac{R^2}{4(v^2\tau/4)} = \frac{R^2}{v^2\tau} = \frac{dN}{\pi v}.$$

□

4 Proposition 3

Proposition 3. *Suppose there are $N(t)$ coins in the room at time t and there are N_0 coins in the room at $t = 0$. Then $N(t)$ is approximated by*

$$N(t) = N_0 \exp(-\frac{vd}{\pi R^2}t)$$

Proof. Let $N(t)$ be the (random) number of coins remaining at time t, and let $A = \pi R^2$ be the room area. We approximate the remaining coins as uniformly distributed at all times, with instantaneous density

$$\rho(t) \approx \frac{N(t)}{A}. \tag{8}$$

In a small interval dt, the ladybird travels distance $v\,dt$ and sweeps out (approximately) a tube of width d, hence area

$$dA_{\text{swept}} \approx v\,dt \cdot d. \tag{9}$$

The expected number of coin centers in this swept area is density × area:

$$\mathbb{E}[\#\text{ hits in }(t, t+dt) \mid N(t)] \approx \rho(t)\,dA_{\text{swept}} \approx \frac{N(t)}{A}\,vd\,dt. \tag{10}$$

Each hit removes exactly one coin, so taking expectations assuming $\mathbb{E}[N(t)] \approx N(t)$ gives the approximation

$$\frac{dN}{dt} \approx -\frac{vd}{A}\,N(t) = -\frac{vd}{\pi R^2}\,N(t). \tag{11}$$

With initial condition $N(0) = N_0$,

$$N(t) \approx N_0 \exp\left(-\frac{vd}{\pi R^2}\,t\right). \tag{12}$$

□

References

[1] S. Redner, *A Guide to First-Passage Processes*, Cambridge University Press (2001).