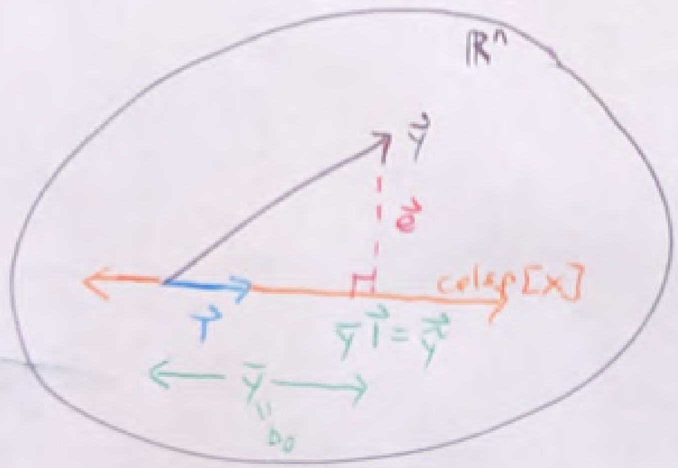
Lecture 10 3/3/21

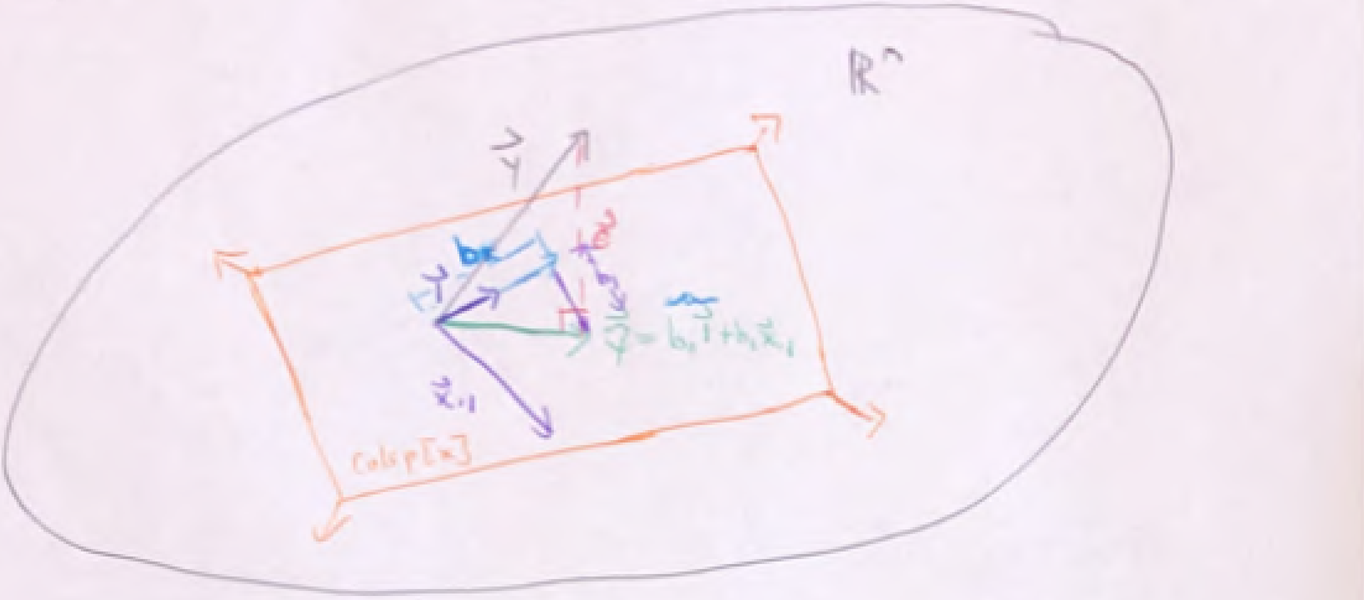
Let's examine the null model, $p = 0$ so that $X = [\vec{1}_n] \Rightarrow \vec{b} = b_0 = \bar{y}$

$$H = X(\underbrace{\underbrace{X^T X}_{n}}_{\frac{1}{n}})^{-1} X^T = \frac{1}{n} \underset{n\times 1}{\vec{1}} \underset{1\times n}{\vec{1}^T} = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$$

$$\hat{\vec{y}} = H\vec{y} = \begin{bmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix} = \bar{y}\vec{1}_n$$



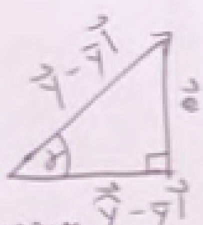
Consider $p=1$ so that $X = [\vec{1}\ \vec{x}_{\cdot 1}]$



pythagorean Thm.

$\Rightarrow \|\hat{\vec{y}}\|^2 + \|\vec{e}\|^2 = \|\vec{y}\|^2, \quad \cos^2(\gamma) = \dfrac{\|\hat{\vec{y}}\|^2}{\|\vec{y}\|^2}$

Is the following illustration accurate? Yes.



$\vec{e} := \vec{y} - \hat{\vec{y}} = \vec{y} - \bar{y}\vec{1} + \bar{y}\vec{1} - \hat{\vec{y}} = (\vec{y} - \bar{y}\vec{1}) - (\hat{\vec{y}} - \bar{y}\vec{1})$

$\text{Proj}_{\text{colsp}[X]}(\vec{y} - \bar{y}\vec{1}) = H(\vec{y} - \bar{y}\vec{1}) = H\vec{y} - \bar{y}\underbrace{H\vec{1}}_{\text{Proj}_{\text{colsp}[X]}(\vec{1}) \,=\, \vec{1}}$

$= \hat{\vec{y}} - \bar{y}\vec{1}$ ✓

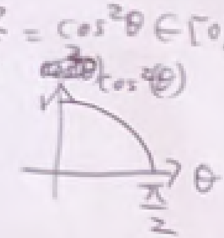
pythagorean thm $\Rightarrow$

$\|\vec{y} - \bar{y}\vec{1}\|^2 = \|\hat{\vec{y}} - \bar{y}\vec{1}\|^2 + \|\vec{e}\|^2,$

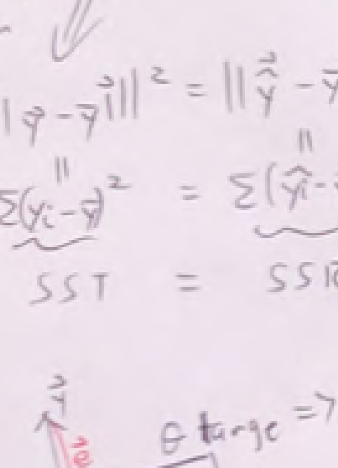
$\underbrace{\sum(y_i - \bar{y})^2}_{SST} = \underbrace{\sum(\hat{y}_i - \bar{y})^2}_{SSR} + \underbrace{\sum e_i^2}_{SSE}$

$SST = SSR + SSE$

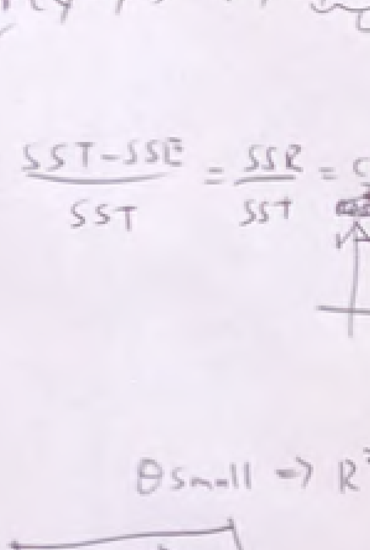
$R^2 = \dfrac{SST - SSE}{SST} = \dfrac{SSR}{SST} = \cos^2\theta \in [0,1]$



θ large $\Rightarrow R^2 \approx 0$



θ small $\Rightarrow R^2 \approx 1$



Back to linear algebra...

By law of cosines,

$$\cos(\theta) = \underbrace{\frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|\|\vec{v}\|}}_{} = \frac{\|\vec{x}\|}{\|\vec{a}\|}$$

def of cosine.

$$\Rightarrow \|\vec{x}\| = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$$

$$\Rightarrow \vec{x} = \|\vec{x}\| \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{\vec{a}^T\vec{v}\vec{v}}{\|\vec{v}\|^2} = \frac{\vec{v}\vec{v}^T\vec{a}}{\|\vec{v}\|^2} = \underbrace{\frac{\vec{v}\vec{v}^T}{\|\vec{v}\|}}_{}\vec{a} = H\vec{a}$$

$$H = \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T = \left[\frac{v_1}{\|\vec{v}\|^2}\vec{v} \,\middle|\, \frac{v_2}{\|\vec{v}\|^2}\vec{v} \,\middle|\, \cdots \,\middle|\, \frac{v_n}{\|\vec{v}\|^2}\vec{v}\right], \quad \text{rank}[H] = 1$$

$$HH = \left(\frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\right)\left(\frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\right) = \frac{1}{\|\vec{v}\|^{\cancel{4}\,2}}\vec{v}\overbrace{\vec{v}^T\vec{v}}^{\|\vec{v}\|^2}\vec{v}^T = \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T = H \checkmark$$

$$V = [\vec{v}_1 | \vec{v}_2] \qquad \underbrace{\text{proj}_V(\vec{a})}_{H\vec{a}} \stackrel{?}{=} \underbrace{\overbrace{\text{proj}_{\vec{v}_1}(\vec{a})}^{H_1\vec{a}} + \overbrace{\text{proj}_{\vec{v}_2}(\vec{a})}^{H_2\vec{a}}}_{} = (H_1 + H_2)\vec{a}$$

sometimes, ...

will always project onto colsp[V] but it may not be the correct length (it can over/under count). The correct length gives you the right angle:

$$\text{proj}_V(\vec{a})^T(\vec{a} - \text{proj}_V(\vec{a})) = 0$$

angle between $\vec{a}$ and $\vec{v}$.

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\cos(\theta)$$

$$\Rightarrow \text{proj}_{\vec{v}}(\vec{a})^T\vec{a} - \text{proj}_V(\vec{a})^T\text{proj}_V(\vec{a})$$

$$= (H_1\vec{a} + H_2\vec{a})^T\vec{a} - (H_1\vec{a} + H_2\vec{a})^T(H_1\vec{a} + H_2\vec{a}) = (\vec{a}^TH_1 + \vec{a}^TH_2)\vec{a} - \|H_1\vec{a} + H_2\vec{a}\|^2$$

$$= \cancel{\vec{a}^TH_1\vec{a}} + \cancel{\vec{a}^TH_2\vec{a}} - \underbrace{\underbrace{\underbrace{\|H_1\vec{a}\|^2}_{(H_1\vec{a})^T(H_1\vec{a})}}_{\vec{a}^TH_1H_1\vec{a}}}_{\vec{a}^TH_1\vec{a}} - \underbrace{\underbrace{\underbrace{\|H_2\vec{a}\|^2}_{(H_2\vec{a})^T(H_2\vec{a})}}_{\vec{a}^TH_2H_2\vec{a}}}_{\vec{a}^TH_2\vec{a}} - 2\overbrace{\|H_1\vec{a}\|}^{>0}\overbrace{\|H_2\vec{a}\|}^{>0}\underbrace{\cos(\theta)}_{\in[0,1]}$$

angle between $\vec{v}_1, \vec{v}_2$

The only way to make this expression zero is if $\cos(\theta) = 0$ i.e. θ = a right angle. Thus, the full projection is a sum of the ~~component~~ projections if the components are orthogonal.

Let $V = [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_d] \in \mathbb{R}^{n \times d}$, $\forall i j \quad \vec{v}_i \cdot \vec{v}_j = 0$

$$V = \left.\begin{bmatrix} \vec{v}_1 \dots \vec{v}_d \end{bmatrix}\right\} n$$

$$\Rightarrow \text{proj}_{\text{colsp}[V]}(\vec{a}) = \text{proj}_{\vec{v}_1}(\vec{a}) + \dots + \text{proj}_{\vec{v}_d}(\vec{a})$$

$$= \frac{\vec{v}_1 \vec{v}_1^T}{\|\vec{v}_1\|^2} \vec{a} + \dots + \frac{\vec{v}_d \vec{v}_d^T}{\|\vec{v}_d\|^2} \vec{a}$$

$$= \left(\frac{\vec{v}_1 \vec{v}_1^T}{\|\vec{v}_1\|^2} + \dots + \frac{\vec{v}_d \vec{v}_d^T}{\|\vec{v}_d\|^2} \right) \vec{a} = \left(\vec{v}_1 \vec{v}_1^T + \dots + \vec{v}_d \vec{v}_d^T \right) \vec{a}$$

$$= \underbrace{\left(\sum_{i=1}^{d} v_i v_i^T \right) \vec{a}}_{H}$$

If $\|\vec{v}_1\| = \|\vec{v}_2\| = \dots = \|\vec{v}_d\| = 1$, i.e. all unit length

$\rightarrow Q = [\vec{v}_1 | \dots | \vec{v}_d]$, which is an "orthonormal matrix"

$$\underbrace{Q^T Q}_{d \times d} = \begin{bmatrix} \leftarrow \vec{v}_1^T \rightarrow \\ \vdots \\ \leftarrow \vec{v}_d^T \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \dots & \vec{v}_d \\ \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix} = I_d$$

$$\underbrace{Q Q^T}_{n \times n} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_d \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow \vec{v}_1^T \rightarrow \\ \leftarrow \vec{v}_2^T \rightarrow \\ \vdots \\ \leftarrow \vec{v}_d^T \rightarrow \end{bmatrix} = \vec{v}_1 \vec{v}_1^T + \vec{v}_2 \vec{v}_2^T + \dots + \vec{v}_d \vec{v}_d^T = H$$

$$= [A_1 A_2 \dots A_d] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_d \end{bmatrix} = A_1 B_1 + A_2 B_2 + \dots + A_d B_d$$

$$\Rightarrow Q Q^T = V (V^T V)^{-1} V^T = H$$

where the columns of Q are the orthonormalized columns of $V = [v_1 | \dots | v_d]$. Further $\text{colsp}[Q] = \text{colsp}[V]$ since the column vectors in Q represents a change of basis of the column vectors of V.

$$\text{proj}_{\text{colsp}[Q]}(\vec{a}) = Q\,\underbrace{(Q^TQ)^{-1}}_{\underbrace{(I)^{-1}}_{I}}\,Q^T = QQ^T$$

How can we convert matrix V to matrix Q? There is a computational algorithm called "Gram-Schmidt" and during the computation, you can collect a matrix that is the change of basis:

$$\underset{n\times d}{V} = \underset{n\times d}{Q}\ \underset{d\times d}{R} \Rightarrow VR^{-1} = Q$$

$\text{colsp}[V] = \text{colsp}[Q]$

This is also called Q-R decomposition of a matrix.
R will be upper triangular and full rank (and invertible).