Lecture 9 3/3/10

$n = 100$, $x = 43$, $P(\theta) = B(1,1) \Rightarrow \underbrace{P(\theta \mid x) = \text{Beta}(44, 58)}$

$\alpha_0 = 5\%$, $\delta = 1\%$

pdf of Beta

$H_0: \theta \in [.49, .51]$

$P_{val} = P(H_0 \mid x) = \int_{.49}^{.51} d\theta = \text{pbeta}(.51, 44, 58) - \text{pbeta}(.49, 44, 58)$

$\quad\quad \|$

$F_{\theta|x}(\theta_0 + \delta) - F_{\theta|x}(\theta_0 - \delta)$

$= 0.06 \Rightarrow$ Retain H_0

$\not< \alpha_0 = 5\%$

Sentence: There is insufficient evidence to prove this coin is unfair.

Last topic before midterm.

$\mathcal{F}: \text{Bin}(n, \theta)$ with n fixed, $P(\theta) = \text{Beta}(\alpha, \beta) \Rightarrow P(\theta | x) = \text{Beta}(\alpha + x, \beta + n - x)$

Laplace $P(\theta) = \text{Beta}(1,1) \Rightarrow n_0 = 2$ pseudotrials $x_0 = 1$ pseudosuccesses.
Laplace's uniform prior is "flat" in an effort to be "objective", i.e. let the data speak for itself and not be "subjective", i.e. allow your personal biases to be part of your inferential conclusion.

Can we be more objective? Can we create a prior that has no part in the inferential conclusion? This would mean $n_0 = 0$. How about $\alpha = \beta = 0$.

$$P(\theta) = \text{Beta}(0,0) = \frac{1}{B(0,0)} \theta^{-1}(1-\theta)^{-1} \Big\} \text{ not a PDF}$$

There is a problem with this. The parameter space for the beta is $\alpha > 0$ and $\beta > 0$. If $\alpha = \beta = 0$, this is not a PDF since its integral over the support diverges. This makes it an "improper prior" since it is not a true variable.
But do we care? Churning through the math, we get the posterior:
$P(\theta | x) = \text{Beta}(x, n-x)$
This posterior is proper as long as $x < n$ and $x > 0$, which means you need to have at least one success and at least one failure in your data. If its proper, you have full bayesian inference: point estimates, CR's, p-vals...

$$\hat{\theta}_{MMSE} = \frac{x}{n} = \hat{\theta}_{MLE} = \bar{x}$$

Also, $\rho = 0$ (no shrinkage). I believe this prior was first introduced by Haldane in 1932 so we'll call it the "Haldane prior."

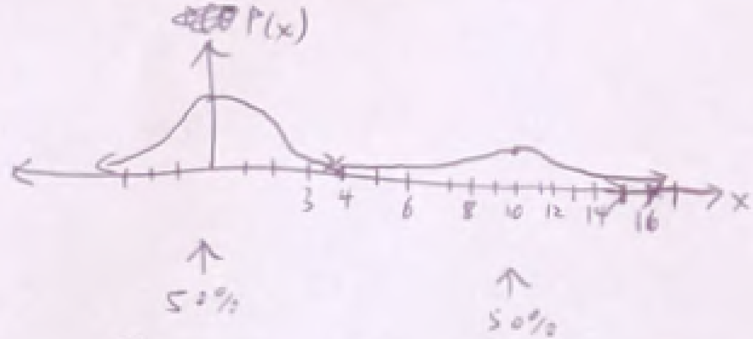
Midterm I ↑

Midterm II

↓

Back to $\frac{\text{Unit II - 341}}{\text{probability}}$ class... We will introduce mixture / compound distributions e.g.

$$X \sim \begin{cases} \underbrace{N(0,1^2)}_{\text{model}} & \text{w.p. } \underbrace{\tfrac{1}{2}}_{\text{mixture}} \\ N(10, 2^2) & \text{w.p. } \tfrac{1}{2} \end{cases}$$



Integrate the whole to get 100%

$$P(x) = \int_{\Theta} P(x, \vec{\theta})\, d\vec{\theta} \text{ if } \vec{\theta} \text{ is continuous}$$

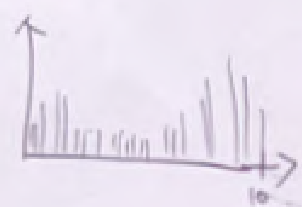
$$\hookrightarrow = \sum_{\vec{\theta} \in \Theta} P(x, \vec{\theta}) \text{ if } \vec{\theta} \text{ is discrete}$$

$$\hookrightarrow = \sum_{\vec{\theta} \in \Theta} P(x \mid \vec{\theta})\, P(\vec{\theta})$$

$$\hookrightarrow = \underbrace{\frac{1}{\sqrt{2\pi(1)^2}} e^{\frac{1}{-2\cdot 1^2}(x-0)^2}(0.5) + \frac{1}{\sqrt{2\pi(2)^2}} e^{\frac{-1}{2\cdot 2^2}(x-10)^2}(0.5)}_{\text{PDF for above graph.}}$$

$$X \sim \begin{cases} \text{Bin}(10, 0.1) & \text{w.p. } \tfrac{1}{4} \\ \text{Bin}(10, 0.8) & \text{w.p. } \tfrac{3}{4} \end{cases}$$

$$P(X) = \binom{10}{x} 0.1^x (0.9)^{10-x} \cdot \left(\tfrac{1}{4}\right) + \binom{10}{x} 0.8^x (.2)^{10-x}$$



Have we seen P(x) before that's the result of a margining, making P(X) a mixture / compound distribution? Yes...

$$P(\theta \mid x) = \frac{P(x \mid \theta)\, P(\theta)}{P(x)} = \text{Beta}(\alpha + x, \beta + n - x)$$

$$P(x) = \int_{\Theta} P(x \mid \theta) P(\theta)\, d\theta$$

F: Bin(n, θ) n fixed, $P(\theta) = \text{Beta}(\alpha, \beta)$

$$P(x) = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta = \frac{\binom{n}{x}}{B(\alpha,\beta)} \overbrace{\int_0^1 \theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1} d\theta}^{\text{beta function}}$$

$$= \frac{\binom{n}{x}}{B(\alpha,\beta)} B(\alpha+x, \beta+n-x) = \text{BetaBinomial}(n, \alpha, \beta)$$ ← took the binomial and compounded it w/ the Beta function.

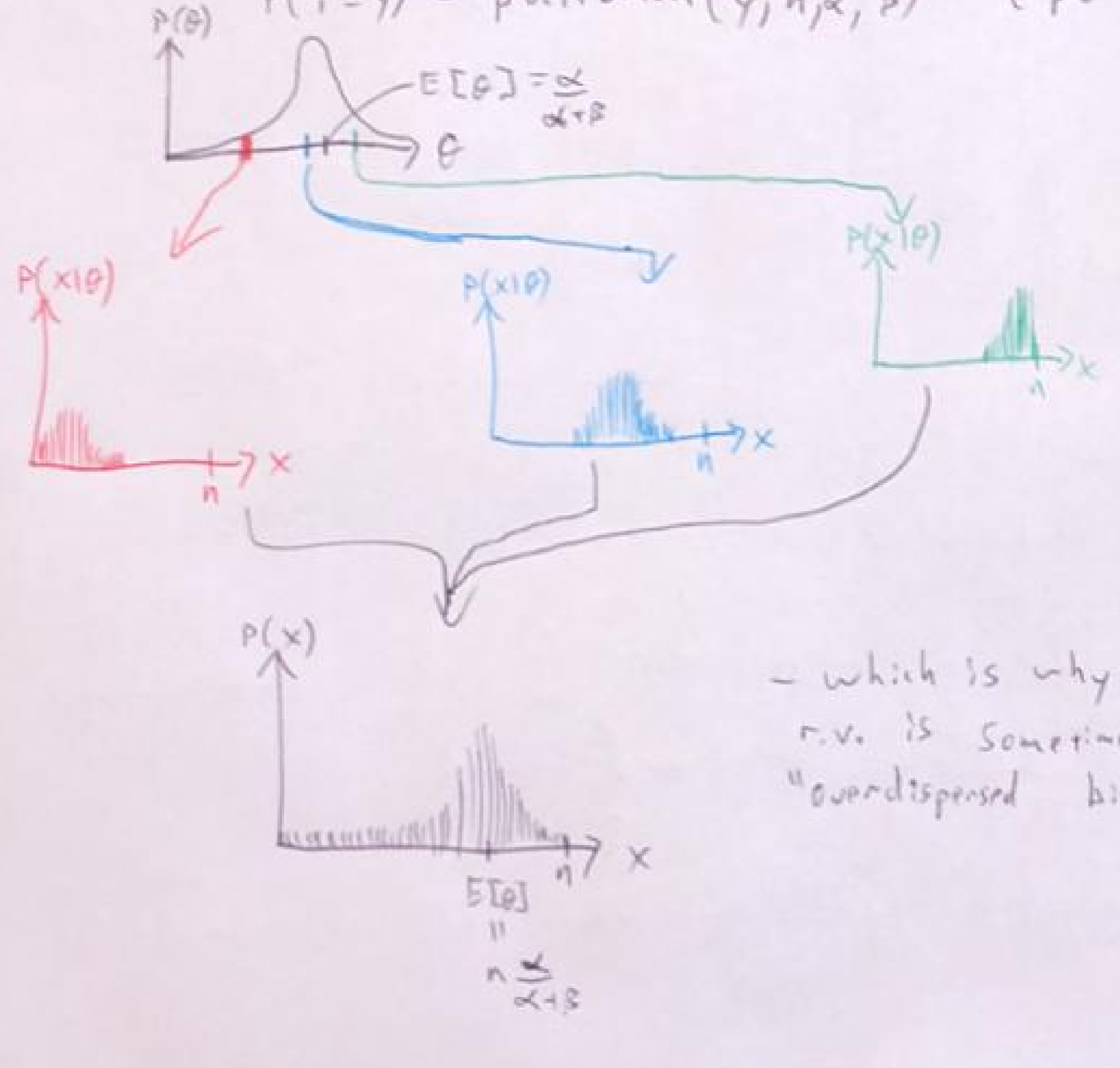
$Y \sim \text{BetaBinomial}(n, \alpha, \beta)$, $\text{Supp}[Y] = \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$

$E[Y] = \ldots = n\frac{\alpha}{\alpha+\beta}$, $\text{Var}[Y] = \ldots = n\frac{\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Since the beta function is not available in closed form, the PMF/CDF are not available in closed form. To compute, you need a computer. Here's the notation we'll use in this class (the R notation):

$P(Y=y) = \text{dbetabinom}(y, n, \alpha, \beta)$ (d = PMF or PDF)

$P(Y \le y) = \text{pbetabinom}(y, n, \alpha, \beta)$ (p = CDF)



– which is why the betabinomial r.v. is sometimes called the "overdispersed binomial."

Let $\theta := \dfrac{\alpha}{\alpha+\beta} \Rightarrow \theta\alpha + \theta\beta = \alpha \Rightarrow (\theta-1)\alpha = -\theta\beta \Rightarrow \beta = \alpha\left(\dfrac{1-\theta}{\theta}\right)$

$\Downarrow$

$E[x] = n\theta$ an ~~[illegible]~~ intuitive formula for the betabinomial expectation since it is the same as binomial expectation.

let $\alpha \to \infty$ but keep $\theta = \dfrac{\alpha}{\alpha+\beta}$ constant

$$\lim_{\alpha\to\infty} Var[x] = \lim_{\alpha\to\infty} \frac{n\alpha\left(\alpha\frac{1-\theta}{\theta}\right)\left(\alpha+\alpha\frac{1-\theta}{\theta}+n\right)}{\left(\alpha+\alpha\frac{1-\theta}{\theta}\right)^2\left(\alpha+\alpha\frac{1-\theta}{\theta}+1\right)} =$$

$$= n\lim_{\alpha\to\infty}\frac{\frac{1-\theta}{\theta}}{\left(1+\frac{1-\theta}{\theta}\right)^2}\lim_{\alpha\to\infty}\overset{\to 1}{\cancel{\frac{\alpha+\alpha\left(\frac{1-\theta}{\theta}\right)+n}{\alpha+\alpha\left(\frac{1-\theta}{\theta}\right)+1}}}$$

$$=$$