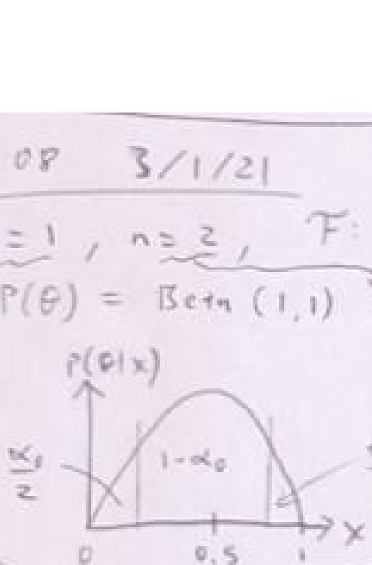
Lecture 08 3/1/21

$x=1$, $n=2$, $\mathcal{F}: \text{Bin}(n,\theta)$

$P(\theta) = \text{Beta}(1,1) \Rightarrow P(\theta \mid x) = \text{Beta}(2,2)$



Produce a 95% credible region for θ.

$$CR_{\theta,1-\alpha_0} := \left[Q\left[\theta \mid x, \tfrac{\alpha_0}{2}\right], Q\left[\theta \mid x, 1-\tfrac{\alpha_0}{2}\right]\right]$$

$$= [\text{qbeta}(2.5\%, 2, 2), \text{qbeta}(97.5\%, 2, 2)]$$

$$= [0.094, 0.906]$$

$$P(\theta \in [.094, .906] \mid x) = 95\%$$

This is a real probability statement! The CI approach cannot give you such a statement! The CR is highly interpretable. Also, the CR is a proper subset of Θ for $\alpha_0 > 0$. This is not always true with CI's e.g. here's the 95% for this data:

$$CI_{\theta,95\%} = \left[0.5 \pm 1.96\sqrt{\frac{(0.5)(0.5)}{2}}\right] = [-0.21, 1.21] \not\subset \Theta = (0,1)$$

The above CR is technically a two-sided CR. You can also create one-sided (i.e. left-sided or right-sided) CR's:

$$CR_{L,\theta,1-\alpha} := [\text{smallest value in } \Theta \text{ or } -\infty, Q[\theta \mid x, 1-\alpha_0]]$$

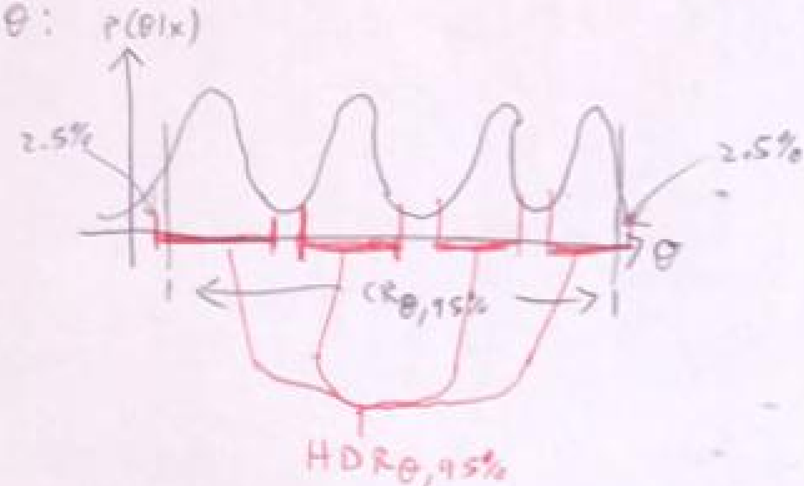
e.g. in our dataset $\cong [0, 0.865] \Rightarrow P(\theta < 0.865 \mid x) = 95\%$

qbeta(.95, 2, 2)

$CR_{x,\theta,1-\alpha_0} := \left[Q[\theta | x, \alpha_0], \text{ largest value in } \Theta \text{ or } +\infty \right]$

e.g. in our dataset $\overset{?}{=} \left[\text{qbeta}(0.5, 2, 2), 1 \right] = [.136, 1] \Rightarrow P(\theta > .136 | x) = 95\%$

Another approach (which we will see but not study further) is called the high density region (HDR) approach. Consider the following posterior for θ:



$\alpha_0 = 5\%$

$P(\theta \in HDR_{\theta,95\%}) = 95\%$ but it has *minimum* width (the sum of the interval pieces is minimum).

Sometimes the CR = HDR (e.g. in unimodal posteriors).

Disadvantages of the HDR approach. (1) it can be non-contiguous i.e. in pieces! (2) it's ~~approach~~ computationally intense (3) no L or R intervals

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Bayesian hypothesis testing. We can immediately compute the following quantities:

Bayesian p-value $:= P(H_0 | x),\ P(H_a | x)$

if $P(H_0 | x) < \alpha_0 \Rightarrow$ Reject H_0

$\uparrow$ threshold of "sufficient evidence"

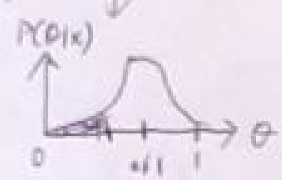
Let's recreate the hypothesis testing example from Lec. 3, $n = 100$ flips of a coin where $x = 61$ were heads. Test if the coin is unfairly weighted towards heads at a 5% significance level.

$H_a: \theta > 0.5 \Rightarrow H_0: \theta \leq 0.5$ Assume $P(\theta) = \text{Beta}(1,1)$

$P(\theta | x) = \text{Beta}(\underset{x+\alpha}{62}, \underset{n-x+\beta}{40})$

$P(H_0 | x) = P(\theta \leq 0.5 | x) = \int_0^{0.5} P(\theta | x)\, d\theta$

$\int_0^{0.5} \frac{1}{B(62,40)} \theta^{61} (1-\theta)^{39} d\theta$



$= \text{pbeta}(0.5, 62, 40) = .014 = 1.4\% \Rightarrow$ Reject H_0 i.e. coin is unfairly weighted towards heads.

Uber driver does $\underbrace{200}_{n}$ rides and gets $\underbrace{37}_{x}$ non-star ratings. If his true proportion of non-5-star ratings is more than 25%, then Uber policy is to fire the driver. Prove he should be fired (or not) at a 5% significance level. $H_a: \theta < 25\% \Rightarrow H_0: \theta \geq 25\%$. Assume. $P(\theta) = \text{Beta}(1,1)$

$$\mathcal{F}: \text{Bin}(n, \theta)$$

$$= P(\theta \mid x) = \text{Beta}(38, 164)$$

$$P(H_0 \mid x) = P(\theta \leq .25 \mid x) = \int_0^{.25} \frac{1}{B(38,164)} \theta^{37}(1-\theta)^{163} d\theta$$

$$= \text{pbeta}(.25, 38, 164)$$

$= \text{pbeta}(.25, 38, 164) = .983 > 5\% \Rightarrow$ Retain H_0 Don't fire him.

Lets test the coin again. Flip 100 times and get 43 heads. Test if the coin is unfair at 5% significance.

$$H_a: \theta \neq 0.5 \Rightarrow H_0: \theta = 0.5$$

$$P(\theta) = \text{Beta}(1,1) \Rightarrow P(\theta \mid x) = \text{Beta}(44, 50)$$

$P(H_0 \mid x) = P(\theta = 0.5 \mid x) = 0 \Rightarrow$ Reject H_0 always?? Yes...

Using this approach, two-sided tests are always rejected (if the posterior is continuous). Does this make sense? Any infinitely precise theory of θ is wrong in the real world. A coin is never exactly 50.0000...% likely to flip heads. So we need to slightly reframe our hypotheses using a notation of "margin of equivalence" called δ (delta).

$$H_a: \theta \notin [\theta_0 + \delta] \Rightarrow H_0: \theta \in [\theta_0 \pm \delta]$$