Modular exponentiation

Hyoungshick Kim

Department of Software

College of Software

Sungkyunkwan University

Computation of $C = M^e$

- Must compute $C = M^e \bmod n$ and $P = C^d \bmod n$ for very large n, e, d, P, and C
  - Example: $3{,}000{,}000^{12007}$
- Problem: M^e is a <u>really big</u> number!
  - Too large for fast computation or even storage on most systems
  - M^e requires exponential time

Big integer implementation

- A big integer will be represented using an array of digits in base B. For example, the following struct can be used:

```
typedef struct {
int sign;
int size;
int *tab; } bignum;
```

  where sign is the sign bit, and size is the size of the dynamic array tab.

- Big number libraries (https://www.openssl.org/docs/crypto/bn.html, https://gmplib.org/, http://www.shoup.net/ntl/)

How can we reduce multiplications?

- We can simply compute $3{,}000{,}000^{12007}$ by multiplying 3,000,000 12007 times.
- However, it requires exponential time.
- Fast modular exponentiation is required.

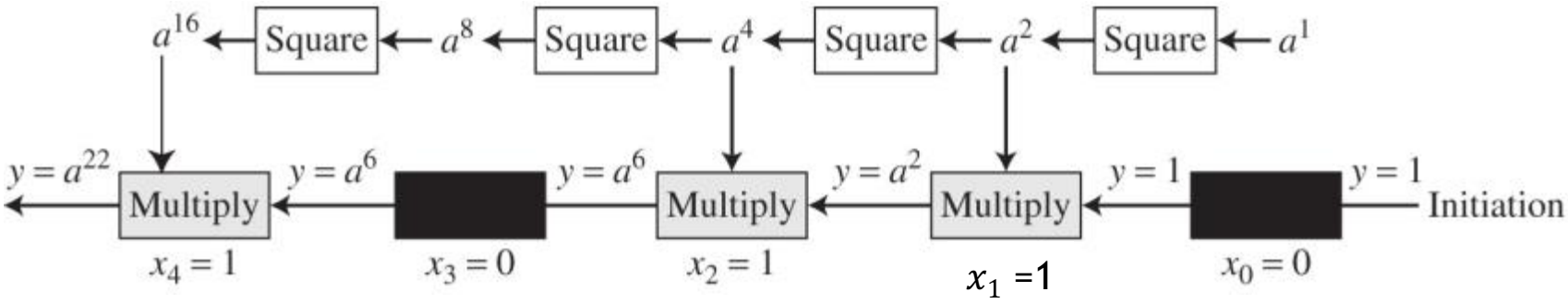
Do you have any idea about this?

How to calculate exponentiation

- Modular exponentiation example
  - $5^{20} \bmod 35 = 95367431640625 \bmod 35 = 25 \bmod 35$
- A better way: **repeated squaring**
  - $20 = 10100$ base 2
  - $(1, 10, 101, 1010, 10100) = (1, 2, 5, 10, 20)$
  - Note that $2 = 1 \cdot 2$, $5 = 2 \cdot 2 + 1$, $10 = 2 \cdot 5$, $20 = 2 \cdot 10$
  - $5^1 = 5 \bmod 35$
  - $5^2 = (5^1)^2 = 5^2 = 25 \bmod 35$
  - $5^5 = (5^2)^2 \cdot 5^1 = 25^2 \cdot 5 = 3125 = 10 \bmod 35$
  - $5^{10} = (5^5)^2 = 10^2 = 100 = 30 \bmod 35$
  - $5^{20} = (5^{10})^2 = 30^2 = 900 = 25 \bmod 35$
- No huge numbers and it's efficient!

Square and multiply method

- Break exponent ***e*** into product of powers of 2
  - Example: $a^{22} = a^{16} \times a^4 \times a^2$
  - Can be represented as "bits": 22 → 10110
- Use **squaring** to compute powers of 2 quickly
  - $a \rightarrow a^2 \rightarrow a^4 \rightarrow a^8 \rightarrow a^{16}$
- Multiply by running total if corresponding bit = 1

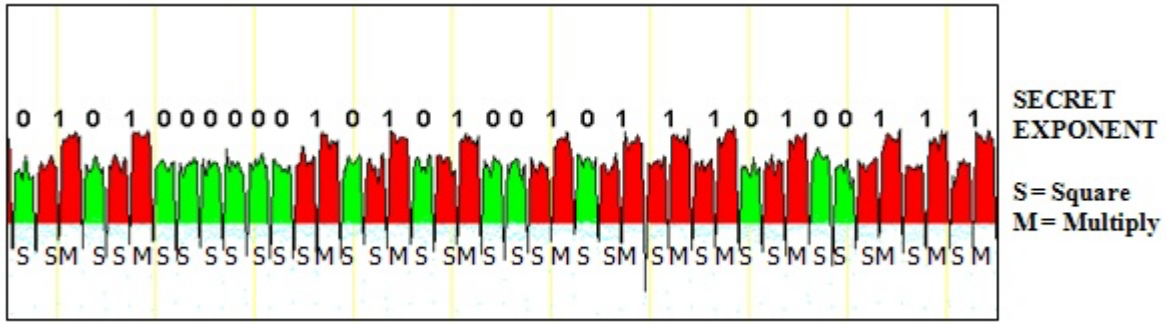


Pseudo code to compute M^e mod n

```
result = 1
for (i = 0 to number of bits in e - 1)
{
   if (i^th bit == 1)
      result = (result * M) mod n
   M = M^2 mod n
}
```

Simple power analysis

A power trace of a portion of an RSA exponentiation operation might recover the secret exponent *d*. Why?



(See https://www.youtube.com/watch?v=cPDDNVKo43w)

See the code!

How to prevent this?

```
result = 1
for (i = 0 to number of bits in e - 1) {
  if (i^th bit == 1)
     result = (result * M) mod n
  M = M^2 mod n
}
```

Questions?

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