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例3.1

(1) 第2行-第3行 $\begin{vmatrix} 0 & 2 & -5 \\ 0 & -2 & -8 \\ 2 & 3 & 5 \end{vmatrix} = 2\times(-1)^{3+1}\times\begin{vmatrix} 2 & -5 \\ -2 & -8 \end{vmatrix} = -52$

(2) 原式 $= a\times(-1)^{1+1}\times\begin{vmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a \end{vmatrix} + b\times(-1)^{4+1}\begin{vmatrix} 0 & 0 & b \\ a & b & 0 \\ b & a & 0 \end{vmatrix}$

$= a\times a\times(-1)^{3+3}\times\begin{vmatrix} a & b \\ b & a \end{vmatrix} - b\times b\times(-1)^{1+3}\times\begin{vmatrix} a & b \\ b & a \end{vmatrix}$

$= (a^2-b^2)(a^2-b^2) = (a^2-b^2)^2$

例3.2 P04. 1.

例3.3 P04 2.

例3.4 证: $A^T=-A$ $\therefore |A^T|=|-A|$

由行列式性质有: $|A^T|=|A|$, $|-A|=(-1)^n|A|$

即 $|A|=(-1)^n|A|$ $\because |A|\neq 0$ $\therefore (-1)^n=1$ 即n为偶数.

例3.5 P04. 4 例3.6 P04. 5 例3.7 P05. 6

例3.8 r_2-r_1, $\begin{vmatrix} x-2 & x-1 & x-2 & x-3 \\ x & x & x & x \\ 3x-3 & 3x-2 & 4x-5 & 3x-5 \\ 4x & 4x-3 & 5x-7 & 4x-3 \end{vmatrix} \xrightarrow[C_4-C_1]{C_2-C_1,\ C_3-C_1} \begin{vmatrix} x-2 & 1 & 0 & -1 \\ 2x-2 & 1 & 0 & -1 \\ 3x-3 & 1 & x-2 & -2 \\ 4x & -3 & x-7 & -3 \end{vmatrix}$

$\xrightarrow{C_4+C_2} \begin{vmatrix} x-2 & 1 & 0 & 0 \\ 2x-2 & 1 & 0 & 0 \\ 3x-3 & 1 & x-2 & -1 \\ 4x & -3 & x-7 & -6 \end{vmatrix} = \begin{vmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{vmatrix} = A_{11}\cdot A_{22}$.

$A_{11}=(x-2)-2(x-1)=-x$ $A_{22}=-6(x-2)+(x-7)=-5(x-1)$

$\therefore f(x)=5x(x-1)$, 即有2个根.

例3.9

$$\xrightarrow{r_1-r_2}\begin{vmatrix} x & -x & 0 & 0 \\ -1 & x+1 & -1 & 1 \\ -1 & 1 & x-1 & 1 \\ -1 & 1 & -1 & x+1 \end{vmatrix} \xrightarrow{r_2-r_3} \begin{vmatrix} x & -x & 0 & 0 \\ 0 & x & -x & 0 \\ -1 & 1 & x-1 & 1 \\ -1 & 1 & -1 & x+1 \end{vmatrix} \xrightarrow{c_2+c_1} \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & -x & 0 \\ -1 & 0 & x-1 & 1 \\ -1 & 0 & -1 & x+1 \end{vmatrix}$$

$$= x\begin{vmatrix} x & -x & 0 \\ 0 & x-1 & 1 \\ 0 & -1 & x+1 \end{vmatrix} = x^2\begin{vmatrix} x-1 & 1 \\ -1 & x+1 \end{vmatrix} = x^2[(x-1)(x+1)+1] = x^4$$

$\therefore f(x)=0$ 的根为 $x_1=x_2=x_3=x_4=0$

例3.10

$$\xrightarrow{r_3-2r_1}\begin{vmatrix} \lambda & -2 & 2 \\ -2 & \lambda-4 & -4 \\ 2-2\lambda & 0 & \lambda-1 \end{vmatrix} \xrightarrow{c_1+2c_3} \begin{vmatrix} \lambda+4 & -2 & 2 \\ -10 & \lambda-4 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$=(\lambda-1)[(\lambda+4)(\lambda-4)-20]=0$. 解得：$\lambda_1=1$, $\lambda_2=6$, $\lambda_3=-6$

例3.11 $f(x)=2x^4-3x^3-7x^2+19x-15$ 这道题学了定义再算

$\therefore x^3$ 的系数为 -3

例3.12 $A+B=(\alpha+\beta, 2r_2, 2r_3, 2r_4)$

$|A+B|=|\alpha\ \ 2r_2\ \ 2r_3\ \ 2r_4|+|\beta\ \ 2r_2\ \ 2r_3\ \ 2r_4|$

$=8|\alpha\ \ r_2\ \ r_3\ \ r_4|+8|\beta\ \ r_2\ \ r_3\ \ r_4|$

$=8|A|+8|B|=40$

例3.13 由题得：$n=1$, $D_1=5$, $n=2$, $D_2=\begin{vmatrix} 5 & 3 \\ 2 & 5 \end{vmatrix}=25-6=19$

$$D_3=\begin{vmatrix} 5 & 3 & \\ 2 & 5 & 3 \\ & 2 & 5 \end{vmatrix}=65$$

对 D_n 按第一行展开：

$$D_n = 5D_{n-1} - 3\times \begin{vmatrix} 2 & 3 & & \\ & 5 & \ddots & \\ & \ddots & \ddots & 3 \\ & & 2 & 5 \end{vmatrix}_{(n-1)}$$

$$= 5D_{n-1} - 3\times 2\cdot D_{n-2} = 5D_{n-1} - 6D_{n-2}$$

构造等比数列: $D_n - x D_{n-1} = (5-x)D_{n-1} - 6D_{n-2}$ ···· A

$$\frac{1}{5-x} = \frac{-x}{-6} \Rightarrow x_1 = 2,\ x_2 = 3$$

将 $x=2$ 代入A式: $D_n - 2D_{n-1} = 3(D_{n-1} - 2D_{n-2})$

$n=3$ 时, $D_{n-1} - 2D_{n-2} = D_2 - 2D_1 = 19 - 2\times 5 = 9 = 3^2$

∴ $D_n - D_{n-1}$ 是以 3^2 为首项, 3为公比的等比数列,

因为下标要从1开始, 即 $n-1 \geq 1$ $n \geq 2$, ∴首项 $n=2$.

∴ $D_n - 2D_{n-1} = 3^2 \times 3^{n-2} = 3^n$ (等比通项: 首项×公比$^{n-首项下标}$) ①

同理 $x=3$ 时可推出 $D_n - 3D_{n-1} = 2^n$ ②

①－②: $D_{n-1} = 3^n - 2^n$,

把 $n-1$ 换为 n: $D_n = 3^{n+1} - 2^{n+1}$

例3.14 解: 爪形行列式 $\xrightarrow[r_1 - \frac{1}{3}r_3,\ r_1 - \frac{1}{2}r_2]{r_1 - \frac{1}{4}r_4} \begin{vmatrix} -\frac{1}{12} & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$

$$= -\frac{1}{12}\times 2\times 3\times 4 = -2$$

例3.15 解: $\xrightarrow{c_1+c_2+\cdots+c_n} \begin{vmatrix} a_1+a_2+\cdots+a_n+b & a_2 & \cdots & a_n \\ a_1+a_2+\cdots+a_n+b & a_2+b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1+a_2+\cdots+a_n+b & a_2 & \cdots & a_n+b \end{vmatrix} \xrightarrow[\substack{\vdots \\ r_n - r_1}]{r_2 - r_1,\ r_3 - r_1}$

$$\begin{vmatrix} a_1+a_2+\cdots+a_n+b & a_2 & \cdots & a_n \\ 0 & b & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{vmatrix}$$

$$=(a_1+a_2+\cdots+a_n+b)\cdot b^{n-1} = b^{n-1}\sum_{i=1}^{n} a_i + b$$

例3.16. 范德蒙行列式

$\therefore$ 原式 $= (x_n - x_{n-1})\cdot(x_n - x_{n-2})\cdots(x_2 - x_1)$

$$= \prod_{1\le j< i\le n}(x_i - x_j)$$

例3.17 解：按第一列展开：

$$原式 = a_1\begin{vmatrix} a_2 & b_2 & \cdots & 0 & 0 \\ 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & a_{n-1} & b_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{vmatrix} + b_1\cdot(-1)^{n+1}\begin{vmatrix} b_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & \cdots & 0 & 0 \\ 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & b_{n-1} \end{vmatrix}$$

$$= a_1\cdot a_2\cdot a_3\cdots a_n + (-1)^{n+1} b_1 b_2\cdots b_n$$

$$= \prod_{i=1}^{n} a_i + (-1)^{n+1}\prod_{i=1}^{n} b_i$$

例3.18

$$解：原式 = (a_1+x)\begin{vmatrix} x & 0 & 0 \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix} - x\cdot(-1)^{2+1}\begin{vmatrix} a_2 & a_3 & a_4 \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix}$$

$$= (a_1+x)x^3 + x^2\begin{vmatrix} a_2 & a_3 \\ -x & x \end{vmatrix} + x^2\begin{vmatrix} a_2 & a_4 \\ -x & 0 \end{vmatrix}$$

$$= (a_1+x)x^3 + x^2(a_2x + a_3x) + x^2(0 + a_4x)$$

$$= (a_1+a_2+a_3+a_4)x^3 + x^4$$

例 3.19

解: $\xrightarrow{C_2+\frac{C_1}{x}} \begin{vmatrix} x & 0 & & & \\ & x & -1 & & \\ & & \ddots & \ddots & \\ & & & x & -1 \\ a_n & a_{n-1}+\frac{a_n}{x} & \cdots & a_2 & a_1+x \end{vmatrix} \xrightarrow{C_3+\frac{C_2}{x}} \begin{vmatrix} x & 0 & & \\ & x & 0 & \\ & & \ddots & \ddots \\ a_n & a_{n-1}+\frac{a_n}{x} & a_{n-2}+\frac{a_n}{x^2}+\frac{a_{n-1}}{x} & \cdots \ a_2 \end{vmatrix}$

$\cdots$

$\longrightarrow \begin{vmatrix} x & & & & & \\ & x & & & & \\ & & x & & & \\ & & & \ddots & & \\ a_n & a_{n-1}+x^{-1}a_n & a_{n-2}+x^{-2}a_n+x^{-1}a_{n-1} & \cdots & \cdots & a_1+a_2x^{-1}+a_3x^{-2}+\cdots a_nx^{-(n-1)}+x \end{vmatrix}$

$= x^{n-1}\cdot(a_1+a_2x^{-1}+a_3x^{-2}+\cdots+a_nx^{-(n-1)}+x)$

$= a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\cdots+a_nx^0+x^n$

$= x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$

$= x^n+\sum_{i=1}^{n} a_ix^{n-i}$