

第一次习题课 群文件《期中 & 期末试题》

期末试题

1.2014~2015(双语)1.

Determine if the following systems are consistent.

$$(a). \begin{cases} x_1 + x_2 - x_3 = 1 \\ x_2 = 1 \\ x_2 - 2x_3 = 3 \end{cases} \quad (b). \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 - 2x_3 = 1 \\ x_2 + 2x_3 = 6 \end{cases}$$

解:

(a) 增广矩阵:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \end{array} \right] &\xrightarrow{r_3-r_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 2 \end{array} \right] &\xrightarrow{r_3 \times (-\frac{1}{2})} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right] &\xrightarrow{r_1+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right] \\ &\xrightarrow{r_1-r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right] \end{aligned}$$

所以解得: $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ 。

(b) 增广矩阵:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 6 \end{array} \right] &\xrightarrow{r_3-r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 5 \end{array} \right] &\xrightarrow{r_3 \times (\frac{1}{4})} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & \frac{5}{4} \end{array} \right] &\xrightarrow{\substack{r_2+2r_3 \\ r_1-r_3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{5}{4} \\ 0 & 1 & 0 & \frac{14}{4} \\ 0 & 0 & 1 & \frac{5}{4} \end{array} \right] \\ &\xrightarrow{r_1-r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{19}{4} \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{5}{4} \end{array} \right] \end{aligned}$$

所以解得: $\mathbf{x} = \begin{bmatrix} -\frac{19}{4} \\ \frac{7}{2} \\ \frac{5}{4} \end{bmatrix}$ 。

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2.2015~2016 三.1.

当 k 为何值时, 线性方程组
$$\begin{cases} kx_1 + x_2 + x_3 = k - 3 \\ x_1 + kx_2 + x_3 = -2 \\ x_1 + x_2 + kx_3 = -2 \end{cases}$$
 有唯一解, 无解和有无穷多解? 当方程组有无穷多解时求出所有解。

解:

增广矩阵

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} k & 1 & 1 & k-3 \\ 1 & k & 1 & -2 \\ 1 & 1 & k & -2 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & k & 1 & -2 \\ k & 1 & 1 & k-3 \\ 1 & 1 & k & -2 \end{array} \right] \xrightarrow{\substack{r_2 - kr_1 \\ r_3 - r_1}} \left[\begin{array}{ccc|c} 1 & k & 1 & -2 \\ 0 & 1-k^2 & 1-k & 3(k-1) \\ 0 & 1-k & k-1 & 0 \end{array} \right] \\
 & \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & k & 1 & -2 \\ 0 & 1-k & k-1 & 0 \\ 0 & 1-k^2 & 1-k & 3(k-1) \end{array} \right] \xrightarrow{r_3 - (1+k)r_2} \left[\begin{array}{ccc|c} 1 & k & 1 & -2 \\ 0 & 1-k & k-1 & 0 \\ 0 & 0 & (1-k)(k+2) & 3(k-1) \end{array} \right]
 \end{aligned}$$

讨论:

(1) 解不存在: 即存在矛盾方程 (增广矩阵主元列在最右列)。即对于 r_3

$$\begin{cases} (1-k)(k+2) = 0 \\ 3(k-1) \neq 0 \end{cases} \Rightarrow k = -2$$

(2) 存在唯一解: 主元列三个元素都不为 0. 即

$$\begin{cases} 1 \neq 0 \\ 1-k \neq 0 \\ (1-k)(k+2) \neq 0 \end{cases} \Rightarrow k \neq 1 \text{ 且 } k \neq -2$$

 $k \neq 1$ 且 $k \neq -2$, 继续对阶梯矩阵进行初等行变换

$$\begin{aligned}
 & \xrightarrow{r_3 \times \frac{1}{(k+2)(1-k)}} \left[\begin{array}{ccc|c} 1 & k & 1 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{(k+2)} \end{array} \right] \xrightarrow{\substack{r_2 + r_3 \\ r_1 - r_3}} \left[\begin{array}{ccc|c} 1 & k & 0 & -2 - \frac{3}{k+2} \\ 0 & 1 & 0 & \frac{3}{k+2} \\ 0 & 0 & 1 & \frac{3}{k+2} \end{array} \right] \\
 & \xrightarrow{r_1 - kr_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{5k+1}{k+2} \\ 0 & 1 & 0 & \frac{3}{k+2} \\ 0 & 0 & 1 & \frac{3}{k+2} \end{array} \right]
 \end{aligned}$$

所以方程组存在唯一解时: $k \neq 1$ 且 $k \neq -2$, 解为

$$\mathbf{x} = \begin{bmatrix} -\frac{5k+1}{k+2} \\ \frac{3}{k+2} \\ \frac{3}{k+2} \end{bmatrix}, \quad k \neq 1 \text{ 且 } k \neq -2$$

(3) 存在无穷解: 至少存在一个自由变量。由阶梯矩阵可以看出

$$\begin{cases} (k-1)(k+2) = 0 \\ 3(k-1) = 0 \end{cases} \Rightarrow k = 1$$

把 $k = 1$ 代入阶梯矩阵:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow x = \begin{bmatrix} -2 - c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix}$$

$$\text{所以 } x \text{ 的解为 } x = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} c_2 \quad c_1, c_2 \in R.$$

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3.2017~2018 二.3.

$$\text{求线性方程组} \begin{cases} 2x_1 - x_2 + 4x_3 - 3x_4 = -4 \\ x_1 + x_3 - x_4 = -3 \\ 3x_1 + x_2 + x_3 = 1 \\ 7x_1 + 7x_3 - 3x_4 = 3 \end{cases} \text{的通解。}$$

解:

增广矩阵

$$\begin{aligned}
 & \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & -4 \\ 1 & 0 & 1 & -1 & -3 \\ 7 & 0 & 7 & -3 & 3 \end{array} \right] \xrightarrow[r_3 - \frac{7}{2}r_1]{r_2 - \frac{1}{2}r_1} \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & -4 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & -1 \\ 0 & \frac{7}{2} & -7 & \frac{15}{2} & 17 \end{array} \right] \xrightarrow{r_3 - 7r_2} \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & -4 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 4 & 24 \end{array} \right] \\
 & \xrightarrow[r_3 \times \frac{1}{4}]{r_1 \times \frac{1}{2}, r_2 \times 2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} & -2 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 6 \end{array} \right] \xrightarrow[r_1 + \frac{3}{2}r_3]{r_2 - r_3} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 2 & 0 & 7 \\ 0 & 1 & -2 & 0 & -8 \\ 0 & 0 & 0 & 1 & 6 \end{array} \right] \xrightarrow{r_1 + \frac{1}{2}r_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -2 & 0 & -8 \\ 0 & 0 & 0 & 1 & 6 \end{array} \right]
 \end{aligned}$$

由最简阶梯型矩阵可以看出:

$$x_1 = 3 - x_3 \quad x_2 = x_3 - 8 \quad x_3 = x_3 \quad x_4 = 6$$

令 $x_3 = C, C \in R$, 则

$$x = \begin{bmatrix} 3-C \\ C-8 \\ C \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} C, \quad C \in R$$

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4.2018~2019 三.1.

$$\text{设 } \begin{cases} \lambda x_1 + x_2 + x_3 = \lambda - 2 \\ x_1 + \lambda x_2 + x_3 = 2 \\ x_1 + x_2 + \lambda x_3 = 2 \end{cases}, \quad \lambda \text{ 为何值时, 该方程组无解、唯一解、无穷解? 并且在有唯一解时求出}$$

解: 有无穷多解时, 求出全部解并用向量表示。

解:

增广矩阵

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} \lambda & 1 & 1 & \lambda-2 \\ 1 & \lambda & 1 & 2 \\ 1 & 1 & \lambda & 2 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & \lambda & 1 & 2 \\ \lambda & 1 & 1 & \lambda-2 \\ 1 & 1 & \lambda & 2 \end{array} \right] \xrightarrow[r_3 - r_1]{r_2 - \lambda r_1} \left[\begin{array}{ccc|c} 1 & \lambda & 1 & 2 \\ 0 & 1-\lambda^2 & 1-\lambda & -\lambda-2 \\ 0 & 1-\lambda & \lambda-1 & 0 \end{array} \right] \\
 & \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & \lambda & 1 & 2 \\ 0 & 1-\lambda & \lambda-1 & 0 \\ 0 & 1-\lambda^2 & 1-\lambda & -\lambda-2 \end{array} \right] \xrightarrow{r_3 - (\lambda+1)r_2} \left[\begin{array}{ccc|c} 1 & \lambda & 1 & 2 \\ 0 & 1-\lambda & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)(-2-\lambda) & -\lambda-2 \end{array} \right]
 \end{aligned}$$

讨论:

(1) 解不存在: 即存在矛盾方程 (增广矩阵主元列在最右列)。即对于 r_3

$$\begin{cases} (\lambda-1)(-2-\lambda) = 0 \\ -\lambda-2 \neq 0 \end{cases} \Rightarrow \lambda = 1$$

(2) 存在唯一解: 主元列三个元素都不为 0. 即

$$\begin{cases} 1 \neq 0 \\ 1-\lambda \neq 0 \\ (\lambda-1)(-2-\lambda) \neq 0 \end{cases} \Rightarrow \lambda \neq 1 \text{ 且 } \lambda \neq -2$$

$\lambda \neq 1$ 且 $\lambda \neq -2$, 继续对阶梯矩阵进行初等行变换

$$\xrightarrow{r_3 \times \frac{1}{(\lambda-1)(-2-\lambda)}} \left[\begin{array}{ccc|c} 1 & \lambda & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{\lambda-1} \end{array} \right] \xrightarrow{\substack{r_2+r_3 \\ r_1-r_3}} \left[\begin{array}{ccc|c} 1 & \lambda & 0 & \frac{2\lambda-3}{\lambda-1} \\ 0 & 1 & 0 & \frac{1}{\lambda-1} \\ 0 & 0 & 1 & \frac{1}{\lambda-1} \end{array} \right] \xrightarrow{r_1-\lambda r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\lambda-3}{\lambda-1} \\ 0 & 1 & 0 & \frac{1}{\lambda-1} \\ 0 & 0 & 1 & \frac{1}{\lambda-1} \end{array} \right]$$

所以方程组存在唯一解时: $\lambda \neq 1$ 且 $\lambda \neq -2$, 解为

$$\mathbf{x} = \begin{bmatrix} \frac{\lambda-3}{\lambda-1} \\ \frac{1}{\lambda-1} \\ \frac{1}{\lambda-1} \end{bmatrix}, \quad \lambda \neq 1 \text{ 且 } \lambda \neq -2$$

(3) 存在无穷解: 至少存在一个自由变量。由阶梯矩阵可以看出

$$\begin{cases} (\lambda-1)(-2-\lambda) = 0 \\ -2-\lambda = 0 \end{cases} \Rightarrow \lambda = -2$$

把 $\lambda = -2$ 代入阶梯矩阵:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{r_2 \times \frac{1}{3}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{r_1+2r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

由最简阶梯型矩阵可以看出:

$$x_1 = 2 + x_3 \quad x_2 = x_3 \quad x_3 = x_3$$

令 $x_3 = C, C \in R$, 则

$$x = \begin{bmatrix} 2+C \\ C \\ C \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} C, \quad C \in R$$

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5.2019~2020 一.4.

若线性方程组 $\begin{cases} x_1 + x_2 = -a_1 \\ x_2 + x_3 = a_2 \\ x_3 + x_4 = -a_3 \\ x_4 + x_1 = a_4 \end{cases}$ 有解, a_1, a_2, a_3, a_4 应满足的条件是 $a_1 + a_2 - a_3 + a_4 = 0$ 。

解:

增广矩阵

$$\begin{aligned} & \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 1 & -a_3 \\ 1 & 0 & 0 & 1 & a_4 \end{array} \right] \xrightarrow{r_4-r_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 1 & -a_3 \\ 0 & -1 & 0 & 1 & a_1+a_4 \end{array} \right] \xrightarrow{r_4+r_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 1 & -a_3 \\ 0 & 0 & 1 & 1 & a_1+a_2+a_4 \end{array} \right] \\ & \xrightarrow{r_4-r_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & 1 & -a_3 \\ 0 & 0 & 0 & 0 & a_1+a_2-a_3+a_4 \end{array} \right] \end{aligned}$$

若方程有解: $a_1 + a_2 - a_3 + a_4 = 0$

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6. 期末 2019-2020 二 1.

设 $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. 求满足 $AX = XA$ 的全部的矩阵 X 。

解:

$$\text{设 } X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

$$AX = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$AX = XA$, 即

$$\begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} \Rightarrow \begin{cases} d=0 & a=e & b=f \\ g=0 & h=d=0 & i=e=a \\ 0=0 & g=0 & h=0 \end{cases}$$

所以 $x = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$, 其中 a, b, c 是任意常数。

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期中试题

2017~2018 二.3. 判断命题是否成立并给出理由

若 $A^2 = B^2$, 则 $A = B$ 或 $A = -B$ 。

解:

判断是否成立: 否。

理由:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

可以看出 $A^2 = B^2$, 但是 $A \neq B, A \neq -B$

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