Cue

4.1 解: 原式 $=\begin{pmatrix}-1\\0.5\\-1.5\end{pmatrix}-\begin{pmatrix}0\\-3\\6\end{pmatrix}+\begin{pmatrix}1\\-1\\-3\end{pmatrix}=\begin{pmatrix}0\\2.5\\-10.5\end{pmatrix}$

4.2 解: A: 行向量 $(-1,0,2,1)$, $(-3,1,1,-2)$, $(0,0,-1,2)$

列向量 $(-1,3,0)^T$, $(0,1,0)^T$, $(2,1,-1)^T$, $(1,-2,2)^T$

B. 行向量 $(-1,2)$, $(3,-2)$, $(4,-5)$ 列向量: $(-1,3,4)^T$, $(2,-2,-5)^T$

4.3 证: $\because\beta$ 可由 $\alpha_i\ (1\le i\le 3)$ 线性表出, 即存在一组不全为0的数 k_1,k_2,k_3, 使得:

$$\beta=k_1\alpha_1+k_2\alpha_2+k_3\alpha_3 \quad ①$$

每个 α_i 都可由 γ_j 线性表出, 即存在一组不全为0的数 $l_{ij}\ (1\le i\le 3, 1\le j\le 2)$, 使得:

$$\begin{aligned}\alpha_1&=l_{11}\gamma_1+l_{12}\gamma_2\\ \alpha_2&=l_{21}\gamma_1+l_{22}\gamma_2\\ \alpha_3&=l_{31}\gamma_1+l_{32}\gamma_2\end{aligned} \quad ②$$

②代入①: $\beta=(k_1l_{11}+k_2l_{21}+k_3l_{31})\gamma_1+(k_1l_{12}+k_2l_{22}+k_3l_{32})\gamma_2$

$\therefore\beta$ 可由 γ_1,γ_2 线性表出.

4.4 对于方程 $\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+\alpha_4x_4=0$ ①, $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\le\min\{3,4\}=3$

$\therefore$ ①有无穷多解, 即3维零向量由 $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ 线性表出方式有无穷多.

4.5. 证: 由题得: $\alpha_1x_1+\alpha_2x_2+\alpha_3x_3=0$ ① $\quad\tilde{\alpha}_1x_1+\tilde{\alpha}_2x_2+\tilde{\alpha}_3x_3=0$ ②

$\because$ 3维零向量由 $\alpha_1,\alpha_2,\alpha_3$ 线性表出唯一, $\therefore$ 方程①只有零解

$\because$ 方程②包括方程①, $\therefore$ ②的解集为①的子集, 即②只有零解, 即…….

4.6 由题可看出 $\beta_1=\beta_2-\beta_3$, 即线性相关.

4.7 解: 例: $\alpha_1=(0,1)^T$, $\alpha_2=(1,0)^T$, $\alpha_3=(0,1)^T$, $\alpha_4=(1,1)^T$

α_1,α_2 线性无关, α_3,α_4 线性无关, 但 $\alpha_4=\alpha_1+\alpha_2+0\alpha_3$, 即 $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ 线性相关.

4.8 (A). $(\alpha_1+\alpha_2)+(\alpha_3+\alpha_4)=(\alpha_2+\alpha_3)+(\alpha_4+\alpha_1)$, 线性相关.

(B). $(\alpha_1-\alpha_2)+(\alpha_2-\alpha_3)=-(\alpha_3-\alpha_4)-(\alpha_4-\alpha_1)$ 相关

(C) 设 $k_1(\alpha_1+\alpha_2)+k_2(\alpha_2+\alpha_3)+k_3(\alpha_3+\alpha_4)+k_4(\alpha_4-\alpha_1)=0$

Summary

即 $(k_1-k_4)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3+(k_3+k_4)\alpha_4=0$

$\because\alpha_1,\alpha_2,\alpha_3,\alpha_4$ 线性无关 $\therefore k_1-k_4=0$, $k_1+k_2=0$, $k_2+k_3=0$ $k_3+k_4=0$

解得 $k_1=k_2=k_3=k_4=0$ 即 …… 线性无关

(D). $(\alpha_1+\alpha_2)+(\alpha_4-\alpha_1)=(\alpha_2+\alpha_3)-(\alpha_3-\alpha_4)$, 线性相关

Cue

4.9 解：由题可列：$A=(\alpha_1,\alpha_2,\alpha_3)$ $B=\beta$

(1) β可由$\alpha_1,\alpha_2,\alpha_3$唯一表出 ~~且表示~~ 则 $r(A|B)=r(A)=n=3$ 即 $|A|\neq 0$

$$|A|=\begin{vmatrix}\lambda+1 & 1 & 1\\ 1 & 1+\lambda & 1\\ 1 & 1 & 1+\lambda\end{vmatrix} \xrightarrow{c_1+c_2+c_3} (\lambda+3)\begin{vmatrix}1 & 1 & 1\\ 1 & 1+\lambda & 1\\ 1 & 1 & \lambda+1\end{vmatrix} \xrightarrow[r_3-r_1]{r_2-r_1} (\lambda+3)\begin{vmatrix}1 & 1 & 1\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{vmatrix} = \lambda^2(\lambda+3)\neq 0$$

即$\lambda\neq 0$且$\lambda\neq 3$

(2) $\lambda=0$时 $r(A)=1$, $B=0$, $r(A:B)=1$, $\therefore\beta$可由$\alpha_1,\alpha_2,\alpha_3$线性表出且表示不唯一

(3) $\lambda=-3$时,

$$(A|B)=\left[\begin{array}{ccc|c}-2 & 1 & 1 & 0\\ 1 & -2 & 1 & -3\\ 1 & 1 & -2 & 9\end{array}\right] \xrightarrow[r_3+\frac{1}{2}r_1]{r_2+\frac{1}{2}r_1} \begin{bmatrix}-2 & 1 & 1 & 0\\ 0 & -\frac{3}{2} & \frac{3}{2} & 3\\ 0 & \frac{3}{2} & -\frac{3}{2} & 9\end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix}-2 & 1 & 1 & 0\\ 0 & -\frac{3}{2} & \frac{3}{2} & 3\\ 0 & 0 & 0 & 12\end{bmatrix}, \quad r(A:B)\neq r(A)$$

$\therefore\beta$不能由$\alpha_1,\alpha_2,\alpha_3$线性表出

4.10 存在不全为0的实数 k_i $(1\le i\le 3)$，使得 $k_1\alpha_1+k_2\alpha_2+k_3\alpha_3=0$

$\because\alpha_2,\alpha_3,\alpha_4$线性无关 $\therefore\alpha_2,\alpha_3\neq 0$ $\therefore k_1\neq 0$

(1) $\therefore\alpha_1$可由α_2,α_3线性表出 $\alpha_1=-\frac{k_2}{k_1}\alpha_2-\frac{k_3}{k_1}\alpha_3$

(2) $\alpha_2,\alpha_3,\alpha_4$线性无关，即$\alpha_4$不能由$\alpha_2,\alpha_3$线性表出

假设α_4可由$\alpha_1,\alpha_2,\alpha_3$线性表出

则 $\alpha_4=l_1\alpha_1+l_2\alpha_2+l_3\alpha_3$ 其中 l_1,l_2,l_3不全为0,

由(1)知 $\alpha_4=l_1(-\frac{k_2}{k_1}\alpha_2-\frac{k_3}{k_1}\alpha_3)+l_2\alpha_2+l_3\alpha_3$，即$\alpha_4$可由$\alpha_2,\alpha_3$线性表出，与前边矛盾，

$\therefore$假设不成立 即——.

4.11 记 $A=(\alpha_1,\alpha_2,\alpha_3)$ $B=\beta$

$$[A:\beta]=\left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\ 4 & 7 & 1 & 10\\ 0 & 1 & -1 & b\\ 2 & 3 & a & 4\end{array}\right] \xrightarrow[r_4-2r_1]{r_2-4r_1} \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & -1 & 1 & -2\\ 0 & 1 & -1 & b\\ 0 & -1 & a & -2\end{bmatrix} \xrightarrow[r_4-r_2]{r_3+r_2}$$

$$\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & -1 & 1 & -2\\ 0 & 0 & 0 & b-2\\ 0 & 0 & a-1 & 0\end{bmatrix} \xrightarrow{r_3\leftrightarrow r_4} \left[\begin{array}{ccc|c}1 & 2 & 0 & 3\\ 0 & -1 & 1 & -2\\ 0 & 0 & a-1 & 0\\ 0 & 0 & 0 & b-2\end{array}\right]$$

(1) $b\neq 2$时, $r(A:B)\neq r(A)$，即β不能表示成$\alpha_1,\alpha_2,\alpha_3$线性组合.

Summary

(2) $b=2$时, $r(A:B)=r(A)$, β可由$\alpha_1,\alpha_2,\alpha_3$线性表示

$a-1=0$时 $\longrightarrow \left(\begin{array}{ccc|c}1 & 0 & 2 & -1\\ 0 & 1 & -1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $\quad x_1=-1-2x_3$, $x_2=2+x_3$ $\quad\therefore\beta=(-1-2k)\alpha_1+(2+k)\alpha_2+k\alpha_3$, k

$a-1\neq 0$时 $\longrightarrow \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$ $\quad x_1=-1$, $x_2=2$, $x_3=0$ $\quad\beta=-\alpha_1+2\alpha_2$

George Willsons

可以看出$a\neq 1$时为$a=1$时中$k=0$的情况，$\therefore$ $\beta=(-1-2k)\alpha_1+(2+k)\alpha_2+k\alpha_3$, $k\in R$.

4.12 解：$A\alpha=\begin{bmatrix}1&2&-2\\2&1&2\\3&0&4\end{bmatrix}\begin{bmatrix}a\\1\\1\end{bmatrix}=\begin{bmatrix}a\\2a+3\\3a+4\end{bmatrix}$

$A\alpha$与α线性相关，即$A\alpha$与α对应下元素成比例

$\therefore \frac{a}{a}=\frac{1}{2a+3}=\frac{1}{3a+4} \Rightarrow a=-1$

4.13. 线性相关，即 $r(\ldots)<4 \Leftrightarrow |\ldots|=0$

$$\begin{vmatrix}2&2&3&4\\1&1&2&3\\1&a&1&2\\1&a&a&1\end{vmatrix}\xlongequal{r_4-r_3}\begin{vmatrix}2&2&3&4\\1&1&2&3\\1&a&1&2\\0&0&a-1&-1\end{vmatrix}\xlongequal{c_2-c_1}\begin{vmatrix}2&0&3&4\\1&0&2&3\\1&a-1&1&2\\0&0&a-1&-1\end{vmatrix}=(a-1)\times(-1)^{3+2}\begin{vmatrix}2&3&4\\1&2&3\\0&a-1&-1\end{vmatrix}$$

$$\xlongequal{r_1-2r_2}(1-a)\begin{vmatrix}0&-1&-2\\1&2&3\\0&a-1&-1\end{vmatrix}=(1-a)\times1\times(-1)^{2+1}\begin{vmatrix}-1&-2\\a-1&-1\end{vmatrix}=(a-1)(1+2(a-1))=0$$

$\because a\neq1$，$\therefore a=\frac{1}{2}$

4.14 (1) ×，相关的前提维数一致

(2) × 如：$\alpha_1=\begin{pmatrix}1\\0\end{pmatrix}$ $\alpha_2=\begin{pmatrix}2\\0\end{pmatrix}$ $\alpha_3=\begin{pmatrix}0\\1\end{pmatrix}$，$\alpha_1,\alpha_2,\alpha_3$相关，但$\alpha_3$无法由$\alpha_1,\alpha_2$表出

(3) √ $\alpha_1\cdots\alpha_4$相关，则由其组成的行列式为0. $\therefore |\alpha_4\ \alpha_2\ \alpha_1\ \alpha_3|=0$，$\therefore$相关.

(4) × 如(2)的例子，去掉α_2后线性无关，但$\alpha_1,\alpha_2,\alpha_3$相关.

4.15 (5) ×，如上题的(2) $\alpha_1+\alpha_2+\alpha_3=\begin{pmatrix}3\\1\end{pmatrix}\neq0$，但$\alpha_1,\alpha_2,\alpha_3$相关.

(6) ×，如 $\alpha_1=(1,0,0)^T$，$\alpha_2=(0,1,1)^T$，$\alpha_3=(1,1,0)^T$

$\alpha_1,\alpha_2,\alpha_3$不成比例，但$\alpha_3=\alpha_1+\alpha_2$，即$\alpha_1,\alpha_2,\alpha_3$相关.

(7). × 向量空间的维数 > 向量维数，一定相关.

4.16. (1) √ 线性表出的传递性

(2) × (I) $\alpha_1=(1,1)^T$ (II) $\beta_1=(1,0)^T$，$\beta_2=(0,1)^T$，$\alpha=\beta_1+\beta_2$，但(I)中向量数小于(II)

(3) × 向量组等价只能得出秩相等，但两向量组个数不一定相等，因此也不一定都相关(无关)

(4) √

(5) ×，如$\alpha_1=\begin{pmatrix}1\\0\end{pmatrix}$ $\alpha_2=\begin{pmatrix}0\\0\end{pmatrix}$，$\beta_1=\begin{pmatrix}0\\0\end{pmatrix}$ $\beta_2=\begin{pmatrix}1\\0\end{pmatrix}$，$x_1\alpha_1+x_2\alpha_2$的解为$t\begin{pmatrix}0\\1\end{pmatrix}$，$t\in R$

$x_1\beta_1+x_2\beta_2=0$的解为$t\begin{pmatrix}1\\0\end{pmatrix}$，$t\in R$，显然两向量组等价但解集不同。