Digital Signal Processing Tutorial

Stephen Coshatt[1] and Dr. WenZhan Song[1]

The University of Georgia[1]

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Outline

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Basic Waves

1. **Sine Wave**

   sine_wave(duration=10, sampling_rate=100, amplitude=1.5, frequency=0.3, show=True)

2. **Square Wave**

   square_wave(duration=10, sampling_rate=100, amplitude=1.5, frequency=.5, show=True)

3. **Triangle Wave**

   triangle_wave(duration=10, sampling_rate=100, amplitude=1, frequency=0.5, show=True)

Basic Waves

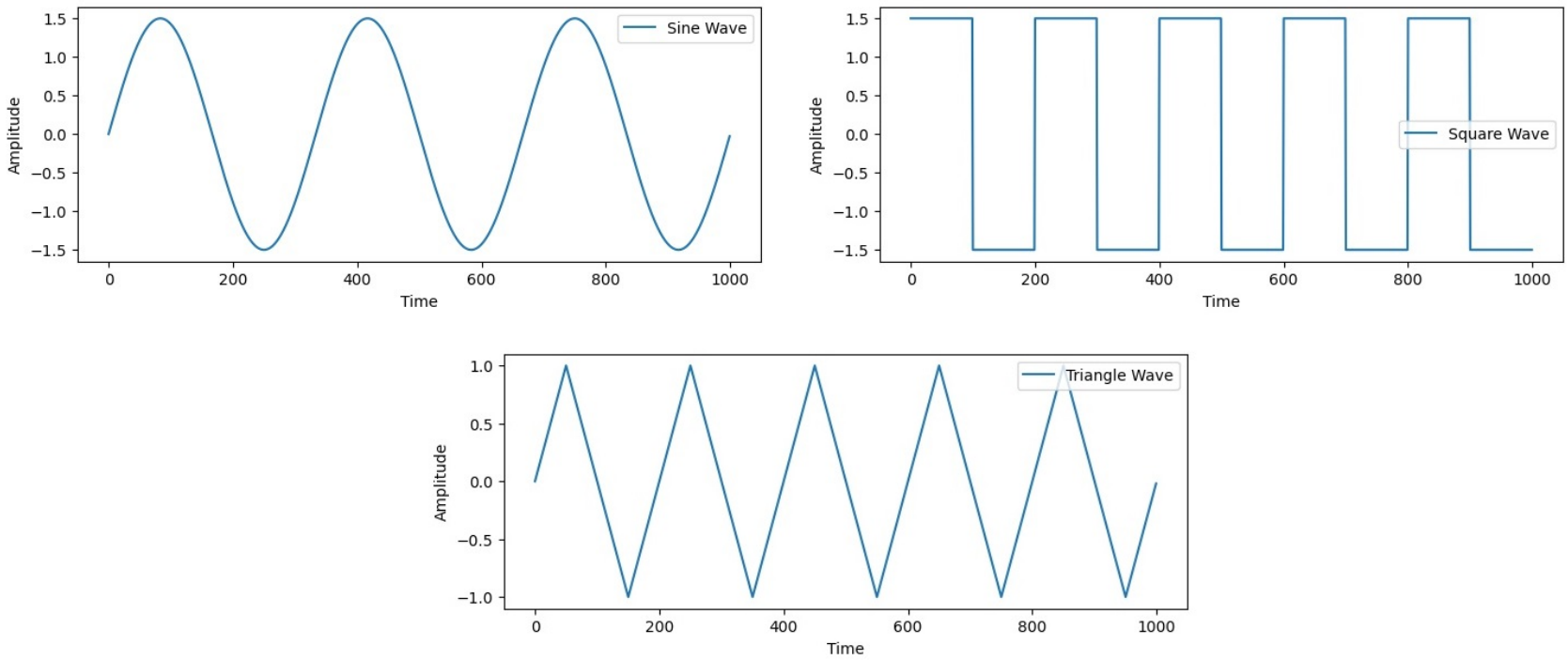

Figure: Sine, Square, and Triangle Waves

Chirps and Pulses

1. **Chirps:**

   Linear: chirp_wave_linear(f0=0.1, c=0.25, show=True)

   Exponential: chirp_wave_exponential(f0=0.1, k=1.5, show=True)

   Hyperbolic: chirp_wave_hyperbolic(f0=0.1, f1=2.6, show=True)

2. **Pulses:**

   pulse_wave(duration=10, sampling_rate=100, amplitude=1, d=0.5, frequency=0.3, expansion=3, show=True)

Chirps and Pulses

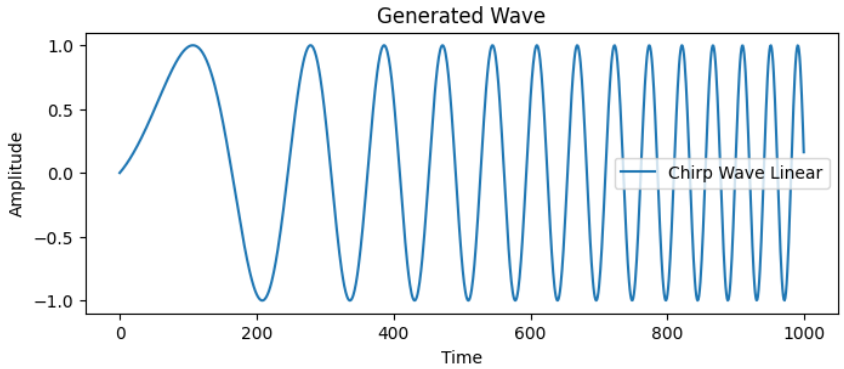

Figure: Linear Chirp

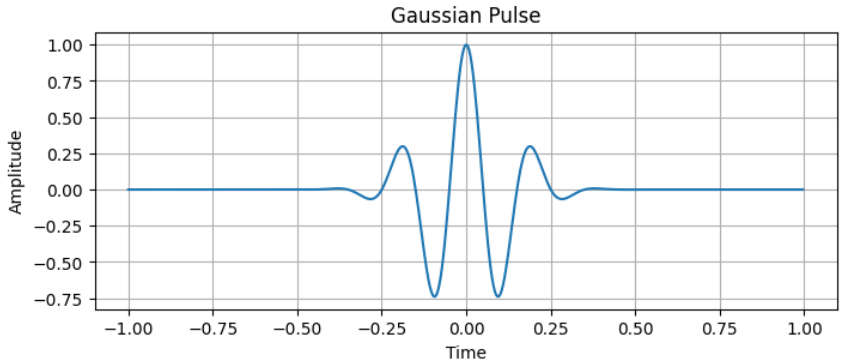

Figure: Pulse

SCG Signal

import DSP

1. **Daubechies Wavelet SCG Signal**

   signal_db = dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="db", heart_rate=(70,71), respiratory_rate=(15,16), systolic=(120,121), diastolic=(90,91))

2. **Morlet Wavelet SCG Signal**

   signal_mor = dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="mor", heart_rate=(70,71), respiratory_rate=(15,16), systolic=(120,121), diastolic=(90,91))

SCG Signal

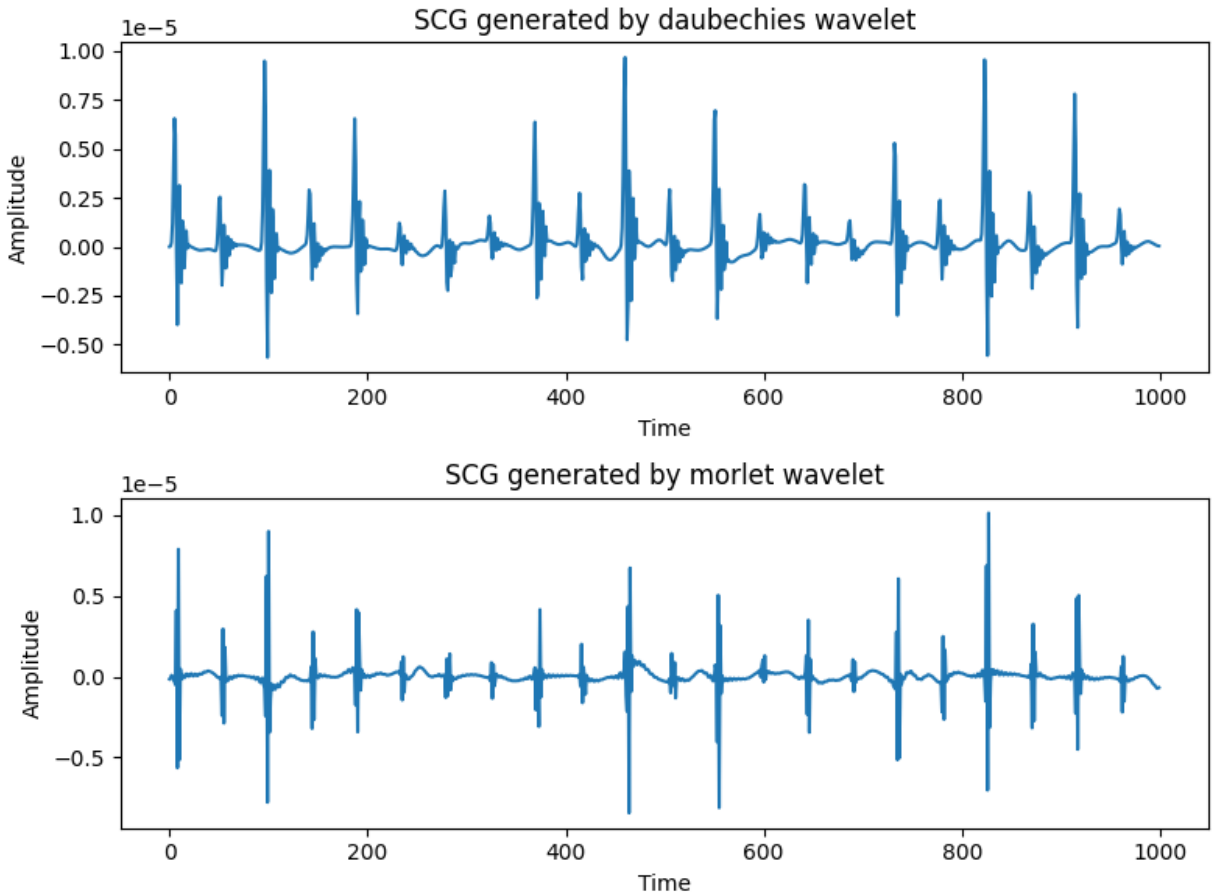

Figure: Two types of SCG generation

SCG Signal

1. **SCG without Respiratory**

   dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="db", heart_rate=(70,71), add_respiratory = False, systolic=(120,121), diastolic=(90,91))

2. **SCG with Respiratory**

   dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="db", heart_rate=(70,71), add_respiratory=True, systolic=(120,121), diastolic=(90,91))

SCG Signal

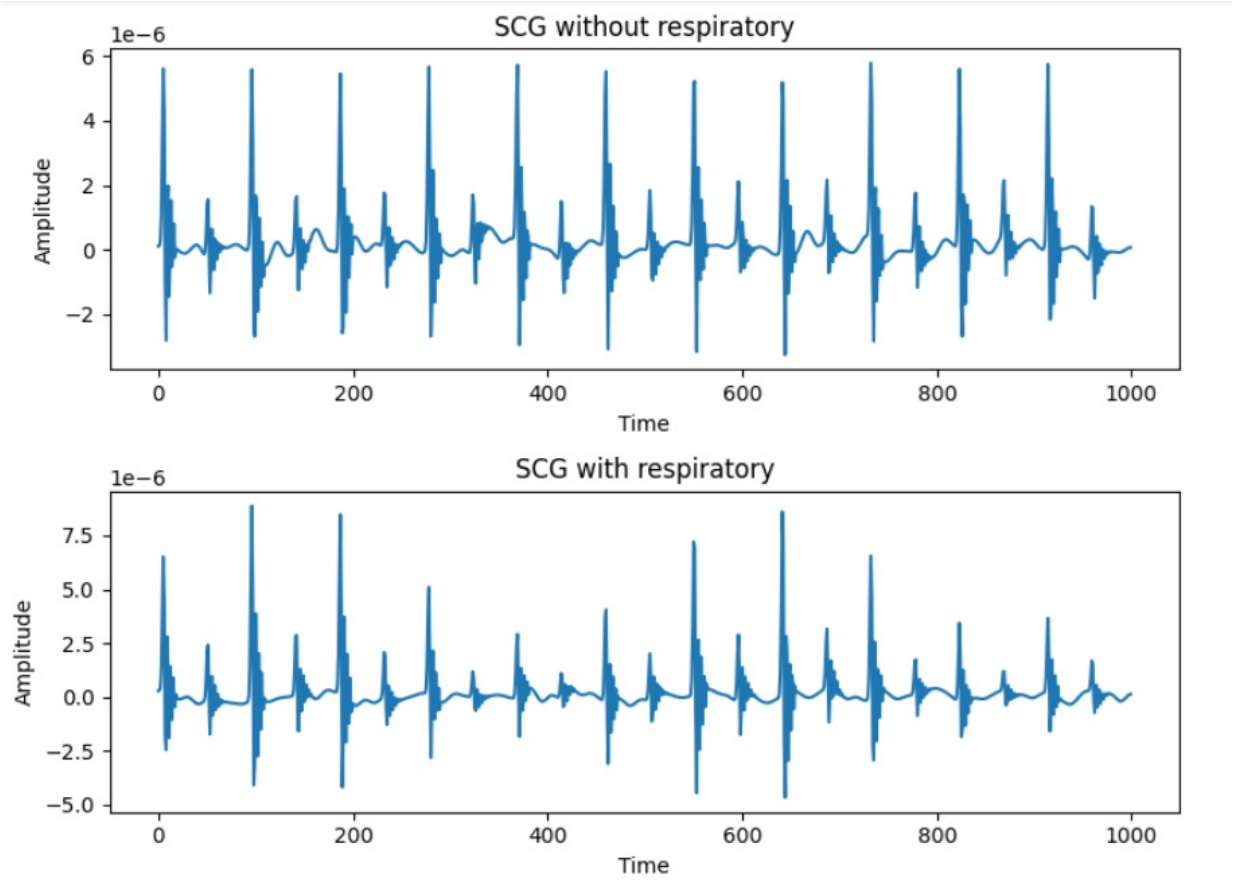

Figure: Adding Respiratory to SCG

SCG Signal

1. **SCG with Systolic of 90**

   dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="db", heart_rate=(70,71), respiratory_rate=(15,16), **systolic=(90,91)**, diastolic=(90,91))

2. **SCG with Systolic of 170**

   dsp.scg_simulate(num_rows=1, duration=10, sampling_rate=100, pulse_type="db", heart_rate=(70,71), respiratory_rate=(15,16), **systolic=(170,171)**, diastolic=(90,91))

SCG Signal

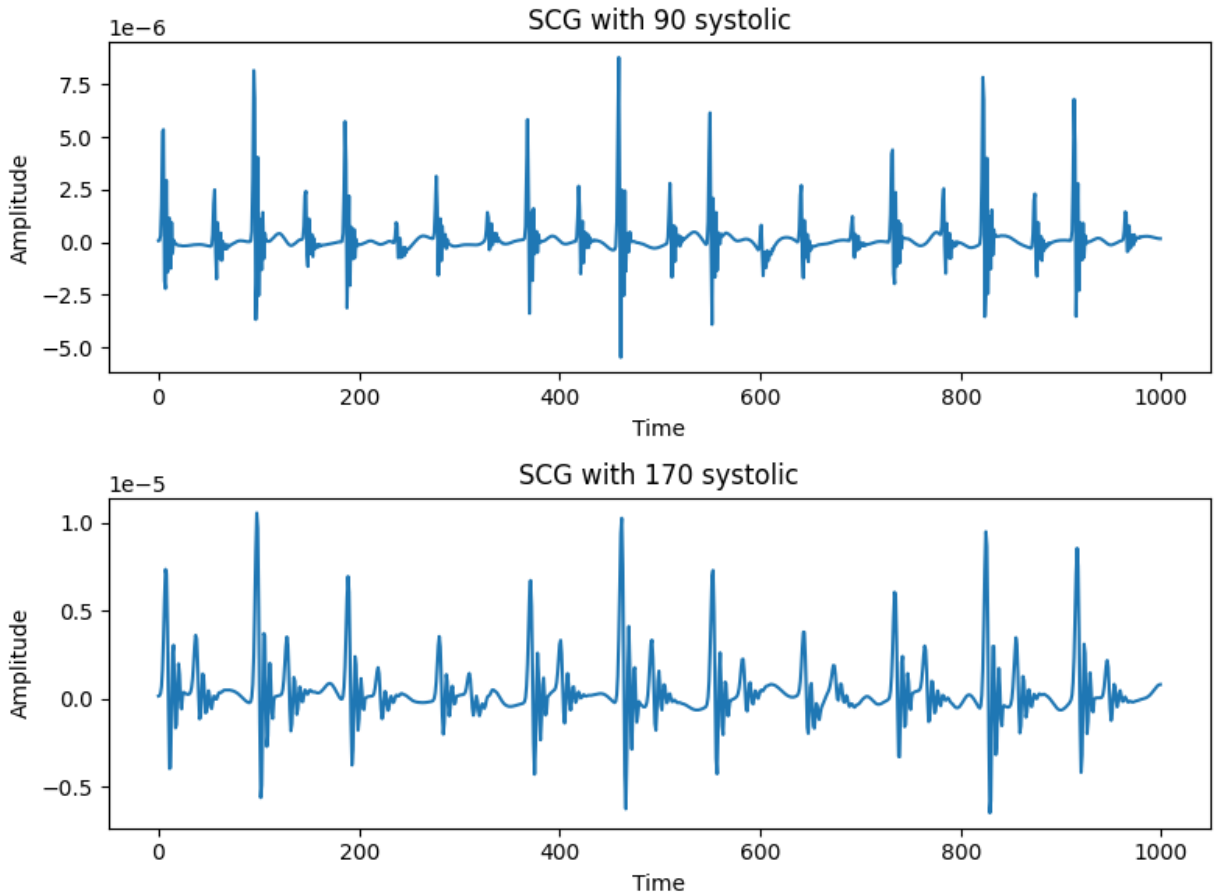

Figure: SCG with Systolic pressures of 90 and 170

Noise

1. **White**

   (Gaussian, Laplacian, and Band-Limited options) - Random signal w/ equal intensity at different frequencies

2. **Impulse**

   sudden spike in signal

3. **Burst**

   sudden intermittent burst of spikes

4. **Brown**

   power spectral density inversely proportional to square of the frequency

Noise

1. **Pink**

   Also known as Flicker noise. equal power in each octave

2. **Blue**

   PSD proportional to the frequency

3. **Power line**

   electrical interference generated by power lines

4. **Echo**

   occurs when portion of a signal reflects back to source

Noise

add_white_noise(signal, noise_amplitude=0.3, model=0, show=True)

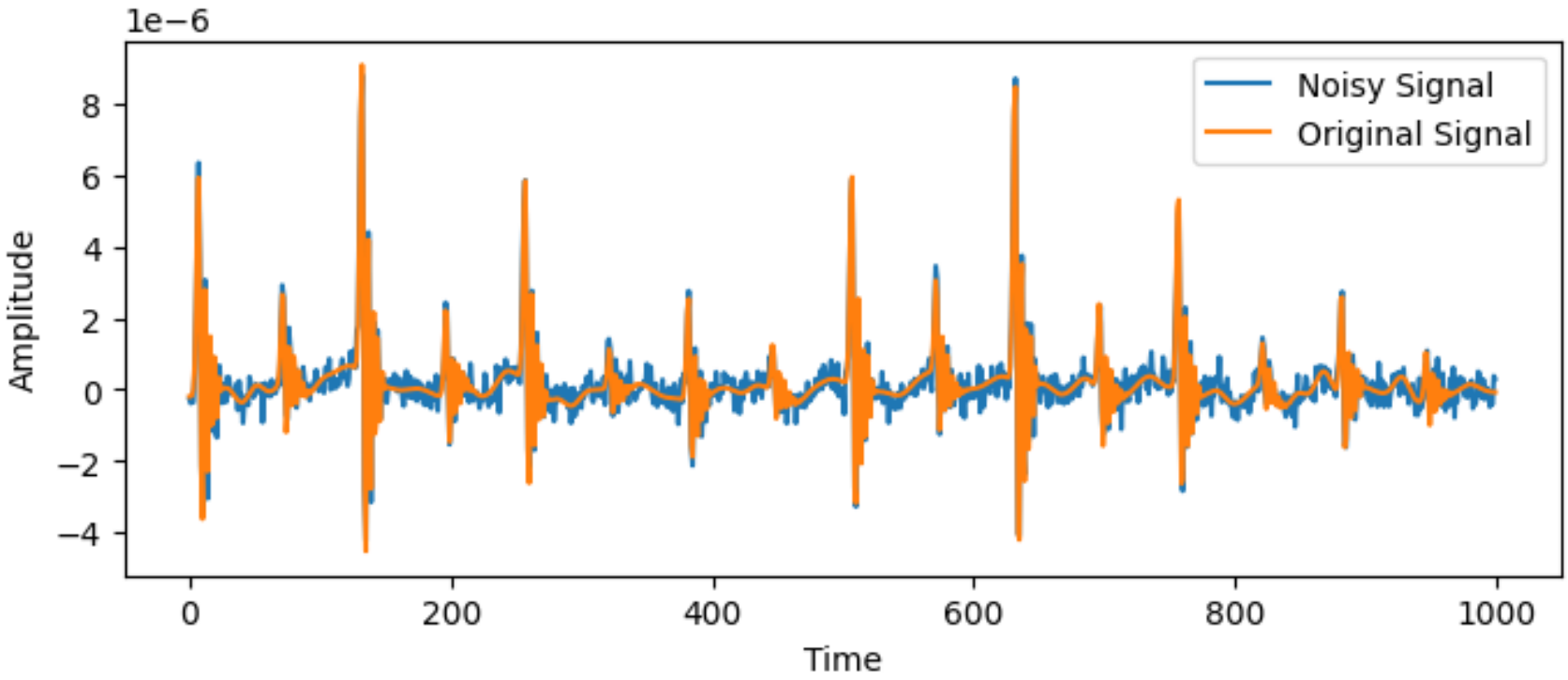

Figure: Signal with Gaussian White Noise

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4. **Wiener**
5. **Notch**
6. **Matched**
7. **Wavelet Denoising**
8. **Adaptive**
9. **Kalman**
10. **Dynamic Time Warping Averaging**

The Butterworth filter is a popular linear filter designed to pass signals within a specified frequency range while attenuating frequencies outside that range.

1. Characterized by a **smooth frequency response**
2. Commonly used in signal processing and communications

Mathematically, the filter transfer function is given by

$$H(s) = \frac{1}{1 + (\frac{s}{\omega_c})^{2n}}$$

where s is the complex frequency variable, ω_c is the cutoff frequency, and n is the filter order.

General Linear Filter Types:

1. **High Pass** - Allows signals above a specified frequency pass
2. **Low Pass** - Allows signals below a specified frequency pass
3. **Band Pass** - Allows signals in a specified frequency band to pass
4. **Band Stop** - Allows signals to pass except those in a specified frequency band

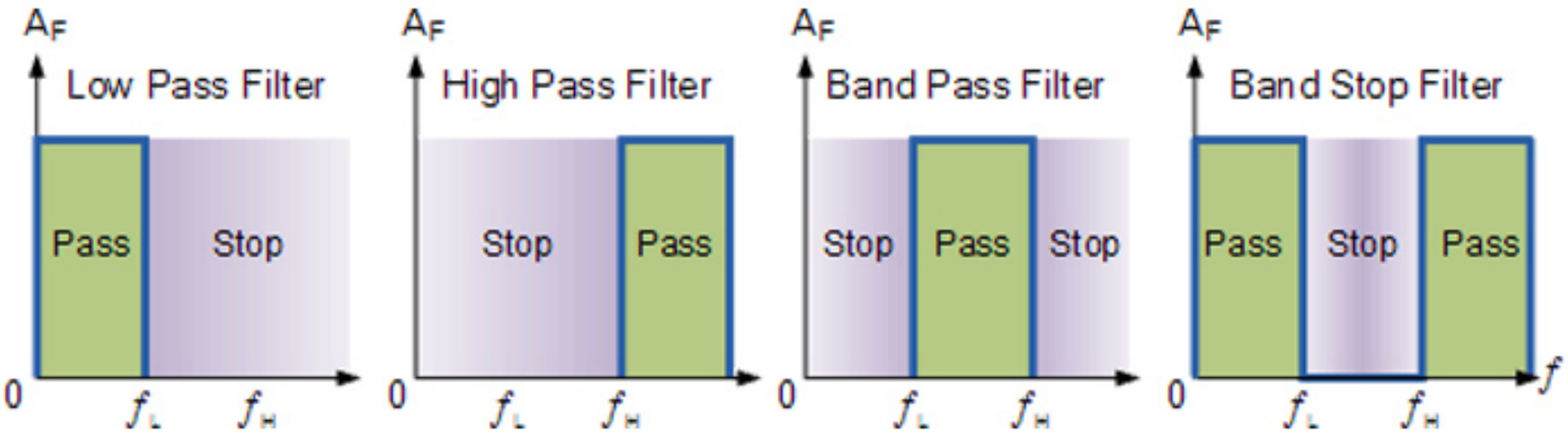

Figure: Linear Filters

Image from Frequency Filter

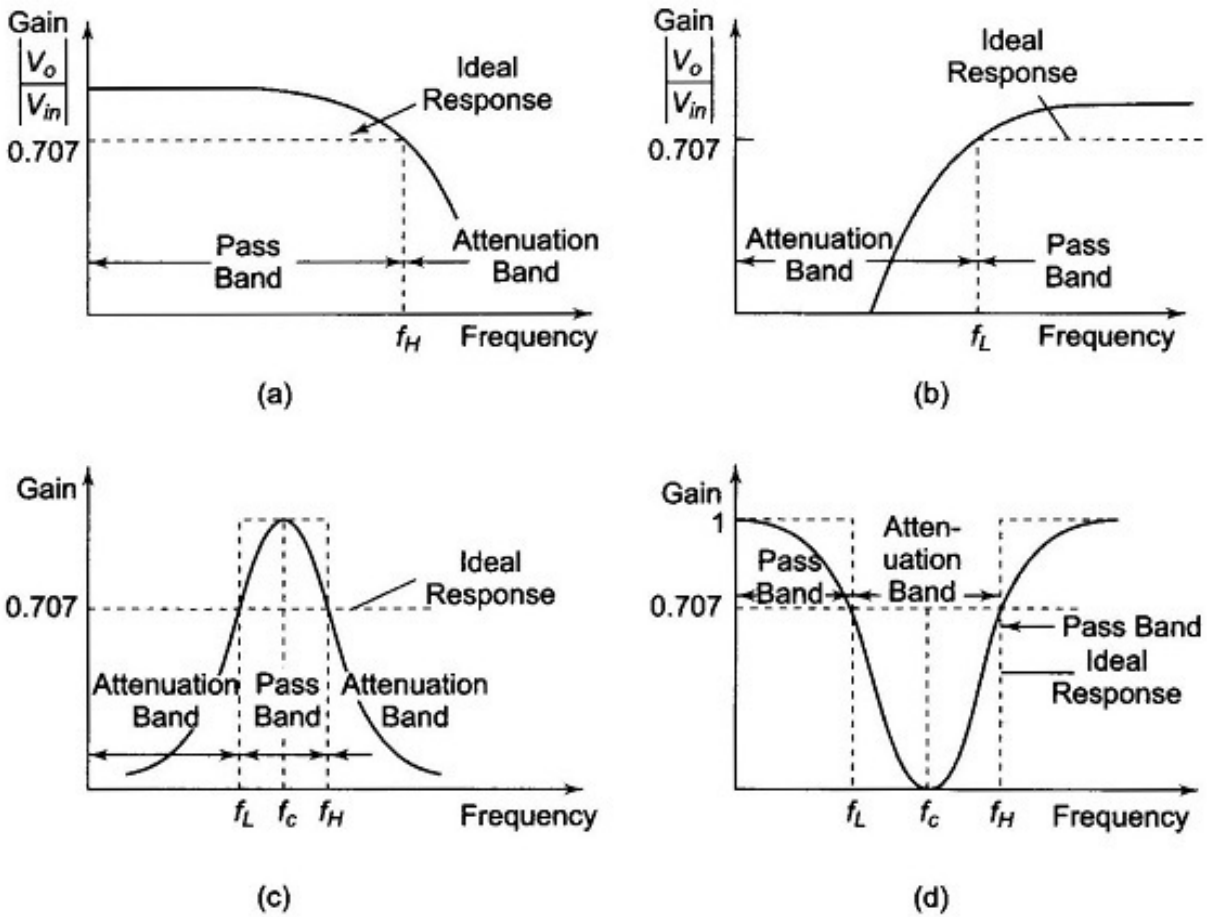

Figure: Butterworth Filter Responses - (a) Low Pass,(b) High Pass, (c) Band Pass, (d) Band Stop

Image from Classification of Active Filters

The **order** parameter in a Butterworth filter specifies the degree of the filter. It determines the frequency response characteristics of the filter, affecting its cutoff frequency and slope.

1. Higher orders result in steeper cutoffs, effectively filtering out frequencies beyond the desired range.
2. Lower orders have wider transition regions but lower phase distortion.
3. The choice of order balances filter performance and computational complexity based on specific application requirements.

1. **Make an SCG signal**

   signal = scg_simulate()[:1000]

2. **Add noise**

   signal_with_3Hz_Noise = add_distort_noise(signal, n_samples=1000, sampling_rate=100,noise_frequency=3, noise_amplitude=0.5, show=True)

3. **Filter Signal**

   filtered_signal = butter_highpass_filter(signal_with_3Hz_Noise, cutoff=5, order=10, show=True)

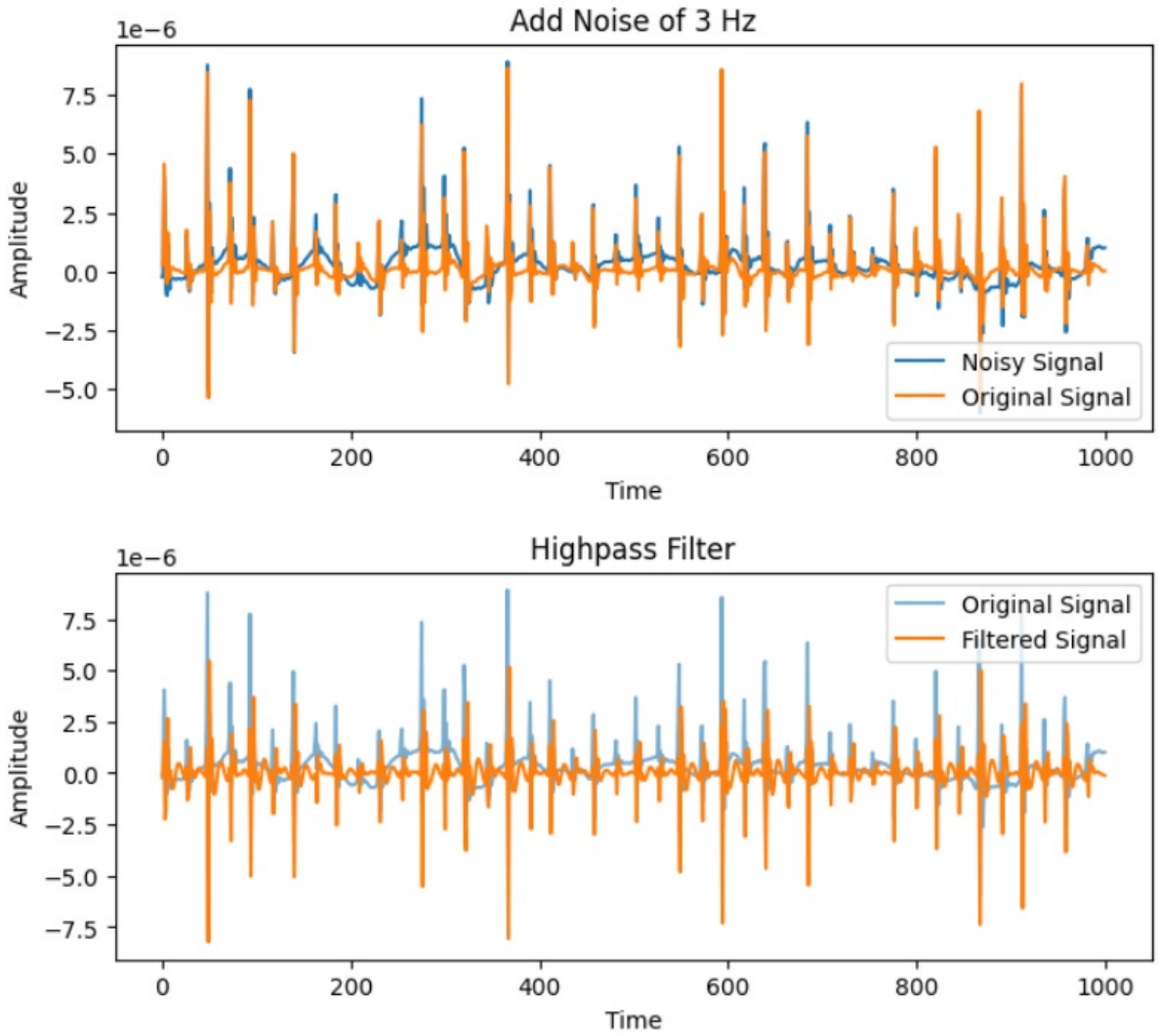

Figure: Filter 3 Hz Noise with Butterworth

Simple Moving Average Filter (SMA)

The SMA filter is a basic time-domain filter that computes the average of a specified number of consecutive data points. It provides a simple means of smoothing a time series to reveal underlying trends.

Mathematically, the SMA for a window size N is given by $y(t) = \frac{1}{N} \sum_{i=1}^{N} x(t - i + 1)$.

Essentially, the SMA filter is a type of **low-pass filter**.

Exponential Moving Average (EMA) Filter

The EMA filter assigns exponentially decreasing weights to past data points, **giving more importance to recent observations**. It is widely used for trend analysis and noise reduction.

Mathematically, the EMA is defined by $y(t) = \alpha x(t) + (1 - \alpha)y(t - 1)$, where α is the smoothing factor.

Essentially, the EMA filter is a type of **low-pass filter** too.

Filters

Moving Average Filters

1. **Simple**

   simple_moving_average_filter(signal_with_40Hz_Noise, length=25, show=True)

2. **Exponential**

   exponential_moving_average_filter(signal_with_40Hz_Noise, length=25, alpha=0.2, show=True)

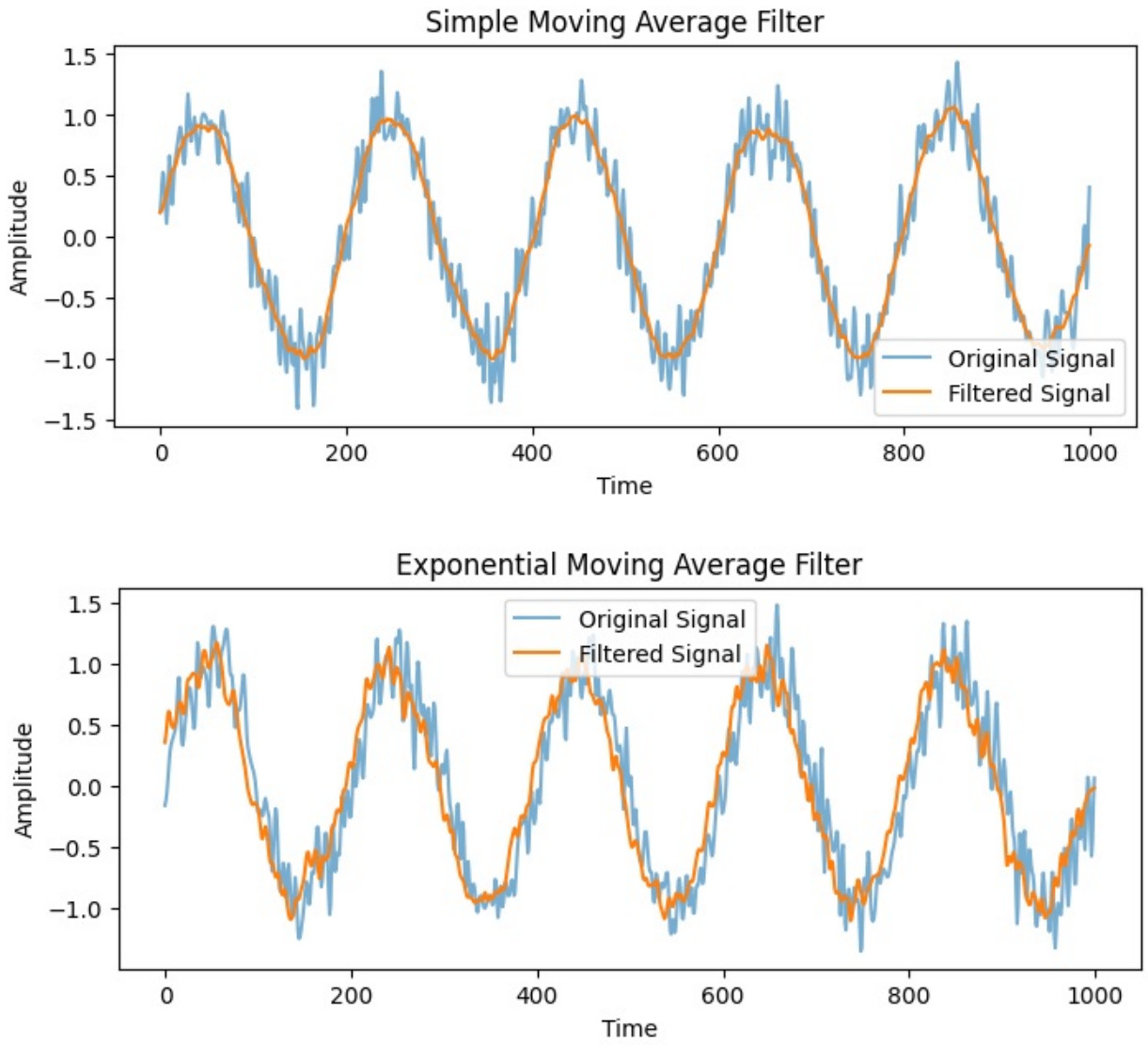

Figure: Moving Average Filters

Savitzky-Golay Filter (savgol_filter)

The **Savitzky-Golay filter** is a smoothing algorithm that preserves important features of a signal while reducing noise. It employs polynomial fitting within a sliding window to smooth the data.

Mathematically, the filter coefficients are determined by least squares fitting, providing a balance between noise reduction and signal preservation.

$$Minimize \ \sum_{i=-m}^{m}(y_{k+i} - \sum_{j=0}^{p} a_j x_{k+i}^2)^2$$

Example:

If polynomial is $y = a_0 + a_1 x + a_2 x^2$

then $Minimize \ \sum_{i=-2}^{2}(y_{k+i} - (a_0 + a_1 x_{k+i} + a_2 x_{k+i}^2))^2$

Reference: Introduction to the Savitzky-Golay Filter: A Comprehensive Guide

Filters

1. **Savitzky-Golay Filter**

savgol_filter(signal_with_5Hz_noise, window_length=15, show=False)

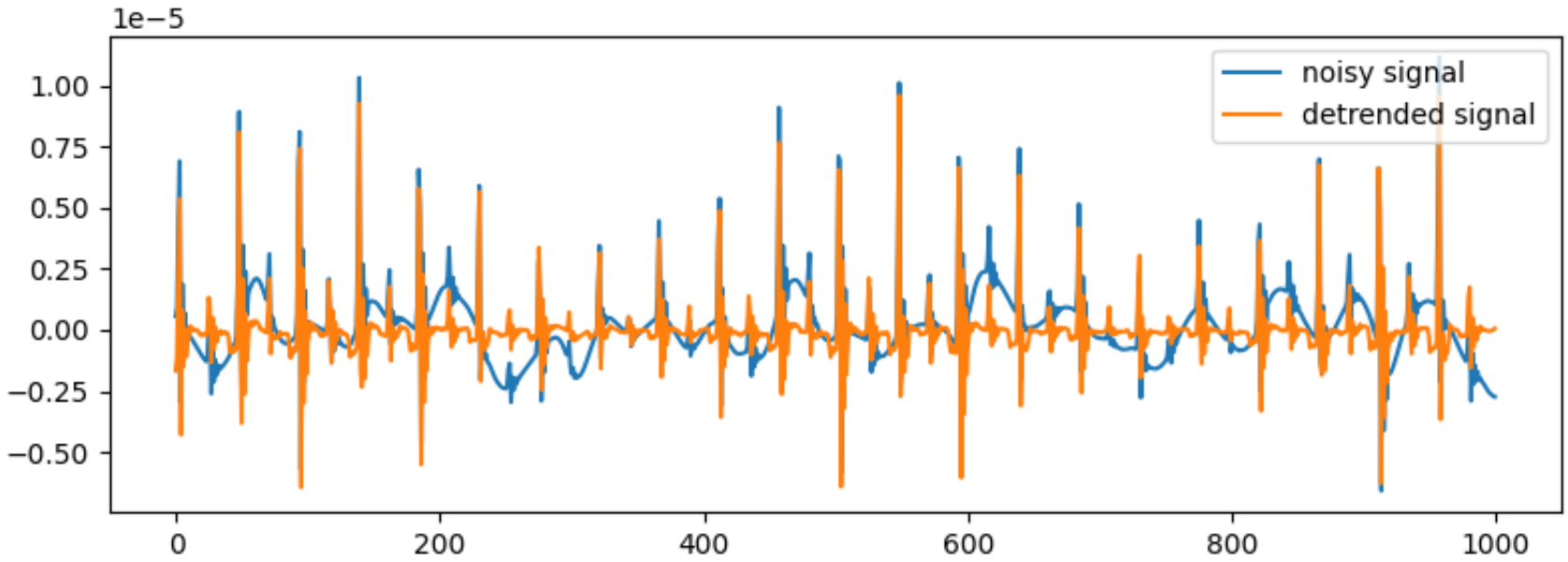

Figure: Savgol Filter

Wiener Filter

The Wiener filter is an optimal linear filter used for signal deconvolution and noise reduction. It minimizes mean-squared error between the estimated signal and the true signal, enhancing signal-to-noise ratio.

Mathematically, the Wiener filter in the frequency domain is given by $H(f) = \frac{S_x(f)}{S_x(f)+S_n(f)}$, where $Sx(f)$ is the signal power spectrum, $Sn(f)$ is the noise power spectrum.

Wiener filters are characterized by the following:

1. **Assumption:** signal and (additive) noise are stationary linear stochastic processes with known spectral characteristics or known autocorrelation and cross-correlation
2. **Requirement:** the filter must be physically realizable/causal (this requirement can be dropped, resulting in a non-causal solution)
3. **Performance criterion:** minimum mean-square error (MMSE).

Filters

1. **Wiener Filter**

wiener_filter(signal + noise, noise, show=True)

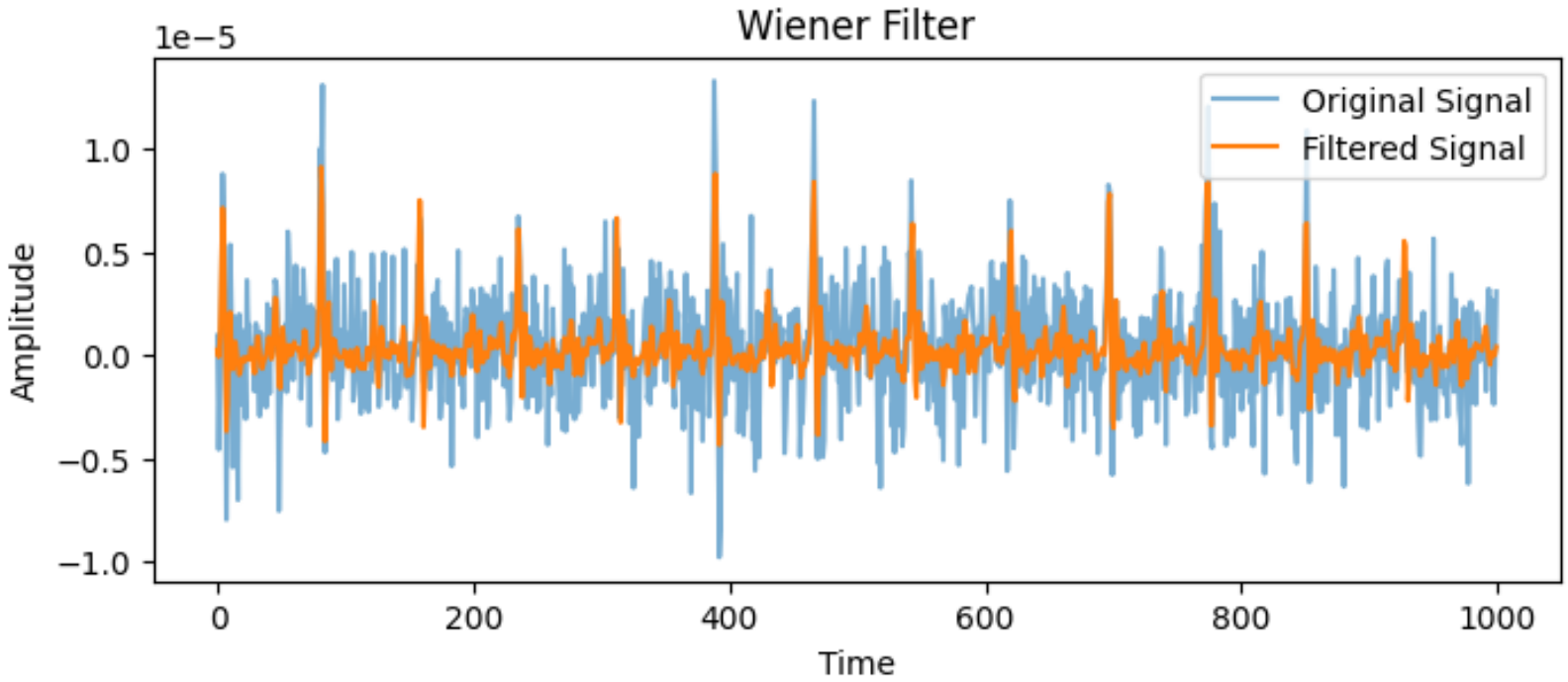

Figure: Wiener Filter

Notch Filter

The notch filter is designed to suppress specific frequencies, often used to eliminate unwanted interference or noise at a particular frequency.

It creates a notch or a dip in the frequency response centered around the target frequency.

Mathematically, the transfer function of a notch filter can be represented as $H(f) = \frac{1}{1+\frac{(f/f_0)^2}{Q}}$, where f_0 is the center frequency, and Q is the quality factor.

A Notch filter is essentially a very narrow band stop filter.

Filters

1. **Notch**

notch_filter(signal_with_5Hz_Noise, cutoff=5, q=3, fs=100, show=True)

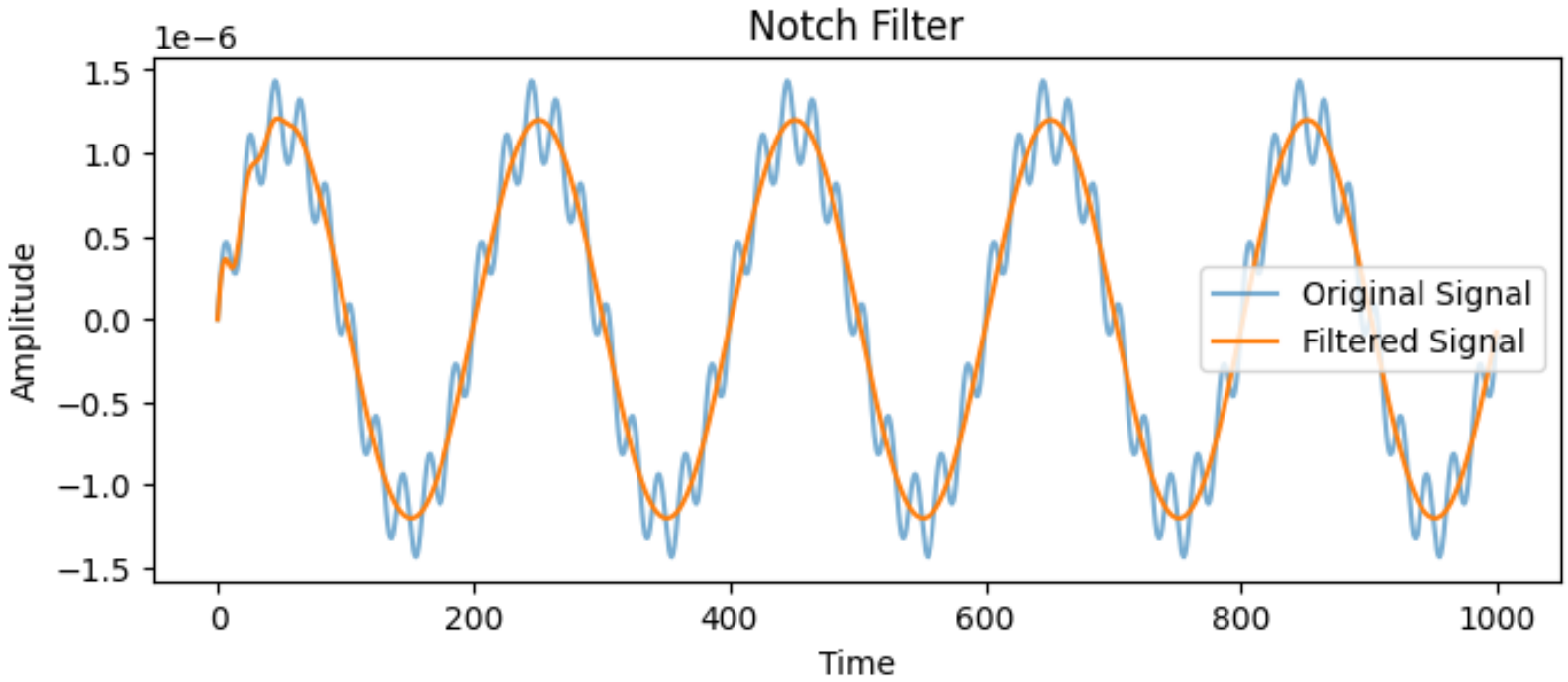

Figure: Notch Filter

Matched Filter

The matched filter is a signal processing filter that maximizes the signal-to-noise ratio for a known signal when embedded in noise.

It is particularly effective in detecting signals with a known template.

Mathematically, the matched filter output is the convolution of the received signal and the time-reversed conjugate of the template signal.

$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$

Where: $x[k]$ is the input function of k, and $y[n]$ is the filtered output

Filters

1. **Matched**

matched_filter(signal_with_40Hz_Noise_, clean_template, show=False)

matched_filter(signal_with_40Hz_Noise_, noisy_template, show=False)

Filters

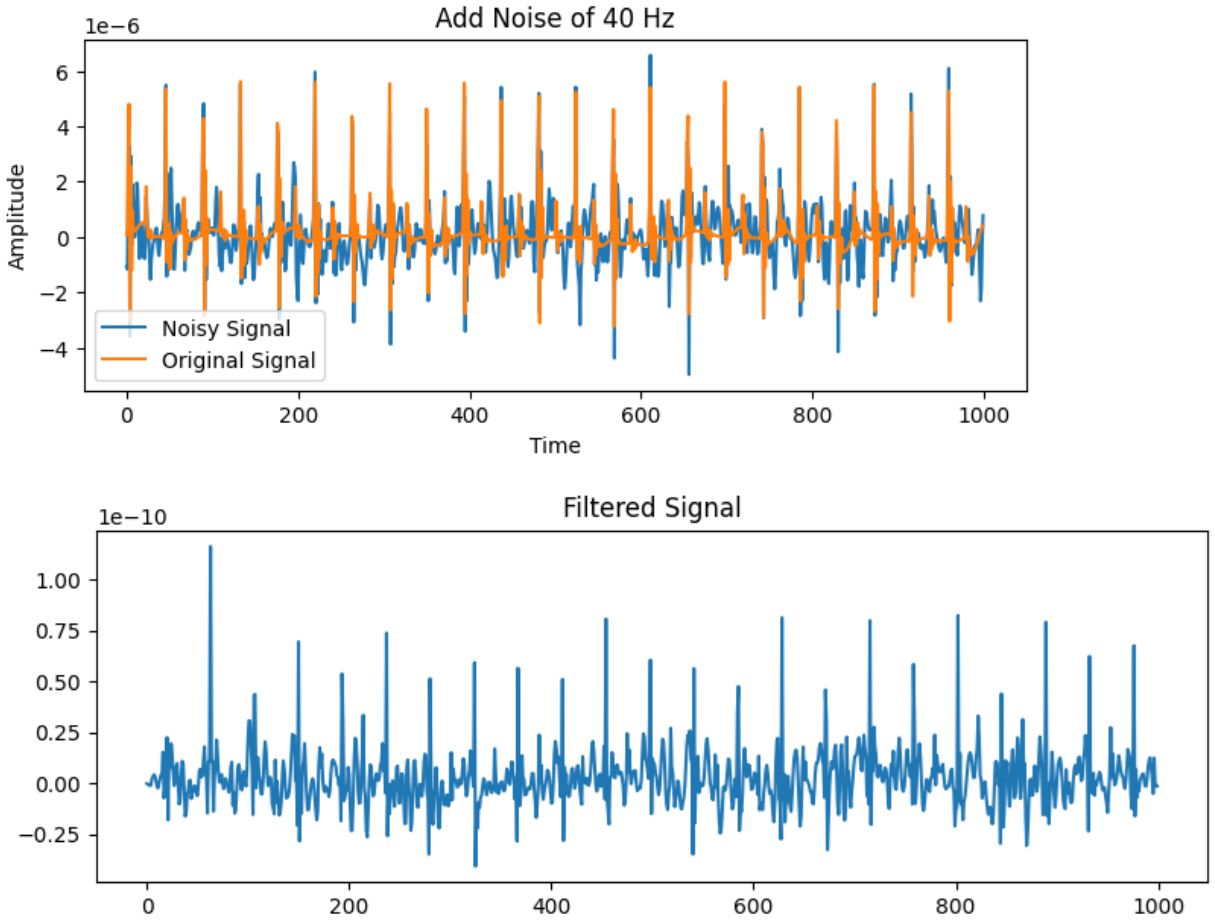

Figure: Matched Filter

Kalman Filter

A Kalman filter is a mathematical algorithm that uses a series of noisy measurements over time to estimate the true state of a system, providing a more accurate representation of the underlying variables by combining a prediction based on a system model with new measurements, effectively "filtering out" noise and uncertainties in the data; it's particularly useful when you can't directly measure the desired variables but have access to related, imperfect measurements.

The filter performs two main operations:

1. **Prediction:** Uses the system model to predict the next state based on the previous estimate.
2. **Update:** Incorporates new measurements to refine the predicted state, weighting the new information based on its reliability

Assumes noise is Gaussian

Kalman Filter Continued

State equation:

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Output equation:

$$y_k = Cx_k + z_k$$

Where:

A, B, and C are matrices

k is the time index

x is the state of the system

w is the process noise

z is the measurement noise

Kalman Filter Continued

Assumptions: k, w_k, and z_k are independent random variables

Process Noise Covariance:

$S_w = E(w_k W_k^T)$

Measurement Noise Covariance:

$S_z = E(z_k z_k^T)$

Where:

w_T is the transpose of w random noise vector

z_T is the transpose of z random noise vector

$E(\cdot)$ is the expected value

Kalman Filter Eqations

$$K_k = AP_kC^T(CP_KC^T + S_z)^{-1}$$

$$\hat{x}_{k+1} = (A\hat{x}_k + Bu_k) + K_k(y_{k+1} - C\hat{x}_k)$$

$$P_{k+1} = AP_kA^T + S_w - AP_kC^TS_z^{-1}CP_kA^T$$

Where:

K is the Kalman gain

P is the estimation error covariance

$\hat{x}$ is the state estimate. The first term is the estimate at time $k+1$ and the second term is the *correction* term

-1 superscript indicates matrix inversion

T superscript indicates matrix transposition

Equations from Kalman Filter Explained (with Equations)

Filters

1. **Kalman Filter**

kalman_filter(x, x_last, p_last, Q, R)

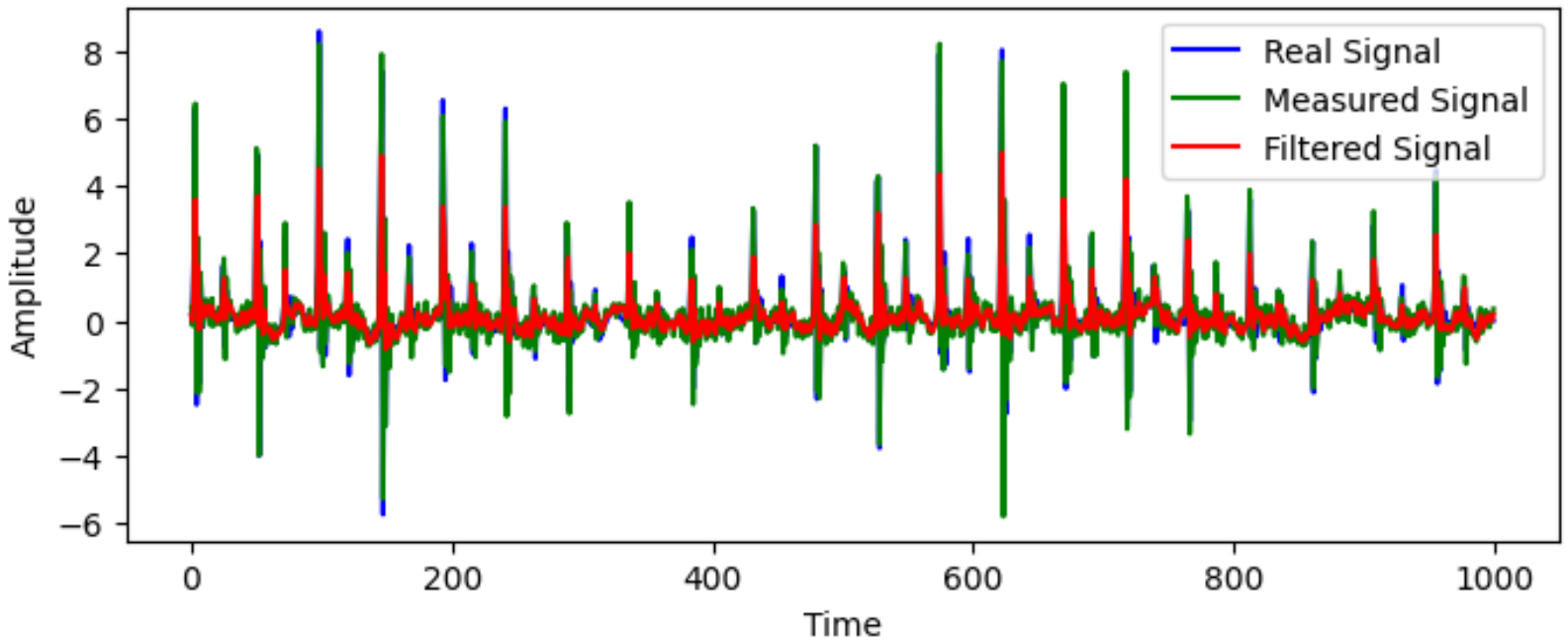

Figure: Kalman Filter

Adaptive Filters

An adaptive filter is a filter with non-constant coefficients. The filter coefficients are adjusted based on an criterium which is often defined to optimize the performance of the filter in its ability to estimate an unknown quantity in an input signal.

In this tutorail, we will limit ourselves to adaptive finite impulse response (FIR) filters.

The filter accepts an input signal **x** and produces an output signal **y**. The FIR coefficients are adjustable, meaning that at every new sample of **x**, the coefficients can take on a new value. The new value of filter coefficients is determined using a coefficient update algorithm, which computes an adjustment for each filter coefficient based on an error signal **e**. The error signal **e** is typically computed as the difference between the actual output signal **y** and a desired output signal **d**.

Adaptive Filters

The desired output signal d depends on the specific application of the adaptive filter. However, the adaptive algorithm will change the coefficients so as to minimize the mean squared value of the error signal e. That is, given that the filter output is defined by filter coefficients $w(n) = [w_0(n), w_1(n), ..., w_{N-1}(n)]^T$. we try to minimize the expected square error:

$\min_{w(n)} E\left[e^2(n)\right]$

There are generally four different configurations common for an adaptive filter:

1. System Identification
2. **Noise Cancellation**
3. Equalization
4. Adaptive Prediction

Basic Adaptive Filter

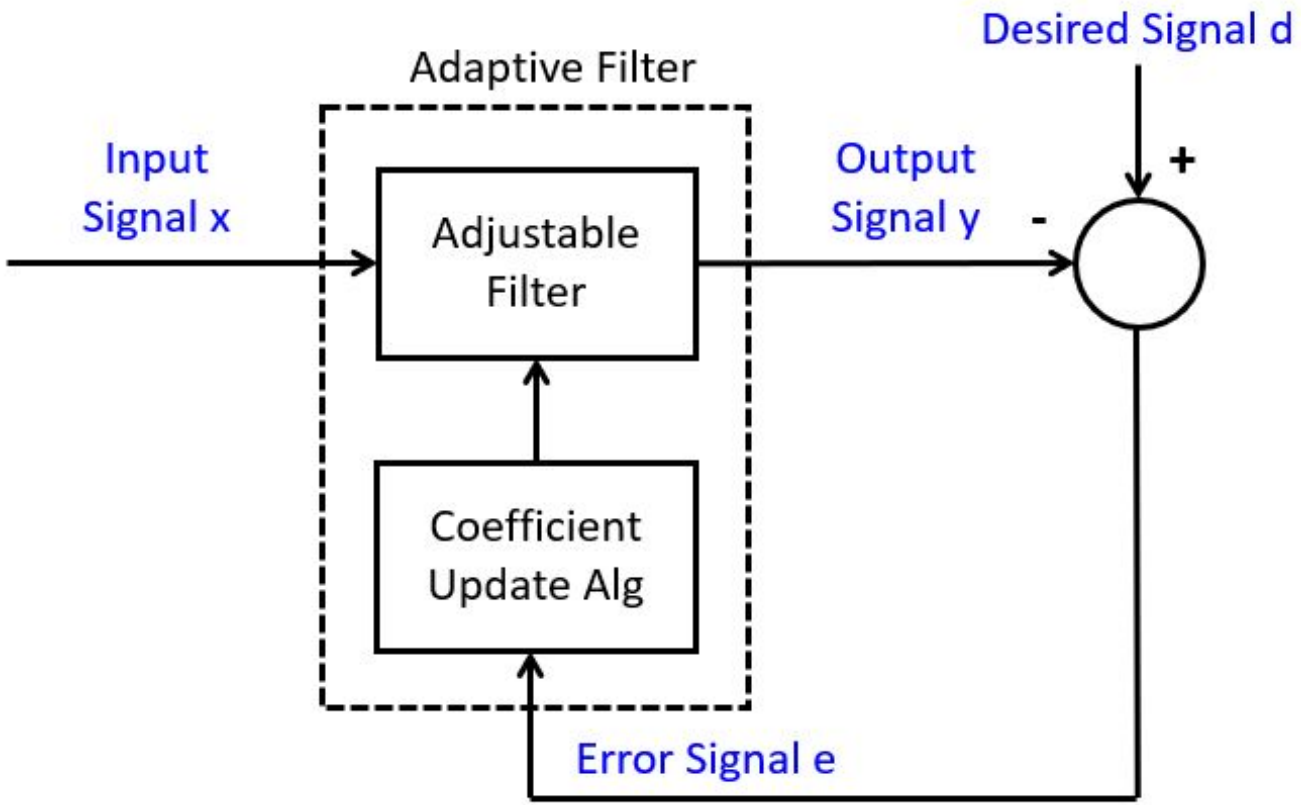

Figure: Adaptive Filter

Image from Adaptive Filters

Adaptive Noise Cancellation

In Noise Cancellation, we are interested in removing a known disturbance n_1 from a signal.

The disturbance is affected by the system dynamics $H(z)$ into n_0, so that we are unsure how much the input signal is affected by the disturbance.

Using an adaptive filter, we estimate the system dynamics, and we remove the filtered disturbance from the output signal.

The output signal, in the case of noise cancellation, is created out of the error signal, which in this case will closely resemble the input signal x.

Adaptive Filter for Noise Cancellation

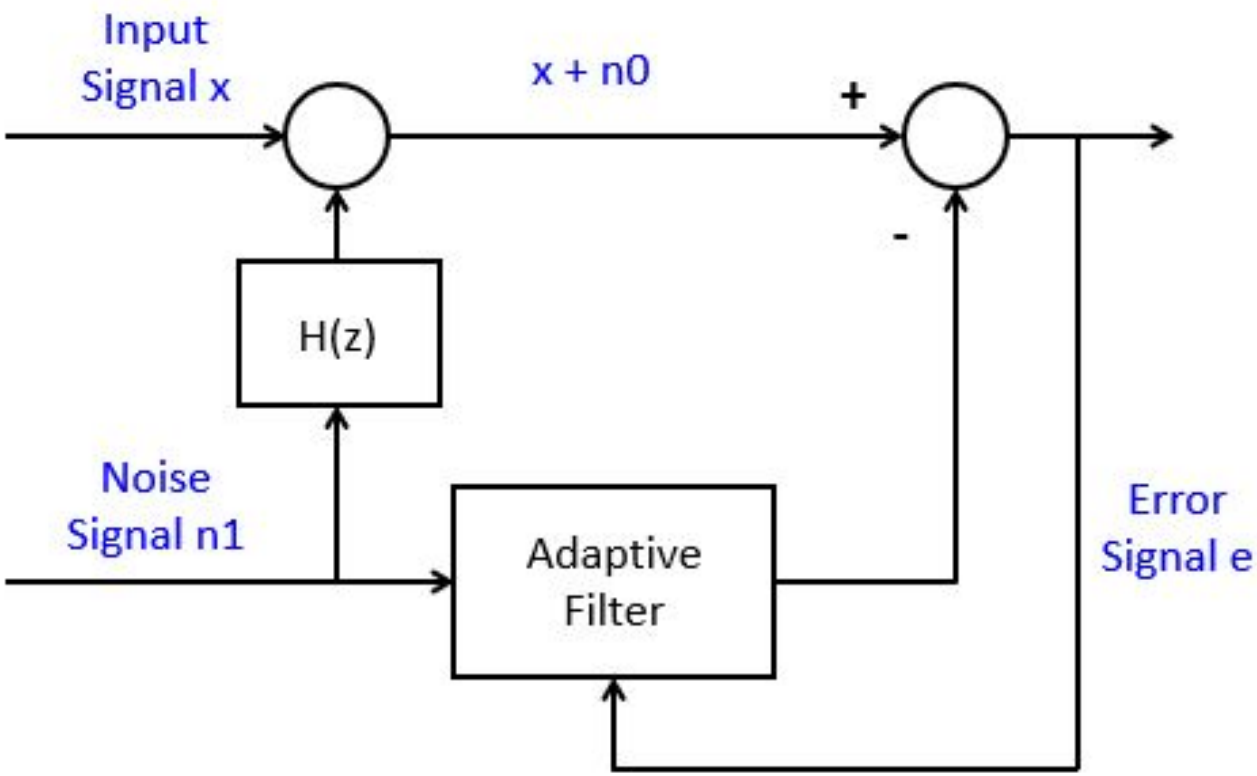

Figure: Adaptive Filter for Noise Cancellation

Image from Adaptive Filters

lms_filter(input, desired_signal, n=1, mu=0.015, show=False)

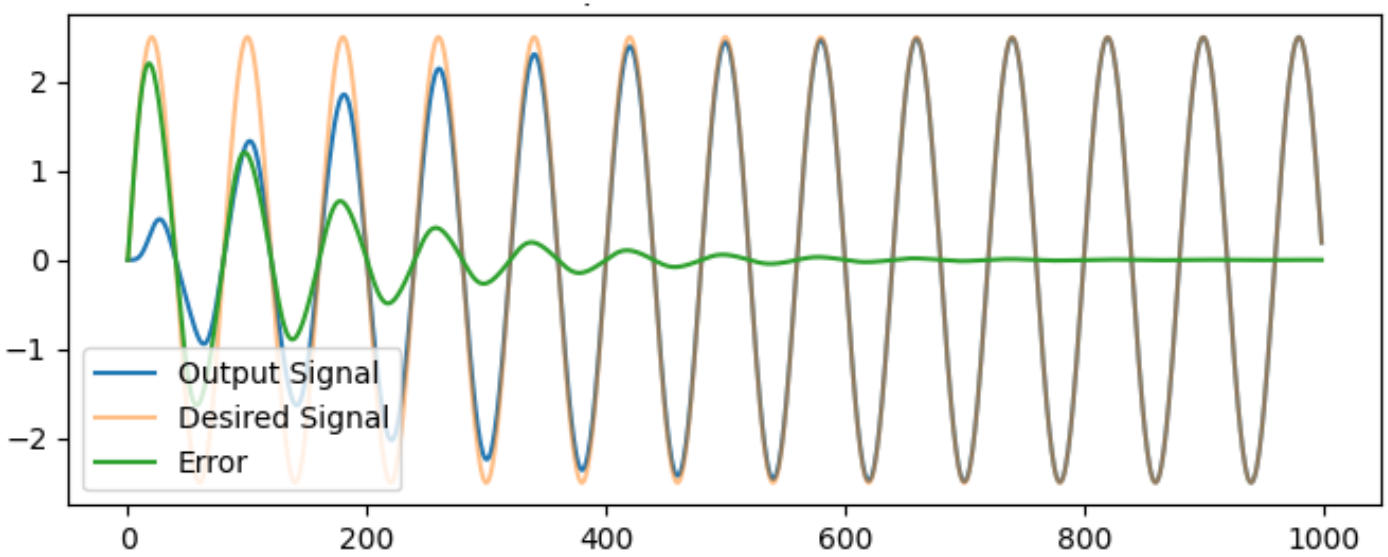

Figure: FIR Filter

Signal Averaging

Signal Averaging is a signal processing technique that improves the signal-to-noise ratio (SNR) by averaging multiple measurements of a signal.

Signal Averaging Steps

1. Take multiple samples of the signal. Each sample may have some degree of noise or error added to the true signal, so they are not identical.
2. **Align the Signals:** If the signal is periodic or repeatable, align measurements in time (e.g., using time synchronization or triggering).
3. **Average the Values:** Average all signal values point-by-point across all the samples. This is done for each corresponding time or measurement point. Mathematically, for N measurements, $x_1(t), x_2(t), ..., x_N(t)$ of a signal, the averaged signal $x(t)$ is calculated as:

$$x(t) = (\frac{1}{N})\Sigma_{i=1}^{N} x_i(t)$$

4. **Reduced Noise:** The noise is expected to have a random nature, thus averaging over many samples should cancel the noise, leaving a clearer representation of the signal. The more measurements averaged, the lower the noise level becomes (in the ideal case), and the clearer the true signal is.

Averaging Pros & Cons

Pros:

1. Improved Signal-to-Noise Ratio (SNR): Signal averaging reduces random noise by averaging it out, which increases the SNR. This makes the underlying signal easier to analyze or measure.
2. **Enhancing Low-Level Signals:** When dealing with low-amplitude signals that are buried in noise, averaging can make these signals more visible by reducing the noise that tends to mask them.
3. **Simple and Effective:** Signal averaging is a straightforward and computationally simple method to improve data quality, especially when the signal is repetitive or periodic.

Cons:

1. **Non-Stationary Signals:** If the signal changes over time or is non-stationary (i.e., it evolves in an unpredictable way), signal averaging may distort or lose important details. This technique works best for signals that are repetitive or periodic.
2. **Slow Process:** Signal averaging can be time-consuming because it requires multiple samples to be taken over time, especially if many samples are needed to effectively reduce noise.

Dynamic Time Warping

Dynamic Time Warping (DTW) is a similarity measure between two temporal sequences, accommodating for variations in their alignment and speed. It finds an optimal warping path by minimizing the accumulated distance between corresponding points.

Mathematically, given sequences $A = [a_1, a_2, ..., a_n]$ and $B = [b_1, b_2, ..., b_m]$, DTW constructs a cost matrix C where $C(i, j)$ represents the local distance between a_i and b_j. It then calculates an optimal path from $C(1, 1)$ to $C(n, m)$ through dynamic programming. DTW is widely used in pattern recognition, speech processing, and signal analysis for aligning time series data.

DTW Concepts

1. **Non-linear Alignment**: DTW can align sequences that have varying speeds. For example, it can compare two sequences where one sequence is "stretched" or "compressed" relative to the other (e.g., a faster and slower version of the same movement or event).
2. **Time Warping:** The idea of warping comes from adjusting the time axis in such a way that corresponding points in the two sequences match as well as possible. This is done by considering all possible alignments of the two sequences, even if they are of different lengths.
3. **Distance Measure:** DTW computes the distance between two time series by considering each point in one series and finding the optimal match (with a possible shift) in the other series. The total distance is the sum of these optimal matches.

DTW Steps

1. **Cost Matrix:** DTW uses a cost matrix to evaluate the alignment between the two sequences. This matrix measures the "cost" of aligning different points from each sequence. It is calculated by comparing each pair of points from the two sequences and computing the difference between them (often squared differences).
2. **Recursive Calculation:** The cost matrix is filled recursively, where each cell contains the minimal cumulative distance required to reach that point from the start of the sequences. The cumulative cost is the sum of the cost to reach the current point and the minimum of the previous possible alignments.
3. **Optimal Path:** The final DTW distance is obtained by tracing back from the end of the cost matrix to the start. This path represents the optimal alignment between the two sequences.
4. **Warping Path:** The warping path describes how the points in one sequence correspond to the points in the other sequence. It allows one sequence to be "warped" (or stretched/compressed) to align with the other.

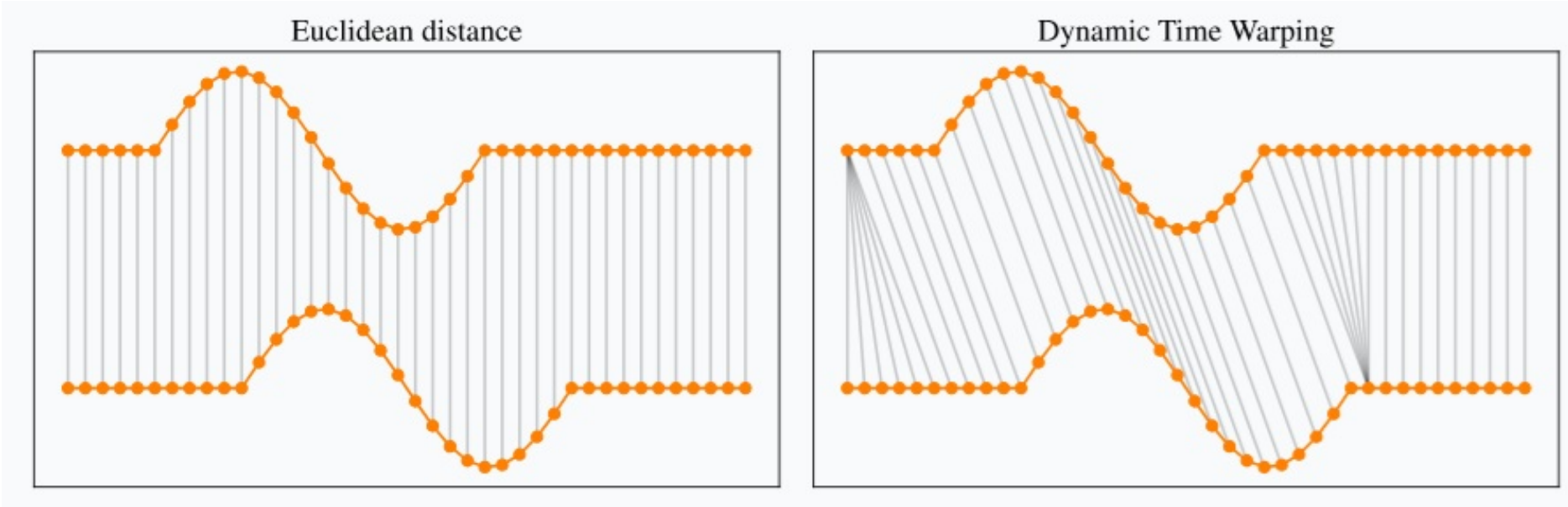

Figure: DTW and Euclidean Distance Comparison.

Image from An Introduction to Dynamic Time Warping by Romain Tavenard

DTW Pros

1. **Handles Temporal Shifts:** DTW is robust to shifts in time. It can align time series even when they are stretched or compressed along the time axis.
2. **Flexible Similarity Measure:** DTW provides a more flexible and accurate measure of similarity for time series compared to simple methods like Euclidean distance, especially when there is temporal misalignment.
3. **General Purpose:** DTW can be applied to a wide range of problems involving time series, from speech and audio processing to medical signal analysis.

DTW Cons

1. **Computationally Expensive:** The basic DTW algorithm has a time complexity of $O(n^2)$, where n is the length of the time series. This can be slow for long sequences.
2. **Overfitting:** If not properly tuned or constrained, DTW can overfit the alignment to the noise in the data, leading to inaccurate comparisons.
3. **No Consideration for Global Shape:** DTW optimizes local alignment, but this might lead to overemphasis on small local features rather than global patterns in the time series.
4. **Distance Matrix Size:** Since DTW requires constructing a full $n\ x\ n$ distance matrix, it can be memory-intensive for large datasets.

Non-Linear Adaptive Averaging Functions

The Non-Linear Adaptive Averaging Function (NLAAF) is a signal processing technique designed to improve the quality of a signal by adaptively adjusting how averaging is applied based on the characteristics of the signal itself. Unlike simple linear averaging methods, which apply the same weight or averaging function to all data points, NLAAF adjusts the averaging process dynamically according to the local behavior or properties of the signal.

NLAAF Continued

The idea behind NLAAF is to use local signal characteristics to determine how much averaging should be applied at each point in the signal. For example, at regions where the signal changes rapidly (edges or transitions), the averaging might be reduced or avoided, so the filter does not blur important features. In regions where the signal is more uniform or contains noise, stronger smoothing or averaging is applied.

The non-linear nature of the filter allows it to better handle complex signals with noise and varying levels of smoothness, making it more adaptable and capable of preserving important features compared to traditional linear filters.

NLAAF Concepts

1. **Non-Linear:** NLAAF is a non-linear filter, meaning that it doesn't apply a linear weighting to each data point in the signal. Instead, the function may apply a more complex, non-linear operation to smooth the signal while preserving important features like edges or rapid transitions that a linear filter might blur or distort.
2. **Adaptive:** The term "adaptive" means that the filter adjusts itself to the characteristics of the data. For example, it might apply stronger smoothing where the signal is relatively stable and less smoothing where the signal changes rapidly. This is important when dealing with signals that contain both noise and sharp transitions, as it can help preserve the important parts of the signal while reducing noise.
3. **Averaging Function:** Averaging refers to combining multiple data points in some way to produce a smoothed value. In NLAAF, this function is not fixed or uniform across the entire signal, but changes depending on the local context of the signal.

NLAAF Tutorial Functions

Note: There are two NLAAF function in this tutorial.

1. **NLAAF 1** - Applies DTW to two small windows (pieces) and determines the average of the pieces. These averages are used to build a center piece.
2. **NLAAF 2** - same as NLAAF 1, except a specified number of pieces greater than 2 are used to build the center piece.

Iterative Constrained Dynamic Time Warping

Iterative Constrained Dynamic Time Warping (ICDTW) is an enhanced DTW algorithm, designed to address some of the limitations of DTW, especially in terms of computational efficiency and accuracy when comparing time series with similar patterns but differing temporal alignments.

STEPS:

1. **Initial Alignment:** Start by performing a basic alignment between the two time series using standard DTW.
2. **Apply Constraints:** During each iteration, impose constraints that restrict the allowable warping paths between the two time series. These constraints might include things like limiting the range of movement in the warping path or ensuring monotonicity.
3. **Refinement:** After applying constraints, refine the alignment by recalculating the DTW distance with the constrained path.
4. **Repeat:** The iterative process is repeated until the alignment converges to an optimal or sufficiently accurate solution.

ICDTW Benefits

1. **Handling Large Variations:** By constraining the warping path and iterating over the alignment, ICDTW can more effectively handle large variations in the time series (e.g., different lengths, tempo variations, or large misalignments) compared to traditional DTW.
2. **Preserving Important Features:** The application of constraints during the iterative process helps to prevent the alignment from distorting important features in the time series, such as sharp transitions or edges. This is especially useful in time series that contain noise or irregularities.
3. **Improved Efficiency:** The iterative refinement approach makes ICDTW more computationally efficient, particularly when dealing with long time series, as it limits the warping search space through constraints.
4. **Flexibility:** ICDTW can be customized with various constraints, making it more adaptable to specific types of time series comparisons.

DTW Barycenter Averaging

DTW Barycenter Averaging (DBA) is a technique used to compute an average of multiple time series using Dynamic Time Warping (DTW). Unlike traditional methods of averaging time series, which simply compute the mean of corresponding points, DBAs use DTW to align the time series before averaging. This allows for the averaging of time series that may have temporal misalignments, speed variations, or shifts. The primary goal of DBA is to find a central tendency or "average" time series that best represents a set of input time series, while accounting for non-linear alignments in the data.

The term **"barycenter"** refers to the center of mass or average of a set of data points. In the context of DBA, the barycenter represents an average time series, which is computed by iteratively aligning and averaging multiple time series.

DTWBA Steps

1. **Initialization:** Start with an initial guess for the barycenter, which can be any of the input time series or some other starting point.
2. **Iterative Process:** In each iteration, the algorithm performs the following steps:

   1. Alignment using DTW: For each time series in the dataset, use DTW to align that series with the current barycenter.

   2. Averaging: The average of all the aligned time series is computed at each time point, creating a new candidate barycenter.
3. **Convergence:** The process is repeated iteratively. In each iteration, the barycenter is updated, and the DTW alignment is recomputed. The algorithm converges when the barycenter stops changing significantly between iterations or after a pre-defined number of iterations.
4. **Final Barycenter:** The final output is a time series that represents the average (or barycenter) of the input time series. This series is aligned in a way that best captures the common underlying pattern of the input sequences.

DTWBA Pros

1. **Handling Temporal Misalignment:** DBA excels in scenarios where the input time series are misaligned or have different lengths. By using DTW to align the series first, DBA can compute a meaningful average despite these misalignments.
2. **Robustness to Noise:** Since DBA aligns the time series before averaging, it tends to be more robust to noise than traditional averaging methods. By aligning the time series, DBA reduces the impact of small fluctuations that may not represent the true underlying patterns.
3. **Flexible for Variable Length Time Series:** Has ability to handle time series of different lengths.
4. **Improved Pattern Recognition:** Can produce more accurate representations of a group of time series, making it useful for pattern recognition tasks.

DTWBA Cons

1. **Computational Complexity:** DTW itself requires $O(N^2)$ time complexity for a pair of time series of length N, and performing this for many series can be quite slow.
2. **Sensitive to Initialization:** DTWBA can be sensitive to its initial guess.
3. **Pairwise DTW Computation:** DTWBA involves computing the pairwise alignment of all time series using DTW, which can be very resource-intensive for large datasets or long time series.
4. **Struggle with Extremely Noisy Data:** While DTWBA can handle some noise, extreme noise or outliers can still affect the result.
5. **Overfitting to Local Variations:** Can overfit to local variations and fluctuations within each time series.
6. **Difficulty with Large Variations in Scale:** If the time series differ greatly in scale (amplitude) in addition to time shifts, DTWBA might not always produce a meaningful average.

DTW Averaging Methods

1. Non-Linear Adaptive Averaging Function (NLAAF) 1 - Applies DTW to two small windows (pieces) and determines the average of the pieces. These averages is used to build a center piece.
2. NLAAF 2 - same as NLAAF 1, except a specified number of pieces greater than 2 are used to build the center piece.
3. Iterative Constrained Dynamic Time Warping (ICDTW) - Repeated DTW comparison over pieces with constraints added over each iteration. Becomes more refined at each step.
4. DTW Barycenter Averaging (DBA) - Randomly determines a "Barycenter". Iteratively uses DTW to compare to each piece and update the Barycenter.

Averageing Code

1. performNLAAF1(list(series), show=True)
2. performNLAAF2(list(series), show=True)
3. performICDTW(list(series), show=True)
4. performDBA(series, show=True)

Averaging Code & Results

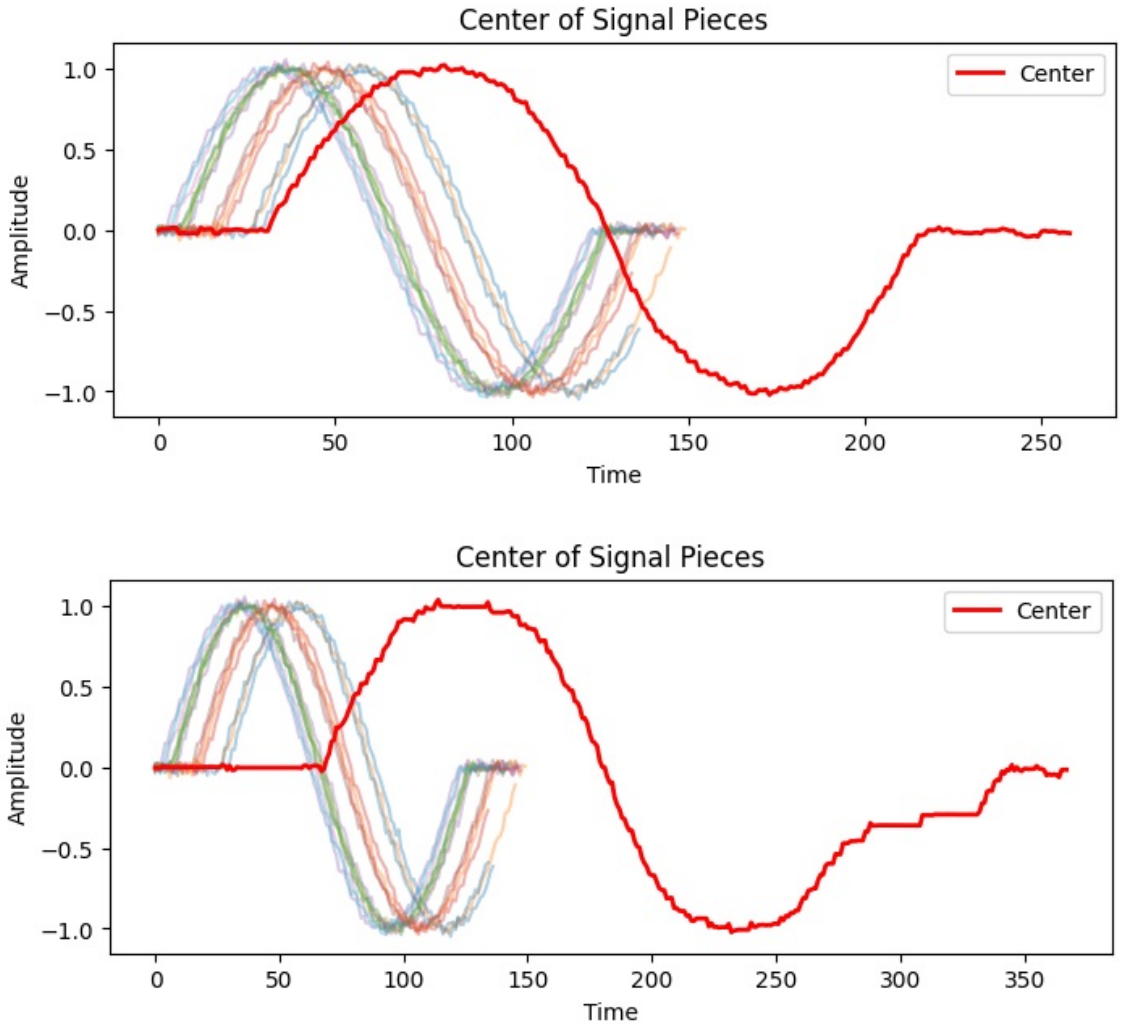

Figure: NLAAF1 and NLAAF2

Averaging Continued

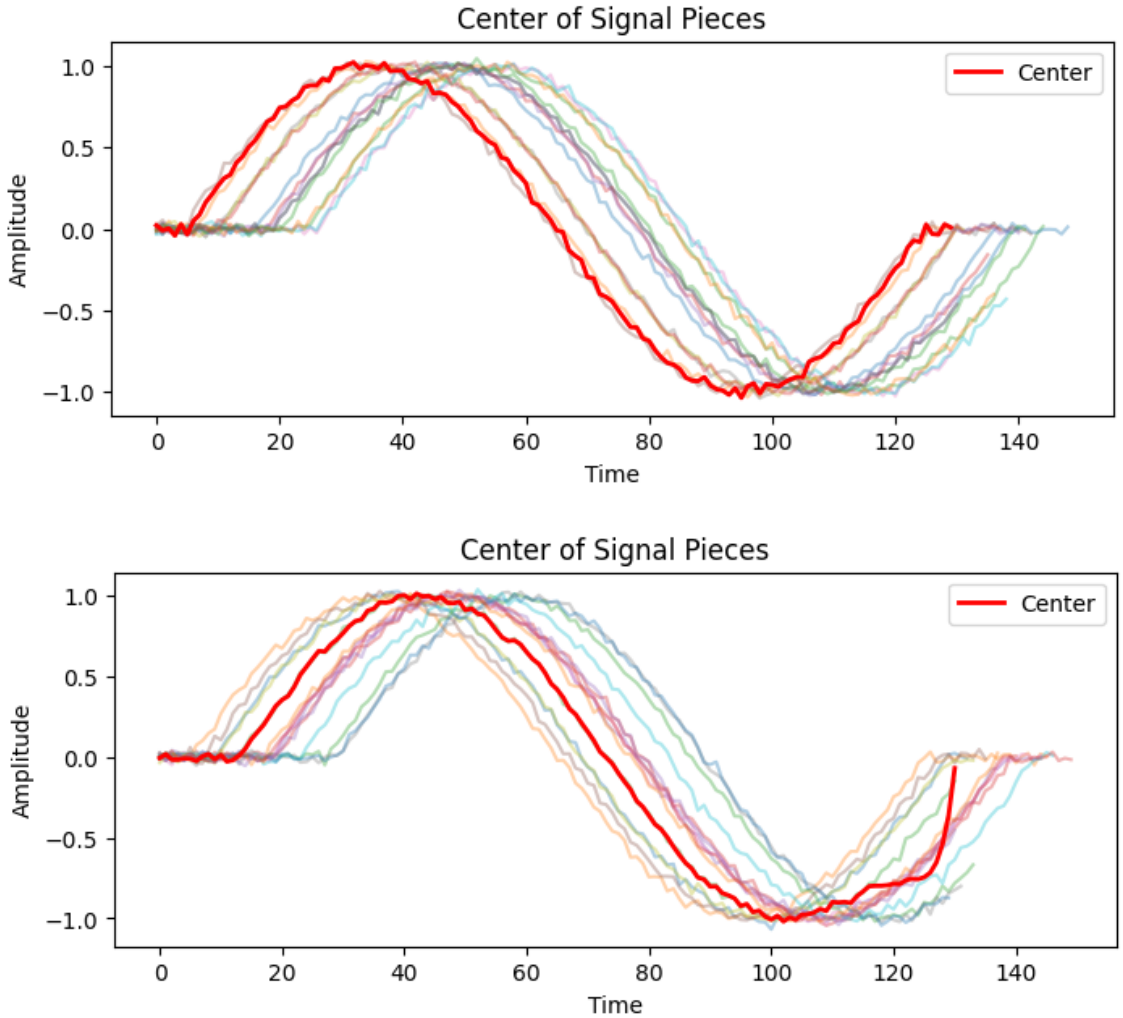

Figure: ICDTW and DBA

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Decomposition

1. Decompposition in signal processing is the breakdown of a complex signal into simpler components.

Generated Signals

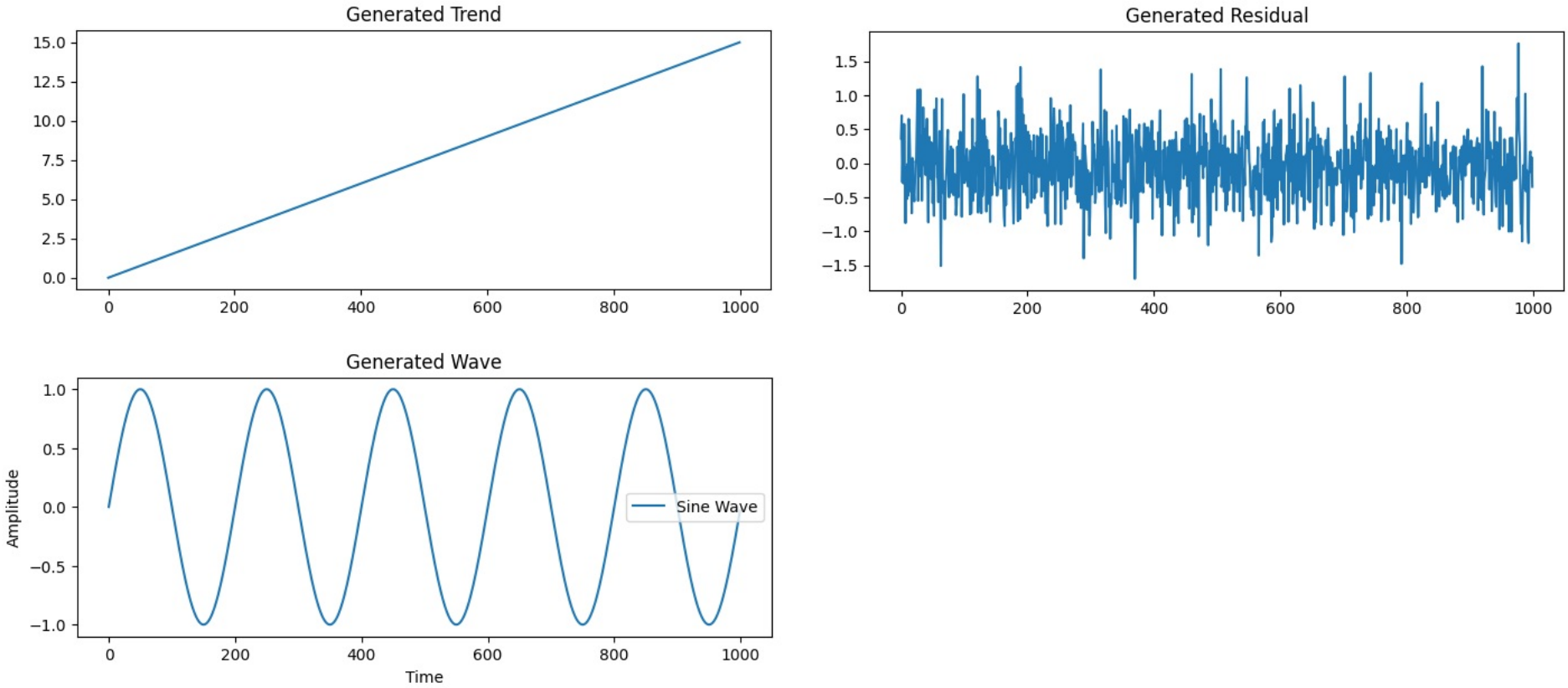

Figure: Generated Signals to Combine for Decomposition

Seasonal Decomposition

Seasonal decomposition is a technique used in time series analysis to break down a time series into its individual components: trend, seasonality, and residual (or noise). This helps to better understand the underlying structure of the data and is often used to detect patterns, forecast future values, and improve the accuracy of predictive models.

The primary goal of seasonal decomposition is to separate the time series into components that represent different underlying behaviors.

Seasonal Decomposition Continued

The model can be expressed as $y(t) = T(t) + S(t) + R(t)$, where $T(t)$ is the trend, $S(t)$ is the seasonal component, and $R(t)$ is the residual:

1. **Trend:** The long-term movement or direction in the data. It represents the general tendency of the data to increase or decrease over time, ignoring short-term fluctuations.
2. **Seasonality:** The repeating and predictable patterns or cycles that occur at regular intervals, such as daily, weekly, monthly, or yearly. This component captures fluctuations that recur over a fixed period.
3. **Residual:** The "leftover" noise or random fluctuations in the data that can't be explained by the trend or seasonal components. This component represents irregular variations, outliers, or random events.

Seasonal Decomposition Code

$$Trend = 1.5 * np.linspace(0, 10, 1000)$$

$$Seasonal = sine_wave(duration = 10, sampling_rate = 100, frequency = 0.5)$$

$$Residual = np.random.normal(0, 0.5, 10 * 100)$$

$$signal = Seasonal + Trend + Residual$$

$$\_ = seasonal\_decomposition(signal, period = 200, show = True)$$

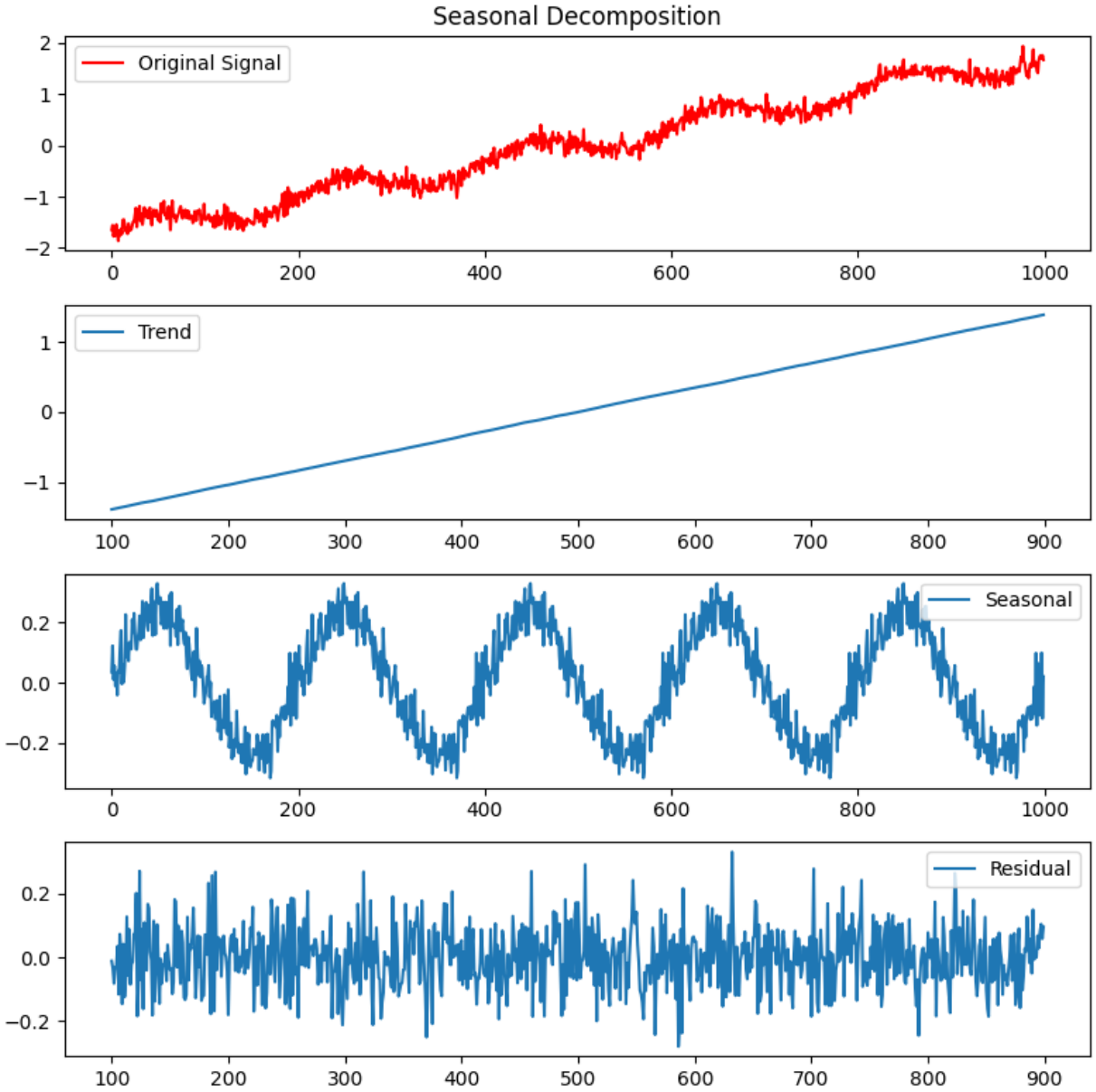

Figure: Seasonal Decomposition: Original, Trend, Seasonal, and Residual

Empirical Mode Decomposition (EMD)

Empirical Mode Decomposition (EMD) is a data-driven method for analyzing non-linear and non-stationary time series data. It decomposes a time series into a set of intrinsic mode functions (IMFs), which represent different oscillatory modes of the data. These IMFs are derived from the data itself, without relying on a pre-defined basis, making EMD particularly useful for analyzing complex, real-world signals that do not follow simple trends or seasonality.

EMD decomposes a signal into intrinsic mode functions (IMFs) based on local extrema. Each IMF represents a specific oscillatory mode. Mathematically, a signal $x(t)$ is decomposed as $x(t) = \sum_{i=1}^{n} C_i(t) + R(t)$, where $C_i(t)$ is the i-th IMF and $R(t)$ is the residual.

EMD Steps

1. **Sifting Process:** The signal is first "sifted" to extract the high-frequency oscillations (IMFs) using the local maxima and minima. This involves:
   1. Identifying the local maxima and minima.
   2. Interpolating between the maxima and minima to create upper and lower envelopes.
   3. Subtracting the mean of the upper and lower envelopes from the signal to obtain the IMF.
   4. Repeating the process on the residual signal until the stopping criterion is met.
2. **Iterative Extraction:** This process continues for several iterations. Each iteration extracts an IMF, and what remains after extracting each IMF is a progressively lower-frequency component of the signal. It stops when the residual signal is a monotonic function (or when other convergence criteria are met).
3. **Final Residual:** The final residual is often considered the trend of the original time series, which has no oscillatory modes left.

EMD Pros

1. **Data-Driven:** EMD does not assume a pre-defined model or basis (like Fourier transforms or wavelets), making it highly flexible for complex, real-world signals.
2. **Adaptability:** It works well for non-stationary, non-linear signals, making it suitable for a wide variety of applications, including those where other methods may struggle.
3. **No Assumptions:** Unlike methods such as Fourier or wavelet transforms, EMD does not require assumptions about the signal's behavior or frequency content.
4. **Multiscale Analysis:** It allows for multiscale decomposition, which helps to capture different frequency components of the signal.

EMD Cons

1. **Mode Mixing:** EMD can sometimes suffer from mode mixing, where different frequency components are incorrectly mixed together in the same IMF. This can occur if the signal is highly irregular or contains sharp discontinuities.
2. **Sensitivity to Noise:** EMD can be sensitive to noise, especially for signals with small-scale variations.
3. **Computational Cost:** The iterative nature of the method can be computationally expensive for large datasets.

EMD Code

emd_decomposition(signal, show=True)

Generated Signals

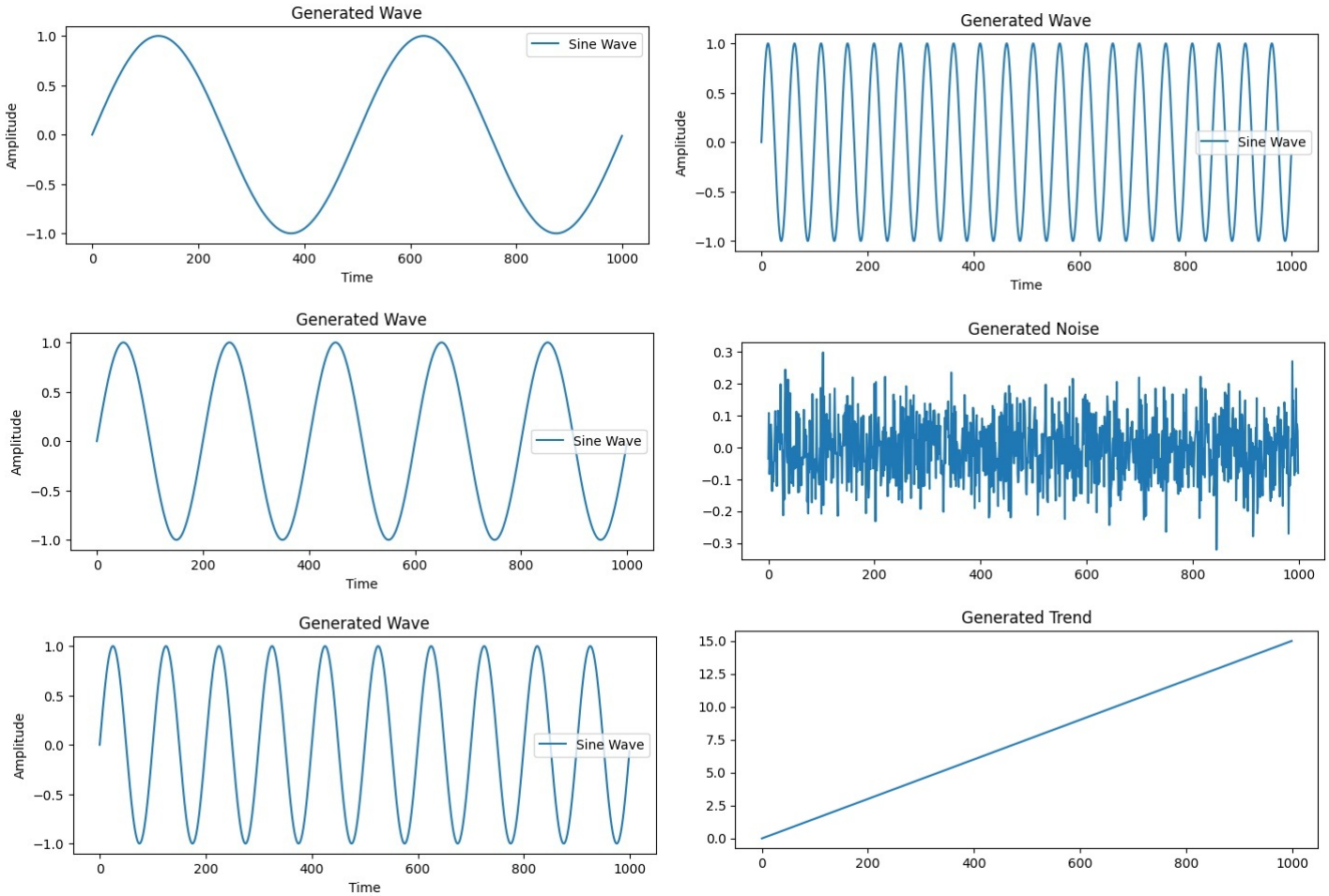

Figure: Generated Waves

EMD Continued

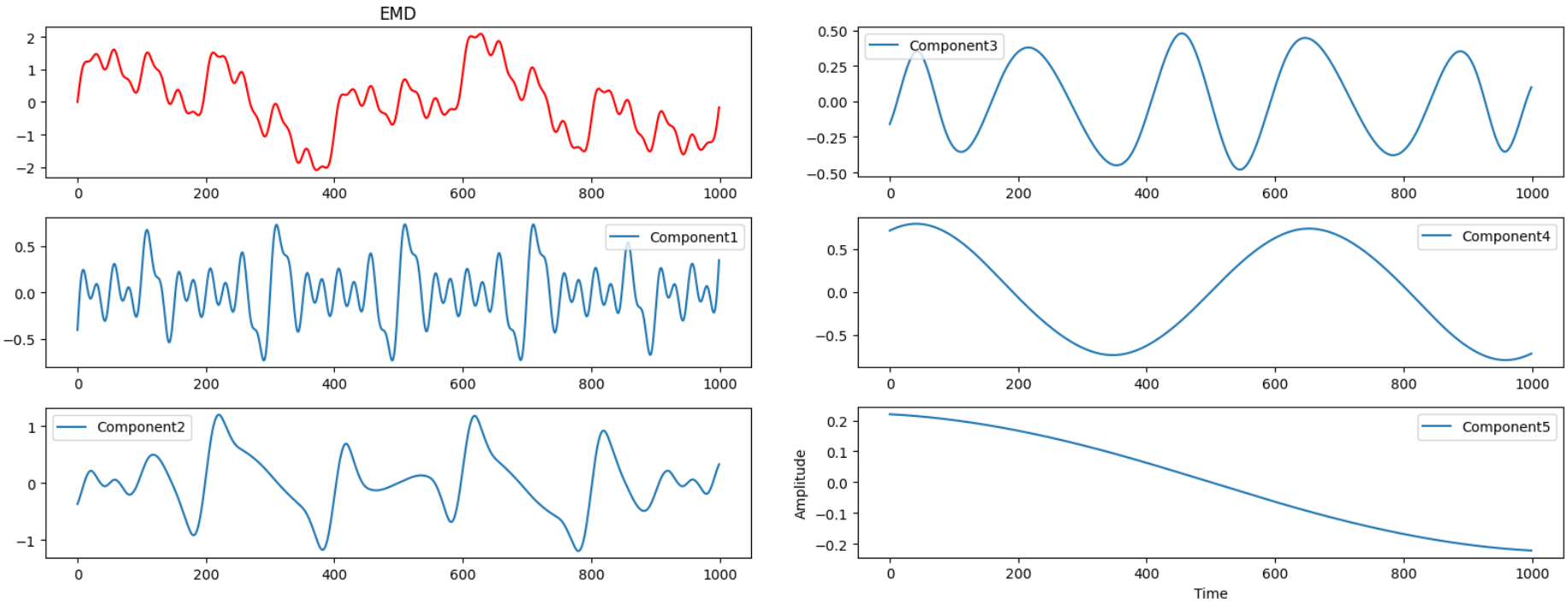

Figure: EMD

Ensemble EMD

EEMD is an extension of EMD that addresses mode mixing by adding noise and performing multiple decompositions. It aims to improve the decomposition results by providing a more stable representation of the signal's components.

Code:

```
eemd_decomposition(signal, show=True)
```

EEMD Steps

1. **Add White Noise:** White noise is added to the original signal. This introduces randomness into the signal, helping to avoid the bias caused by mode mixing and improving the decomposition process.
2. **Decompose with EMD:** Perform Empirical Mode Decomposition on the noisy signal. This step results in a set of IMFs that are influenced by the added noise.
3. **Repeat the Process:** The process is repeated several times, each time adding a different realization of white noise to the signal. For each noisy signal, a separate EMD decomposition is performed, resulting in a different set of IMFs.
4. **Ensemble Averaging:** After completing the decompositions for all white noise realizations, the IMFs from each decomposition are averaged. The averaging process helps to cancel out the effects of the white noise, leaving behind a more accurate representation of the true intrinsic oscillations in the original signal.
5. **Final Decomposition:** The resulting IMFs, averaged over the ensemble of noisy realizations, form the final set of IMFs that are less influenced by noise and mode mixing.

Advantages of EEMD Over EMD

1. **Reduced Mode Mixing:** By adding white noise and performing the decomposition multiple times, EEMD mitigates the problem of mode mixing that can occur in traditional EMD. The averaging process helps to separate components more clearly into different frequency bands.
2. **Noise Robustness:** The introduction of white noise in EEMD helps to "drown out" the effects of real-world noise, making the decomposition more stable and robust, especially in the presence of noise or irregularities in the data.
3. **More Reliable IMFs:** The ensemble averaging step leads to more stable and meaningful IMFs, as the noise and instability from individual decompositions are averaged out, resulting in IMFs that better represent the underlying dynamics of the signal.
4. **Better Handling of Complex Signals:** EEMD is better suited for dealing with complex, non-linear, and non-stationary signals, such as those found in biomedical data, mechanical systems, and financial data, where EMD alone may struggle.

EEMD Continued

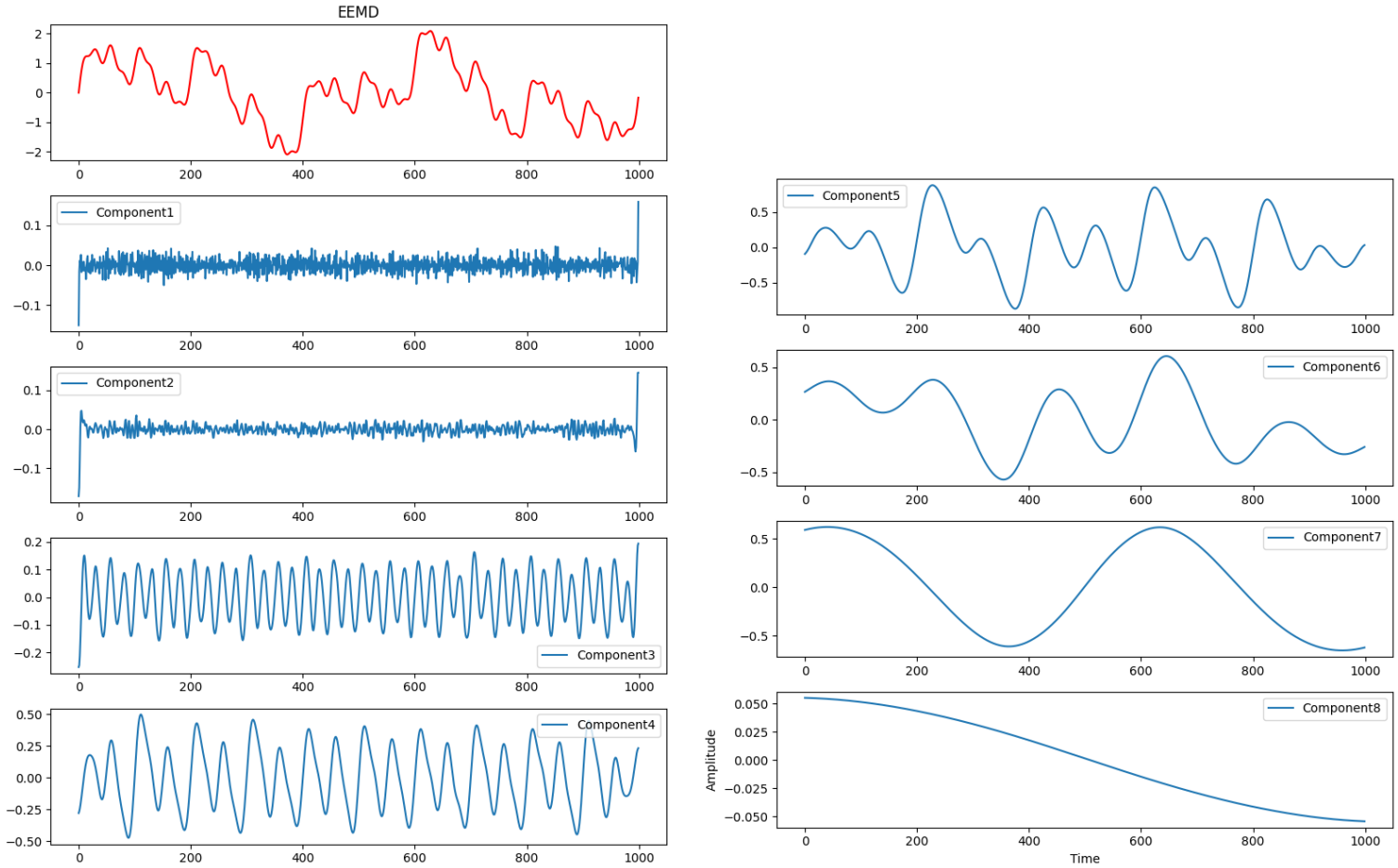

Figure: EEMD

Complete EEMD

CEEMD is a further enhancement of EEMD that uses an ensemble of EMD decompositions with different white noise added at each iteration. It improves upon EEMD by reducing the residual error and enhancing the decomposition's accuracy.

The steps for CEEMD are the same as EEMD, except for Add Noise: Instead of adding just positive white noise (as in EEMD), both positive and negative white noise of equal amplitude are added to the signal. This makes the process more symmetric and helps to cancel out noise artifacts more effectively.

Code: ceemd_decomposition(signal, show=True)

CEEMD Continued

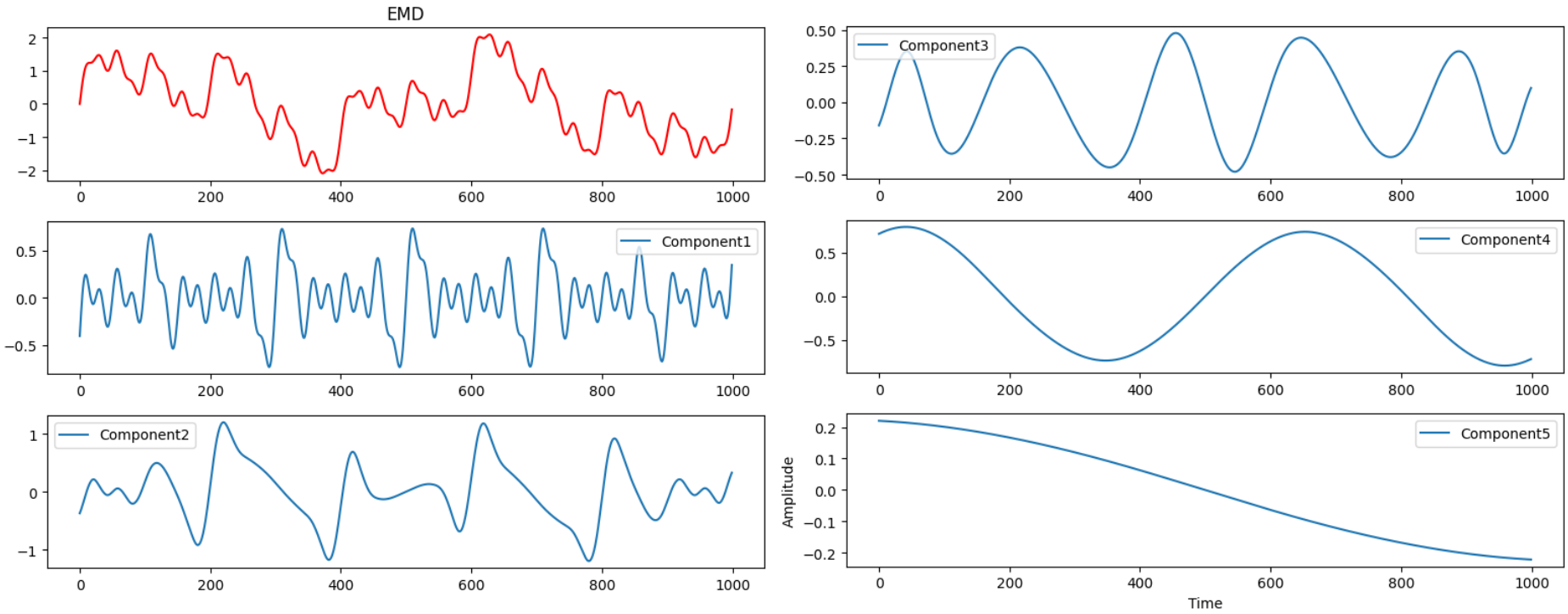

Figure: CEEMD

Variational Mode Decomposition (VMD)

Variational Mode Decomposition (VMD) is a signal processing technique that is used to decompose a complex signal into a set of intrinsic modes or modes of oscillation. It is particularly useful for analyzing non-stationary and non-linear signals, and it has gained popularity as an alternative to methods like Empirical Mode Decomposition (EMD) and its variations (such as CEEMD).

VMD decomposes a signal into a set of oscillatory modes with varying frequencies. It minimizes the mode mixing problem by solving a variational optimization problem. The decomposition can be expressed as $x(t) = \sum_{i=1}^{n} u_i(t) + R(t)$, where $u_i(t)$ are the modes and $R(t)$ is the residual.

Code: vmd_decomposition(signal, show=True)

VMD Steps

VMD is based on an optimization framework where the objective is to find a set of modes such that their sum approximates the original signal. Here's a step-by-step breakdown of how VMD works:

1. Signal Representation: VMD takes a time-domain signal $x(t)$ and aims to represent it as the **sum** of a set of modes $u_k(t)$, each of which has a specific frequency component. The modes are represented as functions with limited frequency bandwidths.

2. Frequency Localization: Each mode is associated with a center frequency F_k, and the objective is to make the **frequency bandwidth** of each mode as narrow as possible. This is done by limiting the frequency range of each mode, which helps in extracting specific frequency components from the signal.

VMD Steps Continued

3. Variational Optimization: VMD solves an optimization problem in the form of a variational principle to find the modes and their associated frequencies. The optimization problem seeks to minimize the following cost function:

$$\frac{min}{u_k(t), f_k} \sum_{k=1}^{k} \left(|\frac{\delta}{\delta t} u_k(t)|^2 + \alpha |f_k - \hat{f}_k|^2 \right)$$

Where:

$u_k(t)$is the k-th mode.

f_k is the central frequency of the k-th mode.

$\hat{f}$ is the reference frequency.

α is a regularization parameter that controls the balance between bandwidth and frequency accuracy.

The first term in the cost function is the **bandwidth** of the mode, while the second term penalizes deviations from the desired central frequency.

VMD Steps Continued

4. Iterative Update: The algorithm uses an iterative process to update the modes and their associated frequencies in order to minimize the cost function. This is done by alternating between updating the modes and adjusting the central frequencies until convergence is achieved.

5. Reconstruction: After obtaining the modes, the original signal can be reconstructed by summing all the extracted modes. The result is a signal representation in terms of different oscillatory components, each with a well-defined frequency range.

VMD Pros

1. **Precise Frequency Separation:** VMD ensures that the extracted modes correspond to distinct frequency bands, reducing mode mixing compared to traditional methods like EMD.
2. **Noise Robustness:** VMD is more robust to noise and is less sensitive to high-frequency noise than methods like EMD, which can struggle in noisy environments.
3. **Flexible and Generalizable:** VMD works well for a wide range of non-linear and non-stationary signals, making it applicable across many domains like biomedical, financial, and geophysical analysis.

VMD Cons

1. **Parameter Sensitivity:** The performance of VMD depends on selecting the right parameters, such as the number of modes K , the regularization parameter α , and the convergence tolerance. Incorrect parameter settings can lead to poor results.
2. **Computational Complexity:** VMD requires solving an optimization problem for each mode, which can be computationally intensive for large datasets or real-time applications.

Generated Waves

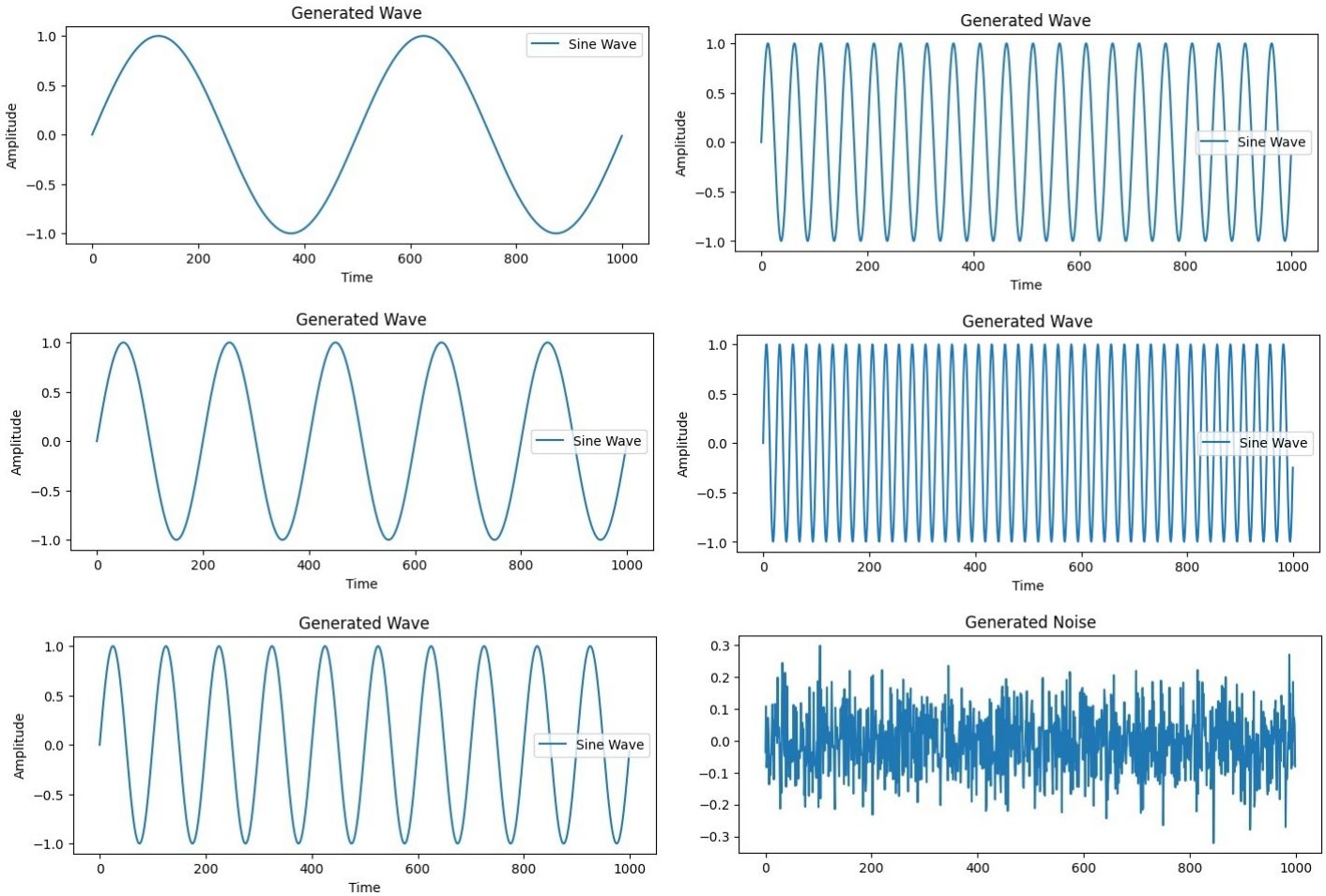

Figure: Generated Waves

VMD Continued

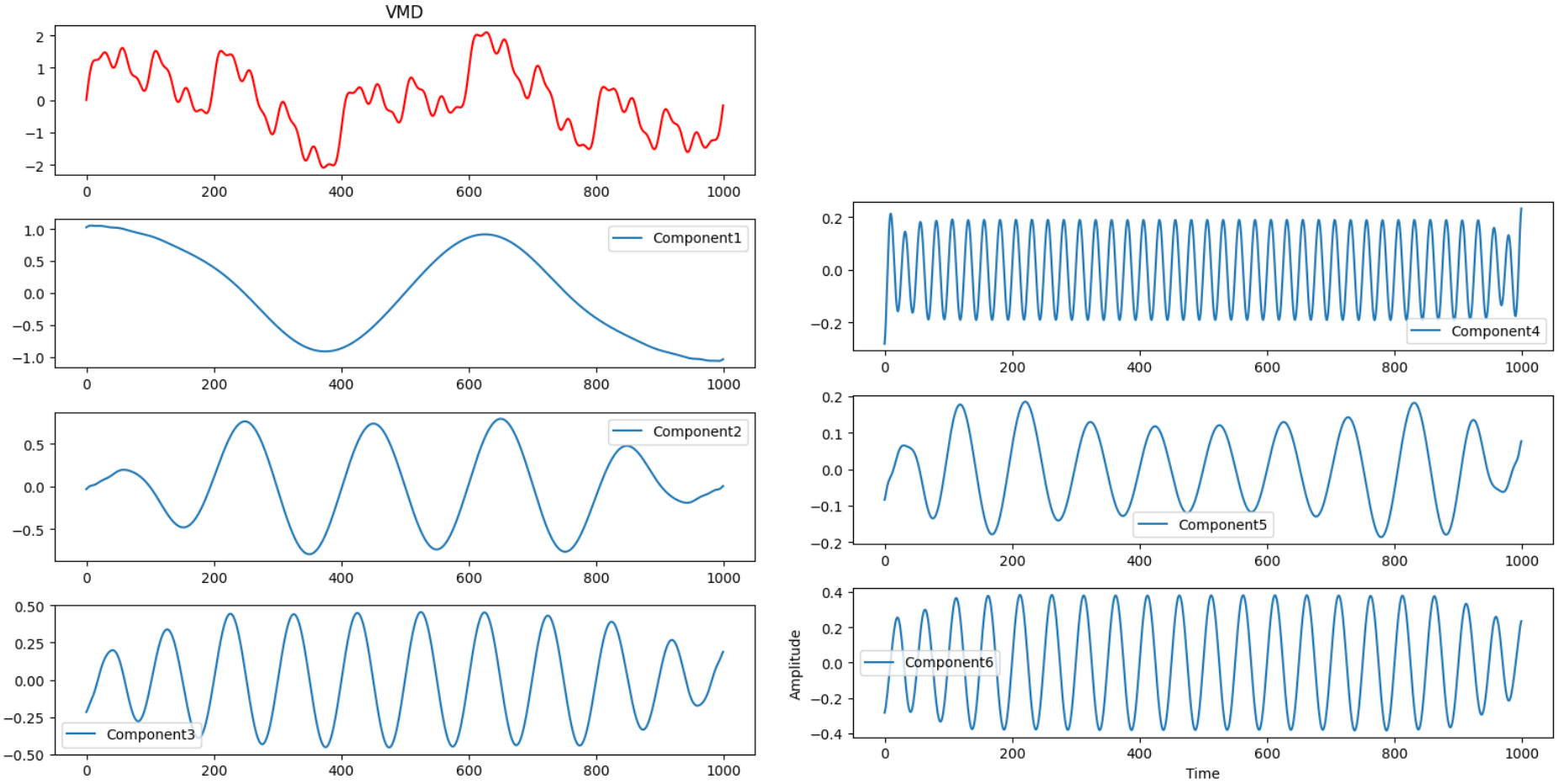

Figure: VMD

Blind Source Seperation (BSS)

BSS is a signal processing technique that aims to extract independent source signals from their observed mixtures. Two widely used methods for BSS are Principal Component Analysis (PCA) and Independent Component Analysis (ICA).

PCA-Based Blind Source Separation - PCA is employed to transform the observed mixed signals into a new set of uncorrelated variables called principal components. In the context of BSS, PCA can be applied to the covariance matrix of the observed signals. The principal components are ordered in terms of their variances, and by selecting a subset of these components, one can achieve a decorrelated representation of the mixed signals. However, PCA does not guarantee independence.

$C * v = \lambda * v$ where C is the covariance matrix, v is an eigenvector, and λ is the corresponding eigenvalue.

Independent Component Analysis BSS

ICA aims to find a linear transformation of the observed signals such that the resulting components are statistically independent. The key assumption is that the sources are statistically independent.

Mathematically, given a matrix representing the observed mixtures, ICA seeks a demixing matrix W such that $S = WX$, where S contains the estimated source signals.

The optimization problem in ICA is often formulated as maximizing the non-Gaussianity of the estimated sources. Using negentropy as a measure of non-Gaussianity is a common approach, leading to objective functions like:

$$J(W) = \sum_{i=1}^{n} [E\{G(u_i)\} - E\{G(v_i)\}]$$

where u_i is the i-th estimated source, v_i is the i-th component of the observed mixtures, $G(\cdot)$ is a nonlinear function, and $E\{\cdot\}$ denotes expectation.

BSS Continued

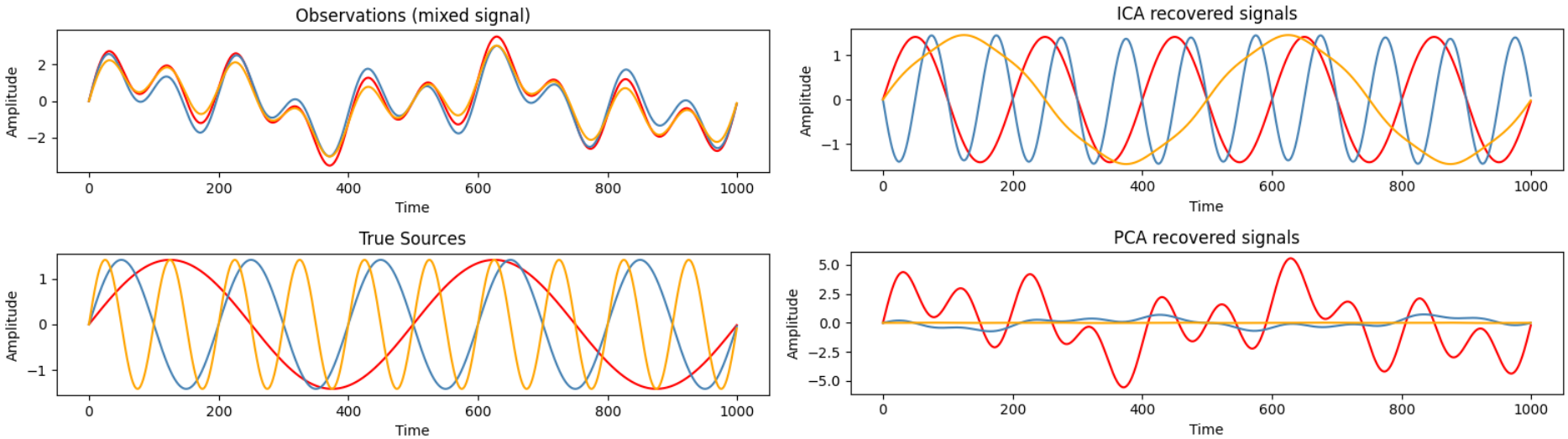

Figure: BSS: Observations, True Sources, ICA Recovery, PCA Recovery

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Ti

Time and Frequency Domain

1. Discussion of Time Domain Features
2. Discussion of Frequency Domain Features
3. Discussion of Time-Frequency Domain Features

Peak Detection & Envelope Extraction

Peak detection in signal processing refers to the process of identifying local maxima (peaks) or minima (troughs) in a signal. These peaks are important features in many applications, as they often correspond to significant events, changes, or patterns in the data.

Envelope extraction is a signal processing technique used to isolate the "envelope" of a signal, which represents the smooth curve that outlines the peaks of a modulated waveform. It's often applied to modulated signals, such as amplitude modulation (AM) in radio signals, to extract the slower-varying component of the signal (the envelope), while ignoring the higher-frequency oscillations.

Definitions

1. **Peak:** A data point is considered a peak if it is higher than its immediate neighbors. For instance, in a time series, a peak at time t satisfies:

   $x(t) > x(t-1)$ and $x(t) > x(t+1)$

   where $x(t)$ is the value of the signal at time t and $t-1$ and $t+1$ are its neighboring values
2. **Trough:** Similarly, a trough is a local minimum, where:

   $x(t) < x(t-1)$ and $x(t) < x(t+1)$
3. **Local Peaks:** These are peaks within a small neighborhood of data points.
4. **Global Peaks:** These refer to the overall highest values within the entire signal.
5. **Thresholding:** A peak might only be considered significant if it exceeds a certain threshold value, which helps filter out smaller, less relevant peaks from noisy data.

Peaks, Valleys, & Envelopes

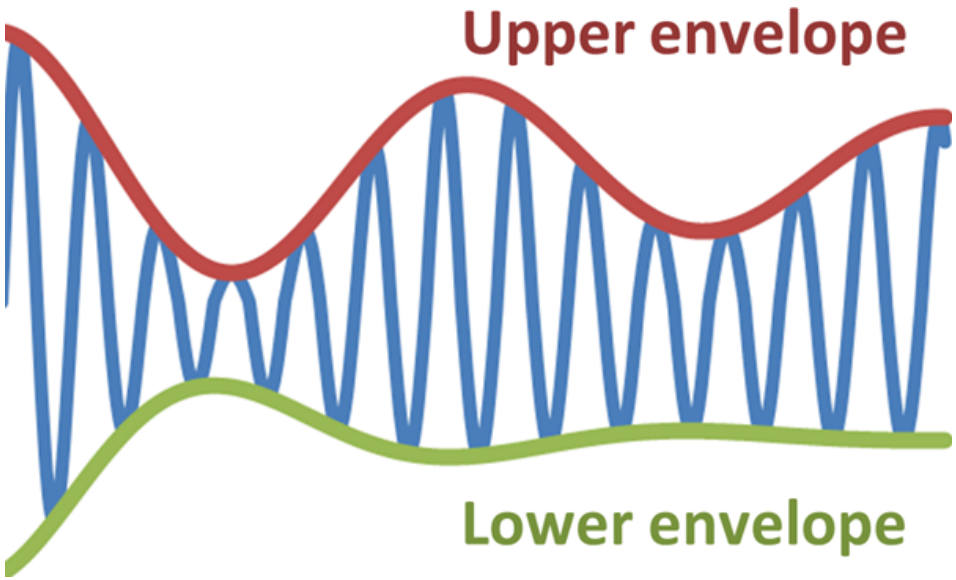


Figure: Peaks, Valleys & Envelopes

Image from Wikipedia

Peaks

get_peaks(signal)

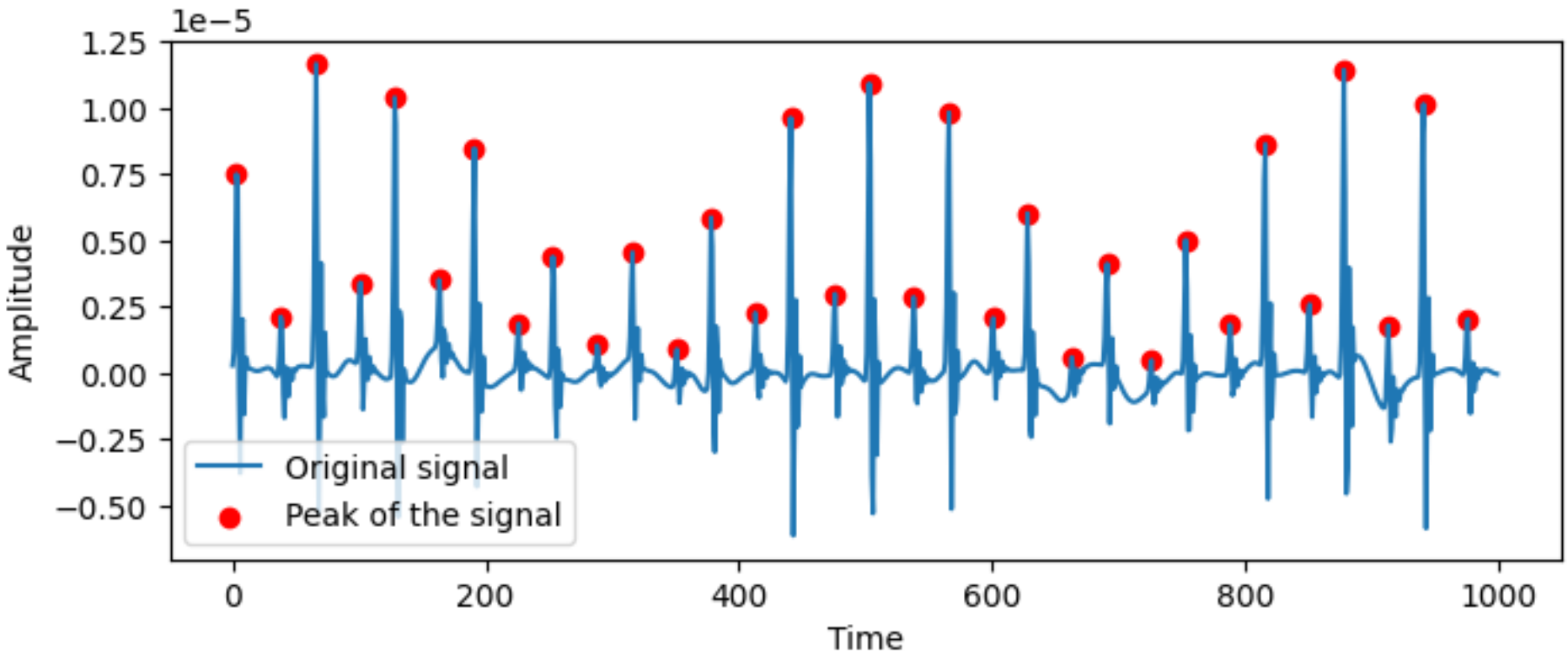


Figure: Get Peaks

Envelope Detection

envelope_from_peaks(signal)

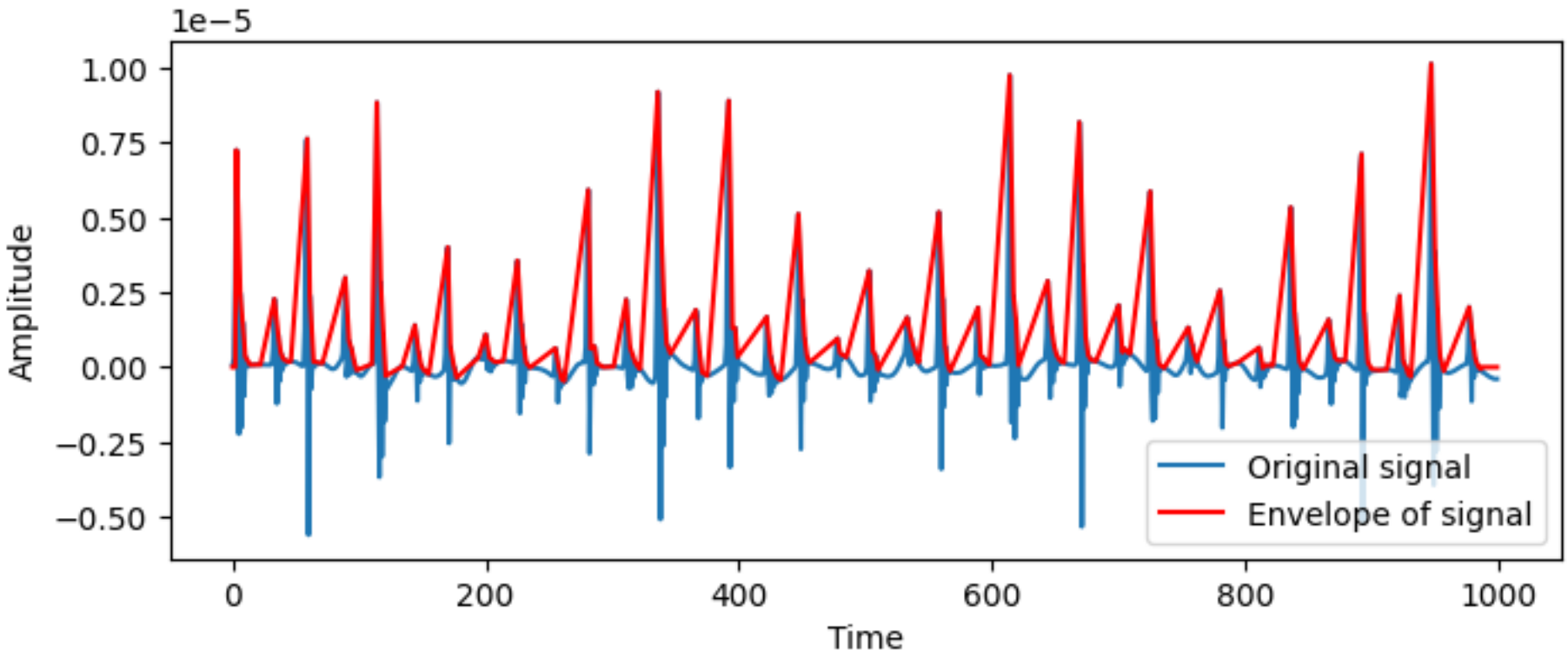


Figure: Get Envelope from Peaks

Average Envelope

We can also use the average window to get the envelope. The average window will calculate the average value of a region to replace the middle timestamp in the region. The longer the window, the smoother the envelope, but the more serious it will change the shape of the original signal.

$$envelope[i] = signal[i - \frac{window\_length}{2}] + ... + signal[i] + ... + signal[i + \frac{window\_length}{2}]$$

Usage Scenario: It is suitable for signals with relatively stable periodicity and slow amplitude variations. This is because the moving average window can eliminate high-frequency noise while preserving the low-frequency characteristics of the signal. However, this envelope will lose the information of peaks' vertical location.

Average Envelope Continued

average_envelope(signal, 3)

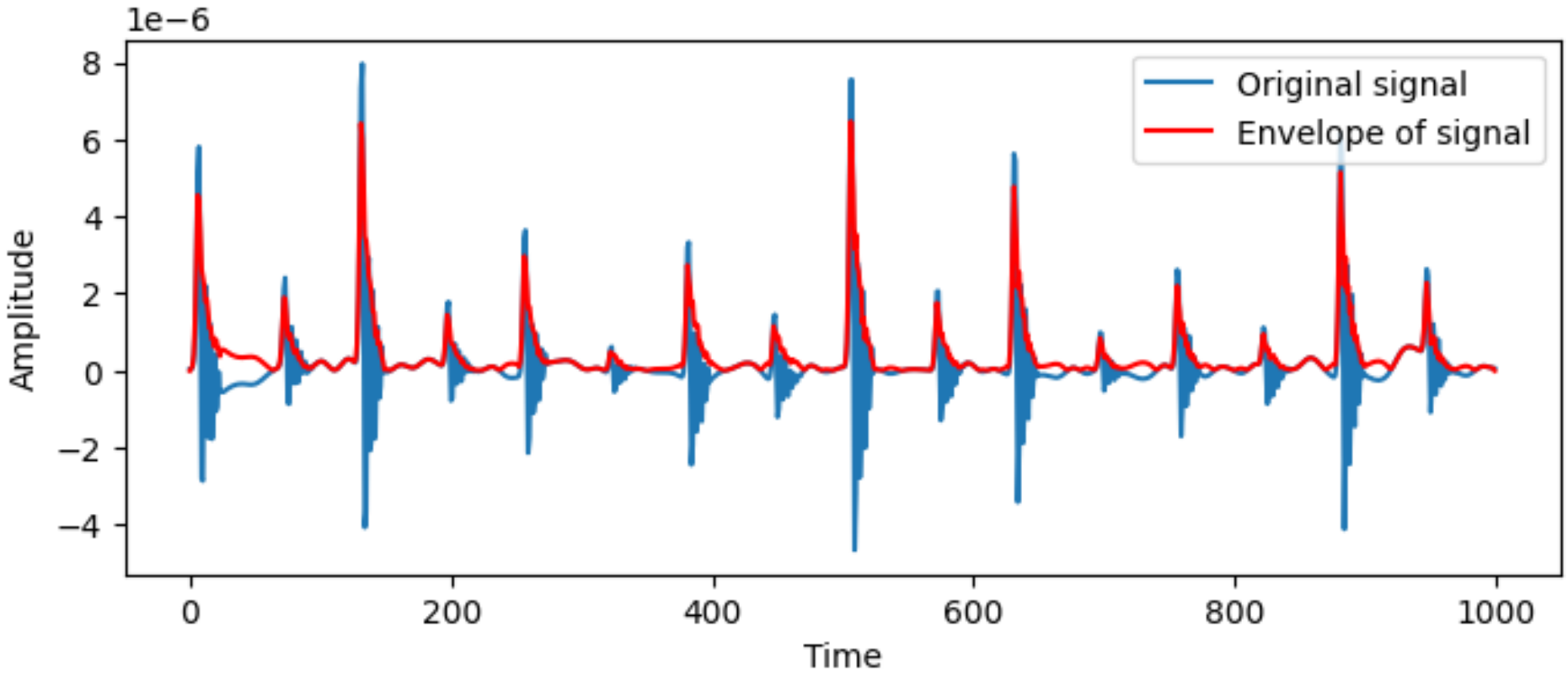


Figure: Get Average Envelope

Hilbert Transformation - Envelope & Phase Extraction

1. If a modulated signal is expressed as $x(t) = a(t)cos[\phi(t)]$
2. The instantaneous amplitude or the envelope of the signal is given by $a(t)$
3. The instantaneous phase is given by $\phi(t)$
4. The instantaneous angular frequency is derived as $\omega(t) = \frac{d}{dt}\phi(t)$
5. The instantaneous temporal frequency is derived as $f(t) = \frac{1}{2\pi}\frac{d}{dt}\phi(t)$

We note that the modulated signal is a real-valued signal. We also take note of the fact that amplitude/phase and frequency can be easily computed if the signal is expressed in complex form. So we can use Hilbert transformation to transform the real-valued signal to a complex version.

Hilbert Transformation Continued

The analytic signal is

$$z(t) = z_r(t) + z_i(t) = x(t) + jHT\{x(t)\}$$

So

$$a(t) = |z(t)| = \sqrt{z_r^2(t) + z_i^2(t)}$$

$$\phi(t) = \angle z(t) = arctan\left[\frac{z_i(t)}{z_r(t)}\right]$$

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\phi(t)$$

where $z(t)$ denotes the analytic signal, the subscripts i and r mean the imaginary and real, j is an imaginary unit, and $HT\{\}$ denotes Hilbert transform.

Hilbert Transformation Continued

inst_amplitude(signal)

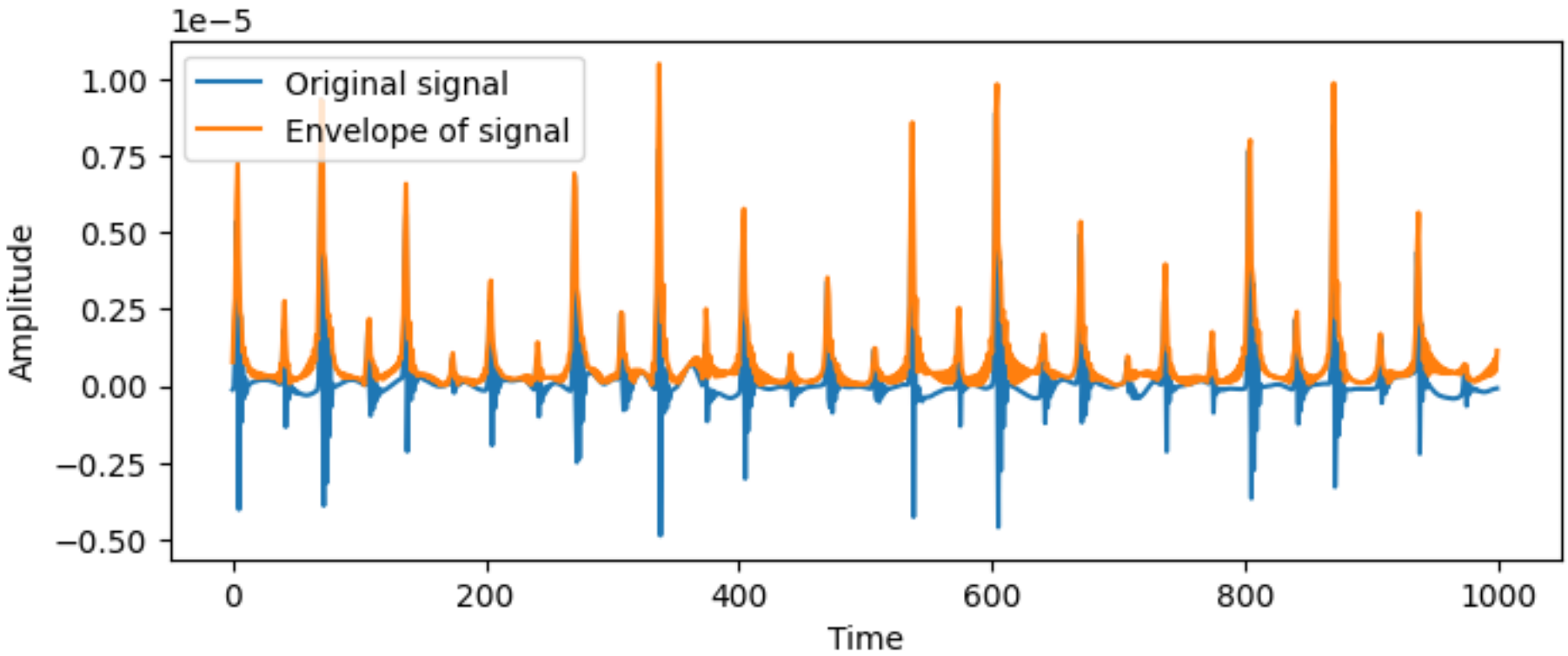


Figure: Envelope from Hilbert Transform

Singular-Spectrum Analysis

Singular-Spectrum Analysis (SSA) is a data analysis method used primarily for time series data. It is a non-parametric technique that transforms the data into a form that is easier to analyze and interpret, particularly for detecting patterns such as trends, cycles, and noise. SSA decomposes a time series into a set of components that can be analyzed individually to better understand underlying structures in the data.

Steps

1. **Windowing** Begins by creating a trajectory matrix from the time series data. It embeds the series in a higher-dimensional space using a sliding window (or lag) of a fixed length. It maps the time series into a matrix, where each row represents a shifted version of the original series.

SSA Continued

Steps Coninued

1. **Singular Value Decomposition (SVD)** The trajectory matrix is then subjected to Singular Value Decomposition, which decomposes it into singular vectors and singular values. SVD is a technique that helps identify the most important patterns in the data, effectively isolating dominant components from noise.

2. **Reconstruction** After SVD, the original time series is reconstructed using the most significant components (singular vectors). By reconstructing the time series with fewer components, SSA filters out noise and highlights key trends and periodic behaviors.

3. **Component Analysis** The resulting components can be analyzed to detect trends, oscillations, and anomalies, and can be used for forecasting, denoising, or feature extraction.

Singular Value Decomposition

The SVD of a matrix A (of size m x n):

$A = U\Sigma V^T$

Where:

1. U: An m x m orthogonal matrix whose columns are the left singular vectors of A.
2. Σ: A diagonal m x n containing the singular values of A in descending order.
3. V^T: The transpose of an n x n orthogonal matrix, where the columns are the right singular vectors of A.

Reference: Singular Value Decomposition (SVD)

SSA Continued

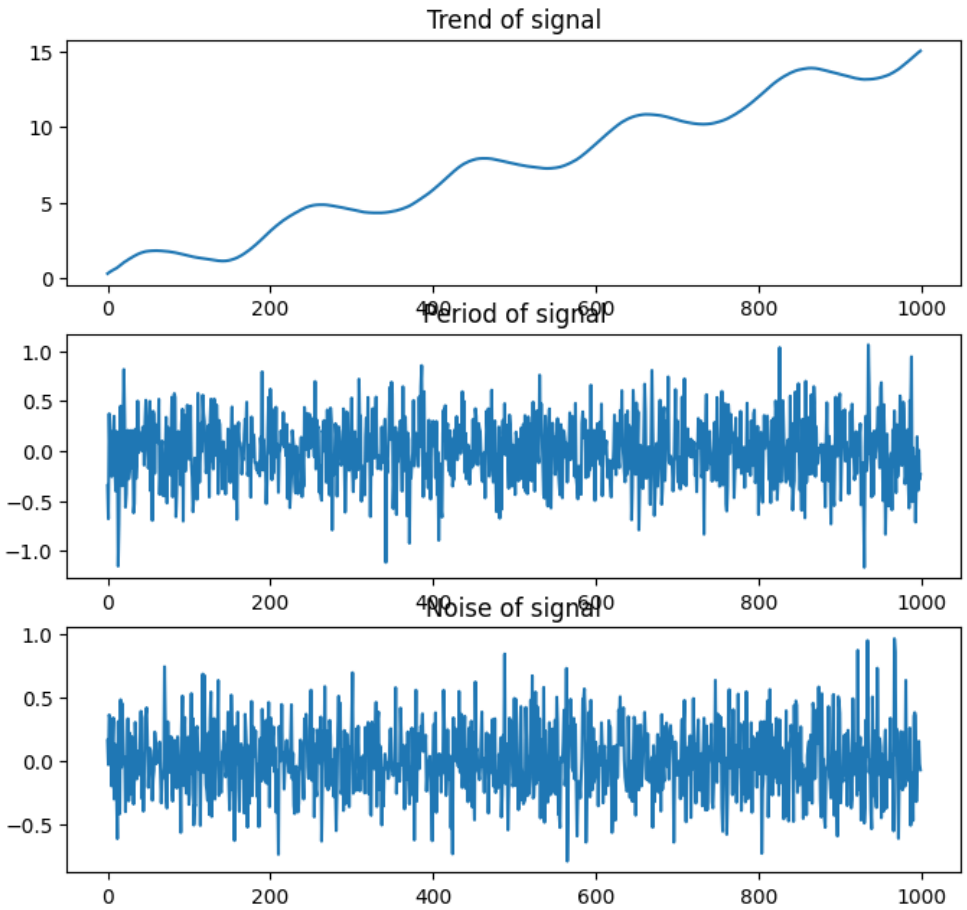


Figure: SSA: Trend, Period, Noise

Singlular Spectrum Transform (SST)

Singular Spectrum Transform (SST) is a signal processing technique based on singular spectrum analysis, used for extracting trends and periodic components in a signal. SST is primarily employed for decomposing signals and reconstructing their components to better understand the structure and features of the signal. This technique is usually used for the change point detection. In the example, we can find that where the signal changes a lot has a relative high score.

SST Steps

1. **Embedding:** Convert the original signal into matrix form. Usually via a Hankel matrix where rows and columns consist of subsequences of the original signal..
2. **Singular Value Decomposition (SVD):** Perform singular value decomposition on the embedded matrix, breaking it down into three matrices: U, Σ, and V^T. Here, U contains the left singular vectors, Σ contains the singular values in a diagonal matrix, and V^T contains the right singular vectors.
3. **Grouping and Reconstruction:** Group the singular values based on their magnitudes into several subsequences. These subsequences correspond to different frequency and trend components of the signal. By selecting relevant combinations of singular values, different components of the original signal can be reconstructed.
4. **Back-Transformation:** Perform the inverse transformation on the decomposed subsequences to obtain an estimate of the original signal. This step involves reversing the embedding operation on the reconstructed subsequences.

SST Results

sst(signal, win_length=50)

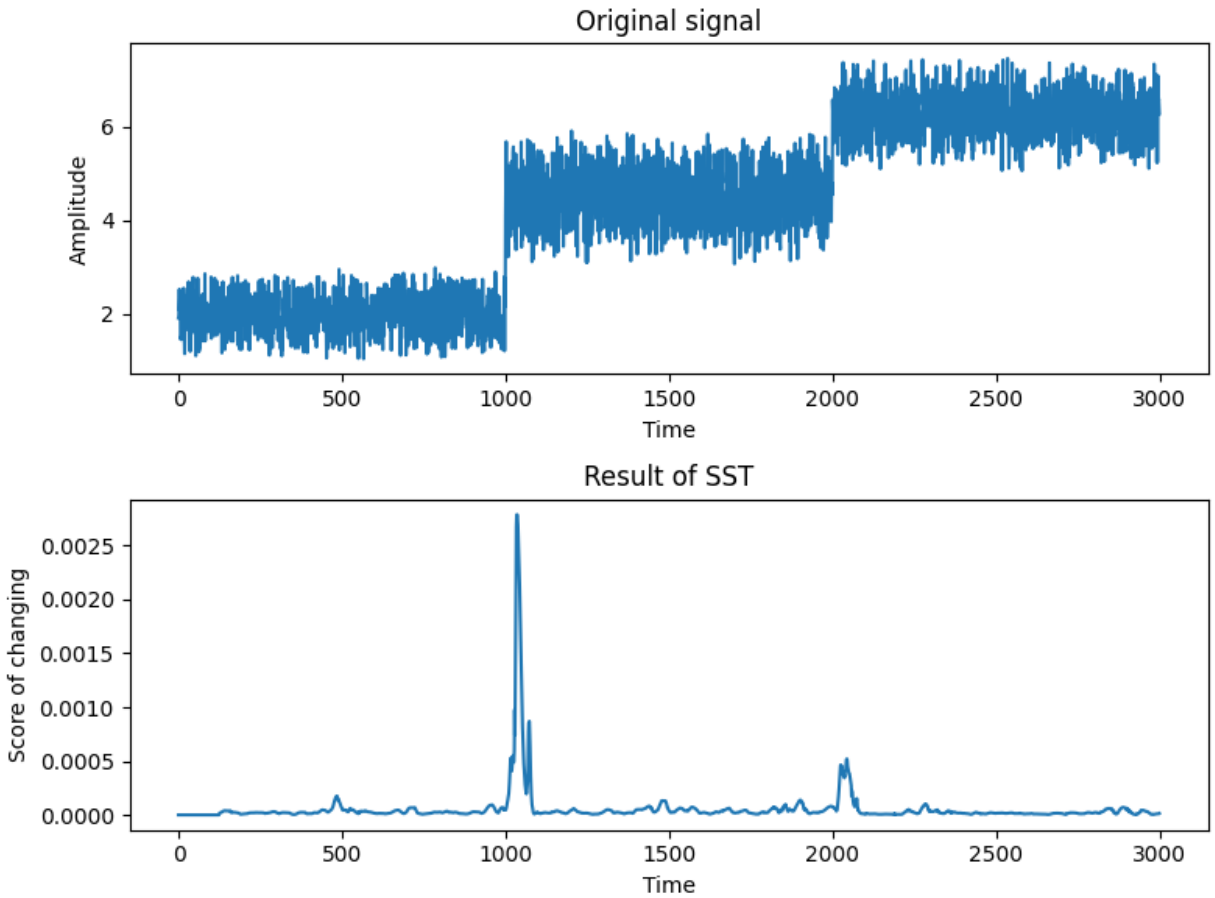


Figure: SST Results

Petrosian Fractal Dimension (PFD)

Petrosian Fractal Dimension (PFD) is a method used to measure the complexity of time-series signals, particularly applicable in biomedical signals such as electrocardiograms (ECG). It calculates the fractal dimension of a signal to describe its complexity and irregularity. A higher Petrosian Fractal Dimension value indicates a more complex signal.

$$PFD = \frac{\log_{10}(N)}{\log_{10}(N) + \log_{10}\left(\frac{N}{N+0.4N_{zc}}\right)}$$

where N is the length of the signal and N_{zc} is the number of zero crossings in the signal derivative.

The example shows that the signal with noise is more complex than the original signal, so that the noisy signal has a bigger PFD than the orignal signal.

PFD Continued

pfd(signal1)); pfd(signal2))

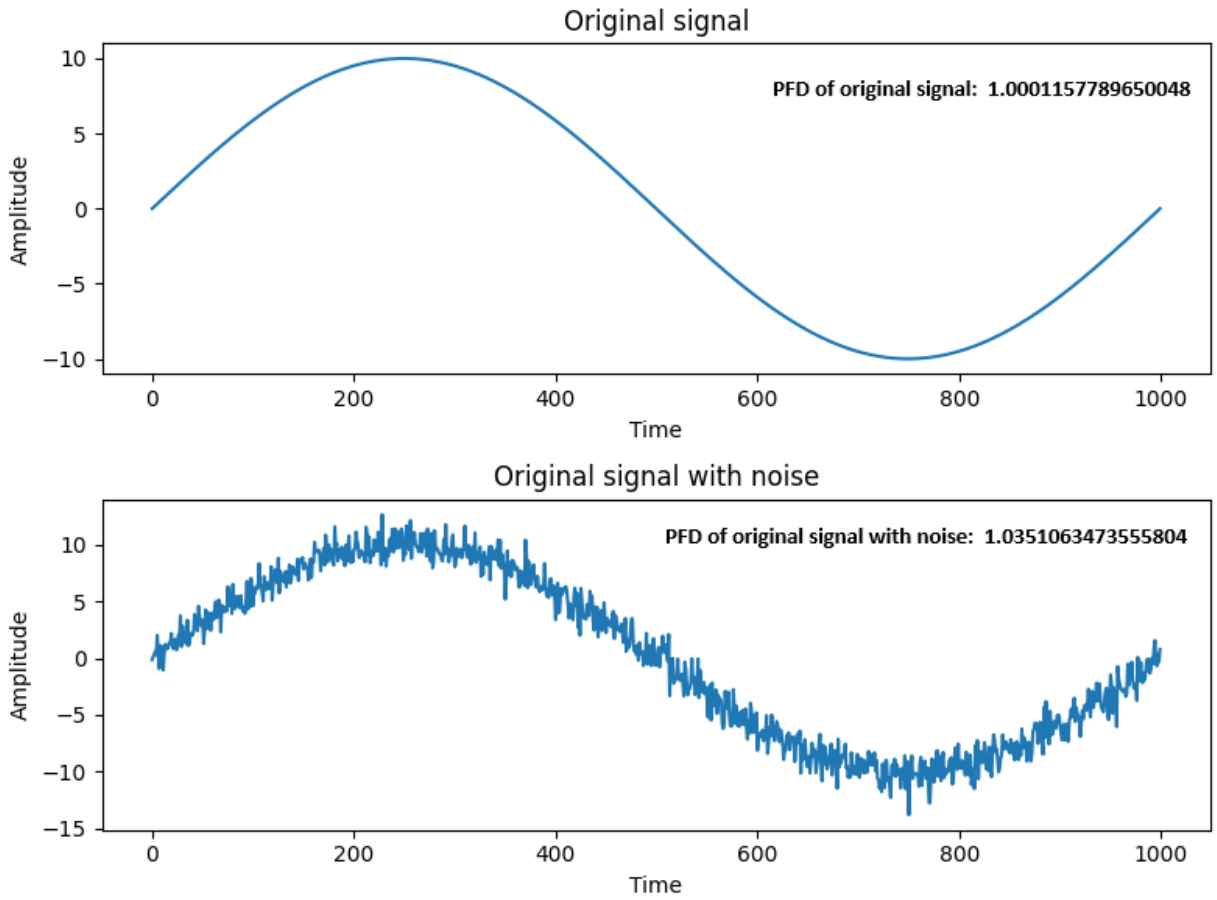


Figure: PFD Scores

Skewness

Skewness is a statistical measure that describes the degree of asymmetry of a distribution. It quantifies the extent to which a distribution deviates from symmetry around its mean.

$$Skewness = E[(X - \mu)^3]/\sigma^3$$

where E is the expected value operator,X is the random variable, μ is the mean of the distribution, σ is the standard deviation of the distribution. A skewness $value > 0$ means there is more weight in the right tail of the distribution. A skewness $value < 0$ means there is more weight in the left tail of the distribution.

Skewness Continued

skew(array1); skew(array2)

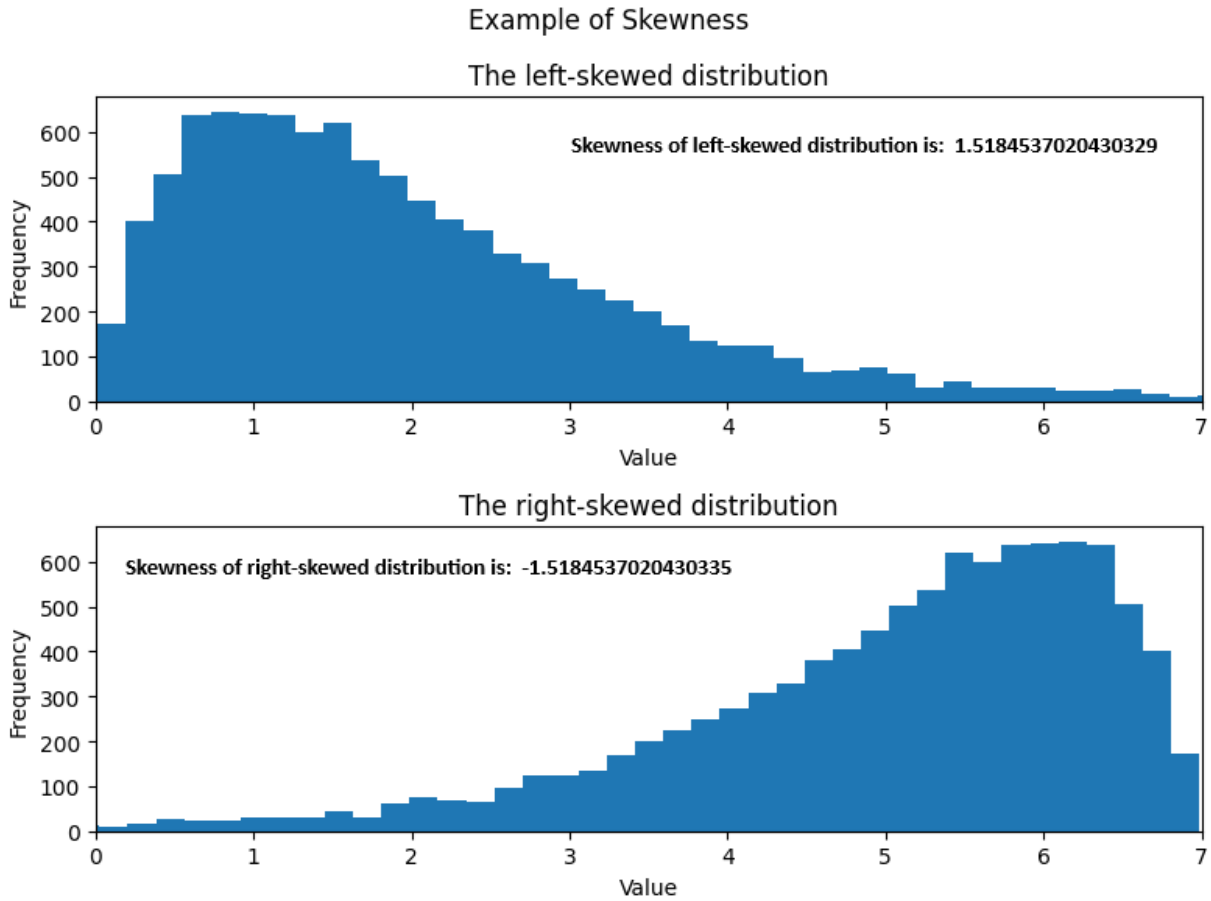


Figure: Skewness: Left Skewed & Right Skewed

Kurtosis

Kurtosis is a statistical measure that describes the "tailedness" of a distribution. It measures the thickness of the tails of a distribution relative to the rest of the data.

$$Kurtosis = E[(X - \mu)^4]/\sigma^4 - 3$$

where E is the expected value operator, X is the random variable, μ is the mean of the distribution, σ is the standard deviation of the distribution. The subtraction of 3 in Fisher's definition adjusts the kurtosis so that the normal distribution has a kurtosis of zero. Positive kurtosis indicates a "leptokurtic" distribution with heavier tails and a sharper peak. Negative kurtosis indicates a "platykurtic" distribution with lighter tails and a flatter peak.

Kurtosis Continued

kurtosis(array3); kurtosis(array4)

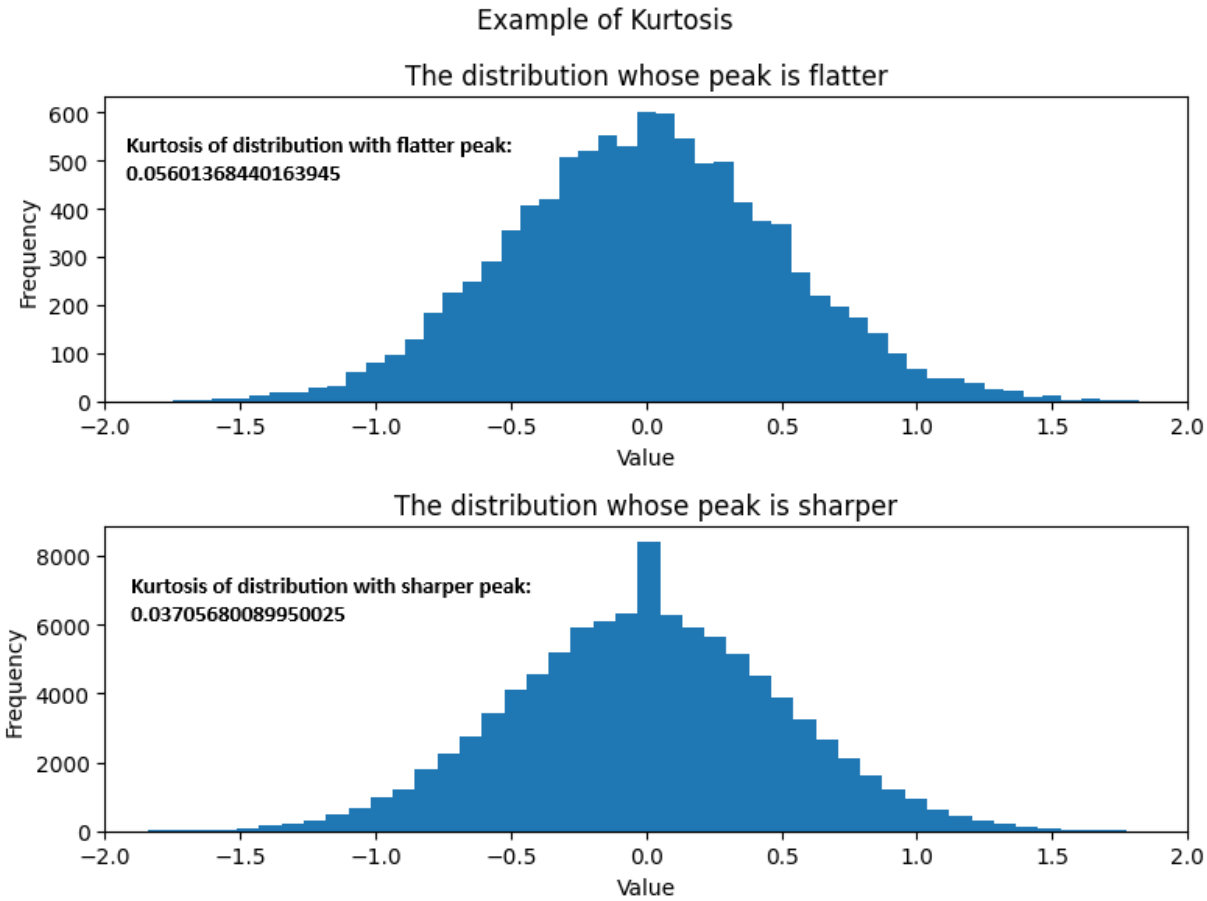


Figure: Kurtosis: Flat & Sharp

Fast Fourier Transform (FFT)

Fast Fourier Transform (FFT) is a fast algorithm to calculate the Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

where X denotes the result of FFT which is the spectrum, x denotes the time-domain signal and j is the imaginary unit.

my_fft function gets the original signal and the sampling rate as input and outputs the frequency and corresponding magnitude (a complex number). The length of the frequency is the half of the fs.

my_ifft function gets the magnitude of the **my_fft** function as input and outputs the original signal.

FFT Results

my_fft(signal, fs); my_ifft(mag)

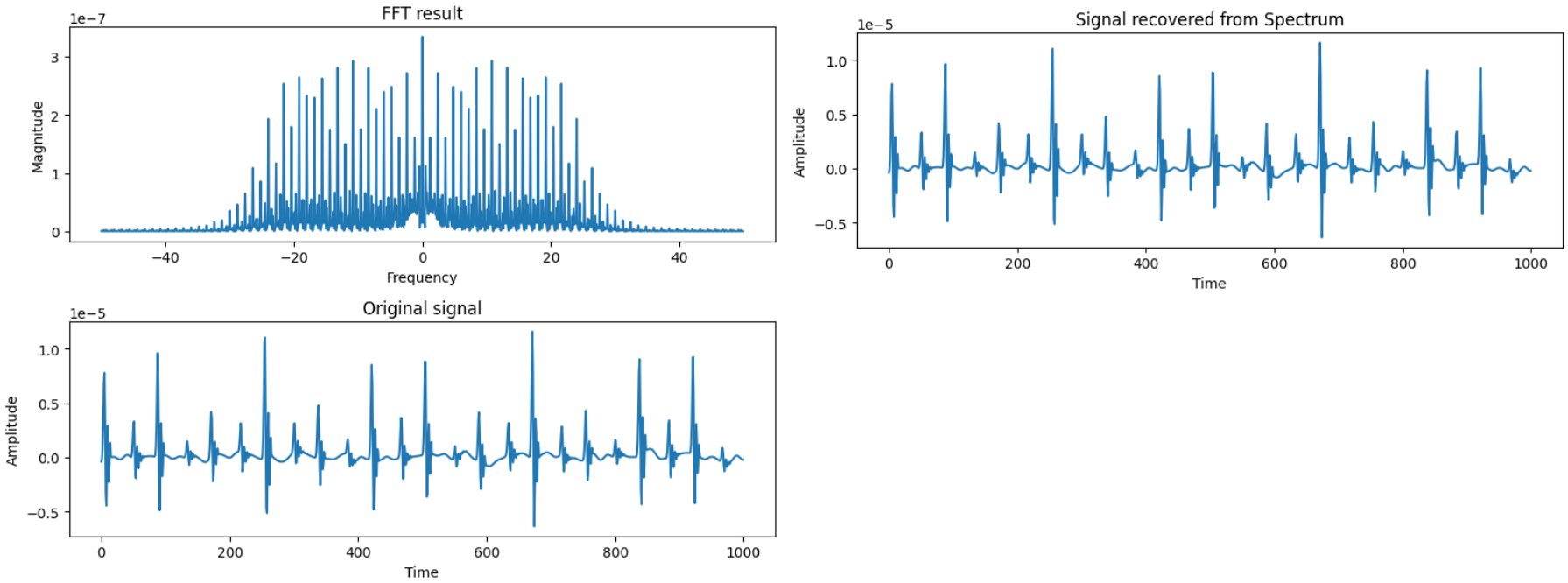


Figure: FFT Results

FFT Denoising

The Fast Fourier Transform (FFT) can be used to analyze and filter out unwanted frequency components from a signal.

By transforming the signal into the frequency domain, one can selectively remove or attenuate specific frequency bands associated with noise.

Mathematically, denoising is achieved by zeroing or attenuating certain frequency components in the Fourier-transformed signal, followed by an inverse FFT to obtain the denoised signal.

The denoising steps are the following

1. Apply the fft to the signal
2. Keep only the coefficients which have a low enough frequency (in absolute)
3. Compute the inverse fft

fft_denoise(noisy_signal, threshold=threshold)

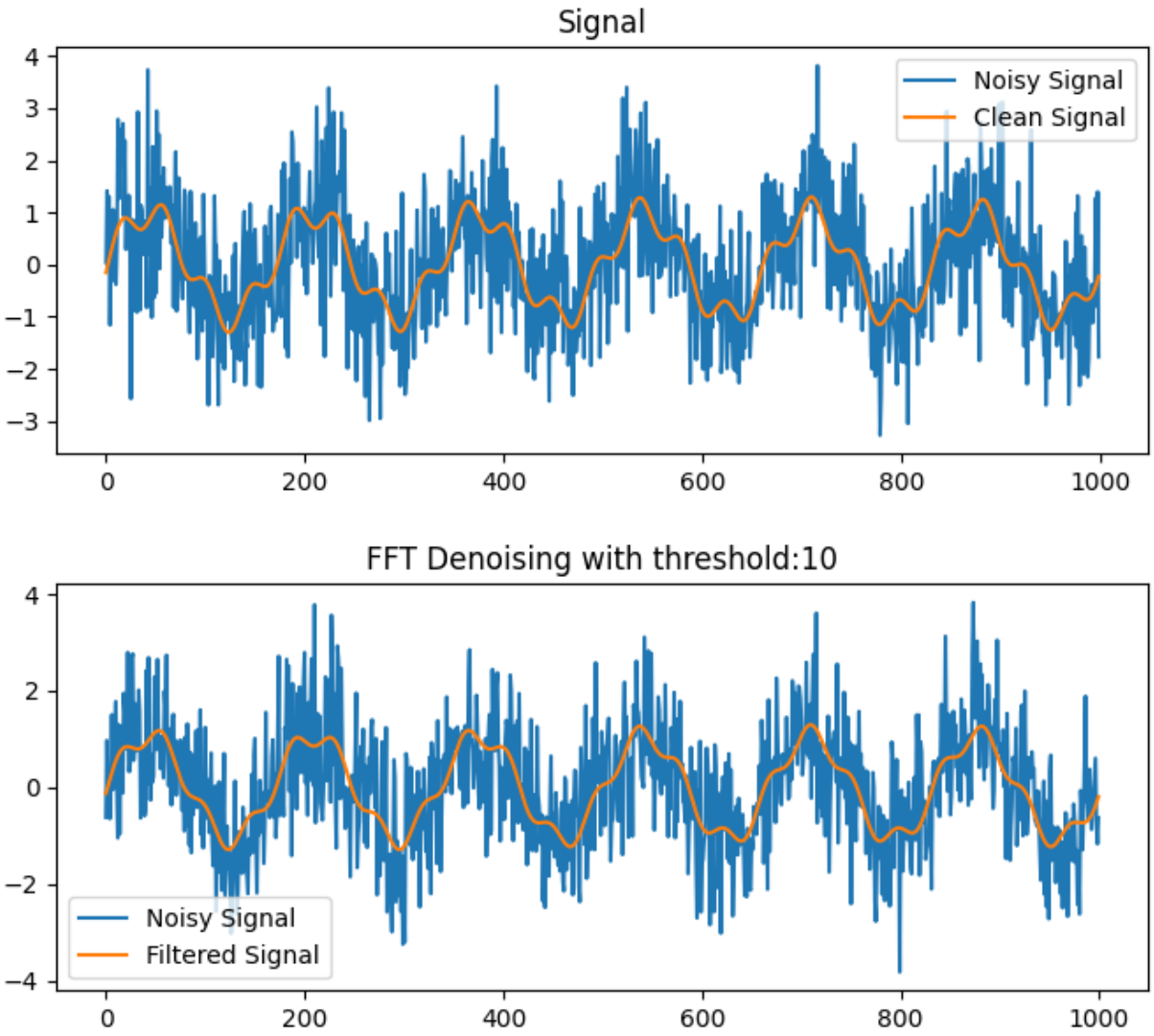


Figure: FFT Denoising

Power Spectral Density (PSD)

Power Spectral Density (PSD) is a tool used for frequency spectrum analysis to describe how the power of a signal varies with frequency.

The PSD represents the distribution of power across different frequencies in a signal, typically measured in power per Hertz (Hz). It can be computed by taking the squared magnitude of the Fourier transform of the signal. Specifically, for a time-domain signal x(t), the Power Spectral Density $S_{xx}(f)$ at frequency f is calculated as:

$$S_{xx}(f) = \lim_{T \to \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} x(t) e^{-j2\pi f t} \, dt \right|^2$$

where x denotes the time-domain signal and j denotes the imaginary unit.

PSD Results

psd(signal, fs)

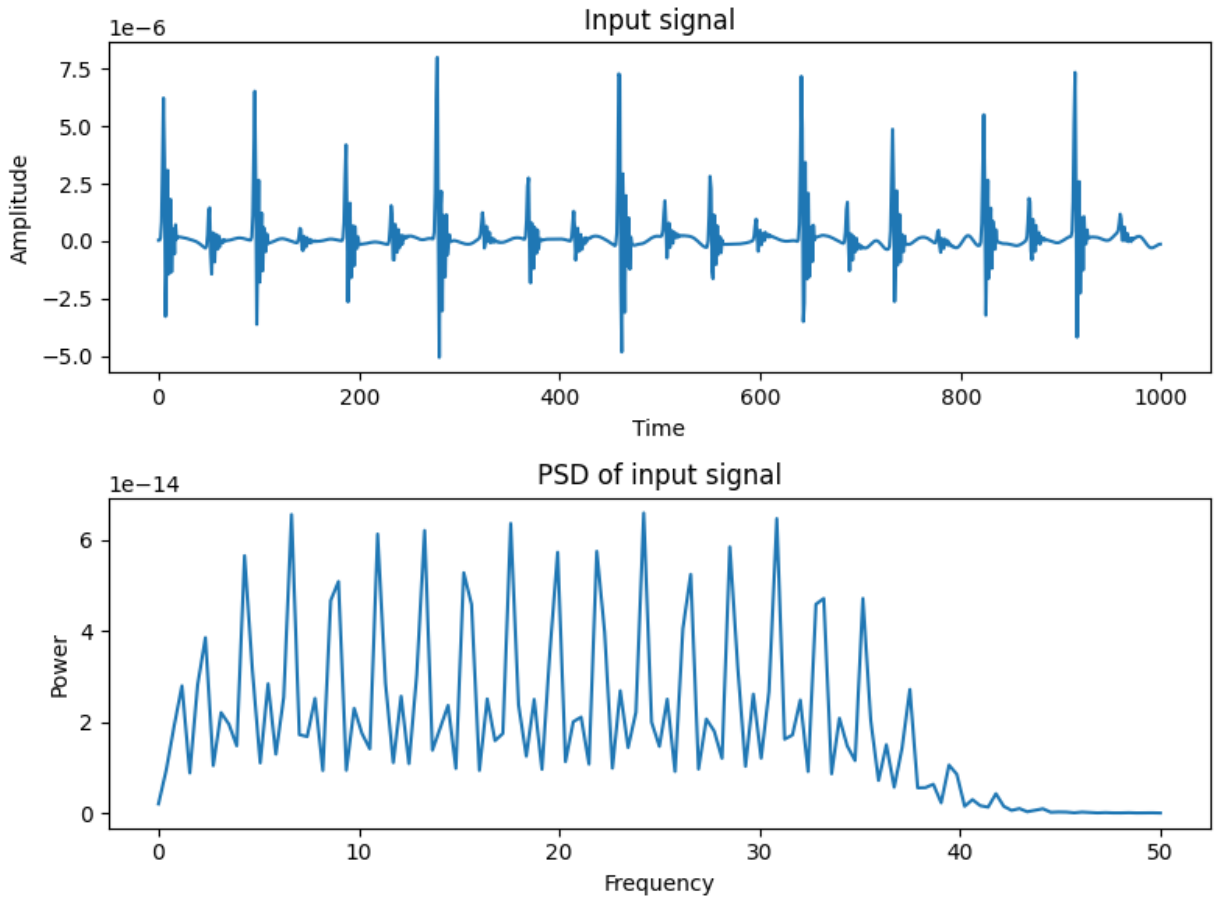


Figure: Input & PSD of Input

Short Time Fourier Transform (STFT)

Short Time Fourier Transform (STFT) - To extract signal information, we segment the signal and view the segment as stationary. However, when the signal does not complete an integer number of cycles, an issue known as Frequency Leakage may occur which can pose challenges in reconstructing the original signal based on the spectrum. To minimize the impact of Frequency Leakage, the original signal can be multiplied by a window function. After framing, we get many framed segment. Then we do the DFT to get frequency information of each frame. The scale of y axis is frequency. Then we concatenate the spectrum horizontally to get the spectrogram. The spectrogram has both the frequency and time frequency of a signal.

STFT Continued

The longer the frame, the frequency information is more accurate, the time information is less accurate. The above is what we call Heisenberg uncertainty principle.

The output of the STFT is y-axis (frequency), x-axis (time) and the value of each point (magnitude). The max number of y-axis (frequency) is the half of the fs which is based on the Nyquist sampling princple, but the number of frequency in y-axis is the half of the nperseg. The number of the x-axis depends on the nperseg and the noverlap and the max number of the x-axis is (length of signal / fs). The output Z is complex-valued.

STFT Results

f, t, Z = my_stft(signal, fs=fs, window=window, nperseg=nperseg)

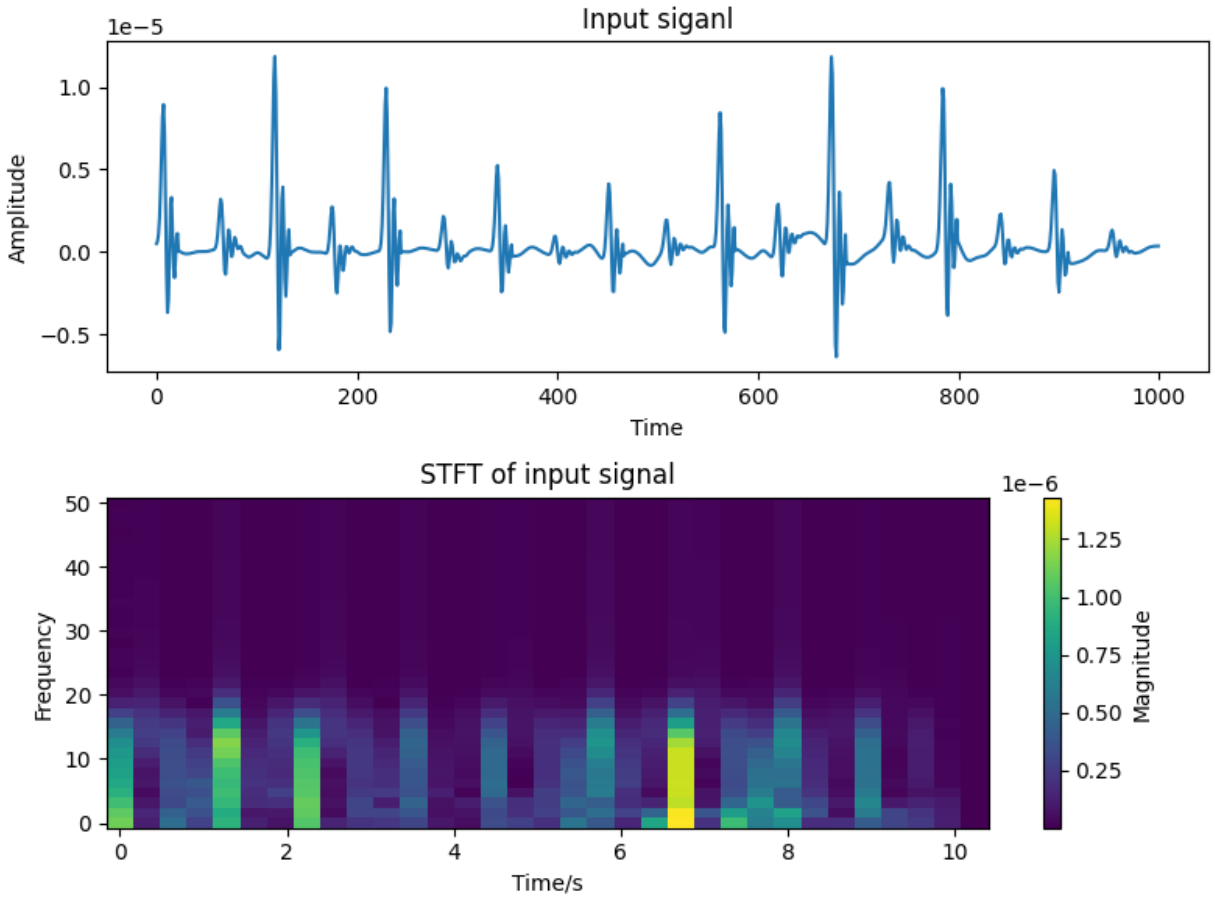


Figure: STFT

Wavelet Analysis

STFT has a serious problem, that is we can't get a good time and frequency resolution. The bigger the nperseg, the better the frequency resolution, but the worse time resolution, vice versa. The above problem is caused by the fact that the nperseg is fixed. So we introduce the wavelet analysis which is a MRA method. MRA (multiresolution analysis) is designed to give good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies. Generally, the basic function of wavelet transform is orthogonal and normalized (normalized makes the transformed signal have the same energy at every scale).

Mexican Hat Wavelet

Mexican Hat Wavelet is the second derivative of the Gaussian function.

Gaussian function:

$$w(t) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{\frac{-t^2}{2\sigma^2}}$$

where σ is the standard deviation and t is time.

Second derivative of the Gaussian function:

$$\psi(t) = \frac{1}{\sqrt{2\pi} \cdot \sigma^3} \left(e^{\frac{-t^2}{2\sigma^2}} \cdot \left(\frac{t^2}{\sigma^2} - 1 \right) \right)$$

Mexican Hat Continued

mexican_hat_wavelet(sigma, length)

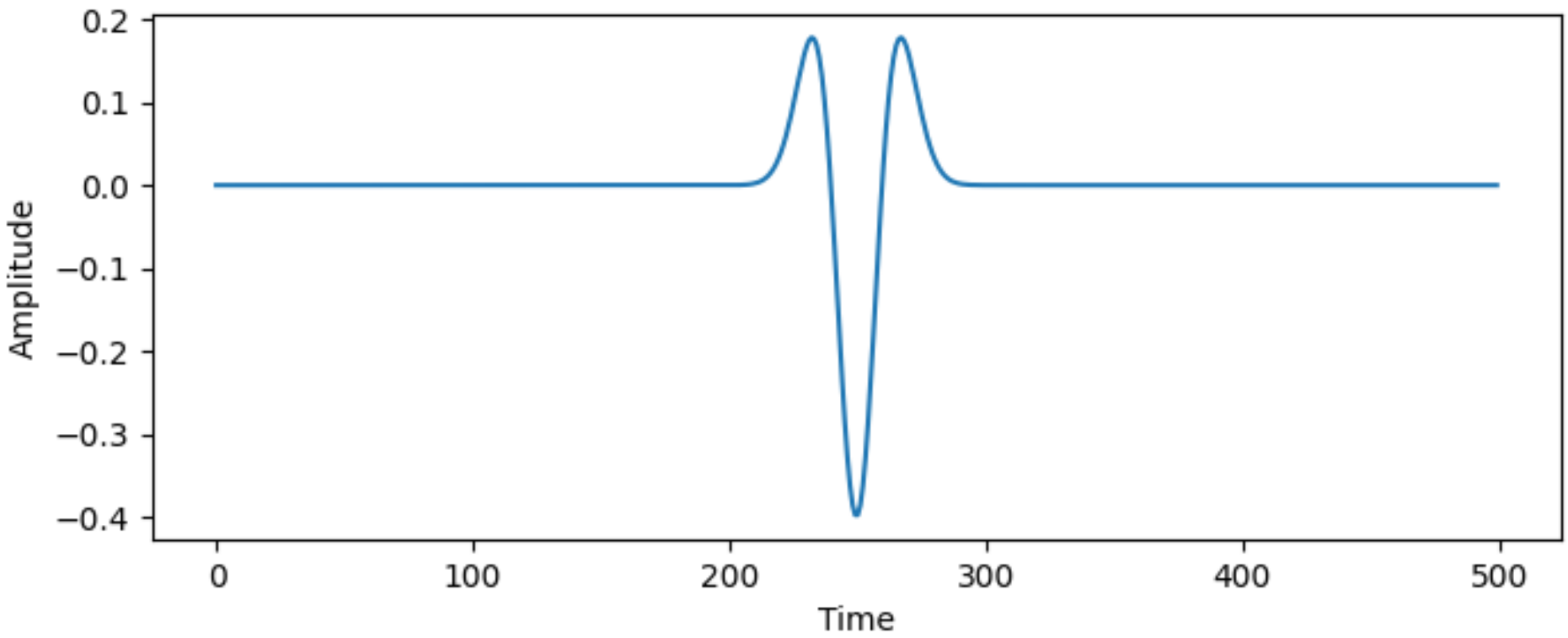


Figure: Mexican Hat Wavelet

Morlet Wavelet

Morlet Wavelet is defined as follows:

$$w(t) = e^{iat} \cdot e^{-\frac{t^2}{2\sigma}}$$

where t is the time variable, a controls the frequency of the sinusoidal oscillation, and σ controls the width of the Gaussian envelope.

Mortlet Wavelet Continued

morlet_wavelet(length, sigma, a)

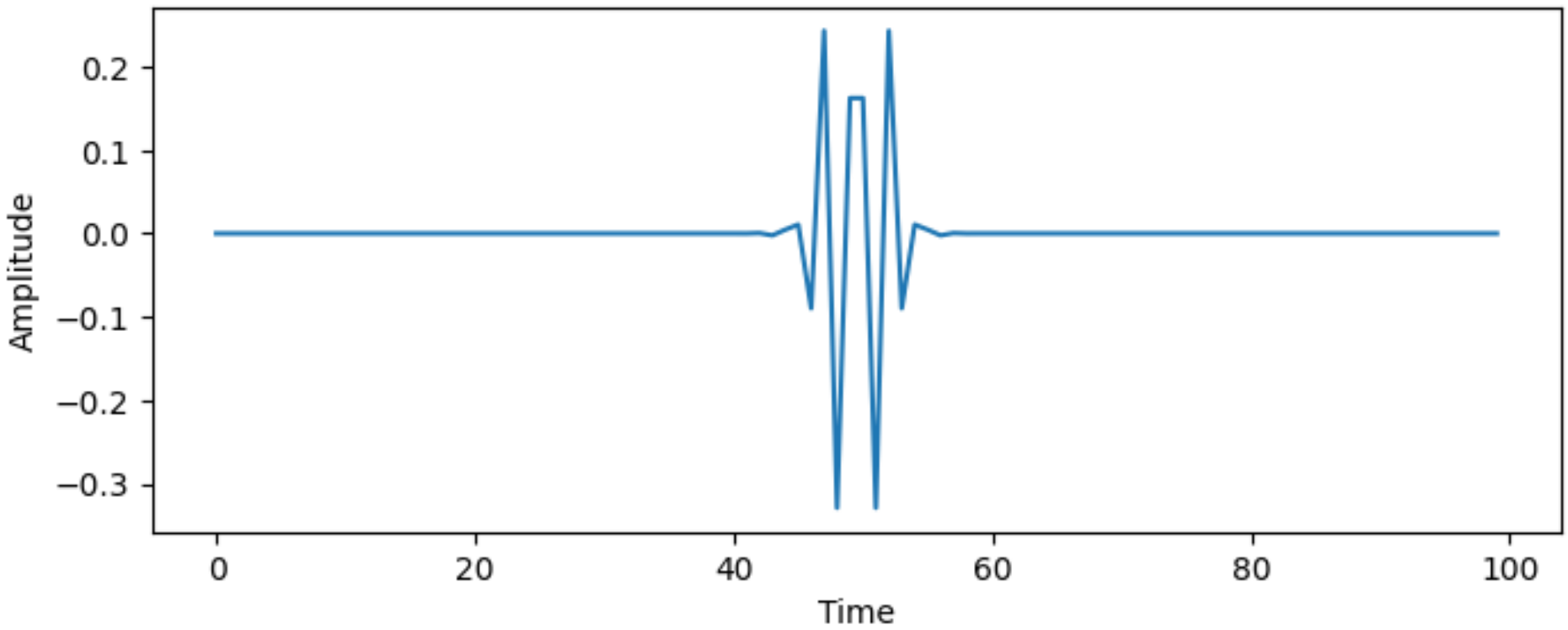


Figure: Morlet Wavelet

Continuous Wavelet Transform (CWT)

Continuous Wavelet Transform (CWT): the signal can be transformed by:

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t)\psi^* \left(\frac{t-\tau}{s} \right) dt$$

where the ψ denotes the mother wavelet, τ denotes the translation of the wavelet, and s denotes the sacle of the wavelet. τ and s are all incremented continuously. However, if this transform needs to be computed by a computer, then both parameters are increased by a sufficiently small step size. This corresponds to sampling the time-scale plane.

Inverse CWT

Inverse CWT:

$$x(t) = \frac{1}{c_\psi^2} \int_s \int_\tau \Psi_x^\psi(\tau, s) \frac{1}{s^2} \psi \left(\frac{t-\tau}{s} \right) d\tau ds$$

where c_ψ is a constant that depends on the wavelet used and the other symbols are described in above cell. c_ψ can be calculated as follows:

$$c_\psi = \left\{ 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi \right\}^{1/2}$$

where $\hat{\psi(\xi)}$ is the FT of $\psi(t)$

CWT Results

coefficients, frequencies = my_cwt(signal, scales, wavelet, fs)

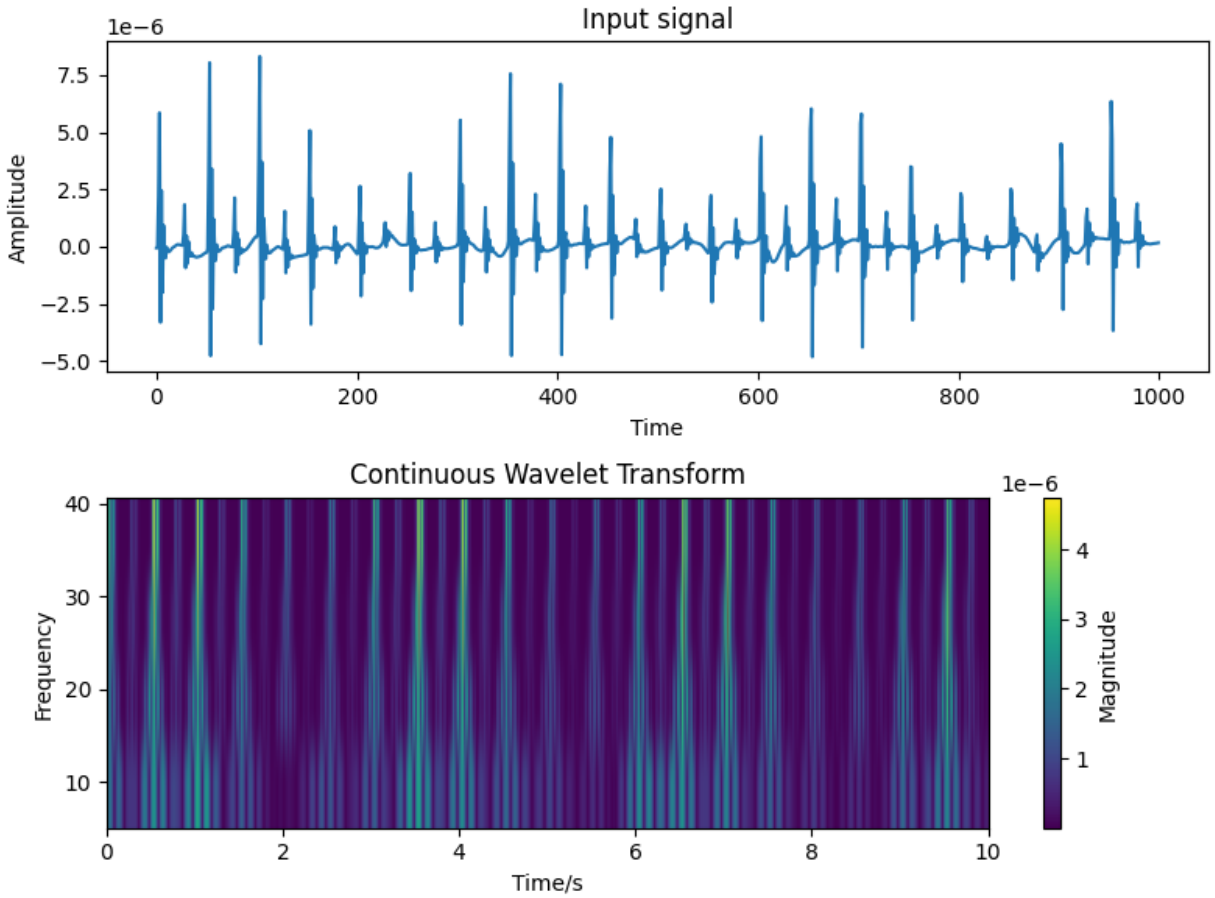


Figure: CWT with a Morlet Wavelet

Wavelet Denoising Transform (DWT)

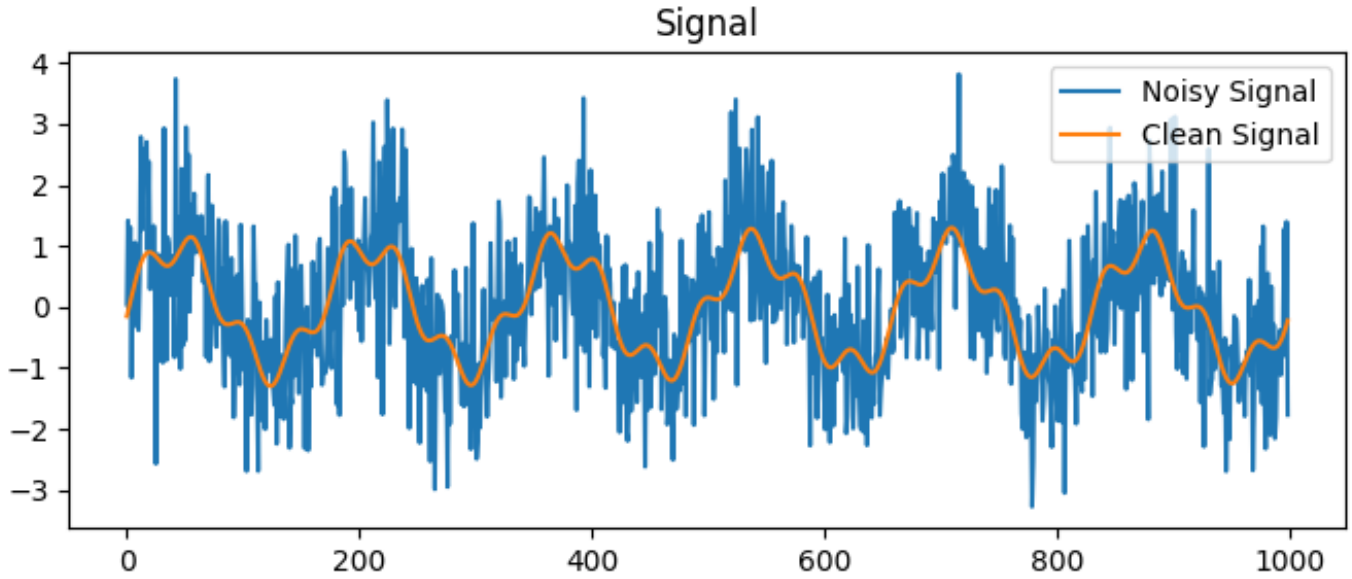
Wavelet denoising is a technique that utilizes wavelet transforms to remove noise from signals or images. It decomposes the signal into different frequency components, thresholds the coefficients, and reconstructs the signal, effectively reducing noise.

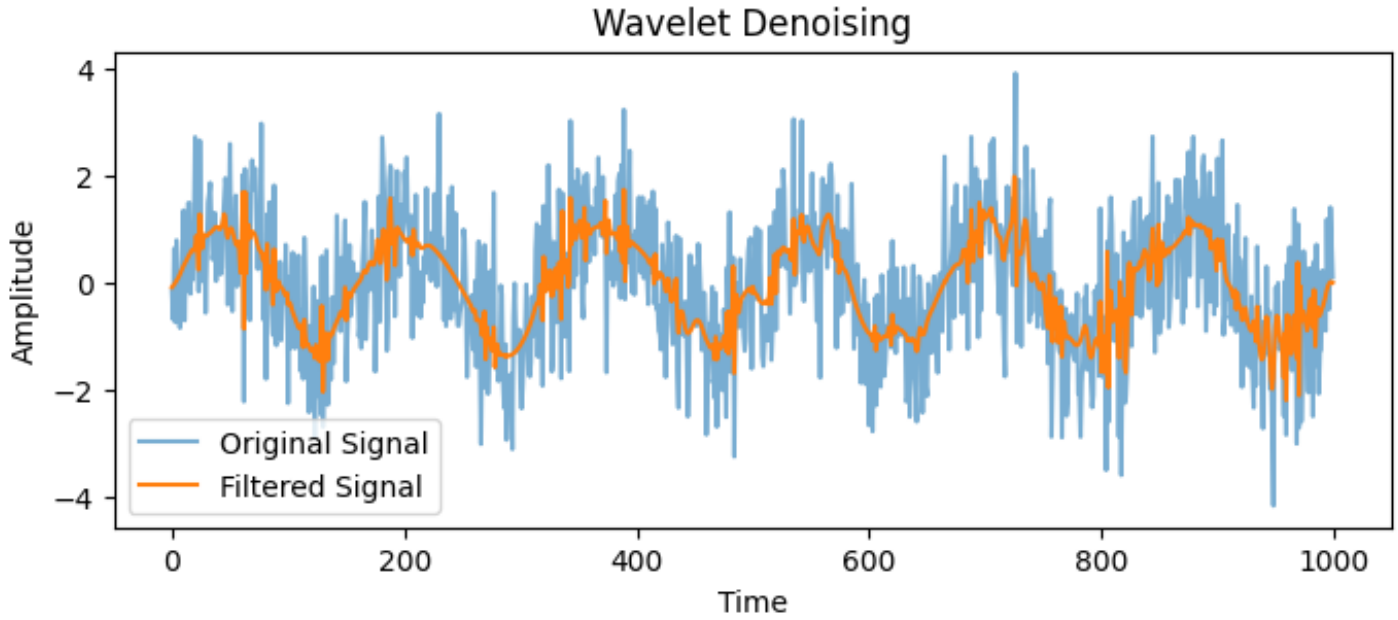
Mathematically, the denoised signal $y(t)$ is obtained by thresholding wavelet coefficients W_j at a certain level: $y(t) = \Sigma_j Threshold(W_j)$.

The denoising steps are the following:

1. Apply the DWT to the signal
2. Compute the threshold corresponding to the chosen level
3. Only keep coefficients with a value higher than the threshold
4. Apply the inverse DWT to retrieve the signal

wavelet_denoise(noisy_signal, wav, 0.5, show=True)


Signal
Noisy Signal
Clean Signal
4
3
2
1
0
−1
−2
−3
0
200
400
600
800
1000


Wavelet Denoising
Amplitude
Time
Original Signal
Filtered Signal
4
2
0
−2
−4
0
200
400
600
800
1000

Polynomial Chirplet Transform (PCT)

Polynomial Chirplet Transform (PCT) is a signal processing technique and a variant of the Chirplet Transform. The Chirplet Transform is a method used to analyze non-stationary modulations in signals, and PCT further introduces polynomial functions to enhance its analytical capabilities.

A "Chirp" refers to a signal with a frequency that varies over time, and a "Chirplet" is a wavelet associated with a Chirp. PCT introduces polynomial modulation functions, allowing the frequency, phase, and amplitude of Chirplets to vary according to the shape of the polynomial.

The core idea of PCT is to use polynomials to describe the nonlinear modulation characteristics of a signal, thereby better capturing the complex structures and non-stationary nature of the signal. This method is effective for analyzing signals where frequency, phase, and amplitude change over time.

Chirplet

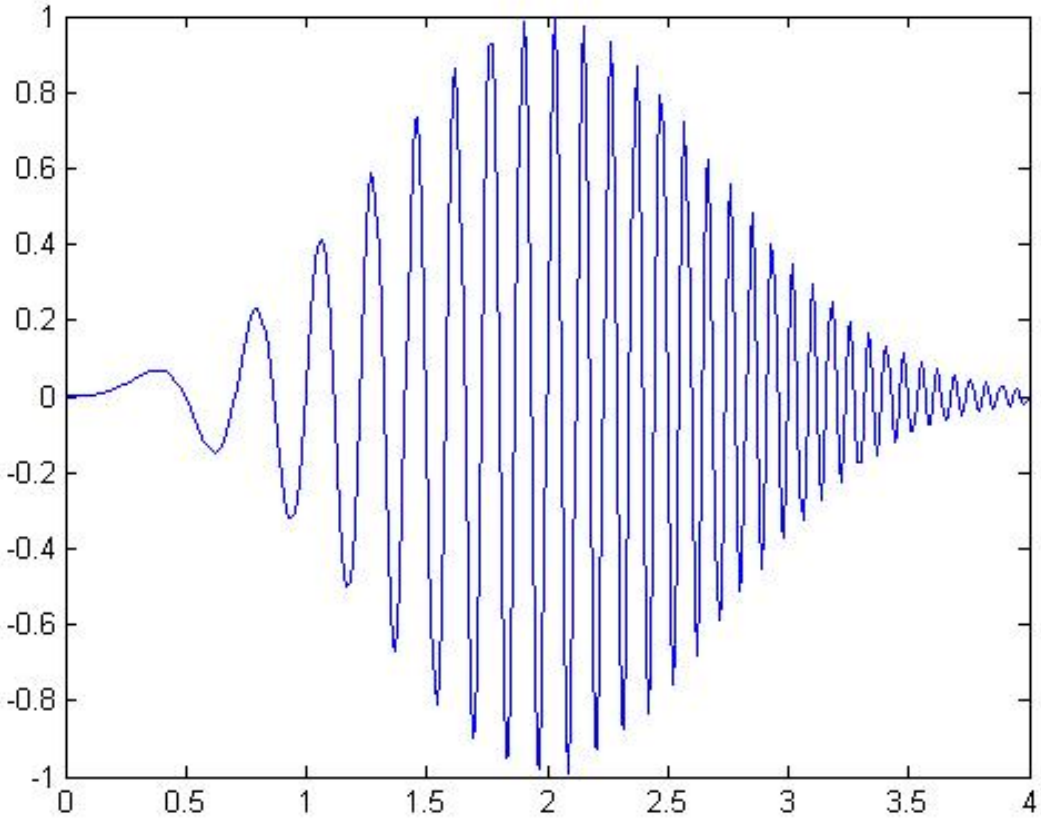


Figure: Sample Chirplet

Image from: Wikimedia Commons: Chirplet

Chirplet Transform

The Chirplet Transform of a signal $x(t)$ is often represented as:

$$C(a, b, \omega, \tau) = \int_{-\infty}^{\infty} x(t) \psi^*_{a,b,\omega,\tau}(t)\, dt$$

where * denotes the complex conjugate and $\psi_{a,b,\omega,\tau}(t)$ is the Chirplet defined as:

$$\psi_{a,b,\omega,\tau}(t) = e^{j(\omega t + \frac{a}{2} t^2 + bt + \tau)}$$

Polynomial Chirplet Transform

The Polynomial Chirplet Transform extends the Chirplet Transform by introducing a polynomial modulation. The general form is:

$$C(a, b, \omega, \tau, P(t)) = \int_{-\infty}^{\infty} x(t) \psi^*_{a,b,\omega,\tau,P(t)}(t)\, dt$$

Here, $P(t)$ is a polynomial function that modulates the Chirplet parameters.

$$\psi_{a,b,\omega,\tau,P(t)}(t) = e^{j(\omega t + \frac{a}{2}t^2 + bt + \tau + P(t))}$$

Chirplet Transform Results

chirplet_transform(signal)

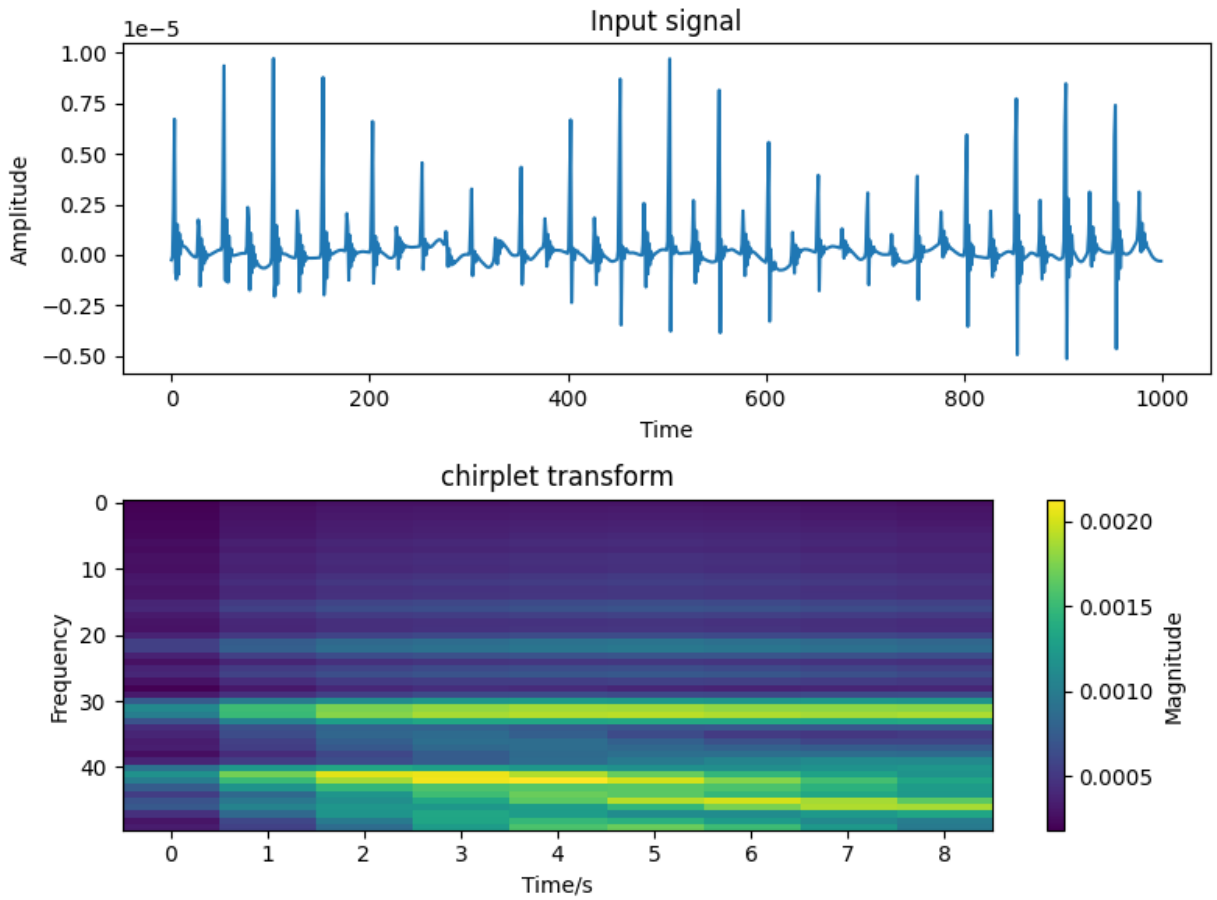


Figure: Chirplet Transform

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Thank you!

Stephen Coshatt, stephen.coshatt@uga.edu
Prof. WenZhan Song, wsong@uga.edu