FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

# Master's Thesis

William Sprent – bsprent@gmail.com

# Prize-Collecting Steiner Trees

## From the Perspective of the Prize-Collecting Travelling Salesman Problem

Supervisors: Pawel Winter

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**Abstract**

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# Chapter 3

# Steiner Trees in Graphs

## 3.1 The Steiner Tree Problem

The Steiner Tree Problem (STP) is defined as follows. Given an undirected graph $G = (V, E, c)$ with vertices $V$, terminal set $N \subseteq V$, edges $E$, and edge weights $c : E \to \mathbb{R}$, a Steiner Tree is a tree, $T \in E$, which spans $N$ with minimal cost. We denote vertices which must be part of a feasible solution, that is $v \in N$, as *terminals*, and vertice which are not required to be part of a solution but are so anyway, that is $v \in T \cap (V \setminus N)$, as *Steiner points* or *Steiner vertices*.

Furthermore, let $S \subset V$ define a cut in $G$. If we have $S \cap N \neq \emptyset$ and $(V \setminus S) \cap \emptyset)$ then we call $S$ a *Steiner cut*.

When $N = V$, the STP is equivalent to the Minimum Spanning Tree Problen, and when $|N| = 2$ the STP is equivalent to finding the shortest path between two vertices. However, while these problems both have polynomial-time solutions, the STP is an NP-Hard problem, the decision variant being part of Edmund Karp's original 21 NP-Complete problems (Karp, 1972).

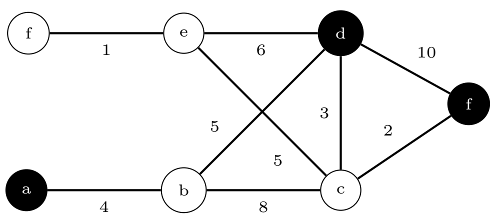



**Figure 3.1:** Instance of the Steiner Tree Problem. Terminals are coloured black and Steiner Points coloured white.

Figure (3.1) shows an instance of the STP with three terminals and four Steiner points. Since vertices A, C, and D are terminals, they must be spanned by any feasible solution. Figure (3.2a) shows a feasible solution

$$T = \{(a, b), (b, d), (d, g)\}$$

with cost

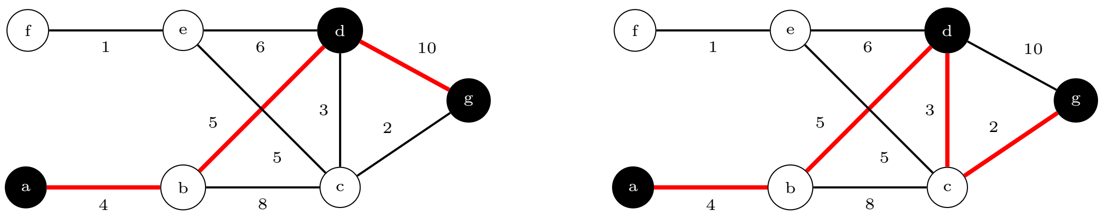
$$c(T) = 4 + 5 + 10 = 19.$$

However, $T$ is not a Steiner tree as there exists at least one feasible solution with lower cost, i.e. the solution

$$T' = \{(a, b), (b, d), (d, c), (c, g)\}$$

in Figure (3.2b) which has cost

$$c(T') = 4 + 5 + 3 + 2 = 14.$$




**(a)** Feasible but not optimal. **(b)** Minimal weight, optimal.

**Figure 3.2:** Solutions to the STP in (3.1). Red edges are part of the solution.

### 3.1.1 ILP Formulations

**Cut Formulation** Let $x$ be a decision vector of length $|E|$ where $x_{ij} = 1$ implies that $(i, j) \in T$ and $x_{ij} = 0$ implies that $(i, j) \notin T$, and let $c$ be a vector of node-weight s.t. $c_{ij} = c((i, j))$. Then define the function $x : E \to \mathbb{Z}$ as

$$x(E') = \sum_{i,j \in E'} x_{ij}$$

that is, $x(E)$ is equal to the number of selected edges in $E$. Finally let,

$$\delta(S) = \{(i, j) \mid i \in S \wedge j \in (E \setminus S)\}$$

be all edges which span the cut defined by $S$. Then we can formulate the STP as in ILP in terms of cuts (see Formulation (3.1)).

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & c^T x && && \text{(3.1a)}\\
\text{subject to} \quad & x(\delta(S)) \geq 1 && \forall S \subset V && \text{(3.1b)}\\
& && S \cap N \neq \emptyset && \\
& && S \cap (V \setminus N) \neq \emptyset && \\
& x \in \mathbb{B}. && && \text{(3.1c)}
\end{aligned}
$$

**Formulation 3.1:** The *Cut Formulation* of the STP (Koch and Martin, 1998).

Constraint (3.1b) ensures that any feasible solution, $T$, must span all terminals in $G$, by requiring that every Steiner cut in $G$ must be crossed by an edge in $T$. This, combined with objective function ensures that any optimal solution to (3.1) must be a Steiner tree in $G$ and vice versa.

## 3.2 Prize-Collecting Steiner Tree Problem

## 3.3 Steiner Aborescence Problem

The Steiner Aborescence Problem (SAP) is the directed version of the Steiner Tree Problem. Given a *directed* graph, $G = (V, A, c)$, with vertices $V$, arcs $A$, a non-empty terminal set $N \subseteq V$, a root terminal $r \in N$, and arc-weights $c : A \to \mathbb{R}$, then a Steiner Aborescence, $T \subseteq A$, is an aborescence in $G$, rooted in $r$, which spans $N$ and has minimal cost.

As with the STP, we denote vertices in $N$ as *terminals*, and any non-terminals spanned by a Steiner Aborescence as *Steiner vertices* or *Steiner Points*.

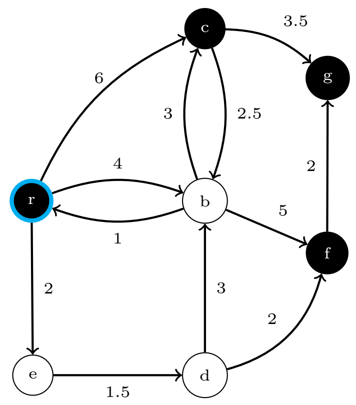



**Figure 3.3:** Instance of the Steiner Aborescence Problem. Terminals are coloured black, Steiner Points are coloured white, and the root terminal has a blue outline.

Figure (3.3) shows an instance of the SAP with four terminals. Figure (3.4) shows a Steiner Aborescence in (3.3) with cost $c(T) = 13.5$. It is worth noting that since all arcs must point away from the root in an aborescence that while the path $c, b, r$ has lower cost than the arc $(r, c)$, it cannot be part of a solution. Similarly, solutions to the SAP in the same graph but with root $c$ have lower cost.

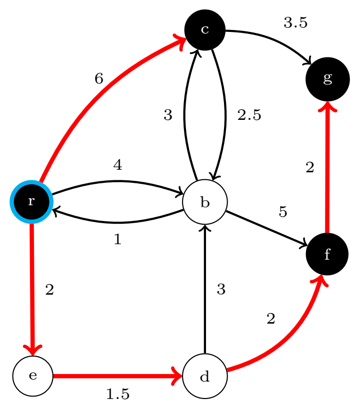
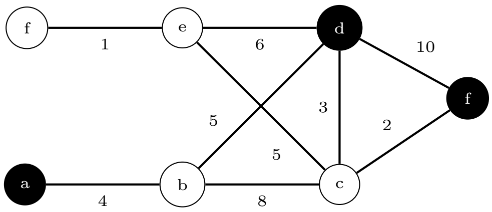


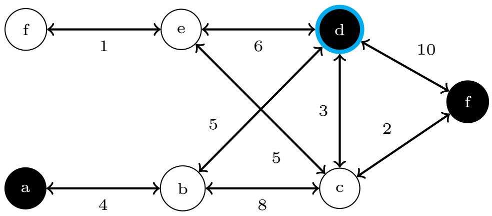

**Figure 3.4:** Steiner Aborescence in (3.3) with cost 13.5.

### 3.3.1 Reduction from Other Variants

#### Steiner Tree Problem




**(a)** Problem Instance (3.1)




**(b)** Instance (3.1) as a SAP instance. Root node is coloured blue.

**Figure 3.5:** Reduction from STP to SAP.

#### Prize-Collecting Steiner Tree Problem

### 3.3.2 ILP Formulations

## 3.4 Methods

### 3.4.1 Preprocessing

### 3.4.2 Primal Heuristics

### 3.4.3 Exact Algorithms

# Bibliography

Richard M Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.

Thorsten Koch and Alexander Martin. Solving steiner tree problems in graphs to optimality. *Networks*, 32(3):207–232, 1998.