On Social Network De-anonymization with Communities: A Maximum A Posteriori Perspective

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***Abstract*—A crucial privacy-driven issue nowadays is re-identifying anonymized social networks by mapping them to correlated cross-domain auxiliary networks. Prior works are typically based on modeling social networks as random graphs representing users and their relations, and subsequently quantify the quality of mappings through cost functions that are proposed without providing sufficient theoretical support. Also, it remains unknown how to algorithmically meet the demand of such quantifications, i.e., to find the minimizer of the cost functions.**

We address those concerns in a social network modeling parameterized by community structures that can be leveraged as side information for de-anonymization. By Maximum A Posteriori (MAP) estimation, our first contribution is an MAP-based cost functions, which, when minimized, enjoy superiority to previous ones in finding the correct mapping with the highest probability. The feasibility of the cost functions is then for the first time algorithmically characterized. While proving the general multiplicative inapproximability, we are able to propose two heuristics, which, respectively, enjoy an ϵ-additive approximation and a conditional optimality in carrying out successful user re-identification. Our theoretical findings are empirically validated, with one of the datasets extracted from rare true cross-domain networks that reproduce genuine social network de-anonymization. Both theoretical and empirical observations also manifest the importance of community information in enhancing privacy inferencing.

I. INTRODUCTION

The proliferation of social networks has led to generation of massive network data. Although users can be anonymized in the released data through removing personal identifiers [3], [4], with their underlying relations preserved, they may still be re-identified by adversaries from correlated cross domain auxiliary networks where user identities are known [5], [6].

Such idea of unveiling hidden users by leveraging their information collected from other domains, or alternatively called social network de-anonymization [6], is a fundamental privacy issue that has received considerable attention. Inspired by Pedarsani and Grossglauser [7], a large body of existing de-anonymization work shares a basic common paradigm: with an underlying network representing social relations between users, both the *published anonymized network* and the *auxiliary un-anonymized network* are generated from that network based on graph sampling that captures their correlation, as observed in many real cross-domain networks. The equivalent node sets they share are corresponded by an unknown correct mapping. With the availability of only structural information, adversaries attempt to re-identify users by establishing a mapping between networks. To quantify such mapping qualities, several global cost functions have been proposed [7]–[9] in favor of exploring the conditions under which the correct matching can be unraveled from the mapping that minimizes the cost function. Typically, these works [7]–[9] adopt Erdős-Rényi random graph or Chung-Lu graph [10], [11], which are both classic models that are widely used in social network analysis. On the other hand, some efforts [12], [14] are also made toward the data science perspective by analyzing the de-anonymization performance with extensive empirical evaluations.

The set of prior works of deanonymization topic, in summary, falls into two categories, i.e., (i) theoretically deriving cost functions without proposing algorithms to show that minimizing such cost functions can be effectively done, and (ii) proposing algorithms without sufficient theoretical rationales supporting the performance guarantee. Instead, in this paper, we are particularly concerned about the following question: **Is it possible to quantify de-anonymization and meanwhile algorithmically meet the demand brought by such quantifications?**

The answer to this question entails appropriate modeling of social networks, well-designed cost functions as metrics of mappings and elaborated algorithms of finding the mapping that is optimal according to the metric, along with data collection that can empirically validate the related claims. Considering the clustering effect that is prevalent in real social networks, we adopt the stochastic block model [15] where nodes are partitioned into disjoint sets representing different communities [16]. Moreover, graphs with various degree distributions can be captured by the stochastic block model since if each community only contains one or a handful of nodes, by varying the probability between communities, the existence probability of most edges can be set independently

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and differently. Based on that, we investigate the problem following the paradigm, as noted earlier, where the published and auxiliary networks serve as two sampled subnetworks. Both of them inherit from the underlying network the community structures that can be leveraged as side structural information for adversaries. Similarly, we assume that other than network structure, there is no additional availability of side information to adversaries as it will only further benefit them.

Varying the amount of availability of community information, here we classify our de-anonymization problem into two categories, i.e., bilateral case, and its counterpart, unilateral case, literally meaning that adversaries have access to community structure of both or only one network. A more formal definition of the two cases information is deferred to Section III. The bilateral case where the adversaries have the community information of both two networks has wide applications in reality. For example, the adversaries can bilaterally de-anonymize an anonymized employment network by obtaining the information of which company (community) the employees belong to without knowing their names; The patient information database also belong to the bilateral case where the names of patients are omitted but the departments that they belong to can be obtained. However, in other situations it may be hard for adversaries to get the community information from both the published anonymized network and the auxiliary network , which makes it also necessary to study the unilateral case. Take some social platforms like facebook, twitter and weibo as an example, they protect users' privacy by omitting both their identities and the community-related information, such as their hobbies, jobs or the social groups they are in, before publishing the anonymized networks so that the adversaries have no access to the community structure of the anonymized networks. Although the attackers can utilize some community detection algorithms to recover the communities, there may still be a number of nodes predicted to be in the wrong communities even by the state-of-art algorithms, which also shows the necessity of studying the unilateral case. More detailed reasons for the detected community information cannot being used in the bilateral case are put into Appendix A.

Subsequently, we summarize our results on metrics, algorithms and empirical validations into three aspects answering the question raised.

Analytical aspect: For both cases, our first contribution is to derive the cost functions as metrics quantifying the structural mismappings between networks based on Maximum A Posteriori (MAP) estimation, which is also adopted by [7] and [13]. However, our metrics is superior to the previous ones in the sense that the minimizers of our cost functions equal to the underlying correct mappings with the highest probability. Also, as we will rigorously prove later, under fairly mild conditions on network density and the closeness between communities, through minimizing the cost function we can perfectly recover the correct mapping.

Algorithmic aspect: Following the derived quantifications, our next contribution is to take an algorithmic look into the demand imposed by the quantifications, i.e., the optimization problems of minimizing the proposed MAP-based cost functions. We find that the induced optimization problems are computationally intractable and highly inapproximable. Therefore, we circumvent pursing exact or multiplicative approximation algorithms, but instead seek for heuristics with other types of guarantees. Our main idea to solve this problem is converting the problems into equivalent formulations that enable some relaxations, through bounding the influence of which, we demonstrate that the proposed heuristics have their respective performance guarantees. Specifically, one algorithm enjoys an ϵ-additive approximation guarantee in both cases, while the other yields optimal solutions for bilateral de-anonymization when the two sub-networks are highly structurally similar but fails to provide such guarantee for the unilateral case due to its lack of sufficient community information. Further comparisons of algorithmic results between the two cases also manifest the importance of community as side information in privacy inferencing.

Experimental aspect: Finally, we empirically verified all our theoretical findings under both synthetic and real datasets. We remark that one dataset is extracted from true cross-domain co-authorship networks [17] serving as published and auxiliary networks without artificial modeling assumptions. The experimental results demonstrate the effectiveness of our algorithms as they correctly re-identify more than 40% of users even in the co-authorship networks that possess the largest deviation from our assumptions. Also, it empirically consolidates our argument that community information can increase the de-anonymization capability.

The rest of this paper is organized as follows: In Section II, we briefly survey the related works. In Section III, we introduce our model for de-anonymization problem of social networks with community structure and characterize the cases of bilateral and unilateral information. In Sections IV and V, we present our results on analytical and algorithmic aspects of bilateral de-anonymization. Following the path of bilateral case, we introduce our results on unilateral de-anonymization and make comparisons between the two cases in Section VI. We present our experiments in Section VII and conclude the paper in Section VIII. Due to the limitation of space, we leave a part of our derivation and experiments in the appendix.

II. Related Works

The issue of social network de-anonymization, which has received considerable attention, was pioneeringly investigated by Narayanan and Shimatikov [6], who proposed the idea that users in anonymized networks can be re-identified through utilizing auxiliary networks with the same set of users from other domains. In that regard, they designed practical de-anonymization schemes that rely on side information in the form of a seed set of "pre-mapped" node pairs, i.e., a subset of nodes that are identified priorly across the two networks. Then the mapping is generated incrementally, starting from the seeds and percolating to the whole node sets.

Following this framework, Pedarsani and Grossglauser developed a succinct modeling that is amiable to theoretical analysis and serves as the paradigm for a family of subsequent related works on social network de-anonymization [7]. They

assumed that the published and auxiliary networks are two graphs that share the same node sets with the edge sets resulted from independent samples of an underlying social network. Additionally, they studied a more challenging but practical version of de-anonymization that are free of prior seed information.

The two seminal works triggered a flurry of subsequent attempts that all fall into the categories of either seeded or seedless de-anonymization, tuning the model of the underlying social networks. Specifically, in terms of seeded de-anonymization, current literature focuses on designing efficient de-anonymization algorithms that are executed by percolating the mapping to the whole node sets starting from the seed set. Yartseva et al. [18], Kazemi et al. [19], and later Chiasserini et al. [20] proposed percolation graph matching algorithms for de-anonymization on Erdős-Rényi graph and scale-free network, respectively. Assuming that the underlying social network is generated following the preferential attachment model, Korula and Lattenzi [21] designed a correspnding efficient de-anonymizaiton algorithm. Chiasserini et al. [22] characterized the impact that clustering imposes on the performance of seeded de-anonymization. Under the classification of both perfect and imperfect seeded de-anonymization, Ji et al. [14] analyzed the two cases both qualitatively and empirically.

While this type of seed-based de-anonymizing methods works well in analysis, it is rather difficult to acquire pre-identified user pairs across different networks as many real situations limit the access to user profiles. Therefore, more often we are faced with adversaries without seeds as side information, which is also the case considered in the present work. A natural alternative, under such circumstance, is to define a global cost function of mappings and unravel the correct mapping through the minimizer of the cost function. For instance, Pedarsani and Grossglauer [7] studied the seedless de-anonymization problem where the underlying social network is an Erdős-Rényi graph, the results of which were further improved by Cullina and Kiyavash [9]. Ji et al. analyzed perfect and partial de-anonymization on Chung-Lu graph [14]. Kazemi et al. [8] focused on the case of de-anonymization problem on Erdős-Rényi graph where the published network and auxiliary network exhibit partial overlapping. As for the Stochastic Block Model, Cullina et al. [35] figure out the regime where the network cannot be deanonymized perfectly, yet the community structure could be learned in case where the communities number is in constant order. When communities number grows with n, the authors also find the regime that the network cannot be deanonymized perfectly, but do not present analysis of the feasibility of recovering the communities. In contrast, our work finds the cost function as well as algorithms to deanonymize the networks in the regime where it can be deanonymized perfectly. A very recent work that shares the highest correlation with ours belongs to that of Onaran et al. [13], who study the situation where there are only two communities in networks, a special case that can be embodied in our bilateral de-anonymization case. Compared with [13], not only do we discuss the unilateral case of de-anonymization problem under SBM, we also generalize our model by allowing any arbitrary number of communities and different edge weights. Furthermore, instead of only deriving the theoretical condition under which our cost function can return true mapping, we also discuss the algorithmic aspect of this problem and prove the validity of our algorithms. As will be disclosed in later sections, our results are free of the constraints that the inter-community edge existence probability and intra-community one must be in the same order, which, however, is required in [13].

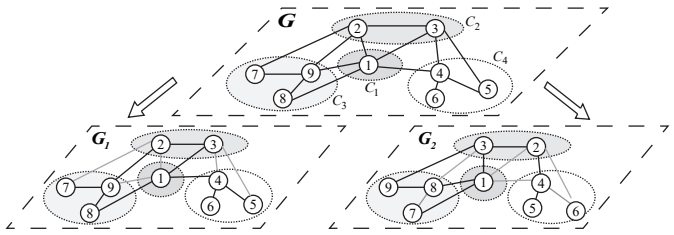


Fig. 1: **An example of underlying social network (G), the published network (G_1) and the auxiliary network (G_2) sampled from G. C_1, C_2, C_3, C_4 represent the four communities in the networks. The correct mapping $\pi_0 = \{(1,1),(2,3),(3,2),(4,4),(5,6),(6,5),(7,9),(8,7),(9,8)\}$.**

III. Models and Definitions

In this section, we introduce the models and definitions of the social network de-anonymization problem. We first present the network models and then formally define the problem of social network de-anonymization.

A. Network Models

The network models consist of the underlying social networks G, the published network G_1 and the auxiliary network G_2 as incomplete observations of G. In reality, the edges of G, for example, might represent the true relationships between a set of people, while G_1 and G_2 characterize the observable interactions between these people such as communication records in cell phones or "follow" relationships in online social networks.

1) Underlying Social Network: To elaborate this, let $G = (V, E)$ be the graph representing the underlying social relationships between network nodes, where V is the set of nodes and E is the set of edges. Denote $\mathbf{M}$[1] to be the adjacency matrix of G. We treat G as an undirected graph and define the number of nodes as $|V| = n$. We assume that G is generated according to the *stochastic block model* [15]. Specifically, the model is interpreted as follows: the set of nodes in V are partitioned into κ disjoint subsets denoted as $C_1, C_2, \ldots, C_\kappa$ indicating their communities with $|C_i| = n_i$ and $\sum_i n_i = n$. The edges between nodes in different communities are drawn independently at random with certain probabilities. Let $c : V \mapsto \{1 \ldots \kappa\}$ be the community assignment function that assigns to each node the label of the community it belongs to, we have

$$Pr\{(u,v) \in E\} = Pr\{\mathbf{M}_{uv} = 1\} = p_{c(u)c(v)},$$

[1]For a matrix $\mathbf{M}$, we use $\mathbf{M}_{ij}$ to denote the element on its ith row and jth column and $\mathbf{M}_i$ to denote its ith row vector. $\mathbf{M}_{ij} = 1$ if $(i,j) \in E$ and $\mathbf{M}_{ij} = 0$ otherwise.

where affinity values $\{p\}_{ab}$ $(1 \leq a, b \leq \kappa)$ are pre-defined parameters that indicate the edge existence probabilities and capture the closeness between communities. By tuning the values of $\{p\}$, this model can generate graphs that have known community structure, but which are essentially random in other respects [23]. It can capture various degree distributions by varying κ and $\{p\}$.

In order to obtain $\{p\}_{ab}$ in reality, we can firstly obtain the approximate community assignment function c with the help of some data mining techniques such as community detection. Some classic community detection algorithms that can be adopted include Clique Percolation Method (CPM), Cluster Affiliation Model for Big Networks (BIGCLAM) and Mixed-Membership Stochastic Block Model (MMSB), all of which can normally handle networks of millions of nodes with fairly good detection accuracy. After approximately finding the communities in the networks, we can then obtain $p_{c(i)c(j)}$ by dividing the existed edges with the edges that are most likely to exist between any two communities. For example, let V_a, V_b be the vertex set of a, b respectively and $\mathcal{E}_{ab}$ be the edge set between a, b. We can perform the estimation by the following formula: When a and b are different communities, $\hat{p}_{ab} = \frac{|E_{ab}|}{|V_a||V_b|}$; When a and b are same communities, $\hat{p}_{ab} = \frac{|E_{ab}|}{\binom{|V_a|}{2}}$. $\hat{p}_{ab}$ is the evaluated edge extience probability between community a and b. We can easily derive that $\mathbb{E}[\hat{p}_{ab}] = p_{ab}$.

2) Published Network and Auxiliary Network: We define $G_1(V_1, E_1)$ as the graph representing the published network and $G_2(V_2, E_2)$ as the graph representing the auxiliary network with E_1, E_2 denoting their edge sets respectively. Denote $\mathbf{A}, \mathbf{B}$ to be the adjacency matrix of G_1 and G_2 respectively. In correspondence to real situations, G_1 represents the publicly available anonymized network where user identities are removed for privacy concern. In contrast, G_2 represents the auxiliary cross-domain un-anonymized network where those users' identities are known, and can be collected by the adversary to re-identify the users in G_1. Following previous literature [7], [14], we assume the node sets in G_1 and G_2 are equivalent and that the published network and the auxiliary network are independent samples obtained from the underlying social network G with sampling probabilities s_1 and s_2, respectively. Specifically, for $i = 1, 2$, we have

$$Pr\{(u, v) \in E_i\} = \begin{cases} s_i & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Technically, G, G_1 and G_2 are defined as the random graph variables for the networks. However, for ease of representation, we will also use them to denote the realizations of the random graph variables without loss of clearance. In the sequel, we will also use $\boldsymbol{\theta}$ as a shorthand of the set of parameters including affinity values $\{p\}$ and sampling probabilities s_1, s_2 in the models of G, G_1, G_2.

In reality, the sampling probabilities can be approximated by some sociological approaches or statistical data. For example, we can approximate the sampling probability of a social network by surveying the number of real life friends that each person has, as well as the number of friends he or she has in the social software.

B. Social Network De-anonymization

Given the published network G_1 and the auxiliary network G_2, the problem of social network de-anonymization aims to find a bijective mapping $\pi : V_1 \mapsto V_2$ that reveals the correct correspondence of the nodes in the two networks. Equivalently, a mapping π[2] can be represented as a permutation matrix $\mathbf{\Pi}$ where $\mathbf{\Pi}_{ij} = 1$ if $\pi(i) = j$ and $\mathbf{\Pi}_{ij} = 0$ otherwise. We naturally extend the definition of mapping of node set to the mapping of edge set, as $\pi(e = (i, j)) = (\pi(i), \pi(j))$.

We define π_0 (or equivalently $\mathbf{\Pi}_0$) to be the correct mapping between the node sets of G_1 and G_2. Note that we do not have access to π_0 or the generator G of G_1 and G_2. In other words, although the node sets of G_1 and G_2 are equivalent, the labeling of the nodes does not reflect their underlying correspondence. We interpret this in the way that the published network G_1 has the same node labeling as the underlying network G while the node labeling of G_2 is permuted. Following this interpretation, the community assignment function of G_1 equals to c. However the community assignment function of G_2, which we further define as c', may be different. We illustrate an example of our network models in Figure 1.

The community assignment functions of the two networks may serve as important structural side information for de-anonymization, which naturally divide the social network de-anonymization problem into two types where the adversary possesses different amount of information on the community assignment. In the first type, the adversary possesses the community assignments of both G_1 and G_2, and the community sets $\{C_1, C_2, \ldots, C_\kappa\}$ of G_1 and G_2 are the same. The corresponding problem is formally defined as follows.

Definition III.1. ***(De-anonymization with Bilateral Community Information)*** *Given the published network G_1, the auxiliary network G_2, the parameters $\boldsymbol{\theta}$, as well as the community assignment function c for G_1 and c' for G_2, the goal is to construct a mapping π that satisfies $\forall i, c(i) = c'(\pi(i))$ and is closest to the correct mapping π_0.*

In this case, since we have the community assignments of both graphs and their community sets are the same, we can perform a relabeling on nodes in G_2 to make its community assignment equals to that of G_1. Hence, without loss of generality, for the case of de-anonymization with bilateral information, we denote c as the community assignment function of both G_1 and G_2 in the sequel.

The second variant corresponds to the case where the adversary only possesses the community assignment of one graph or both two graphs with different community sets, which leads that only the community information of one graph is useful. This case is formally stated as follows.

Definition III.2. ***(De-anonymization with Unilateral Community Information)*** *Given the published network G_1, the auxiliary network G_2, parameters $\boldsymbol{\theta}$, as well as the community assignment function c for G_1 or G_2, the goal is to construct a mapping that is closest to the correct mapping π_0.*

[2]In this paper, all the mappings are assumed to be bijective. Hence, we simply refer to them as mappings for brevity.

TABLE I: **Notions and Definitions**

Notation	Definition
G	Underlying social network
G_1, G_2	Published and auxiliary networks
V, V_1, V_2	Vertex sets of graphs G, G_1 and G_2
E, E_1, E_2	Edge sets of graphs G, G_1, G_2
s_1, s_2	Edge sampling probabilities of graphs G_1, G_2
$\mathbf{M}, \mathbf{A}, \mathbf{B}$	Adjacency matrices of graphs G, G_1, G_2
c	Community assignment function
C_i	Vertex set of community i
n	Total number of vertices
κ	Total number of communities
n_i	Number of vertices in community i
p_{ab}	Affinity value indicating the edge existence probability between communities a and b
$\boldsymbol{\theta}$	Set of parameters in the models of G, G_1 and G_2
π_0	Correct mapping between vertices in G_1 and G_2
π	Mapping between vertices in G_1 and G_2
$\mathbf{\Pi}$	Permutation matrix of mapping π
Δ_π	Cost function of the mappings
$\{w\}$	Set of weights in the cost function

Intuitively, de-anonymization with unilateral information is harder than that with bilateral information due to the lack of side information. We will validate this argument with subsequent theoretical analysis and experiments. In addition, for brevity, we may refer to de-anonymization problem with bilateral community information and with unilateral community information as bilateral de-anonymization and unilateral de-anonymization respectively.

Remark: Till now, we have not given the quantifying metric of the closeness to the correct mapping π_0. A natural choice would be the mapping accuracy, i.e., percentage of nodes that are mapped identically as in π_0. However, as we have no knowledge of π_0, such ground-truth-based metrics do not apply. To tackle this, we leverage the Maximum A Posteriori (MAP) estimator to construct cost functions for measuring the quality of mappings based solely on observable information. The main notations used throughout the paper are summarized in Table I.

IV. Analytical Aspect of Bilateral De-anonymization

First, we investigate the de-anonymization problem with bilateral information, starting with an appropriate metric measuring the quality of mappings. We define our proposed metric in the form of a cost function that derived from Maximum A Posteriori (MAP) estimation.

A. MAP-based Cost Function

According to the definition of MAP estimation, given the published network G_1, auxiliary network G_2, parameters $\boldsymbol{\theta}$ and the community assignment function [3] c, the MAP estimate $\hat{\pi}$ of the correct mapping π_0 is defined as:

$$\hat{\pi} = \arg\max_{\pi\in\Pi} Pr(\pi_0 = \pi \mid G_1, G_2, c, \boldsymbol{\theta}), \tag{1}$$

where $\Pi = \{\pi : V_1 \mapsto V_2 \mid \forall i, c(i) = c(\pi(i))\}$, i.e. the set of bijective mappings that observe the community assignment.

From the results in [13], the MAP estimator in Equation (1) can be computed as

$$\begin{aligned}\hat{\pi} &= \arg\min_{\pi\in\Pi} \sum_{i\le j}^{n} w_{ij} \left|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j))\in E_2\}\right| \\ &\triangleq \arg\min_{\pi\in\Pi} \Delta_\pi,\end{aligned} \tag{2}$$

where $w_{ij} = \log\left(\frac{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)}{p_{c(i)c(j)}(1-s_1)(1-s_2)}\right)$. Based on Equation (2), we have our cost function Δ_π as the metric for the quality of mappings, which can also be interpreted as weighted edge disagreements induced by mappings.

B. Validity of the Cost Function

Since our cost function Δ_π is derived using the MAP estimation, the minimizer of Δ_π, being the MAP estimate of π_0, coincides with the correct mapping with the highest probability [24]. Aside from this, we proceed to justify the use of MAP estimation in de-anonymization problem from another perspective. Specifically, we prove that if the model parameters satisfy certain conditions, then the MAP estimate $\hat{\pi}$ *asymptotically almost surely*[4] coincides with the correct mapping π_0, which means that we can perfectly recover the correct mapping through minimizing Δ_π.Theorem IV.1 illusatrates the conditions under which the cost function is effective. (We use standard Knuth's notations in this paper.)

Theorem IV.1. *Let $\alpha = \min_{ab} p_{ab}, \beta = \max_{ab} p_{ab}$. Assume that $\alpha, \beta \to 0$, s_1, s_2 do not go to 1 as $n \to \infty$ and $\frac{\log\alpha}{\log\beta} \le \gamma$. Suppose that*

$$\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1+s_2} \ge \frac{(6+\epsilon)\gamma^2 \log n}{n},$$

where ϵ can be any constant larger than 0, then $\hat{\pi} = \pi_0$ holds almost surely as $n \to \infty$.

Proof. Due to space limitations, here we only present an outline of the proof and defer the details to **Appendix B**. Recall that for a mapping π, we define $\Delta_\pi = \sum_{i\le j}^{n} w_{ij}|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{\pi(i),\pi(j)\in E_2\}|$. Then the proof can be briefly divided into two major steps. The first one is to derive an upper bound for the expectation of the number of (incorrect) mappings π's with $\Delta_\pi \le \Delta_{\pi_0}$. The second one is to show that the derived upper bound converges to 0 under the conditions stated in the theorem, as $n \to \infty$. Based on that, we denote Π_k as the set of mappings that map k nodes incorrectly and S_k as a random variable representing the the number of mappings $\pi \in \Pi_k$ with $\Delta_\pi \le \Delta_{\pi_0}$. We then define $S = \sum_{k=2}^{n} S_k$ as the total number of incorrect mappings π with $\Delta_\pi \le \Delta_{\pi_0}$ and derive an upper bound on the mean of S as $\mathbb{E}[S] \le \sum_{k=2}^{n} n^k \max_{\pi\in\Pi_k} Pr\{\Delta_\pi - \Delta_{\pi_0} \le 0\}$. We further show that under the conditions stated in the theorem, this upper bound, and consequently $\mathbb{E}[S]$, go to 0 as $n \to \infty$, which implies that π_0 is the unique minimizer of Δ_π and concludes the proof. □

[3] As mentioned after Definition III.1, we can relabel the nodes to make the community assignment c' of G_2 equals to c of G_1

[4] An event *asymptotically almost surely* happens if it happens with probability $1 - o(1)$.

Remark: We now present five further notes regarding Theorem IV.1.

(i) *Validity under other conditions:* In Theorem IV.1, we discuss the validity of the cost function when s_1, s_2 do not go to 1. While this assumption covers most cases of de-anonymization, for sake of comprehensiveness we also study the cases where s_1, s_2 go to or equal to 1, with the corresponding results available in **Appendix C**. The derived conditions in the supplemented cases becomes more mild as the sampling probabilities approach 1, which is in consistent with the intuition.

(ii) *Structure of the condition:* As for the condition in Theorem IV.1, it is only related to the parameter set $\boldsymbol{\theta}$, i.e., $\{p\}, s_1, s_2$, and the network size n. Note that none of the parameters associated with the community structure $(\kappa, |C_i|)$ appear in the condition. The reason is that $|\mathbb{1}\{(i,j) \in E_1\} - \mathbb{1}\{(\pi(i), \pi(j)) \in E_2\}|$ is not affected by whether or not the community structure is known. However, although the theoretical bound is irrelevent to the community structure information, as we will show in the next section, with more knowledge of the community structure, the approximate algorithms will indeed get more accurate mappings.

(iii) *Applicability of the Theorem:* Recall that for a random Erdős-Rényi graph $G(n,p)$ to be connected and free of isolated nodes with high probability, it must satisfy $p = \Omega(\frac{\log n}{n})$ [10]. Conventionally setting the sampling probabilities s_1, s_2 as constants, it is easy to verify that the condition in Theorem IV.1 is looser than that of the graph connectivity even when the expected degree distributions (or equivalently, the closeness between the communities) of G_1 and G_2 are non-uniform (e.g. power law distribution where $\alpha/\beta = O(n)$ and $\log\alpha/\log\beta = O(\log n)$). Therefore we can process successful de-anonymization even if the graph is not connected and has several isolated nodes. From this aspect, the condition is quite mild and thus makes Theorem IV.1 fairly general;

(iv) *Comparison with Previous Result:* The de-anonymization estimator proposed in [7] does not have the weight parameter w_{ij} and can be regarded as an unweighted version of our MAP estimator. Although the two estimators are equivalent when $w_{ij} = 1$ for any node pair, i.e., the affinity values p_{ab} among all communities are equal, we will show in **Appendix D-1** that our MAP Estimate is superior to the estimate in [7] when $\{p\}_{ab}$ are inhomogeneous. Moreover, the de-anonymization estimator proposed in [13] studies the case where there are only two communities in the graph and the sampling probabilities assumed to be equal, i.e., $s_1 = s_2 = s$. Although it adopts the stochastic block model, which is the same model as we adopt, and propose an estimator also based on MAP, we will also show in **Appendix D-2** that our condition under which the true mapping can be found by the MAP estimator is looser and thus our estimator is superior. Besides, we compare the condition in Theorem IV.1 with the bound of the de-anonymizability proposed in [9], and also find that our result is an improved version. The details can be referred to in **Appendix D-3**;

(v) *Extension of the Theorem:* The cost function we design is robust, in the sense that any approximate minimizer Δ_π can map most of the nodes correctly. We formally present the claim in Theorem IV.2.

Theorem IV.2. *Let α, β be the same parameters defined in Theorem IV.1. Assume that $\alpha, \beta \to 0$, and s_1, s_2 do not go to 0 and $\frac{\log\alpha}{\log\beta} \le \gamma$. Additionally, let δ, ϵ be two real numbers with $0 \le \delta, \epsilon \le 1$ with $\epsilon = O(\delta - \frac{\delta^2}{2})\alpha s_1 s_2 \log(1/\alpha)$. If*

$$\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1 + s_2} \ge \frac{(6 + c_0)\gamma^2 \log n}{(1 - \delta/2)n},$$

where c_0 can be any constant larger than 0, then for all π^* with $\Delta_{\pi^*} - \min_{\pi\in\Pi}\Delta_\pi \le \epsilon n^2$, π^* is guaranteed to map at least $(1-\delta)n$ nodes correctly as $n \to \infty$.

Proof. The proof is similar to that of Theorem IV.1. Instead of bounding $\sum_{k=2}^{n}\sum_{\pi\in\Pi_k} Pr\{\Delta_\pi - \Delta_{\pi_0} \le 0\}$, we upper bound $\sum_{k=\delta n}^{n}\sum_{\pi\in\Pi_k} Pr\{\Delta_\pi - \Delta_{\pi_0} \le \epsilon n^2\}$. Using similar technique as in Theorem IV.1, we have that under the conditions stated in the corollary, $\sum_{k=\delta n}^{n}\sum_{\pi\in\Pi_k} Pr\{\Delta_\pi - \Delta_{\pi_0} \le \epsilon n^2\} \to 0$ as $n \to \infty$. Therefore, for a mapping π^* with $\Delta_{\pi^*} - \Delta_{\pi_0} \le \epsilon n^2$, it maps at most $k = \delta n$ nodes incorrectly. Since $\Delta_{\pi_0} \ge \arg\min_{\pi\in\Pi}\Delta_\pi$, we conclude that all π^* with $\Delta_{\pi^*} - \min_{\pi\in\Pi}\Delta_\pi \le \epsilon n^2$ are guaranteed to map at least $(1-\delta)n$ nodes correctly as $n \to \infty$. □

V. Algorithmic Aspect of Bilateral De-anonymization

The quantification in Section IV justified that, under mild conditions, we can unravel the correct mapping through computing its MAP estimate, i.e., the minimizer of Δ_π. This naturally puts forward the optimization problem of computing the minimizer of Δ_π, which reasonably serves as the instantiation of the social network de-anonymization problem (Definition 3.1). To meet the demand of the quantification, in this section, we formally define and investigate this optimization problem, presenting a first look into the algorithmic aspect of social network de-anonymization.

A. The Bilateral MAP-ESTIMATE Problem

Naturally, with some previously defined notations inherited, the optimization problem induced by the cost function can be formulated as follows.

Definition V.1. ***(The BI-MAP-ESTIMATE Problem)*** *Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, community assignment function c and a set of weights $\{w\}$, the goal is to compute a mapping $\hat{\pi} : V_1 \mapsto V_2$ that satisfies*

$$\mathbf{P1}: \quad \hat{\pi} = \arg\min_{\pi\in\Pi}\sum_{i\le j}^{n} w_{ij}\,|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{\pi(i),\pi(j)\in E_2\}| \triangleq \arg\min_{\pi\in\Pi}\Delta_\pi,$$

where $\Pi = \{\pi \mid \forall i, c(i) = c(\pi(i))\}$.

Note that we require the weights $\{w\}$ to be induced by implicit and well-defined community affinity values and sampling probabilities. Also, the BI-MAP-ESTIMATE Problem denoted as $\mathbf{P1}$ above has several equivalent formulations, which will be presented later.

The BI-MAP-ESTIMATE seems to be easy at first glance due to the simplicity of its objective function Δ_π, but as

justified by the following proposition, it is not only computationally intractable but also highly inapproximable.

Proposition V.1. *BI-MAP-ESTIMATE problem is not P-complete. And there is no polynomial time approximation algorithm for BI-MAP-ESTIMATE with any multiplicative approximation guarantee unless $GI \in P$.*[5]

Proof. The proof can be easily constructed by reduction from the graph isomoprhism problem. The reduction is completed by just setting the two graphs in the instance of the graph isomorphism as G_1 and G_2, as well as assigning all $w_{ij} = 1$ and $c(v) = 1$ for all $v \in V_1, V_2$. Obviously, if the two graphs are isomorphic, the value $\Delta_{\hat{\pi}}$ of the optimal mapping $\hat{\pi}$ will be zero. Therefore, in this case, any algorithm with multiplicative approximation guarantee must find a mapping π with $\Delta_\pi = 0$. Furthermore, if G_1 and G_2 are not isomorphic, then any mapping π must induce a Δ_π strictly larger than 0. Hence, a polynomial time approximation algorithm for BI-MAP-ESTIMATE with multiplicative guarantee implies a polynomial time algorithm for the graph isomorphism problem. □

Remark: Although graph isomorphism is not known to be NP-Complete, it does not have any polynomial-time algorithms for now. Since we have reduced the *BI-MAP-ESTIMATE* problem from the graph isomoprhism problem and $GI \in DTIME(n^{polylogn})$, we can conclude that *BI-MAP-ESTIMATE* $\notin P$, which still indicates the prohibitive complexity of solving this problem and necessity of proposing approximate algorithms.

B. Approximation Algorithms

As demonstrated above, the BI-MAP-ESTIMATE problem bears high computational complexity and approximation hardness. It is thus unrealistic to pursue exact or even multiplicative approximation algorithms. To circumvent this obstacle and still find solutions with provable theoretical properties, we propose two algorithms with their respective advantages: one has an ϵ-additive approximation guarantee and the other has lower time complexity and yields optimal solutions under certain conditions. The main idea behind them is to convert **P1** to equivalent formulations which are more amenable to relaxation techniques.

1) Additive Approximation Algorithm: The additive approximation algorithm we propose is based on the following quadratic assignment formulation of the BI-MAP-ESTIMATE Problem which we denote as **P2**.

$$\begin{aligned}
\mathbf{P2}: \quad & \text{maximize} && \textstyle\sum_{i,j,k,l} q_{ijkl} x_{ik} x_{jl} && (3)\\
& \textbf{s.t.} && \textstyle\sum_i x_{ij} = 1, \quad \forall i \in V_1 && (4)\\
& && \textstyle\sum_j x_{ij} = 1, \quad \forall j \in V_2 && (5)\\
& && x_{ij} \in \{0,1\} && (6)
\end{aligned}$$

[5] GI denotes the complexity class Graph Isomorphsim.

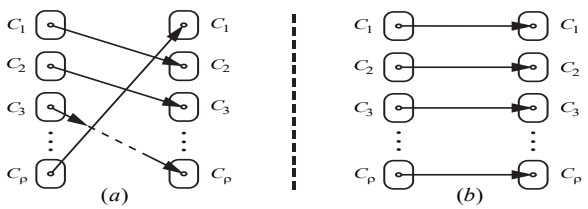


Fig. 2: **Illustration of the reversal of a cycle of community assignment violations: (a) a cycle of community assignment violations in a mapping; (b) reversal of the cycle of violations.**

The coefficients $\{q\}_{ijkl}$ of **P2** are defined as:

$$q_{ijkl} = \begin{cases} \frac{w_{ij}}{\log(1/\alpha)}, & \text{if } (i,j) \in E_1, (k,l) \in E_2 \text{ and} \\ & c(i) = c(k), c(j) = c(l), \\ -1 & \text{if } c(i) \neq c(k) \text{ or } c(j) \neq c(l), \\ 0 & \text{otherwise.} \end{cases}$$

The solutions to **P2** are a set of integers $\{x\}$. We will refer to the value of $\sum_{i,j,k,l} q_{ijkl} x_{ik} x_{kl}$ as the value of $\{x\}$. Based on a solution $\{x\}$, we can construct its equivalent mapping for the BI-MAP-ESTIMATE problem by setting $\pi(i) = j$ iff $x_{ij} = 1$. The following proposition shows the correspondence between **P1** and **P2**.

Proposition V.2. *Given G_1, G_2, c and $\{w\}$, the optimal solutions of* **P1** *and* **P2** *are equivalent.*

Proof. We write the equivalent set of integers $\{x\}$ of a mapping π as $\{x^\pi\}$. First, we prove that the optimal solution $\{x^*\}$ to **P2** must observe the community assignment, i.e., if $x^*_{ij} = 1$, then $c(i) = c(j)$. Indeed, for a solution $\{x\}$ having some $x_{i_0 i_1} = 1$ but $c(i_0) \neq c(i_1)$, we can find a "cycle of community assignment violations" starting from i with $x_{i_0 i_1} = x_{i'_1 i_2} = x_{i'_2 i_3} = \ldots x_{i'_\rho i'_0}$ and $c(i_0) = c(i'_0), c(i_1) = c(i'_1), \ldots, c(i_\rho) = c(i'_\rho)$. Due to the special structure of the coefficients $\{q\}$, this cycle only contributes negative value to the objective function of **P2**. Therefore, by "reversing" the cycle, we obtain a new solution $\{x'\}$ from $\{x\}$ with $x'_{i_0 i'_0} = x'_{i'_1 i_1} = x'_{i'_2 i_2} = \ldots = x'_{i'_\rho i_\rho} = 1$ and $\sum_{i,j,k,l} q_{ijkl} x'_{ij} x'_{kl} > \sum_{i,j,k,l} q_{ijkl} x_{ij} x_{kl}$.The process of reversing cycles of community assignment violations is demonstrated in Figure 2. If follows that the optimal solution to **P2** must observe the community assignment. Then, we proceed to show that the optimal solution to **P1** is equivalent to the optimal solution to **P2**. Notice that for all $\{x^\pi\}$ that observe the community assignment, we have $\sum_{ij} w_{ij} = \log \frac{1}{\alpha} \sum_{ijkl} q_{ijkl} x^\pi_{ik} x^\pi_{jl} + \Delta_\pi$. Therefore, the corresponding $\{x^{\hat{\pi}}\}$ of the optimal solution $\hat{\pi}$ to **P1** is also optimal for **P2** and vice versa. □

The proof of Proposition V.2 also provides the two main stages in our additive approximation algorithm: (i) Convert the instance of the BI-MAP-ESTIMATE problem into its corresponding quadratic assignment formulation **P2** where the solution is then computed. (ii) Reverse all the "cycles of community assignment violations" in the solution and construct the desired mapping based on it.

For the first stage, we adopt the relaxing-rounding based algorithm proposed by Arora at al. [25] as a sub-procedure

referred to as "QA-Rounding" to solve the converted instances of $\mathbf{P2}$. QA-Rounding has additive approximation guarantee when the instances have coefficients $\{q\}$ that do not scale with the size of the problem [25]. Note that the requirement for the coefficients to be independent of the size of the problem is one of the key factors for the seemingly unnatural formulation of $\mathbf{P2}$. For the sake of completeness, we state in the following lemma the related result from [25].

Lemma V.1. *(Theorem 3 in [25]) Given an instance of* $\mathbf{P2}$ *with* $-C \leq q_{ijkl} \leq C$ *for all* $i, j, k, l \in \{1 \dots n\}$ *where* C *is a constant that is independent of* n*, then for any* $\epsilon > 0$*, QA-Rounding finds a solution* $\{x\}$ *with*

$$\sum_{i,j,k,l} q_{ijkl} x_{ik} x_{jl} \geq \sum_{ijkl} q_{ijkl} x^*_{ijkl} - \epsilon n^2$$

in $n^{O(\log n/\epsilon^2)}$ *time, where* $\{x^*\}$ *is the optimal solution.*

The second stage can be completed by repeatedly traversing the solution $\{x\}$ to identify all the cycles of community assignment violations and reversing them. **Algorithm 1** illustrates a whole diagram of our proposed additive approximation algorithm.

Approximation Guarantee: By Lemma V.1, QA-Rounding yields a solution whose value has a gap of less than ϵn^2 from the optimum. Combined with the equality $\sum_{i,j} w_{ij} = \Delta_\pi + \log \frac{1}{\alpha} \sum_{i,j,k,l} q_{ijkl} x_{ik} x_{jl}$ and the fact that the reversal of all the cycles of community assignment violations only incurs an increase on the value of the computed solution $\{x\}$, we have that the mapping π given by **Algorithm 1** has an ϵ-additive approximation guarantee and satisfies $c(i) = c(\pi(i))$ for all i.

Moreover, by Corollary IV.2, we know that when ϵ, δ satisfy the conditions in the corollary, the mappings yielded by **Algorithm 1** map at least $(1-\delta)n$ nodes correctly.

Input: Graphs G_1, G_2, weights $\{w\}$,
community assignment function c.
Output: mapping π.
Initialize: $\pi = \emptyset$, $\forall i, j, k, l \in \{1 \dots n\}, x_{ijkl} = 0$, $i', j' = 0$
Compute the set of coefficients $\{q\}_{ijkl}$ and
form an instance $\mathcal{I}$ of $\mathbf{P2}$.
$\{x\}$:=QA-Rounding($\mathcal{I}$).
for $i = 1$ *to* n **do**
  for $j = 1$ *to* n **do**
    if $x_{ij} = 1$ *and* $c(i) \neq c(j)$ **then**
      $x_{ij} := 0$.
      while $c(j') \neq c(i)$ **do**
        Find i', j' with $x_{i'j'} = 1$ and $c(i') = c(j)$.
        $x_{i'j'} := 0, x_{i'j} := 1, j := j'$.
      end
      $x_{ij'} := 1$.
    end
  end
end
Construct π based on $\{x\}$.
Return π

Algorithm 1: The Additive Approximation Algorithm

Time complexity: The QA-Rounding has a time complexity of $n^{O(\log n/\epsilon^2)}$. The reversal of all the cycles can be completed in $O(n^2)$ time when $\{x\}$ is represented in the form of an adjacency list-like structure. Based on those, the time complexity of **Algorithm 1** is $O(n^{O(\log n/\epsilon^2)} + n^2)$.

The dependence of the time complexity and the accuracy: If the algorithm yields a solution whose value has a gap of less than $\epsilon' n^2$ from the optimum, then based on the equality $\sum_{i,j} w_{i,j} = \log \frac{1}{\alpha} \sum_{i,j,k,l} q_{ijkl} x^\pi_{ik} x^\pi_{jl} + \Delta_\pi$, we have $\Delta_{\pi^*} - \min_{\pi \in \Pi} \Delta_\pi \leq \log \frac{1}{\alpha} \epsilon' n^2$. Moreover, from Theorem IV.2, we know that when $\epsilon = O(\delta - \frac{\delta^2}{2}) \alpha s_1 s_2 \log \frac{1}{\alpha}$, for all π^* with $\Delta_{\pi^*} - \min_{\pi \in \Pi} \Delta_\pi \leq \epsilon n^2$, π^* is guaranteed to map at least $(1-\delta)n$ nodes. Thus the relationship between ϵ' and δ is $\epsilon' = O(\delta - \frac{\delta^2}{2}) \alpha s_1 s_2$. Therefore, when the mappings yielded by **Algorithm 1** map at least $(1-\delta)n$ nodes correctly, the time complexity of the algorithm is $n^{O(\log n/\delta^2 \alpha^2)}$.

2) Convex Optimization-Based Heuristic: Beside the algorithm that provides additive approximation guarantee under general case, it is also useful to pursue algorithms that have stronger guarantee in special cases. In this section, we present one such algorithm that can find the optimal solution in the cases where the structural similarity between the two networks are higher than certain threshold.

The algorithm is based on convex optimization, which relies on a matrix formulation of the BI-MAP-ESTIMATE problem. The main idea is to first solve a convex-relaxed version of the matrix formulation and then convert the solution back to a legitimate one. Specifically, the matrix formulation of the BI-MAP-ESTIMATE problem, which we denote by P3, is formally stated as follows:

$$\begin{aligned}
\mathbf{P3}: \quad & \text{mininize } \|\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}} \mathbf{B} \mathbf{\Pi})\|_{\mathrm{F}}^2 + \mu \|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2 \\
& \textbf{s.t. } \forall i \in V_1, \ \textstyle\sum_i \mathbf{\Pi}_{ij} = 1 && (7) \\
& \qquad \forall j \in V_2, \ \textstyle\sum_j \mathbf{\Pi}_{ij} = 1 && (8) \\
& \qquad \forall i, j, \ \mathbf{\Pi}_{ij} \in \{0, 1\}, && (9)
\end{aligned}$$

where $\mathbf{W}$ is a symmetric matrix with $\mathbf{W}_{ij} = \mathbf{W}_{ji} = \sqrt{w_{ij}}$, $\mathbf{m}$ represents the community assignment vector $(c(1), \dots, c(n))^{\mathrm{T}}$, $\circ$ denotes the matrix Hadamard product with $(\mathbf{W} \circ \mathbf{A})_{ij} = \mathbf{W}_{ij} \cdot \mathbf{A}_{ij}$ and $\| \cdot \|_{\mathrm{F}}$ represents the Frobenius norm. Furthermore, μ is a positive constant that is large enough, and the range of it can be determined according to the following proposition.

Proposition V.3. *μ can be any positive constant larger than $(1 + \epsilon_0) \log \frac{1}{\alpha}$, where ϵ_0 can be any constant.*

Proof. When a node are mapped to a node that is not in the same community, $\mu \|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2$ will be increased by μ, and $\|\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}} \mathbf{B} \mathbf{\Pi})\|_{\mathrm{F}}^2$ will at most be decreased by $\overline{w} = \log \left(\frac{1 - \alpha(s_1 + s_2 - 2 s_1 s_2)}{\alpha (1 - s_1)(1 - s_2)} \right)$, which is equal to $(1 + o(1)) \log \frac{1}{\alpha}$. To ensure that the total cost always increases, μ should satisfies $\mu \geq (1 + \epsilon_0) \log \frac{1}{\alpha}$, where ϵ_0 can be any constant. □

Note that $\mathbf{P3}$ is equivalent to $\mathbf{P1}$ from the perspective of the relation between a mapping and its corresponding permutation matrix, as is stated in the following proposition.

Proposition V.4. *Given G_1, G_2, c and $\{w\}$, the optimal solution of* $\mathbf{P1}$ *and* $\mathbf{P3}$ *are equivalent.*

Proof. The proof is similar to that of Proposition V.2. First, due to the existence of the penalty factor $\mu \|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2$, we have that the optimal solution of $\mathbf{P3}$ must observe the community assignment. Second, as for all the permutation

matrices $\mathbf{\Pi}$'s and their corresponding mappings π's that observe the community assignment, it is easy to show that $\Delta_\pi = \|\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}}\mathbf{B}\mathbf{\Pi})\|_{\mathrm{F}}^2 + \mu\|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2$ (the second term equals to 0 in this case). Hence, the optimal solution of $\mathbf{P1}$ and $\mathbf{P3}$ are equivalent. □

Before introducing the algorithm, we further transform the objective function of $\mathbf{P3}$ into an equivalent but more tractable form. Lemma V.2 gives the main idea of the transformation.

Lemma V.2. *Let $\tilde{\mathbf{A}} = \mathbf{W} \circ \mathbf{A}$ and $\tilde{\mathbf{B}} = \mathbf{W} \circ \mathbf{B}$ be the weighted adjacency matrices of G_1 and G_2 respectively, then for all permutation matrices that observe the community assignment[6],the following equality holds:*

$$\|\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}}\mathbf{B}\mathbf{\Pi})\|_{\mathrm{F}} = \|\mathbf{\Pi}\tilde{\mathbf{A}} - \mathbf{B}\tilde{\mathbf{\Pi}}\|_{\mathrm{F}}.$$

Proof. We prove the lemma by repeatedly using the symmetry of $\mathbf{A}$ and $\mathbf{B}$ and special properties of $\mathbf{W}$ and $\mathbf{\Pi}$. The detailed steps are presented as follows:

$$\begin{aligned}
\|\mathbf{W} \circ (\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}}\mathbf{B}\mathbf{\Pi})\|_{\mathrm{F}} &= \|\mathbf{W} \circ (\mathbf{\Pi}(\mathbf{A} - \mathbf{\Pi}^{\mathrm{T}}\mathbf{B}\mathbf{\Pi}))\|_{\mathrm{F}} && (10)\\
&= \|\mathbf{W} \circ (\mathbf{\Pi}\mathbf{A} - \mathbf{B}\mathbf{\Pi})\|_{\mathrm{F}} && (11)\\
&= \|\mathbf{W} \circ (\mathbf{\Pi}\mathbf{A}) - \mathbf{W} \circ (\mathbf{B}\mathbf{\Pi})\|_{\mathrm{F}} && (12)\\
&= \|\mathbf{\Pi}(\mathbf{W} \circ \mathbf{A}) - (\mathbf{W} \circ \mathbf{B})\mathbf{\Pi}\|_{\mathrm{F}} && (13)\\
&= \|(\mathbf{\Pi}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi})\|_{\mathrm{F}}. && (14)
\end{aligned}$$

Note that Equation (10) holds because multiplying by a permutation matrix does not change the value of element-wise Frobenius norm. Equations (11), (12) and (14) hold due to the definition of Hadamard product and $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$. The validity of Equation (13) is less straightforward and can be interpreted in the following way: For the weight w_{ij} of a node pair (i, j), it is determined only by $p_{c(i)c(j)}, s_1, s_2$. Therefore, if $c(i) = c(j), c(k) = c(l)$ for some nodes i, j, k, l, then we have $\mathbf{W}_{ik} = \mathbf{W}_{jl}$, i.e., the weight is invariant within communities. This crucial property, combined with the fact that $\mathbf{\Pi}$ is permutation matrix that observes the community assignment, makes the Hadamard products and normal matrix multiplication in Equation (13) interchangeable. □

Based on Lemma V.2, we can rewrite the objective function of $\mathbf{P3}$ as $\|(\mathbf{\Pi}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi})\|_{\mathrm{F}}^2 + \mu\|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2$. Then, we further relax constraints (8) and (9) in $\mathbf{P3}$ and obtain the optimization problem $\mathbf{P3'}$ that can be formulated as:

$$\begin{aligned}
\mathbf{P3'} \qquad &\text{minimize } \|(\mathbf{\Pi}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi})\|_{\mathrm{F}}^2 + \mu\|\mathbf{\Pi}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2\\
&\textbf{s.t. } \forall i,\ \textstyle\sum_{i \in V_1} \mathbf{\Pi}_{ij} = 1
\end{aligned}$$

Obviously the objective function and the set of feasible solutions are both convex. Immediately we can conclude that $\mathbf{P3'}$ is a convex-relaxed version of $\mathbf{P3}$, which is stated in the following lemma.

Lemma V.3. $\mathbf{P3'}$ *is a convex optimization problem.*

With all the prerequisites above, we are now ready to present our second convex optimization-based heuristic, which firstly solves for a fractional optimal solution of $\mathbf{P3'}$ and then projects that fractional solution into an integral permutation matrix (and its corresponding mapping). During the projection process, we use an n-dimensional array $Mapped$ to record the projected nodes and a set $Legal_i$ for each node i to record the remaining legitimate nodes to which it can be mapped. The details are illustrated in **Algorithm 2**.

Input: Graphs G_1, G_2, weights $\{w\}$,
community assignment function c.
Output: mapping π.
Initialize: $Mapped[i] = 0$. $Legal_i = \emptyset$ **for all** i,
$\pi = \emptyset$, $\mathbf{\Pi}^p, \mathbf{\Pi}^f = \mathbf{0}$.
Compute the weight matrix $\mathbf{W}$ and
form an instance $\mathcal{I}$ of $\mathbf{P3}$.
Relax $\mathcal{I}$ into an instance $\mathcal{I}'$ of $\mathbf{P3'}$.
$\mathbf{\Pi}^f :=$ the optimal (fractional) solution to $(\mathcal{I}')$.
for $i = 1$ *to* n **do**
  $Legal_i := \{k \mid Mapped[k] = 0 \text{ and } c(k) = c(i)\}$
  $j := \arg\max_{k \in Legal_i} \mathbf{\Pi}^f_{ik}$.
  $\mathbf{\Pi}^p_{ij} := 1$. $Mapped[j]$:=1.
end
Construct π based on $\mathbf{\Pi}^p$.
Return π

Algorithm 2: Convex Optimization-Based Algorithm

Performance Guarantee: Generally, **Algorithm 2** can not yield the optimal solution to the BI-MAP-ESTIMATE problem and the gap between its solution and the optimal one may be large. However, we will demonstrate that when the similarity between G_1 and G_2 are high enough, or equivalently, the difference between the weighted adjacency matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ is sufficiently small, **Algorithm 2** is guaranteed to find the optimal mapping.

Let $\mathbf{\Pi}^p$ be the solution obtained by **Algorithm 2** and $\mathbf{\Pi}^*$ be the optimal solution. Note that $\mathbf{\Pi^*}$ is the optimal solution of MAP estimator, i.e., $\pi^* = \arg\min_{\pi \in \Pi} \Delta_\pi$ and $\mathbf{\Pi^*}$ is the permutation matrix of mapping π^*. $\mathbf{\Pi^*}$ is equal to $\mathbf{\Pi_0}$, the permutation matrix of the true mapping, when the condition in Theorem IV.1 are satisfied. Theorem states the condition that enables $\mathbf{\Pi^p} = \mathbf{\Pi}^*$.

Theorem V.1. *Let $\tilde{\mathbf{B}}'$ be a symmetric matrix that is related with $\tilde{\mathbf{A}}$ by a unique $\hat{\mathbf{\Pi}}$ that observes the community assignment, i.e., $\tilde{\mathbf{B}}' = \hat{\mathbf{\Pi}}\mathbf{A}\hat{\mathbf{\Pi}}^{\mathrm{T}}$. Denote $\tilde{\mathbf{B}}' = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}}$ as its unitary eigen-decomposition with $\epsilon_2 \le \sum_j |\mathbf{U}_{ij}| \le \epsilon_1$ for all i. Define $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the eigenvalues of $\tilde{\mathbf{B}}'$ with $\sigma = \max_i |\lambda_i|$ and $\delta \le |\lambda_i - \lambda_j|$ for all i, j. Assume that there exists a matrix $\mathbf{R}$ that satisfies $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}' + \mathbf{R}$. We denote $\mathbf{E} = \mathbf{U}\mathbf{R}\mathbf{U}^{\mathrm{T}}$ with $\|\mathbf{E}\|_{\mathrm{F}} = \xi$ and $\mathbf{M} = \mathbf{m}^{\mathrm{T}}\mathbf{m}$ with $\|\mathbf{M}\|_{\mathrm{F}} = M$. If $(\sigma^2+1)\xi^2 + \mu^2 M^2 \le \left[\frac{\delta^2}{(2\sqrt{n}+1)(1+\sqrt{n}\epsilon_1/\epsilon_2)(1+2\epsilon_1/\epsilon_2)}\right]^2$, then $\mathbf{\Pi^p} = \mathbf{\Pi}^*$.*

Proof. The proof is divided into three steps: (i) First, similar to the argument in [26], by constructing the Lagrangian function of $\mathbf{P3'}$ and setting its gradient to 0, we obtain the necessary conditions that the optimal fractional solution $\mathbf{\Pi}^f$ to $\mathbf{P3'}$ must satisfy; (ii) Then, combining these with the conditions stated in the theorem and the projection from $\mathbf{\Pi}^f$ to $\mathbf{\Pi}^p$, we show that $\mathbf{\Pi}^p = \hat{\mathbf{\Pi}}$; (iii) Finally, we prove that in this case $\hat{\mathbf{\Pi}} = \mathbf{\Pi}^*$, which concludes the proof. Due to space limitations, we leave the details of the proof in **Appendix E**

□

Remark: Note that the condition in Theorem V.1 requires the knowledge of the adjacency matrices of G, G_1 and G_2, which may not be obtained in advance as they depend on

[6]A permutation matrix $\mathbf{\Pi}$ observes community assignment if for all $\mathbf{\Pi}_{ij} = 1$, $c(i) = c(j)$.

specific realizations of those graphs. Therefore the ranges, in which the condition can be satisfied, of parameter set $\boldsymbol{\theta}$ can hardly be derived directly without knowing the specific realizations of random graph variables.

However, we are still able to obtain simplified conditions under some special cases. For illustration, let us take the classic ER graph $G(n,p)$ for example. This model enjoys a lot of properties that ensure the mathematical tractability. In an ER graph, when $\tilde{\boldsymbol{B}}' = \tilde{\boldsymbol{B}}$, we have $\boldsymbol{R} = \tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}' = \boldsymbol{0}$ and thus $\boldsymbol{E} = \boldsymbol{URU^T} = \boldsymbol{0}$, $\|\boldsymbol{E}\|_F = \xi = \boldsymbol{0}$. When $M = \|\boldsymbol{M}\|_F = \frac{n^2-n}{2}p$ and $\mu = (1+\epsilon_0)\log\frac{1}{\alpha} = 2\log\frac{1}{p}$, the condition will be converted to

$$(n^2-n)p\log\frac{1}{p} \leq \frac{\delta^2}{(1+2\sqrt{n})(1+\sqrt{n}\frac{\epsilon_1}{\epsilon_2})(1+2\frac{\epsilon_1}{\epsilon_2})} \quad (15)$$

Thus when $\frac{\epsilon_2}{\epsilon_1}\delta = \omega(n^{\frac{3}{2}}\sqrt{p\log\frac{1}{p}})$, the condition will be satisfied. One of the sufficient conditions is that $\frac{\epsilon_2}{\epsilon_1} = \Theta(1)$ and $\delta = \omega(n^{\frac{3}{2}}\sqrt{p\log\frac{1}{p}})$, which can be judged given $\tilde{\boldsymbol{B}}$.

Time complexity: In the first stage of **Algorithm 2**, we use the primal interior point algorithm proposed in [28] to solve the instance of $\mathbf{P3}'$, which has a time complexity of $O(N^3) = O(n^6)$ where $N = n^2$ is the number of variables in the instance. The projection process of the second stage can be implemented in $O(n^2)$ time. Thus, the total time complexity of **Algorithm 2** is $O(n^6)$. Note that the result is only the worst case guarantee and the average time complexity of **Algorithm 2** is much lower [28].

VI. De-anonymization with Unilateral Community Information

In this section, we investigate the de-anonymization problem with unilateral community information, i.e., when the adversary only possesses the community assignment function of the published network G_1. Following the path of the bilateral de-anonymization in Sections IV and V, we will give the corresponding results we obtain for the unilateral case. Through comparisons of these results and illustration in our later experiments, we demonstrate that de-anonymization with only unilateral community information is harder than that with bilateral community information, which shows the importance of community assignment as side information.

A. MAP-based Cost Function

We first derive our cost function in the unilateral case. Again, according to the definition of MAP estimation, given the published network G_1, auxiliary network G_2, parameters $\boldsymbol{\theta}$ and the community assignment function c of G_1, the MAP estimate $\hat{\pi}$ of the correct mapping π_0 is defined as:

$$\hat{\pi} = \arg\max_{\pi\in\Pi} Pr(\pi_0 = \pi \mid G_1, G_2, c, \boldsymbol{\theta}), \quad (16)$$

where Π denotes the set of all bijective mappings from V_1 to V_2. Note that in the unilateral case we have no prior knowledge of the community assignment of G_2. Consequently, we can not restrict Π to the set of mappings that observe the community assignment.

Due to the space limit, we omit the processing of the MAP estimator (16) and present the detailed steps in **Appendix F**. After a sequence of manipulations, we arrive at the following equation for calculation of the MAP estimate.

$$\begin{aligned}\hat{\pi} &= \arg\min_{\pi\in\Pi}\left\{\sum_{i<j}^{n} w_{ij}(\mathbb{1}\{(i,j)\notin E_1, (\pi(i),\pi(j))\in E_2)\})\right\}\\ &\triangleq \arg\min_{\pi\in\Pi}\Delta_\pi,\end{aligned}$$

where $w_{ij} = \log\left(\frac{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)}{p_{c(i)c(j)}(1-s_1)(1-s_2)}\right)$. Here we only present the MAP estimate for the case where we know the community assignment c for G_1, and as for the case where we know c for G_2, it can be easily analyzed and solved based on the same approaches shown in the following. Note that different from the bilateral case, the cost function in the unilateral case is equivalent to a single-sided weighted edge disagreement induced by a mapping. This subtle difference has crucial implications to our analysis on the algorithmic aspect of unilateral de-anonymization. Moreover, the cost function above can also serve as an estimator for the bilateral case if we simply ignore the bilateral community information. However, as we will show in subsection VI-C, the feasibility of such cost function will be weaker than the one we adopt in bilateral case.

B. Validity of the Cost Function

Following the same thread of thought, we proceed to justify the MAP estimation used in unilateral de-anonymization. Using similar proof technique, we derive the same result for the cost function in unilateral case as in bilateral one.

Theorem VI.1. *Let $\alpha = \min_{ab} p_{ab}, \beta = \max_{ab} p_{ab}$. Assume that $\alpha, \beta \to 0$, s_1, s_2 do not go to 1 as $n \to \infty$ and $\frac{\log\alpha}{\log\beta} \leq \gamma$. Suppose that*

$$\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1+s_2} \geq \frac{(6+\epsilon)\gamma^2\log n}{n},$$

where ϵ can be any constant larger than 0, then $\hat{\pi} = \pi_0$ holds almost surely as $n \to \infty$.

Proof. The proof is basically identical to the proof of Theorem IV.1. The only difference here is that we redefine X_{ij} as a Bernoulli random variable with mean $p_{ij}s_1(1-p_{\pi(i)\pi(j)}s_2)$ and Y_{ij} as a Bernoulli random variable with mean $p_{ij}s_1(1-s_2)$. Then, by using the same bounding technique for $Pr\{X_\pi - Y_\pi \leq 0\}$, we conclude the same result for the cost function in unilateral case. □

Theorems IV.1 and VI.1 show that the cost function based on MAP estimation is equally effective in de-anonymization with bilateral and unilateral community information. However, as we will show in the sequel, the feasibility of the cost function in unilateral case is weaker than in bilateral case.

C. Algorithmic Aspect

In this section, we investigate the algorithmic aspect of de-anonymization with unilateral community information and propose corresponding algorithms as in the bilateral case.

Dataset	Degree Distribution	Source	# of Nodes	# of Edges	# of Communities
Synthetic Networks	power law	synthetic	500-2000	≈500-100000	10-40
	Poisson	synthetic	500-2000	≈500-100000	10-40
	exponential	synthetic	500-2000	≈500-100000	10-40
Sampled Social Networks		SNAP [3]	500-2000	≈1000-40000	10-40
Co-authorship Networks		MAG [17]	≈ 2000	≈ 8000	≈ 60

TABLE II: **Summary of datasets in experiments**

1) The Unilateral MAP-ESTIMATE Problem: We first formally introduce the combinatorial optimization problem induced by minimizing the cost function in unilateral de-anonymization.

Definition VI.1. *(***The UNI-MAP-ESTIMATE Problem***)* *Given two graphs $G_1(V, E_1)$ and $G_2(V, E_2)$, community assignment function c of G_1 and weights $\{w\}$, the goal is to compute a mapping $\hat{\pi} : V_1 \mapsto V_2$ that satisfies*

$$\hat{\pi} = \arg\min_{\pi\in\Pi} \left\{ \sum_{i<j}^{n} w_{ij}(\mathbb{1}\{(i,j) \notin E_1, (\pi(i),\pi(j)) \in E_2)\} \right\} \triangleq \arg\min_{\pi\in\Pi} \Delta_\pi,$$

where $\Pi = \{\pi : V_1 \mapsto V_2\}$.

Similar to the bilateral de-anonymization, we require the weights $\{w\}$ to be induced by well-defined community affinity values $\{p\}$, s_1 and s_2, though the latter ones are not explicitly given. Due to the asymmetry of Δ_π in unilateral de-anonymization, intuitively, the UNI-MAP-ESTIMATE problem may bear higher approximation hardness than the BI-MAP-ESTIMATE problem in bilateral de-anonymization. The proposition we present below consolidates this intuition.

Proposition VI.1. *UNI-MAP-ESTIMATE problem is NP-hard. Moreover, there is no polynomial time (pseudo polynomial time) approximation algorithm for UNI-MAP-ESTIMATE with any multiplicative approximation guarantee unless $P = NP$ ($NP \in DTIME(n^{\text{polylog}n})$).*

Proof. The proof is done by reduction from k-CLIQUE problem. Given a graph $G(V, E)$, the k-CLIQUE problem asks whether there exists a clique of size no smaller than k in G. The main idea of the reduction is that: Given an instance of k-CLIQUE with $G(V, E)$ and k, we set G_1 as G and G_2 as a graph consisting of a clique of size k and $(|V| - k)$ additional nodes. Setting $w_{ij} = 1$ and $c(v) = 1$ for all v in G_1, we have an instance of UNI-MAP-ESTIMATE. Obviously, if the G contains a clique of size no less than k, the value $\Delta_{\hat{\pi}}$ of the optimal mapping $\hat{\pi}$ in UNI-MAP-ESTIMATE will be zero. Therefore, in this case, any algorithm with multiplicative approximation guarantee must find a mapping π with $\Delta_\pi = 0$. Furthermore, if G does not contain a clique of size no smaller than k, then any mapping π must satisfy $\Delta_\pi > 0$. Hence, a polynomial (pseudo-polynomial) time approximation algorithm for BI-MAP-ESTIMATE with multiplicative guarantee implies a polynomial (pseudo-polynomial) time algorithm for k-CLIQUE. Since k-CLIQUE problem is NP-Complete, we justify the approximation hardness of UNI-MAP-ESTIMATE as stated in the proposition. □

Note that the graph isomorphism problem is at least as hard as the problems in P, which implies that the approximation hardness result for UNI-MAP-ESTIMATE is stronger than that for BI-MAP-ESTIMATE.

2) Additive Approximation Algorithm: We design a similar approximation algorithm with an ϵ-additive approximation guarantee as in the bilateral case, by formulating the UNI-MAP-ESTIMATE problem in quadratic assignment fashion as follows

$$\begin{aligned} \text{minimize } & \textstyle\sum_{i,j,k,l} q_{ijkl} x_{ik} x_{jl} && (17)\\ \textbf{s.t. } & \textstyle\sum_i x_{ij} = 1, \quad \forall i \in V_1 && (18)\\ & \textstyle\sum_j x_{ij} = 1, \quad \forall j \in V_2 && (19)\\ & x_{ij} \in \{0,1\} && (20) \end{aligned}$$

with the coefficients $\{q\}_{ijkl}$ of the formulation defined as:

$$q_{ijkl} = \begin{cases} \frac{w_{ij}}{\log(1/\alpha)}, & \text{if } (i,j) \notin E_1, (k,l) \in E_2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that due to the absence of community assignment constraints, we can directly formulate the problem as a minimization one and omit the penalty factor as in bilateral de-anonymization. By invoking the same QA-Rounding procedure on the formulated instance and convert the resulting solution $\{x\}$ to its equivalent mapping π. Using similar analysis technique as in Section V-B1, we have that the algorithm obtains solutions that have a gap of at most ϵn^2 to the optimal ones in time $O(n^{O(\log n/\epsilon^2)} + n^2)$.

3) Convex Optimization Based Heuristic: We now proceed to present the heursitc based on convex optimization for the UNI-MAP-ESTIMATE problem, which relies on the following matrix formulation.

$$\begin{aligned} \text{mininize } & \|\mathbf{W} \circ (\mathbf{\Pi A} - \mathbf{B\Pi})\|^2_{\lfloor \mathrm{F} \rfloor} \\ \textbf{s.t. } & \forall i \in V_1,\ \textstyle\sum_i \mathbf{\Pi}_{ij} = 1 && (21)\\ & \forall j \in V_2,\ \textstyle\sum_j \mathbf{\Pi}_{ij} = 1 && (22)\\ & \forall i, j,\ \mathbf{\Pi}_{ij} \in \{0,1\}, && (23) \end{aligned}$$

where $\mathbf{W}$ and $\circ$ share the same definitions as those in **P3** and $\|\cdot\|_{\lfloor \mathrm{F} \rfloor}$ is defined to be a variant of Frobenius norm. Specifically, $\|\mathbf{M}\|_{\lfloor \mathrm{F} \rfloor} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbb{1}\{\mathbf{M}_{ij} \le 0\}\mathbf{M}_{ij}^2)}$ for a matrix $\mathbf{M}$, where only negative elements contribute to the value of the norm. By relaxing the integral constraint (23), we again arrive at an optimization problem, which is shown to be convex as follows:

The relaxed formulation is presented as follows:

$$\text{mininize } \|\mathbf{W} \circ (\mathbf{\Pi A} - \mathbf{B \Pi})\|_{\lfloor \mathrm{F} \rfloor}^2$$

$$\textbf{s.t. } \forall i, \ \sum_i \mathbf{\Pi}_{ij} = 1 \tag{24}$$

$$\forall j, \ \sum_j \mathbf{\Pi}_{ij} = 1 \tag{25}$$

Obviously, the set of feasible solutions defined by Constraints (24) and (25) is a convex set. Then, for the objective function $\|\mathbf{W} \circ (\mathbf{\Pi A} - \mathbf{B \Pi})\|_{\lfloor \mathrm{F} \rfloor}^2$, according to the definition of operator $\|\cdot\|_{\lfloor \mathrm{F} \rfloor}$ it can be interpreted as weighted summation of truncated quadratic functions of each element of Π with the weights being positive real numbers. Each truncated function is equivalent to the square of a linear function of an element of Π with the part where the elements take positive values truncated. Therefore, each truncated function is convex. It follows that the whole objective function, being a weighted combination of convex functions, is convex. Thus, we conclude that the relaxed UNI-MAP-ESTIMATE is a convex optimization problem, the global optima of which can be found in $O(n^6)$ time using the same algorithm as in the bilateral case.

Our second algorithm for unilateral de-anonymization is to first solve the relaxed version of the matrix formulation of UNI-MAP-ESTIMATE and then project the fractional solution to an integral one. Unfortunately, due to the asymmetry of the operator $\|\cdot\|_{\lfloor \mathrm{F} \rfloor}$, it is difficult to derive closed form expression for the gradient of the Lagrangian function of UNI-MAP-ESTIMATE. Thus, we cannot prove conditional optimality of the heuristic as we did in BI-MAP-ESTIMATE.

In **Appendix G** we provide a summary of the differences between bilateral and unilateral de-anonymizations from a higher level, along with an illustration of the necessity of studying the unilateral case by disclosing the gap between community detection and node mapping.

4) Discussion on The Time Complexity of Algorithms in Both Bilateral and Unilateral Cases: One might think that the time complexities of the approximate algorithms that we propose in both cases still turn out to be relatively high. However, as noted earlier,the de-anonymization problem that we consider belongs to the seedless one, which means that there is no availability of sufficient prior knowledge of matched nodes to the adversary. The seedless problem, due to its lack of prior known information, makes the efficient algorithm design challenging. To the best of our knowledge, all the existing algorithms proposed for seedless de-anonymization inevitably encounter this bottleneck. As as result, the algorithm design in state-of-art aims for efficient deanonymization are mainly from an heuristic manner. For illustration, let us introduce several references. For instance, the time complexity of a single iteration of RoleSim++ in [31] is $O(n^2d^2)$, where d is the average degree of the nodes. Although the α-RoleSim++ they further explore slightly reduce the complexity, it is an heuristic algorithm, thus with no rationale behind; The SPECTRL proposed in [32] is also based on heuristic methods with no theoretical guarantee. While the complexities of those heuristic algorithms are relatively low, the limit of the scale that they can handle stays on thousands of nodes rather than millions of nodes. Besides, seedless de-anonymization can be intrinsically treated as a graph matching problem, and the state-of-art graph matching algorithms are also proposed with no theoretical guarantee. For example, in [33], the complexity of the LP approach reaches $O(n^7)$, and in PATH algorithm, a single iteration will be done in $O(n^3)$. The complexity of each iteration in FGM algorithm proposed in [34] is also $O(n^3)$. Those algorithms are mainly applied to matching images, which normally contains only hundreds of pixels and thus returns fairly satisfactory performance.

In summary, although the complexity of algorithms may decrease by adopting some heuristic methods, the performance guarantee remains unknown and thus can hardly be further improved. Thus, designing low-complexity algorithms for seedless de-anonymization is still an open problem and needs our further exploration.

VII. Experiments

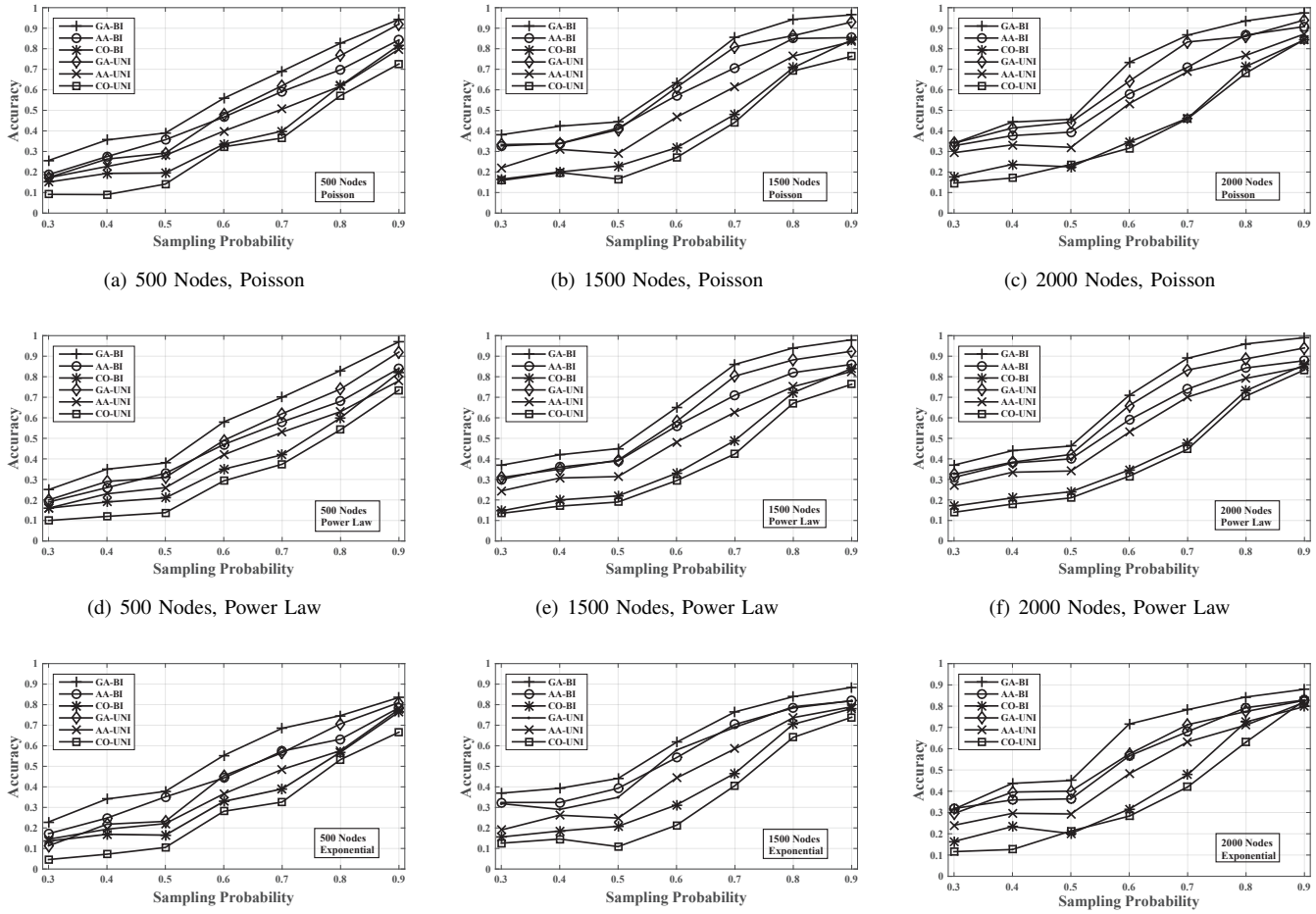
In this section, we present our experimental validation of our theoretical results and the performances of the proposed algorithms. We first introduce our experimental settings and provide detailed results subsequently.

A. Experimental Settings

1) Experiment Datasets: Recall that the two key assumptions made in the modeling are that the underlying social network is generated by the stochastic block model and that the published and the auxiliary networks are sampled from the underlying network. To validate our theoretical findings and meanwhile evaluate the proposed algorithms in real contexts, we conduct experiments on three different types of data sets, with each one closer to the practical situations than the last one by gradually relaxing the assumptions.

(i) **Synthetic Dataset:** Following the stochastic block model, we generate three sets of networks with Poisson, power law and exponential expected degree distributions respectively by properly assigning the community affinity values $\{p\}$. Particularly, as for generating the power law degree distribution, we regard each single node as a community and then the edge existence probability between two nodes can be regarded as the probability between two communities. In this case, the edge existence probability between any two nodes can be set independently and differently, and thus we can generate the power-law degree distribution. As for the Poisson and exponential expected degree distributions, the size of each community is determined by adding a slight variation to the average community size, which equals to the number of nodes divided by the number of communities. For each set of networks, we take the sampling probabilities of the published and the auxiliary networks as $s_1 = s_2$ ranging from 0.3 to 0.9. As this dataset strictly observes the assumptions of our models, it provides direct validations to our theoretical results.

(ii) **Sampled Social Networks:** The underlying social networks are extracted from LiveJournal online social network [29], with the communities following from the ground-truth communities in LiveJournal and the affinity values assigned to be proportional to the ratio of the edges between the communities over the number nodes in the communities. The



(a) 500 Nodes, Poisson (b) 1500 Nodes, Poisson (c) 2000 Nodes, Poisson

(d) 500 Nodes, Power Law (e) 1500 Nodes, Power Law (f) 2000 Nodes, Power Law

(g) 500 Nodes, Exponential (h) 1500 Nodes, Exponential (i) 2000 Nodes, Exponential

Fig. 3: **The accuracy of the algorithms on synthetic datasets with different degree distributions.**

published and the auxiliary networks are sampled from the underlying networks, again, with the sampling probabilities $s_1 = s_2$ ranging from 0.3 to 0.9. This "semi-artificial" dataset lies in the middle of synthetic datasets and true cross-domain networks, which enables us to measure the robustness of our theoretical results against the restrictions imposed on the underlying social network.

(iii) **Co-authorship Networks:** We extract four co-authorship networks in different areas from Microsoft Academic Graph (MAG) [17]. From those, we construct a group of networks with equivalent sets of nodes (2053 nodes in each set) and set up the correspondence of nodes as ground-truth based on the unique identifiers of authors in MAG. The communities are assigned based on the institution information of the authors (the affinity values in this case are assigned as in Sampled Social Networks). The four networks are then combined into six pairs, in which one is set as the published network and the other as the auxiliary network. Without relying on any artificial assumptions of generating the published and auxiliary networks, these procedures enable us to construct most genuine scenarios of de-anonymization from cross-domain social networks, which renders the dataset a touchstone for the applicability of our proposed algorithms.

Note that our empirical results in the first two datasets are respectively obtained by taking the average from 50 repetitive experiments. The statistics of the datasets are summarized in Table II.

2) Algorithms Involved in Comparisons: For both bilateral and unilateral de-anonymization, we run genetic algorithm (**GA-BI,GA-UNI**) in hope of finding exact minimizer of our cost functions, i.e., the optimal solution of BI-MAP-ESTIMATE and UNI-MAP-ESTIMATE problems. The reason why we want to use GA as the optimal solution of BI-MAPESTIMATE and UNI-MAP-ESTIMATE problems is that GA costs more time but can return a better solution. In fact, the performance of the GA algorithm is very unstable, sometimes the time cost of finding the optimal solution is huge, but we can use it as an exact minimizer of our cost functions in the situations where the time is not the concern. The description of how GA works can be referred to **Appendix H**. In both de-anonymization cases, we also evaluate the performance of our two proposed algorithms: the additive approximation algorithm (**AA-BI,AA-UNI**) and the convex optimization-based heuristic (**CO-BI,CO-UNI**).

3) Performance Metrics: The two performance metrics we calculate in the experiments are the **accuracy** of the mappings

yielded by the algorithms and the values of the cost function Δ_π of the mappings. The accuracy of a mapping π is defined as the portion of the nodes that π maps correctly (as the ground-truth correct mapping) over the total number of nodes. Since we are not interested in the absolute values of the cost function of the mapping, we calculate the **relative value** with respect to the cost function of the mappings produced by **GA**, i.e., for a mapping π and the mapping π_{GA} produced by **GA**, π's relative value is computed as $(\Delta_\pi - \Delta_{\pi_{GA}})/\Delta_{\pi_{GA}}$. Since after a huge amount of iterations, $\Delta_{\pi_{GA}}$ is the closest to the minimum value the cost function can achieve. Thus $(\Delta_\pi - \min \Delta_\pi)/\min \Delta_\pi$ can be roughly measured by the relative value. Therefore, the computation of the relative values are supposed to show the gaps between values of other mappings and the minimum cost. Due to space limitations, we defer all the graphical representations of results on the mappings' cost function to **Appendix I**.

B. Experiment Results

In our experiments, we mainly discuss the relationship between the accuracy of the algorithms and the sampling probability s. We also change the network size n and other network parameters to see their influences independently. Since we proved the conditions of our cost function are mild, almost all parameters in the experiment are within the conditions.

1) Synthetic Networks: We plot the performance of the aforementioned algorithms on synthetic networks with $\{500, 1500, 2000\}$ number of nodes in Figures 3 and 8, based on which we have the following observations: (i) Both **GA-BI** and **GA-UNI** exhibit good performance, achieving a de-anonymization accuracy close to 1 when the sampling probability is large in networks with Poisson and power law degree distribution; (ii) The relative value of the correct mapping (**TRUE-BI,TRUE-UNI**) is fairly small. Hence, we conclude that, when the sampling probability is large, the cost function based on MAP estimation is an effective metric in both bilateral and unilateral de-anonymization, and is applicable to a wide range of degree distribution, which justify our theoretical results on the validity of the MAP estimate. However, when the sampling probability is small (e.g. $s = 0.3, 0.4$) or the expected degree distribution has large variation (exponential distribution), the accuracy of **GA** degrades substantially, only achieving a value of less than 0.4. This can be attributed to the fact that when the sampling probability becomes small, the published and the auxiliary networks have lower degree of structural similarity and the parameters deviate from the conditions in our theoretical results.

In terms of the two heuristics we propose, we can see that they obtain good performance with respect to both approximately minimizing the cost function and unraveling the correct mapping, with **AA** superior than **CO** especially in low-sampling-probability area. Note that although the relative value of the two heuristics is large in high-sampling-probability area, this does not imply the poor performance of the heuristics but is mainly due to the optimal Δ_π becoming considerably small as the similarity of G_1 and G_2 grows high.

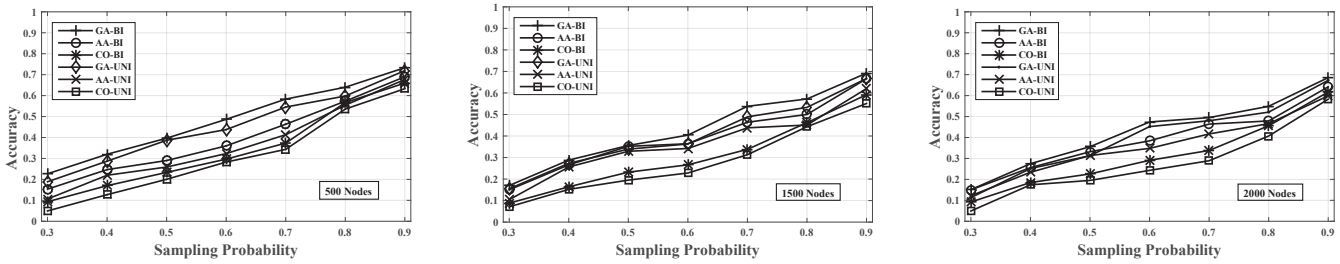
2) Sampled Social Networks: Figures 4 and 9 plot our empirical results on the second datasets where the published and auxiliary networks are sampled from real social networks with the number of nodes set as $\{500, 1500, 2000\}$.

Although in this case the underlying social networks do not follow the stochastic block model, we simply assume the communities in the dataset do not overlap and run our algorithms. As demonstrated by Figures 4 and 9, through minimizing the cost function we can still reveal a large proportion (up to 80%) of the correct mapping, which demonstrates the robustness of the cost function we proposed. Furthermore, the two heuristics **AA** and **CO** still achieve reasonable accuracy of up to 0.7, which is not surprising due to that the cost function they seek to minimize is still effective in this case. However, a little defect is that the accuracy of **AA** can be higher than **GA** at some points. This reflects that the deviation of the real life social networks from the stochastic block model more or less influences the quality of the MAP estimate.

3) Cross-domain Co-authorship Networks: As stated in experimental setup, we extract four groups of cross-domain co-authorship networks named as Networks A, B, C, D and thus construct six scenarios for social network de-anonymization[7]. We evaluate the performance of the algorithms on the six scenarios and show the results in Figures 5 and 10. The figures convey to us several observations and implications: (i) the proposed cost functions still serve as meaningful media for recovering the correct mapping even in realistic scenarios as the relative value of the correct mapping is close to zero, and **GA** achieves an average accuracy of 67.3% in bilateral case and 59.0% in unilateral case; (ii) The two proposed heuristics still enjoy reasonable accuracy, with **AA** successfully de-anonymizing 60.8% of nodes in bilateral case and 51.5% of nodes in unilateral case, and **CO** successfully de-anonymizing 44.4% of nodes in bilateral case and 35.9% of nodes in unilateral case. Therefore, the two heuristics can be qualified as effective methods for seedless social network de-anonymization, which implies that the privacy of current anonymized networks still suffers from attacks of adversaries even when pre-mapped seeds are unavailable; (iii) The performance of **CO** is most susceptible to the structure of networks among all three algorithms as the standard deviation of its accuracy on the six scenarios are above 3.5% (3.51% for **CO-BI**, 3.81% for **CO-UNI**) while the counterparts of the other two algorithms are below 3.0%.

4) Significance of Community Information: A notable phenomenon from all the experiments is that the accuracy of the algorithms in bilateral de-anonymization is higher than that in unilateral de-anonymization, especially for **AA** and **CO**. According to the experimental results, the gap is at least 3.5% in each setting and can reach up to 15% in the worst case. This, from an empirical point of view, demonstrates the importance of the community information on social network de-anonymization.

[7]We do not distinguish the interchange of the published and auxiliary networks as different scenarios.



(a) 500 Nodes (b) 1500 Nodes (c) 2000 Nodes

Fig. 4: **The accuracy of the algorithms on Sampled Social Networks**

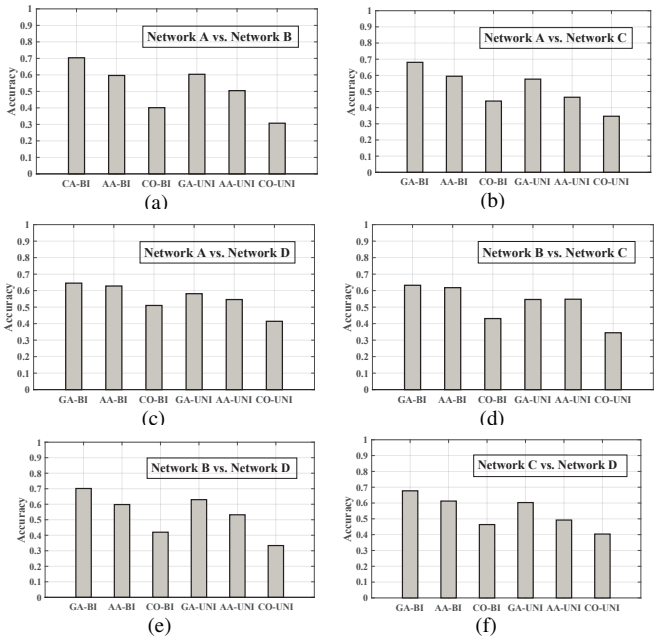


Fig. 5: **The accuracy of the algorithms on Cross-domain Co-authorship Networks**

VIII. Conclusion

In this paper, we have presented a comprehensive study of the community-structured social network de-anonymization problem. Integrating the clustering effect of underlying social network in our models, we have derived a well-justified cost function based on MAP estimation. To further consolidate the validity of such cost function, we have shown that under certain mild conditions, the minimizer of the cost function indeed coincides with the correct mapping. Subsequently, we have investigated the feasibility of the cost function algorithmically by first proving the approximation hardness of the optimization problem induced by the cost function and then proposing two heuristics with their respective performance guarantee by resolving the interweaving of cost function, network topology and candidate mappings through relaxation techniques. All our theoretical findings have been empirically validated through both synthetic and real datasets, with a notable dataset being a set of rare true cross-domain networks that reconstruct a genuine context of social network de-anonymization.

As for the future work, it would be an interesting topic to discuss the de-anonymization problem when the communities are overlapping with each other, which is a model that is closer to reality than SBM.

References

[1] L. Fu , X. Fu , Z. Hu and X.Wang, "De-anonymization of Social Networks with Communities: When Quantifications Meet Algorithms", in *IEEE Globecom*, 2017.

[2] L. Fu, X. Fu, Z. Hu, Z. Xu and X. Wang, "De-anonymization of Social Networks with Communities: When Quantifications Meet Algorithms", arXiv preprint arXiv:1703.09028, 2017.

[3] J. Leskovec and A. Krevl, "SNAP Datasets: Stanford Large Network Dataset Collection", http://snap.stanford.edu/data, 2014.

[4] E. Bakshy, D. Eckles, R. Yan and I. Rosenn, "Social influence in social advertising: evidence from field experiments", in *Proc. ACM EC*, pp. 146-161, 2012.

[5] W. Wang, L. Ying and J. Zhang, "The value of privacy: strategic data subjects, incentive mechanisms and fundamental limits", in *Proc. ACM SIGMETRICS*, pp. 249-260, 2016.

[6] A. Narayanan and V. Shmatikov, "De-anonymizing social networks", in *IEEE Symposium on Security and Privacy*, pp. 173-187, 2009.

[7] P. Pedarsani and M. Grossglauser, "On the privacy of anonymized networks", in *Proc. ACM SIGKDD*, pp. 1235-1243, 2011.

[8] E. Kazemi, L. Yartseva and M. Grossglauser, "When can two unlabeled networks be aligned under partial overlap?", in *IEEE 53rd Annual Allerton Conference on Communication, Control, and Computing*, pp. 33-42, 2015.

[9] D. Cullina and N. Kiyavash, "Improved achievability and converse bounds for Erdős-Rényi graph matching", in *Proc. ACM SIGMETRICS*, pp. 63-72, 2016.

[10] P. Erdős and A. Rényi, "On random graphs", in *Publicationes Mathematicae*, pp. 290-297, 1959.

[11] F. Chung and L. Lu, "The average distance in a random graph with given expected degrees", in *Internet Mathematics*, Vol. 1, No. 1, pp. 91-113, 2003.

[12] S. Ji, W. Li, M. Srivatsa and R. Beyah, "Structural data de-anonymization: Quantification, practice, and implications", in *Proc. ACM CCS*, pp. 1040-1053, 2014.

[13] E. Onaran, G. Siddharth and E. Erkip, "Optimal de-anonymization in random graphs with community structure", arXiv preprint arXiv:1602.01409, 2016.

[14] S. Ji, W. Li, N. Z. Gong, P. Mittal and R. Beyah, "On your social network de-anonymizablity: Quantification and large scale evaluation with seed knowledge", in *NDSS*, 2015.

[15] A. Decelle, F. Krzakala, C. Moore and L. Zdeborov, "Asymptotic analysis of the stochastic block model for modular networks and its algorithmic applications" in *Physical Review E*, No. 84, Vol. 6, pp. 066106, 2011.

[16] M. Newman, "Networks: an introduction", in *Oxford university press*, 2010.

[17] https://www.microsoft.com/en-us/research/project/microsoft-academic-graph/

[18] L. Yartseva and M. Grossglauser, "On the performance of percolation graph matching", in *Proc. ACM COSN*, pp. 119-130, 2013.

[19] E. Kazemi, S. H. Hassani and M. Grossglauser, "Growing a graph matching from a handful of seeds", in *Proc. the VLDB Endowment*, pp. 1010-1021, 2015.

[20] C. F. Chiasserini, M. Garetto and E. Leonardi, "Social network de-anonymization under scale-free user relations", in *IEEE/ACM Trans. on Networking*, Vol. 24, No. 6, pp. 3756-3769, 2016.

[21] N. Korula and S. Lattanzi, "An efficient reconciliation algorithm for social networks", in *Proc. the VLDB Endowment*, pp. 377-388, 2014.

[22] C. F. Chiasserini, M. Garetto and E. Leonardi, "Impact of clustering on the performance of network de-anonymization", in *Proc. ACM COSN*, pp. 83-94, 2015.

[23] M. Girvan and M. Newman, "Community structure in social and biological networks", in *Proc. the National Academy of Sciences*, Vol. 99, No. 12, pp. 7821-7826, 2002.

[24] G. Fanti, P. Kairouz, S. Oh, K. Ramchandran and P. Viswanath, "Rumor Source Obfuscation on Irregular Trees", in *Proc. ACM SIGMETRICS*, pp. 153-164, 2016.

[25] S. Arora, A. Frieze and H. Kaplan, "A new rounding procedure for the assignment problem with applications to dense graph arrangement problems", in *Mathematical programming*, Vol. 92, No. 1, pp. 1-36, 2012.

[26] Y. Aflalo, B. Alexander and R. Kimmel, "On convex relaxation of graph isomorphism", in *Proc. the National Academy of Sciences*, Vol. 112, No. 10, pp. 2942-2947, 2015.

[27] J. R. Bunch, "The weak and strong stability of algorithms in numerical linear algebra", in *Linear Algebra and Its Applications* Nol. 88, pp. 49-66, 1987.

[28] D. Goldfarb and S. Liu, "An $O(n^3L)$ primal interior point algorithm for convex quadratic programming", in *Mathematical Programming*, No. 49, Vol. 1, pp. 325-340, 1990.

[29] J. Yang and J. Leskovec, "Defining and evaluating network communities based on ground-truth", in *Knowledge and Information Systems*, No. 42, Vol. 1, pp. 181-213, 2015.

[30] P. Raghavan, "Probabilistic construction of deterministic algorithms: Approximating packing integer programs", in *Journal of Computer and System Sciences*, No. 37, Vol. 2, pp. 130-143, 1988.

[31] Y. Shao, J. Liu, S. Shi, Y. Zhang and B. Cui, "Fast De-anonymization of Social Networks with Structural Information", in *Data Science and Engineering*, pp. 1-17, 2019.

[32] M. Hayhoe, F. Barreras, H. Hassani and V. M. Preciado, "SPECTRE: Seedless Network Alignment via Spectral Centralities", arXiv preprint arXiv:1811.01056, 2018.

[33] M. Zaslavskiy, F. Bach and J. P. Vert, "A path following algorithm for the graph matching problem", in *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 37, No. 12, pp. 2227-2242, 2008.

[34] F. Zhou and F. De la Torre, "Factorized graph matching", in *IEEE Conference on Computer Vision and Pattern Recognition*, pp. 127-134, 2012

[35] D. Cullina, K. Singhal, N. Kiyavash and P. Mittal, "On the simultaneous preservation of privacy and community structure in anonymized networks", arXiv preprint arXiv:1603.08028, 2016.

APPENDIX A
REASONS FOR FAILURES OF COMMUNITY DETECTION ALGOITHMS

If the attackers know the community assignment function of the auxiliary network, although they may get the community information of the published anonymized network by some community detection algorithms, the detected community information cannot be used in the bilateral case and the reason is twofolds:

- Although several community detection algorithms can recover the communities with probability $1-o(1)$, there may still be $o(n)$ nodes predicted to be in the wrong communites. For illustration, let us consider a certain community a: it can be detected as a_1 in G_1, the published network, and the number of nodes it contains may not be the same as that of a contains. Thus, it violates that for any node i, i and $\pi(i)$ are in the same community, which, however, needs to be guaranteed in our first settings. Moreover, we even cannot judge whether a and a_1 are the same communities.
- When the number of communities grows with n, it is possible to have a large number of isolated communities, i.e. the communities that only contain totally isolated nodes. For such communities, we cannot judge the difference between them, and thus cannot match the communities in G_1 and G_2.

If the attackers do not know the community assignment functions of the auxiliary or the published network but can get them by some community detection algorithms, this detected community information still cannot be used in the bilateral case. Besides the reason mentioned in the case above, there is anouther reason that even if the algorithms can recover the communities with probability 1, when the sampling parameters of s_1 and s_2 are chosen in certain range, the detected communities a_1 and a_2 may not contain the same set of nodes. For example, consider a graph $G(V,E)$ with two communities a and b, whose size are $k+1$ and k with $k\to\infty$. The edge existence probability in community a is p_{aa}, and the probability between two communities is P_{ab}, where $P_{aa}=\Theta(P_{ab})$. G_1 and G_2 are generated by G with sampling probability s_1, s_2. Suppose that the adversary knows the community information of G_1, which is the same as G. When the adversary detects the communities in G_2, suppose he has known the true community assignment $c: V/\{v\}\mapsto\{a',b'\}$ for all nodes except v, a node belongs to a. In G, let $|E_a|,|E_b|$ be the number of nodes v connects with in a and b respectively. Then $|E_a|=\Theta(kP_{aa}), |E_b|=\Theta(kP_{ab})$ almost surely, and $|E_a|=\Theta(|E_b|)$ since $P_{aa}=\Theta(P_{ab})$. In G_2, let $|E'_a|,|E'_b|$ be the number of nodes v connects with in a' and b' respectively. When $s_2=\frac{1}{2}$, we have

$$\begin{aligned} P_{|E'_a|<\frac{|E_b|}{2}} &= \sum_{t=0}^{\frac{|E_b|-1}{2}} \binom{|E_a|}{t} s_2^t(1-s_2)^{|E_a|-t} \\ &= \frac{1}{2^{|E_a|}} \sum_{t=0}^{\frac{|E_b|-1}{2}} \binom{|E_a|}{t} \\ &= \Theta(1), \end{aligned} \tag{26}$$

$$\begin{aligned} P_{|E'_b|\geq\frac{|E_b|}{2}} &= \sum_{t=\frac{|E_b|}{2}}^{|E_b|} \binom{|E_b|}{t} s_2^t(1-s_2)^{|E_b|-t} \\ &= \frac{1}{2^{|E_b|}} \sum_{t=\frac{|E_b|}{2}}^{|E_b|} \binom{|E_b|}{t} \\ &= \Theta(1). \end{aligned} \tag{27}$$

Therefore, $P_{|E'_a|<|E'_b|}\geq P_{|E'_a|<\frac{|E_b|}{2}} P_{|E'_b|\geq\frac{|E_b|}{2}}=\Theta(1)$, which means that the case where $|E'_a|<|E'_b|$ may happen with relatively high probability. In this case, the adversary may assign v into community b', and then b' will be matched with a in G_1 since $|b'|=k+1=|a|\neq|a'|$. Thus in this case, using bilateral community information will still make the de-anonymization fail.

To sum up, if adversaries do not get the community assignment function of any network, even if it can be detected successfully, the detected community information may still not be useful for the bilateral case and thus proposing the unilateral setting is necessary.

APPENDIX B
PROOF OF THEOREM 4.1

The method we use here is similar to that in [7]. Recall that for a mapping π, we define $\Delta_\pi=\sum_{i\leq j}^n w_{ij}|\mathbb{1}\{(i,j)\in E_1\}-\mathbb{1}\{\pi(i),\pi(j)\in E_2\}|$. Then the proof can be briefly divided into two major steps. The first one is to derive an upper bound for the expectation of the number of (incorrect) mappings π's with $\Delta_\pi\leq\Delta_{\pi_0}$. The second one is to show that the derived upper bound converges to 0 under the conditions stated in the theorem, as $n\to\infty$. Based on that, the proof can be concluded as the number of π's with $\Delta_\pi\leq\Delta_{\pi_0}$ goes to 0, i.e., the correct mapping π_0 is th unique minimizer for Δ_π as $n\to\infty$. Now we turn to the first step as follows:

1. **Derivation of the Upper Bound:** We define Π_k as the set of all the mappings in Π that map k nodes incorrectly. Obviously, $\Pi_0=\{\pi_0\}$. Now we have $|\Pi_k|\leq\binom{n}{k}\left(\frac{k!}{2}\right)\leq n^k$. We subsequently define S_k as a random variable representing the number of incorrect mappings in Π_k whose value of cost function is no larger than Δ_{π_0}. Formally, S_k is given by $S_k=\sum_{\pi\in\Pi_k}\mathbb{1}\{\Delta_\pi\leq\Delta_{\pi_0}\}$. Summing over all k, we denote $S=\sum_{k=2}^n S_k$ as the total number of incorrect mappings that induce no larger cost function than the correct mapping π_0. The mean of S can be calculated as:

$$\begin{aligned} \mathbb{E}[S] &= \sum_{k=2}^n \mathbb{E}[S_k] = \sum_{k=2}^n \sum_{\pi\in\Pi_k} \mathbb{E}[\mathbb{1}\{\Delta_\pi\leq\Delta_{\pi_0}\}] \\ &= \sum_{k=2}^n \sum_{\pi\in\Pi_k} Pr\{\Delta_\pi-\Delta_{\pi_0}\leq 0\} \\ &\leq \sum_{k=2}^n n^k \max_{\pi\in\Pi_k} Pr\{\Delta_\pi-\Delta_{\pi_0}\leq 0\}. \end{aligned} \tag{1}$$

For a mapping π, let V_π be the set of vertices that it maps incorrectly. Then, we define $E_\pi=V_\pi\times V$, i.e., the set of unordered node pairs with one or two vertices mapped incorrectly under π. For a $\pi\in\Pi_k$, we have $|E_\pi|=nk-\frac{k^2}{2}-\frac{k}{2}$. As every node pair in $V\times V-E_\pi$ is mapped identically in π

and π_0, they contribute equally to Δ_{π_0} and Δ_π respectively. Next, we define two random variables for π as

$$X_\pi = \sum_{(i,j)\in E_\pi} w_{ij}|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j)\in E_2\}|,$$

$$Y_\pi = \sum_{(i,j)\in E_\pi} w_{ij}|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(i,j)\in E_2\}|.$$

It is easy to verify that $\Delta_\pi - \Delta_{\pi_0} = X_\pi - Y_\pi$ for all π, where Y_π is the value of cost function contributed by node pairs in E_π under the correct permutation. For a node pair (i,j), the probability that it contributes to Y_π equals to $p_{c(i)c(j)}(s_1+s_2-2s_1s_2)$. Therefore, Y_π is the weighted sum of independent Bernoulli random variables.

For X_π, assume that π has $\phi \geq 0$ transpositions[8], then each transposition induces one invariant node pair in E_π. The remaining node pairs are not invariant under π, i.e., they are mapped incorrectly under π. Each node pair (i,j) contributes w_{ij} to X_π if $(i,j)\in E_1$ and $(\pi(i),\pi(j))\notin E_2$ or vice versa. This happens with probability $p_{c(i)c(j)}s_1(1-p_{c(\pi(i))c(\pi(j))}s_2)+p_{c(\pi(i))c(\pi(j))}s_2(1-p_{c(i)c(j)}s_1)$, and since under bilateral settings, we have $p_{c(\pi(i))c(\pi(j))} = p_{c(i)c(j)}$, thus the probability can be reduced to $p_{c(i)c(j)}(s_1+s_2-2p_{c(i)c(j)}s_1s_2)$. As in [8], we conservatively approximate X_π to a weighted sum of independent random Bernoulli variables. Denote X_{ij} as a Bernoulli random variable with mean $p_{c(i)c(j)}(s_1+s_2-2p_{c(i)c(j)}s_1s_2)$, and denote X'_π as the sum of weighted independent random Bernoulli variables $w_{ij}X_{ij}$, i.e., $X'_\pi = \sum_{(i,j)\in E_\pi} w_{ij}X_{ij}$. The approximation we use is summarized in the following lemma.

Lemma B.1. *For the random variable X_π, it can be approximated as*

$$X_\pi \approx \sum_{(i,j)\in E_\pi\setminus\phi} w_{ij}X_{ij} \triangleq X'_\pi \tag{2}$$

Proof. As for the mean of X_π and X'_π, we have

$$\begin{aligned}\mathbb{E}[X_\pi] &= \mathbb{E}\left[\sum_{(i,j)\in E_\pi\setminus\phi} w_{ij}|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j))\in E_2\}|\right]\\ &= \sum_{(i,j)\in E_\pi\setminus\phi} \mathbb{E}\left[w_{ij}|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j))\in E_2\}|\right]\\ &= \sum_{(i,j)\in E_\pi\setminus\phi} \mathbb{E}\left[w_{ij}X_{ij}\right]\\ &= \mathbb{E}\left[X'_\pi\right]\end{aligned}$$

We denote $p_{c(i)c(j)}(s_1+s_2-2p_{c(i)c(j)}s_1s_2)$ as p_{ij} and denote $|\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j))\in E_2\}|$ as $\mathbb{1}_{ij}$. Then for the variance, we can first derive that

$$\begin{aligned}\mathbb{D}[X'_\pi] &= \mathbb{E}[(X'_\pi)^2] - \mathbb{E}^2[X'_\pi]\\ &= \mathbb{E}\Big[\Big(\sum_{(i,j)\in E_\pi} w_{ij}X_{ij}\Big)^2\Big] - \Big(\sum_{(i,j)\in E_\pi} w_{ij}p_{ij}\Big)^2\\ &= \mathbb{E}\Big[\sum_{(i,j)\in E_\pi} w_{ij}^2X_{ij}^2\Big] + 2\mathbb{E}\Big[\sum_{(i,j),(k,l)\in E_\pi} w_{ij}w_{kl}X_{ij}X_{kl}\Big]\\ &\quad - \Big(\sum_{(i,j)\in E_\pi} w_{ij}p_{ij}\Big)^2\\ &= \sum_{(i,j)\in E_\pi} w_{ij}^2p_{ij} + 2\sum_{(i,j),(k,l)\in E_\pi} w_{ij}w_{kl}p_{ij}p_{kl} - \Big(\sum_{(i,j)\in E_\pi} w_{ij}p_{ij}\Big)^2\\ &= \sum_{(i,j)\in E_\pi} w_{ij}^2p_{ij}(1-p_{ij})\end{aligned}$$

Then we can get that

$$\begin{aligned}\frac{\mathbb{D}[X_\pi]-\mathbb{D}[X'_\pi]}{\mathbb{D}[X'_\pi]} &= \frac{\mathbb{E}[(X'_\pi)^2]-\mathbb{E}^2[X'_\pi] - \Big(\mathbb{E}[(X_\pi)^2]-\mathbb{E}^2[X_\pi]\Big)}{\mathbb{D}[X'_\pi]}\\ &= \frac{\mathbb{E}[(X'_\pi)^2]-\mathbb{E}[(X_\pi)^2]}{\mathbb{D}[X'_\pi]}\\ &= \frac{\mathbb{E}\Big[\sum_{(i,j)\in E_\pi} w_{ij}^2X_{ij}^2\Big] + 2\mathbb{E}\Big[\sum_{(i,j),(k,l)\in E_\pi} w_{ij}w_{kl}X_{ij}X_{kl}\Big]}{\mathbb{D}[X'_\pi]}\\ &\quad - \frac{\mathbb{E}\Big[\sum_{(i,j)\in E_\pi} w_{ij}^2\mathbb{1}_{ij}^2\Big] + 2\mathbb{E}\Big[\sum_{(i,j),(k,l)\in E_\pi} w_{ij}w_{kl}\mathbb{1}_{ij}\mathbb{1}_{kl}\Big]}{\mathbb{D}[X'_\pi]}\\ &= \frac{2\mathbb{E}\Big[\sum_{(i,j),(k,l)\in E_\pi} w_{ij}w_{kl}(X_{ij}X_{kl}-\mathbb{1}_{ij}\mathbb{1}_{kl})\Big]}{\mathbb{D}[X'_\pi]}\\ &\overset{1}{=} \frac{2\mathbb{E}\Big[\sum_{(i,j)\in E_\pi} w_{ij}w_{\pi(i)\pi(j)}(X_{ij}X_{\pi(i)\pi(j)}-\mathbb{1}_{ij}\mathbb{1}_{\pi(i)\pi(j)})\Big]}{\sum_{(i,j)\in E_\pi} w_{ij}^2p_{ij}(1-p_{ij})}\\ &= \frac{2\mathbb{E}\Big[\sum_{(i,j)\in E_\pi} w_{ij}^2(p_{ij}^2-\mathbb{1}_{ij}\mathbb{1}_{\pi(i)\pi(j)})\Big]}{\sum_{(i,j)\in E_\pi} w_{ij}^2p_{ij}(1-p_{ij})}\\ &= O\left(\max_{ij} p_{ij}\right)\\ &= O(\beta)\\ &\to 0.\end{aligned}$$

$\overset{1}{=}$ follows from the fact that when $(k,l)\neq(\pi(i),\pi(j))$, $\mathbb{1}_{ij}$ and $\mathbb{1}_{kl}$ are independent and $\mathbb{E}[\mathbb{1}_{ij}\mathbb{1}_{kl}] = \mathbb{E}[\mathbb{1}_{ij}]\mathbb{E}[\mathbb{1}_{kl}] = \mathbb{E}[X_{ij}X_{kl}] = p_{ij}p_{kl}$. The parameter β in the equation above is defined as $\beta = \max_{ij} p_{c(i)c(j)}$.

Above all, $\mathbb{E}[X_\pi] = \mathbb{E}[X'_\pi]$ and $\frac{\mathbb{D}[X_\pi]-\mathbb{D}[X'_\pi]}{\mathbb{D}[X'_\pi]} \to 0$. Therefore, we can approximate X_π to X'_π. □

Moreover, denote Y_{ij} as a Bernoulli random variable with mean $p_{c(i)c(j)}(s_1+s_2-2s_1s_2)$ as Y_{ij}. We can get that:

$$Y_\pi = \sum_{(i,j)\in E_\pi} w_{ij}Y_{ij} \triangleq Y'_\pi. \tag{3}$$

Based on Equation 2 and 3, we can use the probability of event $\{X_{\pi'}-Y_{\pi'}\}\leq 0$ to approximate the probability of event $\{X_\pi - Y_\pi\}\leq 0$. Denoting λ_X as the expectation of X'_π and λ_Y as the expectation of Y'_π, the bound we use for $Pr\{X_\pi - Y_\pi \leq 0\}$ is summarized in the following lemma.

Lemma B.2. *For all mapping π, random variables X_π and Y_π satisfy that*

$$Pr\{X_\pi - Y_\pi \leq 0\} \leq 2\exp\left(\frac{-(\lambda_X-\lambda_Y)^2}{12(\lambda_X+\lambda_Y)}\right) \tag{4}$$

[8]If a mapping π has a transposition on i,j, it means that $\pi(i)=j$ and $\pi(j)=i$.

Proof. First, we have that for all π

$$Pr\{X_\pi - Y_\pi \le 0\} \approx Pr\{X'_\pi - Y'_\pi \le 0\}$$
$$\le Pr\Big\{Y'_\pi \ge \frac{\lambda_X+\lambda_Y}{2}\Big\} + Pr\Big\{X'_\pi \le \frac{\lambda_X+\lambda_Y}{2}\Big\}$$

Then we invoke Lemma B.3 (Theorems 1 and 2 in [30]), which presents Chernoff-type bounds for weighted sum of independent Bernoulli variables.

Lemma B.3. *(Theorems 1 and 2 in [30]) Let $a_1, a_2, \ldots, a_r$ be positive real numbers and let $X_1, \ldots, X_n$ be independent Bernoulli trials with $\mathbb{E}[X_j] = p_j$. Defining random variable $\Psi = \sum_{j=1}^r a_j X_j$ with $\mathbb{E}[\Psi] = \sum_{j=1}^r a_j p_j = m$, we have*

$$Pr\{\Psi \ge (1+\delta)m\} \le \exp\left(-m\delta^2/3\right),$$
$$Pr\{\Psi \le (1-\delta)m\} \le \exp\left(-m\delta^2/2\right).$$

Using Lemma B.3 by treating X'_π and Y'_π as the weighted (w_{ij}) sum of random variables X_{ij} (Y_{ij}), we obtain that

$$Pr\Big\{Y'_\pi \ge \frac{\lambda_X+\lambda_Y}{2}\Big\} \le \exp\left(-(\lambda_X-\lambda_Y)^2/12(\lambda_X+\lambda_Y)\right),$$
$$Pr\Big\{X'_\pi \le \frac{\lambda_X+\lambda_Y}{2}\Big\} \le \exp\left(-(\lambda_X-\lambda_Y)^2/8(\lambda_X+\lambda_Y)\right).$$

Hence, we have

$$Pr\{X_\pi - Y_\pi \le 0\} \le 2\exp\left(\frac{-(\lambda_X-\lambda_Y)^2}{12(\lambda_X+\lambda_Y)}\right). \quad \square$$

$\square$

We now proceed to derive lower bound for the numerator and upper bound for the denominator in the exponent of the RHS of Inequality (4) to obtain the upper bound of the RHS. By standard calculation, we have

$$\begin{aligned}
&(\lambda_X-\lambda_Y)^2 \\
\ge&\Big(2\sum_{(i,j)\in E_\pi\setminus\phi} w_{ij}p_{c(i)c(j)}(1-p_{c(i)c(j)})s_1s_2 - \frac{k\overline{w}\beta(s_1+s_2-2s_1s_2)}{2}\Big)^2 \\
\ge&\frac{k^2}{4}\left[4\left(n-\frac{k}{2}-1\right)\underline{w}\alpha(1-\beta)s_1s_2 - \overline{w}\beta(s_1+s_2-2s_1s_2)\right]^2,
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_X+\lambda_Y \\
\le&\sum_{(i,j)\in E_\pi}[w_{ij}p_{c(i)c(j)}(s_1+s_2-2s_1s_2) \\
&\quad + w_{ij}p_{c(i)c(j)}(s_1+s_2-2p_{c(i)c(j)}s_1s_2)] \\
\le&2\sum_{(i,j)\in E_\pi} w_{ij}p_{c(i)c(j)}(s_1+s_2) \\
\le&2\left(nk-\frac{k^2}{2}-k\right)\overline{w}\alpha(s_1+s_2).
\end{aligned}$$

Therefore, by Lemma B.2, $Pr\{X_\pi - Y_\pi \le 0\}$ can be upper bounded by

$$\begin{aligned}
&Pr\{X_\pi - Y_\pi \le 0\} \le 2\exp\left[-(\lambda_X-\lambda_Y)^2/12(\lambda_X+\lambda_Y)\right] \\
\le&2\exp\left\{-\frac{k^2\left[4(\frac{2n-k+2}{2})\underline{w}\alpha(1-\beta)s_1s_2 - \overline{w}\beta(s_1+s_2-2s_1s_2)\right]^2}{96(nk-\frac{k^2}{2}-k)\overline{w}\alpha(s_1+s_2)}\right\} \quad (5) \\
\le&\exp\left\{-\frac{k^2\left[(n-\frac{k}{2}-1)\underline{w}\alpha(1-\beta)s_1s_2\right]^2}{6(nk-\frac{k^2}{2}-k)\overline{w}\alpha(s_1+s_2)}\right\}, \quad (6)
\end{aligned}$$

where Inequality (6) follows from the conditions stated in the theorem.

2. **Convergence of the Upper Bound:** Now, we further show that the derived upper bound converges to 0 as $n \to \infty$. Due to the monotonicity of w_{ij} with respect to $p_{c(i)c(j)}$, we easily obtain that $\overline{w} = \log\left(\frac{1-\alpha(s_1+s_2-2s_1s_2)}{\alpha(1-s_1)(1-s_2)}\right)$ and $\underline{w} = \log\left(\frac{1-\beta(s_1+s_2-2s_1s_2)}{\beta(1-s_1)(1-s_2)}\right)$. Hence, $\overline{w}$ and $\underline{w}$ can be determined by α, β, s_1, s_2. Plugging Inequality (6) into Inequality (1), we have

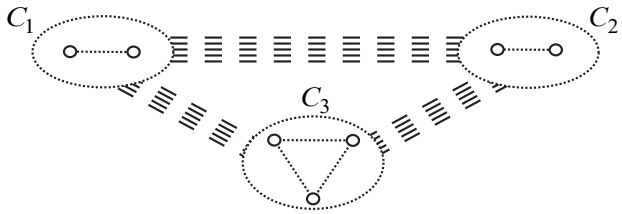


Fig. 6: **An example demonstrating the superiority of our cost function: the sizes of the communities $|C_1|, |C_2|$ equal to some constant C and $|C_3| = n - 2C$, the affinity values $p_{11} = p_{22} = p_{33} = p_{12} = p_{23} = 5\log n/n$, $p_{13} = \log n/\sqrt{n}$, the sampling probabilities $s_1 = s_2 = 2/3$**

$$\begin{aligned}
\mathbb{E}[S] &\le 2\sum_{k=2}^n n^k \cdot \exp\left(-\frac{k^2\left[(n-\frac{k}{2}-1)\underline{w}\alpha(1-\beta)s_1s_2\right]^2}{6(nk-\frac{k^2}{2}-k)\overline{w}\alpha(s_1+s_2)}\right) \\
&\le \sum_{k=2}^\infty \exp\left\{k\left(-\frac{\left[(n-\frac{k}{2}-1)\underline{w}\alpha(1-\beta)s_1s_2\right]^2}{6(n-\frac{k}{2}-1)\overline{w}\alpha(s_1+s_2)} + \log n\right)\right\} \\
&\le \sum_{k=2}^\infty \exp\left\{k\left(-\frac{\left[(n-\frac{k}{2}-1)\underline{w}^2\alpha^2(1-\beta)^2s_1^2s_2^2\right]}{6\overline{w}\alpha(s_1+s_2)} + \log n\right)\right\}
\end{aligned}$$

Since $\alpha, \beta \to 0$, $\frac{\log\alpha}{\log\beta} \le \gamma$, and s_1, s_2 do not go to 1, we have

$$\begin{aligned}
\frac{\overline{w}}{\underline{w}} &= \log\left(\frac{1-\alpha(s_1+s_2-2s_1s_2)}{\alpha(1-s_1)(1-s_2)}\right) \Big/ \log\left(\frac{1-\beta(s_1+s_2-2s_1s_2)}{\beta(1-s_1)(1-s_2)}\right) \\
&= \frac{-\log\alpha - \log(1-s_1)(1-s_2) + \log(1-\Theta(\alpha))}{-\log\beta - \log(1-s_1)(1-s_2) + \log(1-\Theta(\beta))} \\
&= \frac{(1+o(1))\log\alpha}{(1+o(1))\log\beta} \\
&= (1+o(1))\frac{\log\alpha}{\log\beta} \\
&\le (1+o(1))\gamma,
\end{aligned}$$

and $\overline{w} = (1+o(1))\log\frac{1}{\alpha}$. Hence we have

$$\mathbb{E}[S] \le \sum_{k=2}^\infty \exp\left\{k\left(-\frac{(1+o(1))\left[n\alpha s_1^2s_2^2\log\frac{1}{\alpha}\right]}{6\gamma^2(s_1+s_2)} + \log n\right)\right\}.$$

Therefore, if $\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1+s_2} \ge \frac{(6+\epsilon)\gamma^2\log n}{n}$, for a certain constant $\epsilon > 0$, the exponential term of the above summation will approach negative infinity, making the sum of the above geometric series goes to zero as n goes to infinity. Therefore, $\mathbb{E}[S] \to 0$. Hence, with the above conditions in Theorem IV.1 satisfied, the MAP estimate $\hat{\pi}$ coincides with the correct mapping π_0 with probability goes to 1 as n goes to infinity.

APPENDIX C
VALIDITY OF THE COST FUNCTION UNDER OTHER CONDITIONS

In this section, we analyze the validity of the cost function under the cases where s_1, s_2 go to or equal to 1. The conditions under which $\hat{\pi} = \pi_0$ holds, along with the corresponding proof are given as follows. Particularly, Theorem C.1 below considers the case where s_1 and s_2 go to 1 but are not equal

to 1, while Theorem C.2 states the result under the case where s_1 and s_2 are equal to 1.

Theorem C.1. *Let* $\alpha = \min_{ab} p_{ab}, \beta = \max_{ab} p_{ab}$, $\overline{w} = \max_{ij} w_{ij}$, $\underline{w} = \min_{ij} w_{ij}$ *and* $\lambda = \log(1-s_1)(1-s_2)$. *Assume that* $\alpha, \beta \to 0$, s_1, s_2 *go to 1 as* $n \to \infty$ *and* $\frac{\log \alpha}{\log \beta} \leq \gamma$. *Suppose that*

$$\alpha \log \frac{1}{\alpha} \geq \eta_{\gamma,\lambda} \frac{\gamma^2 \log n}{n},$$

where

$$\eta_{\gamma,\lambda} = \begin{cases} 12+\epsilon & \lambda = o(\log \beta) \\ \frac{12(1+c)+\epsilon}{(1+c\gamma)^2} & \gamma = \Theta(1), \lambda = c \log \alpha \\ \frac{12+\epsilon}{(1+c)^2} & \gamma = \omega(1), \lambda = c \log \beta \\ \frac{12(1+c)+\epsilon}{c^2\gamma^2} & \gamma = \omega(1), \lambda = c \log \alpha \\ -\frac{(12+\epsilon)\log(1/\alpha)}{\lambda \gamma^2} & \lambda = \omega(\log \alpha) \end{cases},$$

and $c = \Theta(1)$, ϵ *can be any constant larger than* 0. *Then* $\hat{\pi} = \pi_0$ *holds almost surely as* $n \to \infty$.

Proof. When s_1, s_2 go to 1, following the same derivation as the cases where s_1 and s_2 do not go to 1, we can get the same expression of $\mathbb{E}[S]$ as

$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(-\frac{\left[(n-\frac{k}{2}-1)\underline{w}^2\alpha^2(1-\beta)^2 s_1^2 s_2^2\right]}{6\overline{w}\alpha(s_1+s_2)} + \log n\right)\right\}$$

Now we will prove that $\mathbb{E}[S] \to 0$ by dividing the ranges of λ and γ into the following 5 cases.

- When $\lambda = o(\log \beta)$, we have
$$\frac{\overline{w}}{\underline{w}} = (1+o(1))\frac{\log \alpha}{\log \beta} \leq (1+o(1))\gamma,$$
which is the same as the case where s_1, s_2 do not go to 1. Therefore, if $\alpha \log \frac{1}{\alpha} \geq \frac{(12+\epsilon)\gamma^2 \log n}{n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$.
- When $\gamma = \Theta(1)$ and $\lambda = c \log \alpha$, where $c = \Theta(1)$, we have
$$\frac{\overline{w}}{\underline{w}} = \frac{(1+c+o(1))\log \alpha}{(1+o(1))(c\log \alpha + \log \beta)} = (1+o(1))\frac{(1+c)\log \alpha}{c\log \alpha + \log \beta}$$
$$\leq (1+o(1))\frac{\gamma+\gamma c}{1+\gamma c},$$
and $\overline{w} = (1+o(1))\,(1+c)\log \frac{1}{\alpha}$. Then
$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(-\frac{(1+o(1))\left[n\alpha s_1^2 s_2^2 (1+\gamma c)^2 \log \frac{1}{\alpha}\right]}{6(1+c)\gamma^2(s_1+s_2)} + \log n\right)\right\}.$$
Therefore, if $\alpha \log \frac{1}{\alpha} \geq \frac{(12(1+c)+\epsilon)\gamma^2 \log n}{(1+\gamma c)^2 n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$.
- When $\gamma = \omega(1)$ and $\lambda = c \log \beta$, where $c = \Theta(1)$, we have
$$\frac{\overline{w}}{\underline{w}} = \frac{(1+o(1))\log \alpha}{(1+c+o(1))\log \beta} \leq (1+o(1))\frac{\gamma}{1+c},$$
and $\overline{w} = (1+o(1))\log \frac{1}{\alpha}$. Then
$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(-\frac{(1+o(1))\left[n\alpha s_1^2 s_2^2 (1+c)^2 \log \frac{1}{\alpha}\right]}{6\gamma^2(s_1+s_2)} + \log n\right)\right\}.$$
Therefore, if $\alpha \log \frac{1}{\alpha} \geq \frac{(12+\epsilon)\gamma^2 \log n}{(1+c)^2 n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$.
- When $\gamma = \omega(1)$ and $\lambda = c \log \alpha$, where $c = \Theta(1)$, we have
$$\frac{\overline{w}}{\underline{w}} = \frac{(1+c+o(1))\log \alpha}{(1+o(1))c\log \alpha} \leq (1+o(1))\frac{1+c}{c},$$
and $\overline{w} = (1+o(1))\,(1+c)\log \frac{1}{\alpha}$. Then
$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(-\frac{(1+o(1))\left[n\alpha s_1^2 s_2^2 c^2 \log \frac{1}{\alpha}\right]}{6(1+c)(s_1+s_2)} + \log n\right)\right\}.$$
Therefore, if $\alpha \log \frac{1}{\alpha} \geq \frac{(12(1+c)+\epsilon)\log n}{c^2 n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$.
- When $\lambda = \omega(\log \alpha)$, we have $\frac{\overline{w}}{\underline{w}} = 1 + o(1)$, and $\overline{w} = (1+o(1))\,\lambda$. Then
$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(\frac{(1+o(1))n\alpha s_1^2 s_2^2 \lambda}{6(s_1+s_2)} + \log n\right)\right\}.$$
Therefore, if $-\alpha\lambda \geq \frac{(12+\epsilon)\log n}{n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$.

□

To compare with Theorem IV.1, the condition above can be rewrite as $\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1+s_2} \geq \frac{\eta_{\gamma,\lambda}}{2} \cdot \frac{\gamma^2 \log n}{n}$. Note that $1 \leq \frac{\log \alpha}{\log \beta} \leq \gamma$ and $c > 0$, we can get that $\frac{\eta_{\gamma,\lambda}}{2} \leq 6 + \epsilon$. Thus when s_1, s_2 go to 1, the condition that ensures $\hat{\pi} = \pi_0$ is looser. Besides, as for the case where s_1, s_2 go to 1, when $|\lambda|$ increases, i.e., s_1, s_2 are closer to 1, $\eta_{\gamma,\lambda}$ decreases, which means that the condition also becomes looser.

As for the case where s_1, s_2 are equal to 1, we present Theorem C.2 as follows.

Theorem C.2. *Let* $\alpha = \min_{ab} p_{ab}, \beta = \max_{ab} p_{ab}$. *Assume that* $\alpha, \beta \to 0$ *as* $n \to \infty$, $s_1, s_2 = 1$ *and* $\frac{\log \alpha}{\log \beta} \leq \gamma$. *Suppose that*

$$\alpha \log \frac{1}{\alpha} = \frac{12\gamma^2 \log n + \omega(1)}{n},$$

where ϵ *can be any constant larger than* 0, *then* $\hat{\pi} = \pi_0$ *holds almost surely as* $n \to \infty$.

Proof. With the similar derivations as the first part of Appendix D, we can get the weight $w_{i,j} = \log \frac{1}{p_{c(i)c(j)}}$. Then we can easily get that

$$\mathbb{E}[S] \leq \sum_{k=2}^{\infty} \exp\left\{k\left(-\frac{(n-\frac{k}{2}-1)\alpha \log \frac{1}{\alpha}(1-\beta)^2}{12\gamma^2} + \log n\right)\right\}.$$

Therefore, if $\alpha \log \frac{1}{\alpha} = \frac{12\gamma^2 \log n + \omega(1)}{n}$, for a certain constant $\epsilon > 0$, $\mathbb{E}[S] \to 0$. □

The condition above can be rewriten as $\frac{\alpha s_1^2 s_2^2 \log(1/\alpha)}{s_1+s_2} = \frac{6\gamma^2 \log n + \omega(1)}{n}$. Comparing Theorem C.2 and Theorem IV.1, we can get that the condition in Theorem C.2 is looser.

Appendix D
Superiority of Our Cost Function

In this section, we compare our cost functions over previous works proposed in the literature. Specifically, we demonstrate the superiority of our cost function in bilateral case over similar previous cost functions proposed by Pedarsani et al. [7] and Onaran et al. [13]. Moreover, we also show that our validity condition is an improved version compared with the bound in [9]

1. Comparison with cost function in [7]

As for the cost function derived in [7], which we denoted as Δ'_π, it can be written as:

$$\Delta'_\pi = \sum_{i\le j}^{n} |\mathbb{1}\{(i,j)\in E_1\} - \mathbb{1}\{(\pi(i),\pi(j))\in E_2\}|.$$

The advantages of our cost function is two-fold. First, Δ'_π, as an unweighted version of our proposed Δ_π, corresponds to the MAP estimator in bilateral de-anonymization when the underlying social network is an Erdős-Rényi graph. Therefore, our cost function in a sense, subsumes the cost function in [7] as a special case in bilateral de-anonymization, and has more generality when the underlying network is non-uniform or the adversary only possesses unilateral community information. Second, we show that in certain cases, the correct mapping π_0 is the unique minimizer of Δ_π, while it is not the unique minimizer of Δ'_π. Indeed, when the underlying social network is as shown in Figure 6, and the sampling probabilities $s_1 = s_2 \le \frac{\gamma'}{2}$, with γ' defined as in the proof of Theorem IV.1, we have that the unique minimizer of Δ_π asymptotically almost surely coincides with π_0 by Theorem IV.1. However, as Δ'_π does not count the weight of node pairs, in each realization of G_1 and G_2, there exists a mapping π' that permutes $\pi_0 = \arg\min_{\pi\in\Pi}\Delta_\pi$ on some nodes in C_3 with $\Delta'_{\pi'} \le \Delta_{\pi_0}$. Therefore, in this case, the minimizer of Δ'_π does not equals to π_0, which demonstrates that Δ_π has wider application.

2. Comparison with cost function in [13]

As for the cost function in [13], the condition under which it can reveal the true mapping is:

$$\begin{aligned} s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_2}{n_1}q\right) &= \frac{3\log n_1}{n_1} + \omega(n_1^{-1}) \\ s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_1}{n_2}q\right) &= \frac{3\log n_2}{n_2} + \omega(n_2^{-1}) \end{aligned} \tag{7}$$

This condition requires $p = \Theta(q)$ (Although this assumption is not given in the theorem in [13], the proof of this condition adopts it.). As for the community sizes n_1, n_2, we assume $n_1 = \Theta(n_2)$, since in most cases the size of each community will not have a huge difference. Moreover, we also assume $s = \Theta(1)$, since when s goes to 0, the condition in both [13] and ours can not be satisfied.

To make a comparison between these conditions and ours, we set $s_1 = s_2 = s$, $\alpha = q$ and $\beta = p$ (According to our model setting, when there are only two communities, p represents the probability of intra-community edge existence between any two nodes and q is the probability of inter-community edge existence.). The condition in Theorem IV.1 can be rewritten as

$$s^3 q\log(1/q) \ge \frac{(12+\epsilon)\log^2 q\log n}{n\log^2 p} \tag{8}$$

To prove that our bound is an improved version, we should verify that our condition can be achieved more easily, that is:

$$\frac{s^3 q\log^2 p}{\log(1/q)}\frac{n}{\log n} \ge 4s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_2}{n_1}q\right)\frac{n_1}{\log n_1}, \tag{9}$$

$$\frac{s^3 q\log^2 p}{\log(1/q)}\frac{n}{\log n} \ge 4s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_1}{n_2}q\right)\frac{n_2}{\log n_2}. \tag{10}$$

Since $n_1 = \Theta(n_2)$ and $p = \Theta(q)$, we have

$$4s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_2}{n_1}q\right) = \Theta(q) \tag{11}$$

Therefore, we can derive that:

$$\frac{\frac{s^3 q\log^2 p}{\log(1/q)}}{4s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_2}{n_1}q\right)} = \Theta(\log\frac{1}{q}) = o(1) \tag{12}$$

Moreover, since $n > n_1$, we can get $\frac{n}{\log n} > \frac{n_1}{\log n_1}$. Thus $\frac{s^3 q\log^2 p}{\log(1/q)}\frac{n}{\log n} \ge 4s\left(1-\sqrt{1-s^2}\right)\left(p+2\frac{n_2}{n_1}q\right)\frac{n_1}{\log n_1}$. With the same idea, we can verify the correctness of Inequality 10. Above all, we can claim that our bound is an improved version. Therefore, there exists some conditions where minimizing the estimator proposed in [13] would not find the underlying correct mapping while minimizing ours can.

3. Comparison with the bound in [9]

As for the bound in [9], it reveals the condition under which the ER graph can be de-anonymized. It shows that if $p \to 0$ and

$$ps_1s_2 \ge 2\frac{\log n + \omega(1)}{n}, \tag{13}$$

then there is a de-anonymizer that succeeds with probability $1 - o(1)$. In our work, the stochastic block model can be reduced to an ER graph when $\alpha = \beta = p$. In such case, our result yields to that when

$$\frac{s_1s_2\log(1/p)}{s_1+s_2}ps_1s_2 \ge \frac{(6+\epsilon)\log n}{n}, \tag{14}$$

$\hat\pi = \pi_0$ holds almost surely. Comparing the results in Inequalities (13) and (14), we can get that when $\frac{s_1s_2\log(1/p)}{s_1+s_2} > 3$, Inequality (14) can be achieved more easily. Furthermore, the result in [9] holds when $p \to 0$, thus $\frac{s_1s_2\log(1/p)}{s_1+s_2} \to \infty$ when $s_1, s_2 = \omega(\frac{1}{\log(1/p)})$, which can be achieved in most cases. Therefore, our MAP estimator can de-anonymize graphs under looser conditions and is thus an improved bound.

Appendix E
Proof of Theorem V.1

1. Derivation of the Necessary Conditions: We start the first step with rewriting $\mathbf{P3'}$ as an optimization problem with respect to $\mathbf{Q} = \mathbf{\Pi}\hat{\mathbf{\Pi}}^{\mathrm{T}}$. Since

$$\mathbf{\Pi}\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{\Pi} = (\mathbf{\Pi}\hat{\mathbf{\Pi}}^{\mathrm{T}}\tilde{\mathbf{B}}' - \tilde{\mathbf{B}}\mathbf{\Pi}\hat{\mathbf{\Pi}}^{\mathrm{T}})\hat{\mathbf{\Pi}} = (\mathbf{Q}\tilde{\mathbf{B}}' - \tilde{\mathbf{B}}\mathbf{Q})\hat{\mathbf{\Pi}},$$

and $\mathbf{\Pi}\mathbf{m} - \mathbf{m} = (\mathbf{Q}\mathbf{m} - \mathbf{m})\hat{\mathbf{\Pi}}$, we can reformulate the objective function of $\mathbf{P3'}$ with $\mathbf{Q}$ as variable and divide it by two for ease of further manipulation as $\frac{1}{2}\|\mathbf{Q}\tilde{\mathbf{B}} - \mathbf{B}\mathbf{Q}\|_{\mathrm{F}}^2 + \frac{\mu}{2}\|\mathbf{Q}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2$. The constraint $\sum_j \mathbf{\Pi}_{ij}$ for all i can be expressed as $\mathbf{Q}\mathbf{1} = \mathbf{1}$. The solution of the reformulated version can be associated with the original one by $\mathbf{\Pi} = \mathbf{Q}\hat{\mathbf{\Pi}}$. Next, by introducing multiplier $\boldsymbol{\alpha}$ for the equality constraint of $\mathbf{P3'}$, we construct its Lagrangian function as

$$L(\mathbf{Q},\boldsymbol{\alpha}) = \frac{1}{2}\|\mathbf{Q}\tilde{\mathbf{B}} - \mathbf{B}\mathbf{Q}\|_{\mathrm{F}}^2 + \frac{\mu}{2}\|\mathbf{Q}\mathbf{m} - \mathbf{m}\|_{\mathrm{F}}^2 + \mathrm{tr}(\mathbf{Q}\mathbf{1} - \mathbf{1})\boldsymbol{\alpha}^{\mathrm{T}}.$$

The key element of the proof of the lemma is the sufficient conditions for $\mathbf{Q}$ to be the optimal (fractional) solution to

$\mathbf{P3}'$. To yield the sufficient conditions, we take the gradient of $L(\mathbf{Q}, \boldsymbol{\alpha})$ with respect to $\mathbf{Q}$ and set it as 0. Then we have $\bigtriangledown_{\mathbf{Q}} L(\mathbf{Q}, \boldsymbol{\alpha}) = \mathbf{Q}\mathbf{B}^2 + \tilde{\mathbf{B}}^2\mathbf{Q} - 2\tilde{\mathbf{B}}\mathbf{Q}\mathbf{B} + \boldsymbol{\alpha}\mathbf{1}^{\mathrm{T}} + \mu(\mathbf{Q}\mathbf{M} - \mathbf{M}) = \mathbf{0}$.

Multiplying $\mathbf{U}^{\mathrm{T}}$ to the left side of $\bigtriangledown_{\mathbf{Q}} L(\mathbf{Q}, \boldsymbol{\alpha})$ and $\mathbf{U}$ to the right side we get

$$(\mathbf{F}\boldsymbol{\Lambda}^2 + \boldsymbol{\Lambda}^{\mathbf{2}}\mathbf{F} - 2\boldsymbol{\Lambda}\mathbf{F}\boldsymbol{\Lambda}) + (\mathbf{F}\mathbf{E}\boldsymbol{\Lambda} + \mathbf{F}\boldsymbol{\Lambda}\mathbf{E} - 2\boldsymbol{\Lambda}\mathbf{F}\mathbf{E}) + \boldsymbol{\gamma}\mathbf{v}^{\mathrm{T}} + \mathbf{F}\mathbf{G} + \mu\mathbf{F}\mathbf{M}' - \mu\mathbf{M}' = \mathbf{0},$$

where $\mathbf{F} = \mathbf{U}^{\mathrm{T}}\mathbf{Q}\mathbf{U}$, $\mathbf{v} = \mathbf{U}^{\mathrm{T}}\mathbf{1}$, $\boldsymbol{\gamma} = \mathbf{U}^{\mathrm{T}}\boldsymbol{\alpha}$, $\mathbf{G} = \mathbf{E}^2$ and $\mathbf{M}' = \mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}$.

Rewriting the equation coordinate-wise, we have

$$\mathbf{F}_{ij}(\lambda_i - \lambda_j)^2 + \mathbf{v}_j\boldsymbol{\gamma}_i - \mu\mathbf{M}'_{ij} + \textstyle\sum_k \mathbf{F}_{ik}(\mathbf{E}_{kj}(\lambda_j + \lambda_k - 2\lambda_i) + \mathbf{G}_{kj} + \mu\mathbf{M}'_{kj}) = 0$$

Substituting $i = j$ into the above equation and plugging the results back to eliminate variables $\boldsymbol{\gamma}_i$'s, it follows that

$$\begin{aligned}\mathbf{F}_{ij}\mathbf{v}_i(\lambda_i - \lambda_j)^2 &+ \textstyle\sum_k \mathbf{F}_{ik}(\mathbf{v}_i\mathbf{G}_{kj} - \mathbf{v}_j\mathbf{G}_{ki} + \mu\mathbf{v}_i\mathbf{M}'_{kj} - \mu\mathbf{v}_j\mathbf{M}'_{ki}) \\ &+ \textstyle\sum_k \mathbf{F}_{ik}(\mathbf{v}_i\,\mathbf{E}_{kj}(\lambda_j + \lambda_k - 2\lambda_i) - \mathbf{v}_j\mathbf{E}_{kj}(\lambda_k - \lambda_i)) \\ &+ \mu(\mathbf{v}_j\mathbf{M}'_{ii} - \mathbf{v}_i\mathbf{M}'_{ij}) = 0\end{aligned}$$

We further define the following variables

$$\begin{aligned} r_{ij} &= \frac{\mu}{(\lambda_i - \lambda_j)^2}(\mathbf{v}_j\mathbf{M}'_{ii} - \mathbf{v}_i\mathbf{M}'_{ij}) \\ s^i_{jk} &= \frac{1}{(\lambda_i - \lambda_j)^2}\left(\mathbf{E}_{kj}(\lambda_j + \lambda_k - 2\lambda_i) - \frac{\mathbf{v}_j}{\mathbf{v}_i}\mathbf{E}_{ki}(\lambda_k - \lambda_i)\right) \\ t^i_{jk} &= \frac{1}{(\lambda_i - \lambda_j)^2}\left(\mathbf{G}_{kj} - \frac{\mathbf{v}_j}{\mathbf{v}_i}\mathbf{G}_{ki}\right) \\ w^i_{jk} &= \frac{\mu}{(\lambda_i - \lambda_j)^2}\left(\mathbf{M}'_{kj} - \frac{\mathbf{v}_j}{\mathbf{v}_i}\mathbf{M}'_{ki}\right), \end{aligned}$$

for $i \neq j$. And $s^j_{ik} = t^j_{ik} = w^i_{jk} = r_{ij} = 0$ for $i = j$. Then, we arrive at the following linear system

$$\mathbf{F}_{ij} + \textstyle\sum_k \mathbf{F}_{ik}(s^i_{jk} + t^i_{jk} + w^i_{jk} + \frac{r_{ij}}{n}) = 0, \quad i \neq j \tag{15}$$

$$\textstyle\sum_k \mathbf{F}_{ik}\mathbf{v}_k = \mathbf{v}_i, \tag{16}$$

where the second set of equations come from the constraint $\mathbf{Q}\mathbf{1} = \mathbf{1}$. Equations (15) and (16) represent conditions that the optimal solution $\mathbf{Q}$ (or equivalently $\mathbf{F}$) needs to satisfy.

2. The Equivalence of $\boldsymbol{\Pi}^p$ and $\hat{\boldsymbol{\Pi}}$: Based on the conditions above, we move on to the second step. Recall that in this step our goal is to prove that $\boldsymbol{\Pi}^p$, which is a projection of the optimal fractional solution $\boldsymbol{\Pi}^f$, equals to $\hat{\boldsymbol{\Pi}}$. We formalize this notion in Lemma E.1, the proof of which carries on the main idea of the second step.

Lemma E.1. *Let $\boldsymbol{\Pi}^p$ be the solution computed by **Algorithm 2** and $\hat{\boldsymbol{\Pi}}$ be defined as in Theorem V.1. Under the conditions stated in the theorem, $\boldsymbol{\Pi}^p = \hat{\boldsymbol{\Pi}}$.*

Proof. As the optimal fractional solution $\boldsymbol{\Pi}^f = \mathbf{Q}\hat{\boldsymbol{\Pi}}$, we first show that $\mathbf{Q}$ (or $\mathbf{F}$) is sufficiently close to the identity matrix $\mathbf{I}$, from which using the property of the projection process we obtain that $\boldsymbol{\Pi}^p$ is identical to $\hat{\boldsymbol{\Pi}}$. We achieve this by treating linear system consisting of Equations (15) and (16) as a perturbed version of

$$\mathbf{F}_{ij} = 0, \quad i \neq j \tag{17}$$

$$\textstyle\sum_k \mathbf{F}_{ik}\mathbf{v}_k = \mathbf{v}_i, \tag{18}$$

the solution of which is clearly $\mathbf{I}$. Then using the results from stability of perturbed linear system [27] that is presented in Lemma E.2 below and the conditions in Theorem V.1, we can bound the difference between $\mathbf{F}$ and $\mathbf{I}$.

Lemma E.2. *(Theorem 1 in [27]) Let $\|\cdot\|$ be any p-norm. For two linear systems $\mathbf{D}\mathbf{x} = \mathbf{b}$ and $\tilde{\mathbf{D}}\mathbf{x} = \tilde{\mathbf{b}}$, let $\mathbf{x}_0$ and $\mathbf{x}$ be their solutions, if $\|\mathbf{D} - \tilde{\mathbf{D}}\|\|\mathbf{D}^{-1}\| < 1$, then we have*

$$\frac{\|\mathbf{x} - \mathbf{x_0}\|}{\|\mathbf{x}_0\|} \leq \frac{\|\mathbf{D}\|\|\mathbf{D}^{-1}\|}{1 - \|\mathbf{D} - \tilde{\mathbf{D}}\|\|\mathbf{D}^{-1}\|}\left\{\frac{\|\mathbf{D} - \tilde{\mathbf{D}}\|}{\|\mathbf{D}\|} + \frac{\|\mathbf{b} - \tilde{\mathbf{b}}\|}{\mathbf{b}}\right\}.$$

Denoting by $\mathbf{f} = (\mathbf{F}_{11}, \ldots, \mathbf{F}_{1n}, \ldots, \mathbf{F}_{n1}, \ldots, \mathbf{F}_{nn})^{\mathrm{T}}$ the row stack vector representation of $\mathbf{F}$, we can rewrite the perturbed system as $(\mathbf{D} + \mathbf{N})\mathbf{f} = \mathbf{b}$, and the original unperturbed system as $\mathbf{D}\mathbf{f} = \mathbf{b}$, with $\mathbf{D} = \mathrm{diag}\{\mathbf{D_1}, \ldots, \mathbf{D}_n\}$ being an $n^2 \times n^2$ block-diagonal matrix, where each $\mathbf{D}_i$ is an $n \times n$ block consisting of identity matrix with the ith row replaced by vector $\mathbf{v}^{\mathrm{T}}$. $\mathbf{N}$ is also an $n^2 \times n^2$ block-diagonal matrix with the $n \times n$ blocks $\mathbf{N}_i$ being a matrix with elements $(\mathbf{N}_i)_{jk} = s^i_{jk} + t^i_{jk} + w^i_{jk} + r_{ij}/n$. And $\mathbf{b}$ is an $n^2 \times 1$ vector with the $[(i-1)(n+1)+1]$-st element as v_i and other element as 0. Using Lemma E.2 on the perturbed system and the unperturbed one with $\|\cdot\|$ taken as 2-norm (Euclidean norm), we obtain that

$$\|\mathbf{f} - \mathbf{f}_0\| \leq \|\mathbf{f}_0\|\frac{\|\mathbf{D}^{-1}\|\|\mathbf{N}\|}{1 - \|\mathbf{D}^{-1}\|\|\mathbf{N}\|}, \tag{19}$$

where $\mathbf{f}_0$ is the row stack vector representation of $\mathbf{I}$. Therefore, to derive the upper bound for the difference between $\mathbf{F}$ and $\mathbf{I}$, we need to further upper bound the RHS of Inequality (19). The technique we use here is harnessing the special structure of $\mathbf{D}$ and $\mathbf{N}$ so that we can derive bounds for $\|\mathbf{D}^{-1}\|$ and $\|\mathbf{N}\|$, which are represented in functions of variables $\{s\}, \{t\}, \{w\}$ and $\{r\}$. By further associating the variables with the spectral parameters $\delta, \epsilon_1, \epsilon_2$, etc. defined in the theorem, we yield an upper bound for the RHS of Inequality (19) that depends on those spectral parameters. To present the upper bound of Inequality (19), we begin with bounding $\|\mathbf{D}^{-1}\|$ and $\|\mathbf{N}\|$. First, by the special block-diagonal structure of $\mathbf{D}$, we readily have that $\mathbf{D}^{-1}$ is also block diagonal with each $n \times n$ diagonal block as $\mathbf{D}_i^{-1}$, which is the identity matrix with the ith row replaced by $\frac{1}{\mathbf{v}_i}(-\mathbf{v}_1, \ldots, -\mathbf{v}_{i-1}, 1, -\mathbf{v}_{i+1}, \ldots, -\mathbf{v}_n)$. We have

$$\|\mathbf{D}_i^{-1}\| \leq 1 + \frac{\sqrt{n}\epsilon_1}{\epsilon_2}, \quad \text{for all } i.$$

Hence we have,

$$\|\mathbf{D}^{-1}\| \leq \max_{i=1\ldots n}\|\mathbf{D}_i^{-1}\| \leq 1 + \frac{\sqrt{n}\epsilon_1}{\epsilon_2}. \tag{20}$$

Similarly, we obtain

$$\begin{aligned}\|\mathbf{N}\|^2 \leq \max_{i,j=1\ldots n}\|\mathbf{N}_i\|_{\mathrm{F}}^2 &= \max_{i=1\ldots n}\sum_{jk}(s^i_{jk} + t^i_{jk} + w^i_{jk} + \frac{r_{ij}}{n})^2 \\ &\leq 4\left(\max_{i=1\ldots n}\sum_k\left(s^i_{jk}\right)^2 + \max_{i,j=1\ldots n}\sum_{jk}\left(t^i_{jk}\right)^2\right. \\ &\quad \left. + \max_{i,j=1\ldots n}\sum_k\left(w^i_{jk}\right)^2 + \max_{i,j=1\ldots n}\sum_k\left(\frac{r_{ij}}{n}\right)^2\right).\end{aligned}$$

Next, we bound these the terms s^i_{jk}, t^i_{jk}, w^i_{jk} and r_{ij} one by

one in the following inequalities.

$$\begin{aligned}
&\max_{i=1\dots n}\sum_{jk}\left(s_{jk}^{i}\right)^{2}\\
=&\max_{i=1\dots n}\sum_{k}\frac{1}{(\lambda_i-\lambda_j)^4}\\
&\qquad\cdot\left(\mathbf{E}_{kj}(\lambda_j+\lambda_k-2\lambda_i)-\frac{v_j}{v_i}\mathbf{E}_{ki}(\lambda_k-\lambda_i)\right)^{2}\\
\le&\max_{i=1\dots n}\frac{1}{\delta^4}\left(4\sigma\sum_{jk}|\mathbf{E}_{kj}|+2\sigma\frac{\epsilon_1}{\epsilon_2}\sum_{kj}|\mathbf{E}_{kj}|\right)^{2}\\
\le&\frac{4\sigma^2}{\delta^4}\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\xi^2\\
&\qquad\max_{i=1\dots n}\sum_{jk}\left(t_{jk}^{i}\right)^{2}\\
&\qquad=\max_{i=1\dots n}\sum_{jk}\frac{1}{(\lambda_i-\lambda_j)^4}\left(\mathbf{G}_{kj}-\frac{v_j}{v_i}\mathbf{G}_{ki}\right)^{2}\\
&\qquad\le\sum_{jk}\frac{1}{\delta^4}\left(\mathbf{G}_{kj}+2\frac{\epsilon_1}{\epsilon_2}\mathbf{G}_{ki}\right)^{2}\\
&\qquad\le\frac{1}{\delta^4}\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\|\mathbf{G}\|_{\mathrm{F}}^{2}\\
&\qquad\le\frac{1}{\delta^4}\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\xi^4.\\
&\max_{i=1\dots n}\sum_{jk}\left(w_{jk}^{i}\right)^{2}\\
=&\max_{i=1\dots n}\sum_{jk}\frac{\mu^2}{(\lambda_i-\lambda_j)^4}\left(\mathbf{M}'_{kj}-\frac{\mathbf{v}_j}{\mathbf{v}_i}\mathbf{M}'_{ki}\right)^{2}\\
\le&\frac{\mu^2}{\delta^4}\max_{i=1\dots n}\sum_{jk}\left(\sum_{k}\mathbf{M}'_{kj}+2\frac{\epsilon_1}{\epsilon_2}\sum_{k}\mathbf{M}'_{ki}\right)^{2}\\
\le&\frac{\mu^2}{\delta^4}\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\|\mathbf{M}'\|_{\mathrm{F}}^{2}\\
\le&\frac{\mu^2}{\delta^4}\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)M^2.\quad\text{(by the orthonomality of }\mathbf{U})\\
&\qquad\max_{i,j=1\dots n}\sum_{k}\left(\frac{r_{ij}}{n}\right)^{2}\\
&\qquad=\max_{i,j=1\dots n}\frac{\mu^2}{n(\lambda_i-\lambda_j)^4}\left(\mathbf{v}_j\mathbf{M}'_{ii}-\mathbf{v}_i\mathbf{M}'_{ij}\right)^{2}\\
&\qquad\le\frac{4\epsilon_1^2\mu^2}{n\delta^4}\|\mathbf{M}'\|_{\mathrm{F}}^{2}\\
&\qquad\le\frac{4\epsilon_1^2\mu^2}{n\delta^4}M^2.
\end{aligned}$$

From the above manipulations, we have

$$\begin{aligned}
\|\mathbf{N}\|^2&\le 4\left[\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\left(\frac{\sigma^2}{\delta^4}\xi^2+\frac{1}{\delta^4}\xi^4+\frac{\mu^2}{\delta^4}M^2\right)+\frac{4\epsilon_1^2\mu^2M^2}{n\delta^4}\right]\\
&\le 5\left[\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}\left(\frac{\sigma^2}{\delta^4}\xi^2+\frac{1}{\delta^4}\xi^4+\frac{\mu^2}{\delta^4}M^2\right)\right],\qquad(21)
\end{aligned}$$

for sufficiently large n. Substituting Inequalities (20) and (21) into (19), it follows that

$$\|\mathbf{F}-\mathbf{I}\|_{\mathrm{F}}=\|\mathbf{f}-\mathbf{f}_0\|\le\frac{1-\left(1+\frac{\sqrt{n}\epsilon_1}{\epsilon_2}\right)\sqrt{\frac{5}{\delta^4}\left[\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}(\sigma^2\xi^2+\xi^4+\mu^2M^2)\right]}}{\sqrt{n}\left(1+\frac{\sqrt{n}\epsilon_1}{\epsilon_2}\right)\sqrt{\frac{5}{\delta^4}\left[\left(1+2\frac{\epsilon_1}{\epsilon_2}\right)^{2}(\sigma^2\xi^2+\xi^4+\mu^2M^2)\right]}}.$$

Based on the upper bound, we have that if the conditions in the theorem are satisfied, then

$$\|\mathbf{F}-\mathbf{I}\|_{\mathrm{F}}=\|\mathbf{f}-\mathbf{f_0}\|\le\frac{1}{2}.$$

Since $\|\mathbf{\Pi}^f-\hat{\mathbf{\Pi}}\|_{\mathrm{F}}=\|\mathbf{Q}\hat{\mathbf{\Pi}}-\hat{\mathbf{\Pi}}\|_{\mathrm{F}}=\|(\mathbf{Q}-\mathbf{I})\hat{\mathbf{\Pi}}\|_{\mathrm{F}}=\|\mathbf{Q}-\mathbf{I}\|_{\mathrm{F}}=\|\mathbf{F}-\mathbf{I}\|_{\mathrm{F}}\le 1/2$, the entry-wise difference between $\mathbf{\Pi}^f$ and $\hat{\mathbf{\Pi}}$ is less than $1/2$. Thus, the projection process in **Algorithm 2** is bound to project $\mathbf{\Pi}_f$ as $\hat{\mathbf{\Pi}}$, which concludes the second step, i.e., the proof of Lemma E.1. □

3. Optimality of $\hat{\mathbf{\Pi}}$: Now we proceed to the final step and prove that $\hat{\mathbf{\Pi}}=\mathbf{\Pi}^*$ by contradiction. If there exists some permutation matrix $\mathbf{\Pi}'\neq\hat{\mathbf{\Pi}}$ with $\|\mathbf{\Pi}'\tilde{\mathbf{A}}-\mathbf{\Pi}'\tilde{\mathbf{B}}\|_{\mathrm{F}}<\|\hat{\mathbf{\Pi}}\tilde{\mathbf{A}}-\hat{\mathbf{\Pi}}\tilde{\mathbf{B}}\|_{\mathrm{F}}$. Then, we consider $\tilde{\mathbf{B}}=\mathbf{B_0}+\mathbf{R}'$ with $\mathbf{B}_0=\mathbf{\Pi}'^{\mathrm{T}}\mathbf{A}\mathbf{\Pi}'$. Obviously, $\mathbf{R}'$ satisfies the conditions in Theorem V.1. Hence, by Lemma E.1, we should have that the the solution $\mathbf{\Pi}^p$ computed by **Algorithm 2** equals to $\mathbf{\Pi}'$. However, we also have $\mathbf{\Pi}^p=\hat{\mathbf{\Pi}}$, which leads to a contradiction. Thus, $\hat{\mathbf{\Pi}}$ is the optimal solution to $\mathbf{P3}$, which finishes the proof of the theorem. □

Appendix F
MAP estimation of Unilateral De-anonymization

In this section, we derive the MAP estimator for unilateral de-anonymization. Recall that given G_1, G_2, c, $\boldsymbol{\theta}$, the MAP estimate $\hat{\pi}$ of the correct mapping π_0 is defined as follows

$$\hat{\pi}=\arg\max_{\pi\in\Pi}Pr(\pi_0=\pi\mid G_1,G_2,c,\boldsymbol{\theta}),\qquad(22)$$

The MAP estimator can be further written as:

$$\hat{\pi}=\arg\max_{\pi\in\Pi}\sum_{G\in\mathcal{G}_\pi}p(G,\pi\mid G_1,G_2,c,\boldsymbol{\theta}),\qquad(23)$$

where $\mathcal{G}_\pi$ is the set of all realizations of the underlying social network that are consistent with G_1, G_2 and π. By Bayesian rule, we have

$$\begin{aligned}
&\arg\max_{\pi\in\Pi}\sum_{G\in\mathcal{G}_\pi}p(G,\pi\mid G_1,G_2,c,\boldsymbol{\theta})\\
=&\arg\max_{\pi\in\Pi}\sum_{G\in\mathcal{G}_\pi}\frac{p(G_1,G_2\mid G,\pi)p(G,\pi)}{p(G_1,G_2)}\\
=&\arg\max_{\pi\in\Pi}\sum_{G\in\mathcal{G}_\pi}p(G_1,G_2\mid G,\pi)p(G)p(\pi)\\
=&\arg\max_{\pi\in\Pi}\sum_{G\in\mathcal{G}_\pi}p(G_1\mid G)p(G_2\mid G,\pi)p(G).
\end{aligned}$$

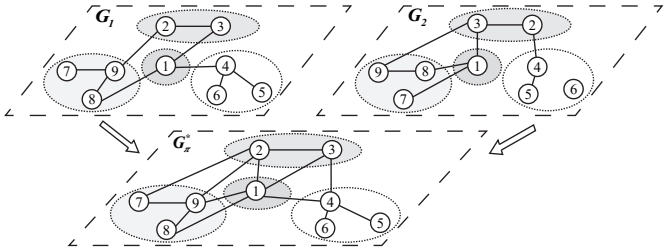


Fig. 7: **An example of G_π^* that has the minimum number of edges in $\mathcal{G}_\pi$, which is the set of all realizations of G that are consistent with G_1, G_2, π. In this case $\pi = \pi_0$.**

Note that we drop parameters c and $\boldsymbol{\theta}$ for brevity since their values are fixed. From the definitions of the models, we have:

$$
\begin{aligned}
&\arg\max_{\pi\in\Pi} \sum_{G\in\mathcal{G}_\pi} p(G_1 \mid G=g)p(G_2 \mid G,\pi)p(G) \\
=&\arg\max_{\pi\in\Pi} \sum_{G\in\mathcal{G}_\pi} \prod_{i<j}^{n} (1-s_1)^{|E^{ij}|-|E_1^{ij}|} s_1^{|E_1^{ij}|} \\
&\cdot \prod_{i<j}^{n} (1-s_2)^{|E^{ij}|-|E_2^{\pi(i)\pi(j)}|} s_2^{|E_2^{\pi(i)\pi(j)}|} \\
&\cdot \prod_{i<j}^{n} p_{c(i)c(j)}^{|E^{ij}|} (1-p_{c(i)c(j)})^{1-|E^{ij}|} \\
=&\arg\max_{\pi\in\Pi} \left(\prod_{i<j}^{n} \left(\frac{s_1}{1-s_1}\right)^{|E_1^{ij}|} \left(\frac{s_2}{1-s_2}\right)^{|E_2^{\pi(i)\pi(j)}|} \right) \\
&\cdot \left(\sum_{g\in\mathcal{G}_\pi} \prod_{i<j}^{k} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{|E^{ij}|} \right) \\
=&\arg\max_{\pi\in\Pi} \prod_{i<j}^{n} \left(\frac{s_2}{1-s_2}\right)^{|E_2^{\pi(i)\pi(j)}|} \\
&\cdot \sum_{g\in\mathcal{G}_\pi} \prod_{i<j}^{k} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{|E^{ij}|} \\
=&\arg\max_{\pi\in\Pi} \sum_{g\in\mathcal{G}_\pi} \prod_{i<j}^{k} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{|E^{ij}|},
\end{aligned}
$$

where $|E^{ij}|, |E_1|^{ij}, |E_2^{ij}|$ take value 0 or 1 indicating whether there exists an edge between nodes i and j in G, G_1, G_2 respectively. Note that in the above manipulations, we frequently eliminate the terms that do not depend on π. Particularly, in the last step, although the term $\left(\frac{s_2}{1-s_2}\right)^{|E_2^{\pi(i)\pi(j)}|}$ depends on π, the value of the whole product $\prod_{i<j}^{n} \left(\frac{s_2}{1-s_2}\right)^{|E_2^{\pi(i)\pi(j)}|}$ is independent of π itself since it is a bijective mapping.

Now, let G_π^* be the graph having the smallest number of edges in $\mathcal{G}_\pi$, which is equivalent to that $G_\pi^* = (V, E_1 \cup \pi(E_2))$. An illustration of G_π^* is provided in Figure 7. Denote the set of edges in G_π^* as E_π^*, with $|E_{\pi^*}^{ij}|$ indicating the number of edges between i and j. By the definition we have that in $\mathcal{G}_\pi$, all the graphs have edge sets that are supersets of G_π^*. By summing over all the graphs in $\mathcal{G}_\pi$, we have that

$$
\begin{aligned}
\hat{\pi} = \arg\max_{\pi\in\Pi} &\prod_{i<j}^{n} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{|E_{\pi^*}^{ij}|} \\
&\cdot \prod_{i<j}^{n} \left(1 + \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right) \right)^{1-|E_{\pi^*}^{ij}|},
\end{aligned}
$$

where the above equality follows from that

$$
\begin{aligned}
&\sum_{g\in\mathcal{G}_\pi} \prod_{i<j}^{n} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{|E^{ij}|-|E_{\pi^*}^{ij}|} \\
=&\sum_{0\le k_{ij}\le 1-|E_{\pi^*}^{ij}|} \prod_{i<j}^{n} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right)^{k_{ij}} \\
=&\prod_{i<j}^{n} \left(1 + \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}} \right) \right)^{1-|E_{\pi^*}^{ij}|}.
\end{aligned}
$$

Then, from the above equation we can further write the MAP estimator as:

$$
\begin{aligned}
&\arg\max_{\pi\in\Pi} \prod_{i<j}^{n} \left(\frac{p_{c(i)c(j)}(1-s_1)(1-s_2)}{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)} \right)^{|E_{\pi^*}^{ij}|} \\
=&\arg\min_{\pi\in\Pi} \prod_{i<j}^{n} \left(\frac{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)}{p_{c(i)c(j)}(1-s_1)(1-s_2)} \right)^{|E_{\pi^*}^{ij}|} \\
=&\arg\min_{\pi\in\Pi} \left[\sum_{i<j}^{n} |E_{\pi^*}^{ij}| \log \left(\frac{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)}{p_{c(i)c(j)}(1-s_1)(1-s_2)} \right) \right].
\end{aligned}
$$

Next, for the case where the community assignment c of G_1 is available, we notice that

$$
|E_{\pi^*}^{ij}| = E_1^{ij} + \mathbb{1}\{(i,j) \notin E_1, (\pi(i),\pi(j)) \in E_2\}.
$$

This is because in the process of generating G_π^*, we should firstly label nodes and communities in G_π^* as G_1 does, since only the community assignment function of G_1 is known. After adding edges to G_π^* based on E_1, we then add an edge between i, j if $(i,j) \notin E_1$ but $(\pi(i),\pi(j)) \in E_2$.

Hence, by setting $w_{ij} = \log\left(\frac{1-p_{c(i)c(j)}(s_1+s_2-s_1s_2)}{p_{c(i)c(j)}(1-s_1)(1-s_2)}\right)$ we have

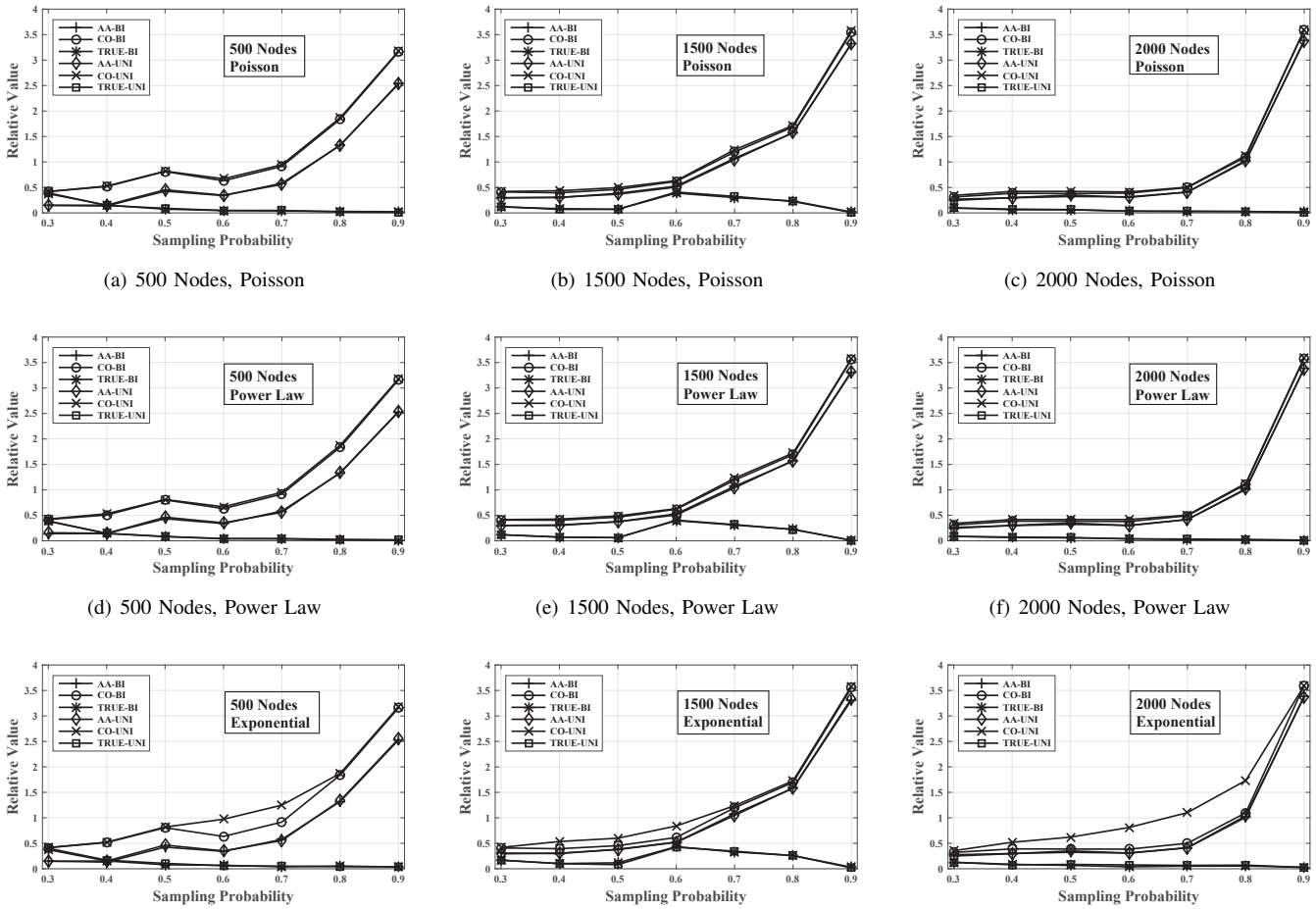
$$
\begin{aligned}
\hat{\pi} &= \arg\min_{\pi\in\Pi} \sum_{i<j}^{n} w_{ij} |E_{\pi^*}^{ij}| \\
&= \arg\min_{\pi\in\Pi} \left[\sum_{i<j}^{n} w_{ij} \left(E_1^{ij} + \mathbb{1}\{(i,j) \notin E_1, (\pi(i),\pi(j)) \in E_2\} \right) \right] \\
&= \arg\min_{\pi\in\Pi} \left[\sum_{i<j}^{n} w_{ij} \left(\mathbb{1}\{(i,j) \notin E_1, (\pi(i),\pi(j)) \in E_2\} \right) \right].
\end{aligned}
$$

Similarly, for the case where we know the community assignmnent of G_2, we can get that

$$
|E_{\pi^*}^{ij}| = E_1^{ij} + \mathbb{1}\{(i,j) \in E_1, (\pi(i),\pi(j)) \notin E_2\}
$$

$$
\hat{\pi} = \arg\min_{\pi\in\Pi} \left[\sum_{i<j}^{n} w_{ij} \left(\mathbb{1}\{(i,j) \in E_1, (\pi(i),\pi(j)) \notin E_2\} \right) \right].
$$

Note that the MAP estimator is not symmetric with regard to G_1 and G_2. This stems from the fact that the adversary in



(a) 500 Nodes, Poisson (b) 1500 Nodes, Poisson (c) 2000 Nodes, Poisson

(d) 500 Nodes, Power Law (e) 1500 Nodes, Power Law (f) 2000 Nodes, Power Law

(g) 500 Nodes, Exponential (h) 1500 Nodes, Exponential (i) 2000 Nodes, Exponential

Fig. 8: The relative value of the cost function of the mappings produced by the algorithms on synthetic datasets with different degree distributions.

this case only has knowledge on the community assignment function of G_1.

APPENDIX G
ANALYSIS OF UNILATERAL DE-ANONYMIZATION

1. Necessity of Unilateral Case

In practice, there are several cases proposed in Section I where community structure is difficult to get. Although there is a huge line of works on the feasibility of community detection on the stochastic block model even under very sparse random graph regimes ([1]–[4]), which enables adversaries to recover the communities by community dectection algorithms, the beneficial effect of dectected community information on de-anonymization process will be blocked for the following reasons.

The first reason is that although several community detection algorithms can recover the communities with probability $1 - o(1)$, there are also $o(n)$ nodes are predicted to be in the wrong community. For a certain community a, it will be detected as a_1 and a_2 in G_1, G_2 respectively, and the number of nodes they contain may not be the same. Thus it violates that nodes i and $\pi(i)$ are in the same community, which is required in our algorithms, what's more, we even cannot judge whether a_1 and a_2 are the same communities in G.

Second, when s_1 and s_2 are in certain range, even though the community detection algorithm can recover the communities with probability 1, the detected communities a_1 and a_2 may not contain the same set of nodes. For example, consider a graph $G(V, E)$ with two communities a and b, whose size is $k+1$ and k, where $k \to \infty$. The edge existence probability in community a is p_{aa}, and the probability between two communities is P_{ab}, where $P_{aa} = \Theta(P_{ab})$. G_1 and G_2 are generated by G with sampling probability s_1, s_2. Suppose the adversary know the community information of G_1, which is the same as G. When the adversary detects the communities in G_2, suppose he or she have known the true community assignment $c : V/\{v\} \mapsto \{a', b'\}$ for all nodes except v, a node belongs to a. In G, let $|E_a|, |E_b|$ be the number of nodes v connects with in a and b respectively. Then $|E_a| = \Theta(kP_{aa}), |E_b| = \Theta(kP_{ab})$ almost sure, and $|E_a| = \Theta(|E_b|)$ since $P_{aa} = \Theta(P_{ab})$. In G_2, let $|E'_a|, |E'_b|$ be the number of nodes v connects with in a' and b' respectively.

When $s_2 = \frac{1}{2}$, we have

$$\begin{aligned} P_{|E'_a|<\frac{|E_b|}{2}} &= \sum_{t=0}^{\frac{|E_b|-1}{2}} \binom{|E_a|}{t} s_2^t (1-s_2)^{|E_a|-t} \\ &= \frac{1}{2^{|E_a|}} \sum_{t=0}^{\frac{|E_b|-1}{2}} \binom{|E_a|}{t} \\ &= \Theta(1) \\ P_{|E'_b|\geq\frac{|E_b|}{2}} &= \sum_{t=\frac{|E_b|}{2}}^{|E_b|} \binom{|E_b|}{t} s_2^t (1-s_2)^{|E_b|-t} \\ &= \frac{1}{2^{|E_b|}} \sum_{t=\frac{|E_b|}{2}}^{|E_b|} \binom{|E_b|}{t} \\ &= \Theta(1) \end{aligned}$$

Therefore, $P_{|E'_a|<|E'_b|} \geq P_{|E'_a|<\frac{|E_b|}{2}} P_{|E'_b|\geq\frac{|E_b|}{2}} = \Theta(1)$, which means that the case where $|E'_a| < |E'_b|$ may happens with relatively large probability. In this case, the adversary may assign v into community b', and then b' will be matched with a in G_1 since $|b'| = k+1 = |a| \neq |a'|$. Thus in this case, using bilateral commmunity information will fail the de-anonymization process.

The last reason is that when the number of communities is growing with n, it is possible to have a large number of total isolated communities, i.e. the community that only contains a total isolated nodes. For such communities, we cannot judge the difference between them, thus cannot find the true community mapping function.

To sum up, the community information may not be useful in some cases, and therefore the estimator based on unilateral case becomes necessary.

2. Differences Between Bilateral and Unilateral De-anonymization

In this subsection, we summarize, from a higher level, the differences existing in the essence of bilateral and unilateral de-anonymization and the results we obtain for the two problems.

- The extra knowledge on the community assignment function in bilateral de-anonymization enables us to restrict the feasible mappings to the ones that observe the community assignment, thus decreases the number of possible candidates and makes the problem intuitively easier than unilateral one.
- The community assignment as side information is the main reason behind the difference of the posterior distribution of the optimal mapping, which leads to different MAP estimates, and thus different cost functions in the two cases. Note that the cost function for bilateral de-anonymization cannot be calculated in unilateral case since we have no knowledge on the community assignment of G_2.
- Although under similar conditions, minimizing the cost function asymptotically almost surely recovers the correct mapping in both cases, the lack of community assignment in unilateral de-anonymization impose asymmetry in its cost function and render the cost function harder to (approximately) minimize, as justified by our stronger complexity-theoretic result.
- In terms of the proposed algorithms, the additive approximation algorithms for both bilateral and unilateral de-anonymization share the same guarantee. However, the convex optimization-based heuristic has been shown to yield conditionally yield optimal solutions only for bilateral de-anonymization.
- The empirical results demonstrate that in all the contexts, our algorithms successfully de-anonymize larger portion of users when provided with bilateral community information.

APPENDIX H
WORKING MECHANISM OF GA

We first denote a random assignment function R, with $R(V)$ being a vertice randomly chosen in V. Here $R_i(V) \neq R_j(V)$ when $i \neq j$. The algorithm **GA-BI** works as follows:

1. Initialize $2k$ mappings randomly.

2. Set these mappings into k pairs. For each pair (π_a, π_b), set a vertice set V', and let V_a, V_b be $V - (\pi_a(V - V') \cup \pi_b(V'))$ and $V - (\pi_b(V - V') \cup \pi_a(V'))$ respectively. Generate the mappings π'_a, π'_b:

$$\pi'_a(i) = \begin{cases} \pi_b(i) & i \in V' \\ \pi_a(i) & i \notin V' \bigcap \pi_a(i) \notin (\pi_a(V - V') \cap \pi_b(V')) \\ R_i(V_a) & i \notin V' \bigcap \pi_a(i) \in (\pi_a(V - V') \cap \pi_b(V')) \end{cases},$$

$$\pi'_b(i) = \begin{cases} \pi_a(i) & i \in V' \\ \pi_b(i) & i \notin V' \bigcap \pi_b(i) \notin (\pi_b(V - V') \cap \pi_a(V')) \\ R_i(V_b) & i \notin V' \bigcap \pi_b(i) \in (\pi_b(V - V') \cap \pi_a(V')) \end{cases}.$$

3. Randomly choose a vertice set V'' for each newly generated mapping. Update the newly generated mapping π'_a according to the following formula:

$$\pi'_a(i) = \begin{cases} R_i(\pi'_a(V''_a)) & i \in V''_a \\ \pi'_a(i) & i \notin V''_a \end{cases}$$

4. Reverse all the "cycles of communities assignment violations" in each mapping by the method proposed in our algorithm **The Additive Approximation Algorithm**.

5. Select $2k$ mappings that have lower cost based on the cost function we propose.

6. Run steps 2 to 5 until the lowest cost of these mappings converges. Output the corresponding mapping π.

APPENDIX I
GRAPHICAL RESULTS ON RELATIVE VALUE OF COST FUNCTION

In this section we present graphical results on the relative value of the cost function of the mappings produced by the algorithms, as shown in Fig. 9 and Fig. 10. Recall that for a mapping π and the mapping π_{GA} produced by **GA** algorithm, the relative value of the cost function of π equals to $(\Delta_\pi - \Delta_{\pi_{GA}})/\Delta_{\pi_{GA}}$.

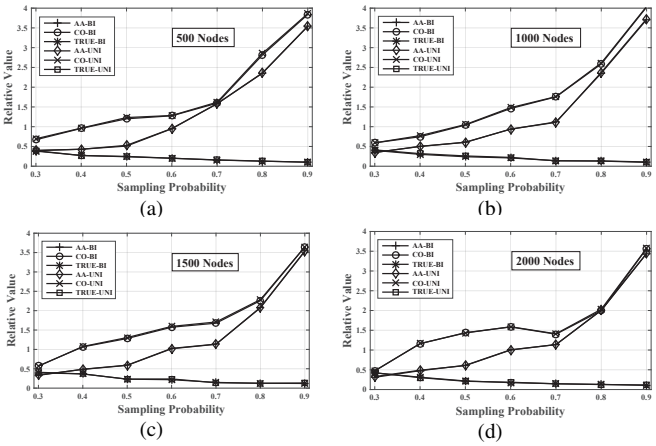


Fig. 9: **The relative value of the cost function of the mappings p S**

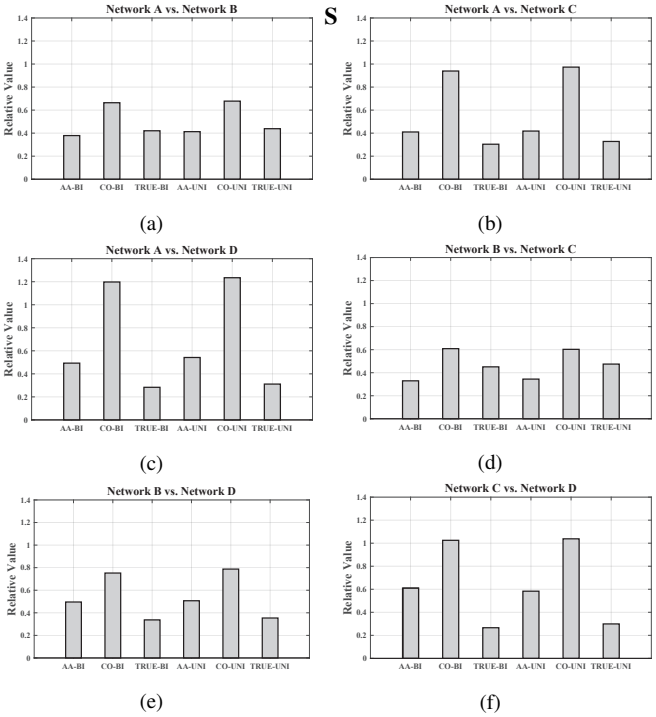


Fig. 10: **The relative value of the cost function of the mappings produced by the algorithms on Cross-domain Co-authorship Networks**

REFERENCES

[1] L. Massoulié, "Community detection thresholds and the weak Ramanujan property", in *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pp. 694-703, 2014.

[2] E. Mossel, J. Neeman, and A. Sly, "A proof of the block model threshold conjecture", in *Combinatorica*, Vol. 38, No. 3, pp. 665-708, 2018.

[3] E. Abbe, "Community detection and stochastic block models: recent developments", in *The Journal of Machine Learning Research*, Vol. 18, No. 1, pp. 6446-6531, 2017.

[4] C. Bordenave, M. Lelarge, and L. Massoulié, "Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs", in *IEEE 56th Annual Symposium on Foundations of Computer Science*, pp. 1347-1357, 2015.