Chapter 9: Radiating Systems, Multipole Fields and Radiation

An Overview of Chapters on EM Waves : (covered in this course)

	Source term in wave equation	Boundary
Ch. 7	none	plane wave in ∞ space or two semi-∞ spaces (as in reflection/refraction)
Ch. 8	none	conducting walls
Ch. 9	$\mathbf{J}, \rho \sim e^{-i\omega t}$ prescribed, as in an antenna	outgoing wave to ∞
Ch. 10	$\mathbf{J}, \rho \sim e^{-i\omega t}$ induced by an incident EM wave, as in the case of scattering of a plane wave by a dielectric object	outgoing wave to ∞
Ch. 14	a moving charge, such as electrons in a synchrotron	outgoing wave to ∞

9.6 Spherical Wave Solutions of the Scalar Wave Equation

Spherical Bessel Functions and Spherical Hankel functions :

This chapter deals with EM fields generated by harmonic $\mathbf{J}$ and ρ.

We first solve the scalar, source-free wave equation in *spherical* coordinates. The purpose is to obtain a complete set of spherical Bessel functions and spherical Hankel functions, which will be used to expand the radiated fields.

The scalar, source-free wave equation in *free space* is [see (6.32)]

$$\nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = 0 \quad (9.77)$$

Let $\psi(\mathbf{x},t) = \int_{-\infty}^{\infty}\psi(\mathbf{x},\omega)e^{-i\omega t}d\omega$ (9.78)

$\Rightarrow$ Each Fourier component satisfies the Helmholtz wave equation:

$$(\nabla^2 + k^2)\psi(\mathbf{x},\omega) = 0 \;\; [k \equiv \tfrac{\omega}{c}] \quad (9.79)$$

Question: The problem is much simpler in free space. Why?

In spherical coordinates, $(\nabla^2 + k^2)\psi(\mathbf{x},\omega) = 0$ is written

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + k^2\psi = 0$$

Let $\psi = \sum_{lm} f_\ell(r)P_l^m(\cos\theta)e^{im\varphi}$ $\begin{bmatrix}\text{1. Each term is linearly independent.}\\ \text{2. } f_\ell(r) \text{ is indep. of } m \text{ [see (9.81)].}\end{bmatrix}$

$$\Rightarrow \quad P_l^m(\cos\theta)e^{im\varphi}\frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{\partial}{\partial r}f_\ell(r)]$$
$$+\frac{f_\ell(r)}{r^2}e^{im\varphi}\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\frac{\partial}{\partial\theta}P_l^m(\cos\theta)] - \frac{m^2}{\sin^2\theta}P_l^m(\cos\theta)\}$$
$$+k^2 f_\ell(r)P_l^m(\cos\theta)e^{im\varphi} = 0$$

Use $\frac{1}{\sin\theta}\frac{d}{d\theta}[\sin\theta\frac{dP_l^m(\cos\theta)}{d\theta}] + [l(l+1) - \frac{m^2}{\sin^2\theta}]P_l^m(\cos\theta) = 0$ [(3.6)]

$$\Rightarrow \quad [\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}]f_\ell(r) = 0 \quad [\Rightarrow f_\ell(r) \text{ is indep. of } m.] \quad (9.81)$$

Note: $P_l^m(\cos\theta)$ is finite in the interval $-1 \le \cos\theta \le 1$ only when $l = 0,\ 1,\ 2\cdots$ and $m = -l,\ -(l-1),\ldots,\ -1,\ 0,\ 1,\ldots,\ (l-1),\ l$ (p. 107).

Rewrite $[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}]f_\ell(r) = 0$ [(9.81)]

Let $f_\ell(r) = \frac{1}{r^{1/2}}u_l(r) \Rightarrow [\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(l+\frac{1}{2})^2}{r^2}]u_l(r) = 0$ (9.83)

$\Rightarrow u_l(r) = J_{l+\frac{1}{2}}(kr),\ N_{l+\frac{1}{2}}(kr)$ [Bessel functions of fractional order]

$\Rightarrow f_\ell(r) = \frac{1}{r^{1/2}}J_{l+\frac{1}{2}}(kr),\ \frac{1}{r^{1/2}}N_{l+\frac{1}{2}}(kr)$

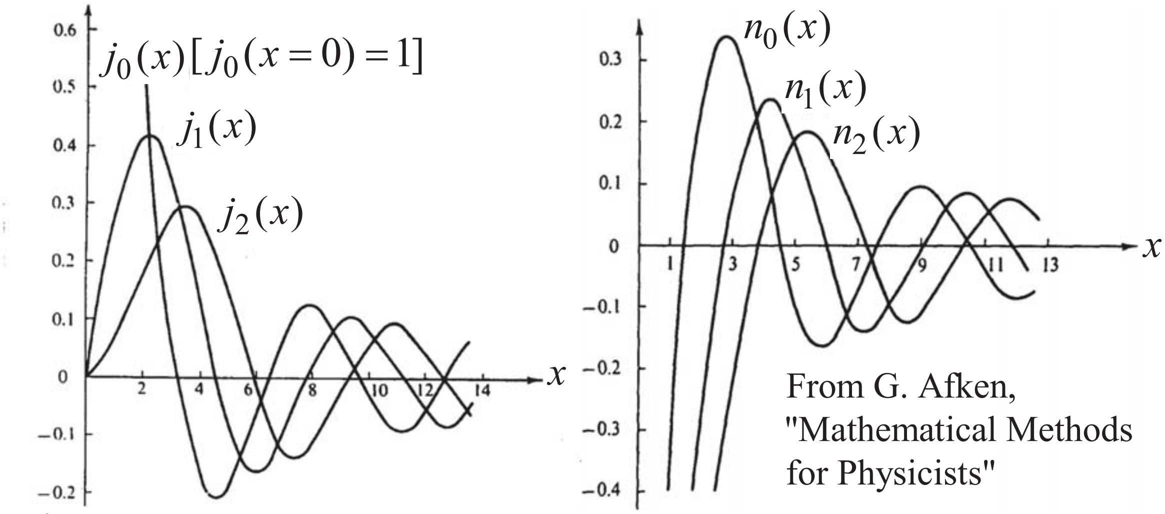
$$\text{Define } \begin{cases} j_l(kr) = (\frac{\pi}{2kr})^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr) \\ \underset{\uparrow}{n_l}(kr) = (\frac{\pi}{2kr})^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr)\end{cases} \& \begin{cases} h_l^{(1)}(kr) = j_l(kr) + in_l(kr) \\ \underset{\uparrow}{h_l^{(2)}}(kr) = j_l(kr) - in_l(kr)\end{cases} \quad (9.85)$$

spherical Bessel functions spherical Hankel functions

$\Rightarrow \psi(\mathbf{x},\omega) = \sum_{lm} f_\ell(r)P_l^m(\cos\theta)e^{im\varphi}$ $\leftarrow$ Use $h_l^{(1)}$ & $h_l^{(2)}$ to represent $f_\ell(r)$

$$= \sum_{lm}[A_{lm}^{(1)}h_l^{(1)}(kr) + A_{lm}^{(2)}h_l^{(2)}(kr)]Y_{lm}(\theta,\phi) \quad [k = \frac{\omega}{c}] \quad (9.92)$$

where $Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$ [(3.53)]



From G. Afken, "Mathematical Methods for Physicists"

$$\begin{cases} j_l(x) \xrightarrow{x \ll 1 \text{ or } l} \dfrac{x^l}{(2l+1)!!} \\ n_l(x) \xrightarrow{x \ll 1 \text{ or } l} -\dfrac{(2l-1)!!}{x^{l+1}} \end{cases} \quad (9.88) \qquad \begin{cases} j_l(x) \xrightarrow{x \gg 1 \ \& \ l} \frac{1}{x}\sin(x-\frac{l\pi}{2}) \\ n_l(x) \xrightarrow{x \gg 1 \ \& \ l} -\frac{1}{x}\cos(x-\frac{l\pi}{2}) \end{cases} \quad (9.89)$$

$$(9.89) \Rightarrow h_l^{(1)}(x) \xrightarrow{x \gg 1 \ \& \ l} (-i)^{l+1}\frac{e^{ix}}{x}; \quad h_l^{(2)}(x) \xrightarrow{x \gg 1 \ \& \ l} (i)^{l+1}\frac{e^{-ix}}{x}$$

See pp. 426-427 for more properties of j_l, n_l, and $h_l^{(1,2)}$.

Expansion of the Green Function: Sec. 6.4 shows the the Greens eq. $(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ $[k=\frac{\omega}{c}]$ [(6.36)] has the solution:

$$G(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad \left[\begin{array}{l}\text{in infinite space; for outgoing} \\ \text{wave b.c. } (\because e^{-i\omega t} \text{ dependence})\end{array}\right] \quad [(6.40)]$$

We now solve (6.36) by dividing the space into $r<r'$ & $r>r'$ regions.

Write $G(\mathbf{x},\mathbf{x}') = \sum\limits_{lm} g_l(r,r')Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$ and let

$$\boxed{\begin{array}{l}\mathbf{x} = (r,\theta,\varphi) \\ \mathbf{x}' = (r',\theta',\varphi')\end{array}}$$

$$\begin{cases} g_l(r,r') = A_l j_l(kr) \text{ for } r<r' \ [n_l(kr) \to \infty \text{ as } r \to 0] \\ g_l(r,r') = B_l h_l^{(1)}(kr) \text{ for } r>r' \ [h_l^{(2)}(kr) \to \text{incoming wave as } r \to \infty] \end{cases}$$

then, b.c.'s at $r=r'$ give A_l, B_l in terms of r' (as in Sec. 3.9). The result is

$$G(\mathbf{x},\mathbf{x}') = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

where $r_<$ and $r_>$ are the smaller and larger of r and r'.

Equating the two expressions above for $G(\mathbf{x},\mathbf{x}')$, we obtain

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (9.98)$$

Summary of Differential Equations and Solutions:

Source-free D.E.	Laplace eq. $\nabla^2\phi = 0$	Helmholtz eq. $\left(\nabla^2 + k^2\right)\psi = 0$
Solutions: Cartesian	$e^{i\alpha x},\ e^{i\beta y},\ e^{\sqrt{\alpha^2+\beta^2}z}$, etc. (Sec. 2.9)	$e^{ik_x x},\ e^{ik_y y},\ e^{ik_z z}$, etc. (Sec. 8.4)
cylindrical	$J_m(kr),\ e^{im\theta},\ e^{kz}$, etc. (Sec. 3.7)	$J_m\left(\sqrt{\frac{\omega^2}{c^2}-k_z^2}\ r\right), e^{im\theta}, e^{ik_z z}$, etc. (Sec. 8.7)
spherical	$Y_{lm}(\theta,\phi),\ r^l$, etc. (Secs. 3.1, 3.2)	$Y_{lm}(\theta,\phi),\ j_l(kr),\ n_l(kr)$, etc. (Sec. 9.6)
D.E. with a point source	$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: $G(\infty)=0$	$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave
Solutions (Green functions)	$G = \frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$G = \frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]
Series expansion of Green function	Eqs. (3.70), (3.148), (3.168)	Eq. (9.98)

9.1 Radiation of a Localized Oscillating Source

Review of Inhomogeneous Wave Eqs. and Solus. in Ch. 6:

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\frac{\rho}{\varepsilon_0} & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

[The medium is free space. Φ, $\mathbf{A}$ satisfy Lorenz cond.: $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$ [(6.14)]]

Basic form of (6.15), (6.16): $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t)$ (6.32)

The solution of (6.32) with outgoing-wave b.c. is

$$\psi(\mathbf{x},t) = \underbrace{\psi_{in}(\mathbf{x},t)}_{\text{due to incoming wave}} + \int d^3x' \int dt' \underbrace{G^+(\mathbf{x},t,\mathbf{x}',t')}_{\text{due to } \rho,\ \mathbf{J} \text{ in (6.16), (6.16)}} f(\mathbf{x}',t'), \quad (6.45)$$

where $G^+(\mathbf{x},t,\mathbf{x}',t') = \dfrac{\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|}$ [$\Rightarrow f(\mathbf{x}',t')$ in (6.45) is evaluated at the retarded time t'.] (6.44)

is the solution of

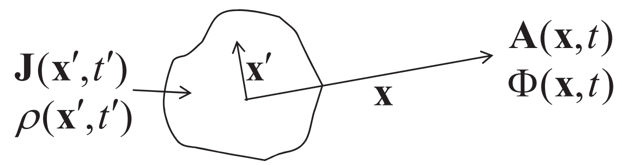
$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad (6.41)$$

with outgoing-wave b.c. $\psi_{in} = 0$ if there is no incoming wave.

Using (6.45) on (6.15) & (6.16) and letting $\psi_{in} = 0$, we obtain the gereral solutions for $\mathbf{A}$ and Φ, valid for arbitrary $\mathbf{J}$ and ρ.

$$\begin{Bmatrix} \mathbf{A}(\mathbf{x},t) \\ \Phi(\mathbf{x},t) \end{Bmatrix} = \frac{1}{4\pi}\int d^3x' \int dt' \frac{\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \mu_0 \mathbf{J}(\mathbf{x}',t') \\ \dfrac{\rho(\mathbf{x}',t')}{\varepsilon_0} \end{Bmatrix} \tag{9.2}$$

Note: (9.2) is also derived in lecture notes, Ch. 6, Eq. (7).



Question: It is stated on p. 408 that (9.2) is valid provided no boundary surfaces are present. Why? [See discussion on (6.44) in Ch. 6 of lectures notes.]

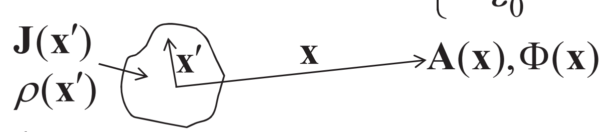
If either $\mathbf{J}$ or ρ is static or contains a static part, i.e. $\mathbf{J}(\mathbf{x}',t') = \mathbf{J}(\mathbf{x}')$ or $\rho(\mathbf{x}',t') = \rho(\mathbf{x}')$, the delta function in (9.2) can be easily integrated to result in the static $\mathbf{A}(\mathbf{x})$ [(5.32)] or $\Phi(\mathbf{x})$ [(1.17)].

Fields by Harmonic Sources : Only time-dependent sources can radiate. Radiation from moving charges are treated in Ch. 14. Here, we specialize to sources of the form (as in an attenna):

$$\begin{cases} \mathbf{J}(\mathbf{x},t) = \mathrm{Re}[\mathbf{J}(\mathbf{x})e^{-i\omega t}] \\ \rho(\mathbf{x},t) = \mathrm{Re}[\rho(\mathbf{x})e^{-i\omega t}] \end{cases} \quad \left[\begin{matrix} \mathbf{J}(\mathbf{x}),\ \rho(\mathbf{x}) \text{ are complex} \\ \omega\text{-space quantities.} \end{matrix}\right] \tag{9.1}$$

Sub. (9.1) into $\begin{Bmatrix} \mathbf{A}(\mathbf{x},t) \\ \Phi(\mathbf{x},t) \end{Bmatrix} = \frac{1}{4\pi}\int d^3x' \int dt' \frac{\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \mu_0 \mathbf{J}(\mathbf{x}',t') \\ \dfrac{\rho(\mathbf{x}',t')}{\varepsilon_0} \end{Bmatrix}$

and carry out the integration over t', we obtain



$$\begin{cases} \mathbf{A}(\mathbf{x},t) = \mathrm{Re}[\mathbf{A}(\mathbf{x})e^{-i\omega t}] \\ \Phi(\mathbf{x},t) = \mathrm{Re}[\Phi(\mathbf{x})e^{-i\omega t}] \end{cases} \text{with} \begin{cases} \mathbf{A}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \\ \Phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}') \end{cases} \tag{9.3}$$

where $k = \frac{\omega}{c}$ and $\mathbf{A}(\mathbf{x})$, $\Phi(\mathbf{x})$ are complex ω-space quantities.

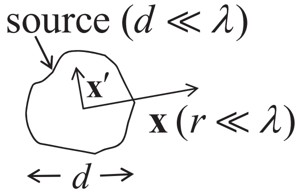
Property of near fields (kr & $kr' \ll 1$):

Before going into algebraic details, we may readily observe a very important property of fields near the source. Rewrite

$$\begin{cases} \mathbf{A}(\mathbf{x},t) = \mathrm{Re}[\mathbf{A}(\mathbf{x})e^{-i\omega t}] \\ \Phi(\mathbf{x},t) = \mathrm{Re}[\Phi(\mathbf{x})e^{-i\omega t}] \end{cases} \text{with} \begin{cases} \mathbf{A}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x' \dfrac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \\ \Phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3x' \dfrac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}') \end{cases} \quad (9.3)$$

If r & d (d : source dimension) $\ll \lambda$, then kr & $kr' \ll 1$ and we have

$$e^{ik|\mathbf{x}-\mathbf{x}'|} \approx 1 \Rightarrow \begin{cases} \mathbf{A}(\mathbf{x}) \approx \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x' \dfrac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ \Phi(\mathbf{x}) \approx \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3x' \dfrac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \end{cases} \quad (1)$$



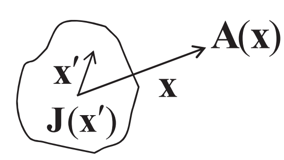
$\Rightarrow$ Under r & $d \ll \lambda$ (or kr & $kr' \ll 1$), the *spatial* profiles of $\mathbf{A}(\mathbf{x})$ & $\Phi(\mathbf{x})$ are approx. the same as the *static* fields due to $\mathbf{J}(\mathbf{x})$ [(5.32)] & $\rho(\mathbf{x})$ [(1.17)], but both are multiplied by $e^{-i\omega t}$ (see p. 408, bottom).

General formalism: Rewrite $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$ [(9.3)]

$$\text{Maxwell eqs.} \Rightarrow \begin{cases} \mathbf{H} = \dfrac{1}{\mu_0}\nabla\times\mathbf{A} \quad \text{(everywhere)} & (9.4) \\ \mathbf{E} = \dfrac{iZ_0}{k}\nabla\times\mathbf{H} \quad \text{(outside the source, no } \mathbf{J}) & (9.5) \end{cases}$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$ (free space impedance, p. 297).

Thus, given a *prescribed* $\mathbf{J}(\mathbf{x})$ [indep. of $\mathbf{A}(\mathbf{x})$], we may evaluate $\mathbf{A}(\mathbf{x})$ from (9.3), then obtain $\mathbf{H}$, $\mathbf{E}$ from (9.4), (9.5). *Note*: If $\mathbf{J}(\mathbf{x})$ depends on $\mathbf{A}(\mathbf{x})$, then (9.3) is an integral equation for $\mathbf{A}(\mathbf{x})$.



Question: Show that the charge density (ρ) and scalar potential (Φ) are implicit in $\mathbf{J}$ and $\mathbf{A}$, hence not required in determining $\mathbf{H}$ & $\mathbf{E}$.

Ans: With $e^{-i\omega t}$ dependence, $\nabla\cdot\mathbf{J} + \frac{\partial}{\partial t}\rho = 0 \Rightarrow \rho = \frac{\nabla\cdot\mathbf{J}}{i\omega}$. Similarly, $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$ [Lorenz condition, (6.14)] $\Rightarrow \Phi = \frac{c^2\nabla\cdot\mathbf{A}}{i\omega}$.

We may expand $\mathbf{A}(\mathbf{x})$ exactly in spherical coordinates by using

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}=4\pi ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \quad [(9.98)]$$

For $\mathbf{x}$ outside the source, we have $r_> = |\mathbf{x}| = r,\ r_< = |\mathbf{x}'| = r'$. Hence,

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}=4\pi ik\sum_{l=0}^{\infty} j_l(kr')h_l^{(1)}(kr)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Sub. this equation into $\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$ [(9.3)]

$$\Rightarrow \mathbf{A}(\mathbf{x})=\mu_0 ik\sum_{l,m} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')j_l(kr')Y_{lm}^*(\theta',\phi') \quad (9.11)$$

We will skip (9.6)-(9.10) which are not general, e.g. (9.6) needs to be modified by (9.12) to be valid for any $\mathbf{x}$. Instead, we make use of

$$\begin{cases} h_l^{(1)}(kr)=\dfrac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}}\sum\limits_{n=0}^{l} a_n(ikr)^n & (2a) \\ \text{with } a_n=\dfrac{(-1)^n(2l-n)!}{(2l-1)!!(l-n)!2^{\ell-n}n!} & (2b) \end{cases}$$

[an exact expression for $h_l^{(1)}(kr)$ derived on next page]

9.1 Radiation of a Localized Oscillating Source (*continued*)

Exercise: Derive (2a,b) from $h_l^{(1)}(kr)=(-i)^{\ell+1}\frac{e^{ikr}}{kr}\sum_{s=0}^{l}\frac{i^s}{s!(2kr)^s}\frac{(\ell+s)!}{(\ell-s)!}$

The eq. above is (11.152) in Arfken's "Math. Meth. for Phys." (3rd ed.) or (10.1.16) in Abramowitz/Stegun's "Handbook of Math. Funcs." We need to convert it to (2a,b) to derive Jackson's results.

Write $h_l^{(1)}(kr)$ as $h_l^{(1)}(kr)=\frac{e^{ikr}}{(kr)^{l+1}}\sum_{s=0}^{l}\frac{(-i)^{\ell+1}i^s}{s!2^s(kr)^{s-\ell}}\frac{(\ell+s)!}{(\ell-s)!}$

Let $s=\ell-n$ and use $\left[\begin{array}{l}(-i)^{\ell+1}i^{\ell-n}=(-1)^{\ell+1}i^{\ell+1}i^{\ell-n}=i(-1)^{\ell+1}i^{2\ell-2n+n} \\ =i(-1)^{\ell+1}(-1)^{\ell-n}i^n=-i(-1)^{2\ell}(-1)^{-n}i^n=-i(-1)^n i^n\end{array}\right]$

$$\Rightarrow h_l^{(1)}(kr)=\frac{e^{ikr}}{(kr)^{l+1}}\sum_{n=0}^{l}\frac{(-i)^{\ell+1}i^{\ell-n}(kr)^n}{(\ell-n)!2^{\ell-n}}\frac{(2\ell-n)!}{n!}$$

[Multiply the RHS by $\frac{(2l-1)!!}{(2l-1)!!}$ (=1) to get the a_n in (9.12)] $\Rightarrow$

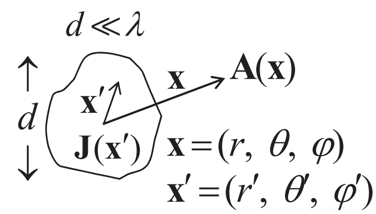
$$\begin{cases} h_l^{(1)}(kr)=\dfrac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}}\sum\limits_{n=0}^{l} a_n(ikr)^n & [(2a)] \\ \text{with } a_n=\dfrac{(-1)^n(2l-n)!}{(2l-1)!!(l-n)!2^{\ell-n}n!} & [(2b)] \end{cases}$$

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr) Y_{lm}(\theta,\phi) \int d^3x' \mathbf{J}(\mathbf{x}') j_l(kr') Y_{lm}^*(\theta',\phi') \text{ in (9.11)}$$

is exact for any size ($\sim d$) of the source and any $\mathbf{x}$ outside the source.

To apply the small-argument limit of $j_l(kr')$, we assume $d \ll \lambda$ and

$$\text{sub} \begin{cases} h_l^{(1)}(kr) = \dfrac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}} \sum\limits_{n=0}^{l} a_n (ikr)^n \ \text{[(2a)]} \\ j_l(kr')\big|_{kr' \ll 1} \approx \dfrac{(kr')^l}{(2l+1)!!} \ \text{[(9.88)]} \end{cases}$$



into (9.11) to obtain

$$\mathbf{A}(\mathbf{x}) \underset{d \ll \lambda}{=} \mu_0 \sum_{l,m} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} \sum_{n=0}^{l} a_n (ikr)^n \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y_{lm}^*(\theta',\phi'), \quad (3)$$

where $a_n = \dfrac{(-1)^n (2l-n)!}{(2l-1)!!(l-n)! 2^{\ell-n} n!}$ [(2b)]

Note: (3) is valid for $d \ll \lambda$ and any $\mathbf{x}$ outside the source. In comparison, (9.6) in Jackson is valid for $d \ll \lambda$ and $x \ll \lambda$. It needs to be modified by (9.12) to get (3), but a_n in (9.12) is unspecified.

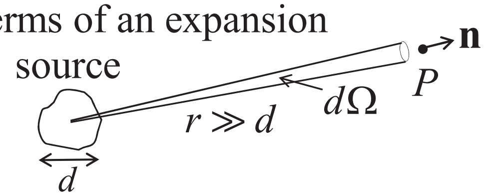
Three zones: Divide the region outside the source into 3 zones:

$$\begin{cases} \text{The } \underline{\text{near (static) zone}}: & d \ll r \ll \lambda \ (\Rightarrow kr \ll 1) \\ \text{The } \underline{\text{intermediate (induction) zone}}: & d \ll r \sim \lambda \ \ (\Rightarrow kr \sim 1) \\ \text{The } \underline{\text{far (radiation) zone}}: & d \ll \lambda \ll r \ (\Rightarrow kr \gg 1) \end{cases}$$

More generally, the far zone is $r \gg d, \lambda$ and the near zone is $r, d \ll \lambda$. The near zone features spatial profiles of the static fields [see (1)]. The far zone feature radiation fields (next section).

Secs. 9.1-9.3 (but not Sec. 9.4) assume $d \ll \lambda$ (not a general far-zone requirement) for all zones for the convenience of using $j_l(kr')\big|_{kr' \ll 1} \approx \dfrac{(kr')^l}{(2l+1)!!}$ [(9.88)], hence (3) $\left[\begin{array}{l}\text{inapplicable to large} \\ \text{antennas } (d \gg \text{ or } \sim \lambda\end{array}\right]$

All Secs. assume $d \ll r$ (not a general near-zone requirement) for all zones, so the distance from any source pt. to the observation pt. P can be approximated by 1 or 2 terms of an expansion [see (4)]. Also, the direction of $d\Omega$ from all source points to P can be approximated by the same $\mathbf{n}$.



9.2 Electric Dipole Fields and Radiation

Rewrite the expression valid for $d \ll \lambda$ and any $\mathbf{x}$ outside the source:

$$\mathbf{A}(\mathbf{x}) \underset{d \ll \lambda}{=} \mu_0 \sum_{l,m} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} \sum_{n=0}^{l} a_n (ikr)^n \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y_{lm}^*(\theta',\phi') \quad [(3)]$$

Take the $l=0$ term [$a_0 = 1$ by (2b); $Y_{00} = \frac{1}{\sqrt{4\pi}}$] and denote it by $\mathbf{A}^p(\mathbf{x})$.

$$\mathbf{A}^p(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{l=0} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \mathbf{J}(\mathbf{x}')$$

$$= -\frac{i\mu_0\omega}{4\pi} \mathbf{p} \frac{e^{ikr}}{r}, \qquad (9.16)$$

where $\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'$ (4.8) & (9.13)

Note: 1. $\mathbf{A}^p$ (due to $\mathbf{p}$) represents electric dipole radiation.

2. There is no monopole radiation (see p. 410).

3. (9.16) is valid in all 3 zones.

$$\iiint J_x dxdydz \quad \left(\int J_x dx = xJ_x - \int x dJ_x\right)$$

$$= \iint dydz\left[\cancel{xJ_x\Big|_{-d}^{d}} - \int x \frac{\partial J_x}{\partial x} dx\right]$$

$$= -\iiint x\left(\frac{\partial J_x}{\partial x} + \underbrace{\frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}\right) dxdydz$$

give no contribution because $\mathbf{J}$ is localized: $\int \frac{\partial J_y}{\partial y} dy = J_y\Big|_{-d}^{d} = 0$

$$= -\int x \nabla \cdot \mathbf{J} d^3x \qquad = -\frac{\partial \rho}{\partial t} = i\omega\rho$$

$$\Rightarrow \int \mathbf{J} d^3x = -\int \mathbf{x} \overbrace{\nabla \cdot \mathbf{J}} d^3x$$

$$= -i\omega \int \mathbf{x} \underbrace{\rho(\mathbf{x})} d^3x = -i\omega\mathbf{p}$$

ρ comes from $\mathbf{J}$ (see above)

9.2 Electric Dipole Fields and Radiation (*continued*)

Rewrite $\mathbf{A}^p(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p} \frac{e^{ikr}}{r}$ [(9.16)] $\quad \boxed{\mathbf{p} = const}$

$$\Rightarrow \mathbf{H}^p = \frac{\nabla \times \mathbf{A}^p}{\mu_0} = -\frac{i\omega}{4\pi}\left[\left(\nabla \frac{e^{ikr}}{r}\right) \times \mathbf{p} + \frac{e^{ikr}}{r} \cancel{\nabla \times \mathbf{p}}\right]$$

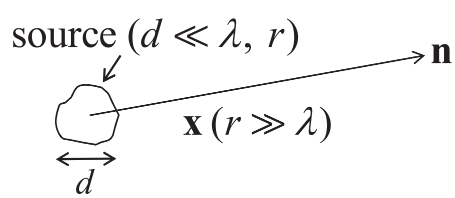
$$\left[\text{use } \nabla r = \frac{\partial r}{\partial r}\mathbf{e}_r = \mathbf{e}_r = \mathbf{n}\right]$$

$$\boxed{\mathbf{E}^p = \frac{iZ_0}{k} \nabla \times \mathbf{H}^p \ [(9.5)]}$$

$$\Rightarrow \begin{cases} \mathbf{H}^p = \frac{ck^2}{4\pi}(\mathbf{n} \times \mathbf{p}) \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \\ \mathbf{E}^p = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n} \times \mathbf{p}) \times \mathbf{n} \frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n} \cdot \mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr}\right\} \end{cases} \qquad (9.18)$$

In the far zone ($kr \gg 1$), (9.18) reduces to a <u>spherical wave</u>

$$\begin{cases} \mathbf{H}^p \approx \frac{ck^2}{4\pi}(\mathbf{n} \times \mathbf{p}) \frac{e^{ikr}}{r} \\ \mathbf{E}^p \approx Z_0 \mathbf{H}^p \times \mathbf{n} \end{cases} \left[\begin{matrix}\text{far} \\ \text{zone}\end{matrix}\right] \qquad (9.19)$$

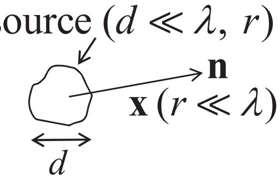


In (9.19), we see that $\mathbf{E}^p$, $\mathbf{H}^p$ are in phase, and $\mathbf{E}^p$, $\mathbf{H}^p$, $\mathbf{n}$ are mutually perpendicular. This is the general property of EM waves in unbounded, uniform space. The $e^{ikr-i\omega t}/r$ dependence represents an outgoing spherical wave.

$$\left\{\begin{aligned} \mathbf{H}^p &= \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \mathbf{E}^p &= \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}\end{aligned}\right. \quad [(9.18)]$$

In the near zone ($kr \ll 1$), (9.18) reduces to

$$\left\{\begin{aligned} \mathbf{H}^p &\approx \frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r^2} \\ \mathbf{E}^p &\approx \frac{1}{4\pi\varepsilon_0}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\frac{1}{r^3}\end{aligned}\right. \quad \begin{bmatrix}\text{near} \\ \text{zone}\end{bmatrix} \quad (9.20)$$



$\Rightarrow$
- (i) $\mathbf{E}^p$, $\mathbf{H}^p$ in (9.20) are 90° out of phase $\Rightarrow$ average power = 0.
- (ii) $\mathbf{E}^p$ has the same spatial pattern as the static electric dipole in (4.13), but with $e^{-i\omega t}$ dependence. This is expected from (1).
- (iii) $\mu_0|H|^2 \sim (kr)^2\varepsilon_0|E|^2 \Rightarrow$ **E**-field energy $\gg$ **B**-field energy.

Questions: 1. To obtain the near-zone field [(9.20)] from (9.18), 3 small terms are neglected in (9.18). But 2 of the neglected terms are physically important. Which two? In what sense are they important?

2. E, B have different dimensions. How to compare their strength?

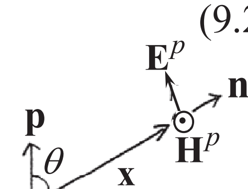
$\left\langle \frac{dP}{d\Omega}\right\rangle_t$ = time-averaged power in the far zone/unit solid angle

$$= \tfrac{1}{2}\mathrm{Re}[r^2\mathbf{n}\cdot(\mathbf{E}^p\times\mathbf{H}^{p*})] \quad (9.21)$$

$\boxed{(9.19)} \rightarrow$

$$= \frac{c^2Z_0}{32\pi^2}k^4\,|(\mathbf{n}\times\mathbf{p})\times\mathbf{n}|^2 \quad \text{[far zone]} \quad (9.22)$$

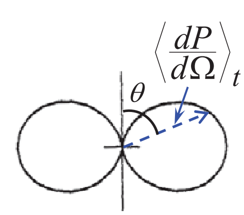
In general, $\mathbf{p} = p_x e^{i\alpha}\mathbf{e}_x + p_y e^{i\beta}\mathbf{e}_y + p_z e^{i\gamma}\mathbf{e}_z$. If $\alpha=\beta=\gamma$, then $\mathbf{p}$ has a fixed direction and the vector $(\mathbf{n}\times\mathbf{p})\times\mathbf{n}$ in (9.22) lies on the $\mathbf{n}$-$\mathbf{p}$ plane, giving the direction of $\mathbf{E}^p$ [see (9.19)], i,e, the polarization of the radiation. With $\alpha=\beta=\gamma$, (9.22) gives



source ($d \ll \lambda$)
e.g. radio station

$$\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta \quad \begin{bmatrix}\theta: \text{angle btn.} \\ \mathbf{p} \text{ and } \mathbf{n}\end{bmatrix} \quad (9.23)$$

$$\Rightarrow \langle P\rangle_t = \text{total power radiated} = \frac{c^2Z_0k^4}{12\pi}|\mathbf{p}|^2 \quad (9.24)$$



dipole radiation pattern
(maximal at $\theta = 90^\circ$)

Note: (9.24) [but not (9.23)] is valid even if $\alpha=\beta=\gamma$ is not true (p.412, top).

9.3 Magnetic Dipole and Electric Quadrupole Field

Rewrite the expression valid for $d \ll \lambda$ and any $\mathbf{x}$ outside the source:

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \tfrac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} \sum_{n=0}^{l} a_n (ikr)^n \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y_{lm}^*(\theta',\phi') \text{ [(3)]}$$

Take the $l=1$ ($m=-1,0,1$) terms and use $a_0 = 1,\ a_1 = -1$ [from (2b)]

$$\Rightarrow \mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{3}\frac{e^{ikr}}{r^2}(1-ikr) \sum_{m=-1,0,1} Y_{1m}(\theta,\phi) \int d^3x' \mathbf{J}(\mathbf{x}') r' Y_{1m}^*(\theta',\phi')$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\tfrac{1}{r}-ik) \int d^3x' \mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') \qquad (9.30)$$

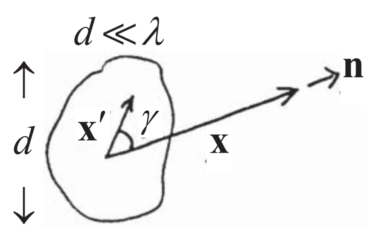
p. 109

$$\sum_{m=-1,0,1} Y_{1m}(\theta,\phi) Y_{1m}^*(\theta',\phi') = \tfrac{3}{8\pi}\sin\theta\sin\theta' e^{i(\phi-\phi')} + \tfrac{3}{4\pi}\cos\theta\cos\theta' + \tfrac{3}{8\pi}\sin\theta\sin\theta' e^{-i(\phi-\phi')}$$

$$= \tfrac{3}{4\pi}[\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta']$$

$$= \tfrac{3}{4\pi}\cos\gamma = \tfrac{3}{4\pi r'}\mathbf{n}\cdot\mathbf{x}' \text{ [}\gamma\text{: angle between } \mathbf{x},\ \mathbf{x}'\text{]}$$

set $l=1$ in (3.68). Also see (1) of Ch. 3.



Use $(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} = (\mathbf{n}\cdot\mathbf{J})\mathbf{x}' + (\mathbf{x}'\times\mathbf{J})\times\mathbf{n}$ [by standard vector formula]

$$= \tfrac{1}{2}(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + \tfrac{1}{2}[(\mathbf{n}\cdot\mathbf{J})\mathbf{x}' + (\mathbf{x}'\times\mathbf{J})\times\mathbf{n}] \qquad (9.31)$$

We obtain $\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\frac{1}{r}-ik) \int d^3x' \mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}')$ [(9.30)]

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\tfrac{1}{r}-ik)\Big\{ \underbrace{\int d^3x' \tfrac{1}{2}(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}}_{\text{magnetic dipole radiation}} + \underbrace{\int d^3x' \tfrac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\mathbf{x}']}_{\text{electric quadrupole radiation}} \Big\}$$

$$= \mathbf{A}^m + \mathbf{A}^Q,$$

where $\mathbf{A}^m(\mathbf{x}) = \frac{ik\mu_0}{4\pi}(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$ $\left[\begin{array}{l}\text{for } kd \ll 1 \text{ and any} \\ \mathbf{x} \text{ outside the source}\end{array}\right]$ (9.33)

with $\mathbf{m} = \frac{1}{2}\int(\mathbf{x}\times\mathbf{J})d^3x$ [magnetic dipole moment] (5.54) and (9.34)

$\mathbf{A}^m$ gives the magnetic dipole fields through (9.4) and (9.5):

$$\begin{cases} \mathbf{H}^m = \frac{1}{4\pi}\left\{k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}](\frac{1}{r^3} - \frac{ik}{r^2})e^{ikr}\right\} & (9.35) \\ \mathbf{E}^m = -\frac{Z_0}{4\pi}k^2(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}(1-\frac{1}{ikr}) & (9.36) \end{cases}$$

In the far zone ($kr \gg 1$), we have the spherical wave sloution:

$$\begin{cases} \mathbf{H}^m \approx \frac{k^2}{4\pi}(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r} \\ \mathbf{E}^m \approx Z_0\mathbf{H}^m \times\mathbf{n} \end{cases} \Rightarrow \begin{cases} \left\langle \frac{dP}{d\Omega}\right\rangle_t \approx \frac{Z_0}{32\pi^2}k^4 |\underbrace{\mathbf{m}\times\mathbf{n}}|^2 \\ \langle P\rangle_t \approx \frac{Z_0}{12\pi}k^4 |\mathbf{m}|^2 \end{cases}$$

$\boxed{\Rightarrow \text{direction of } \mathbf{E}^m}$

In the near zone ($kr \ll 1$),

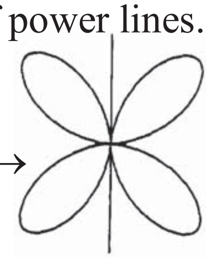
$$\begin{cases} \mathbf{H}^m \approx \frac{1}{4\pi}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}]\frac{1}{r^3} \\ \mathbf{E}^m \approx \frac{Z_0 k}{4\pi i}(\mathbf{n}\times\mathbf{m})\frac{1}{r^2} \end{cases} \Rightarrow$$

(i) $\mathbf{E}^m$ and $\mathbf{H}^m$ are 90^{o} out of phase
$\Rightarrow$ average power $= 0$.
(ii) $\mathbf{H}^m$ has the same spatial pattern as that of the static magnetic dipole in (5.56), but with $e^{-i\omega t}$ dependence.
(iii) B-field energy $\gg$ E-field energy.

Note: $\mathbf{H}^m$ is the dominant field in the near zone of power lines.

The electric quadrupole radiation [discussed in (9.37)-(9.52)] is more complicated. Here, we only illustrate its radiation pattern (right figure).

electric quadrupole radiation pattern $\rightarrow$

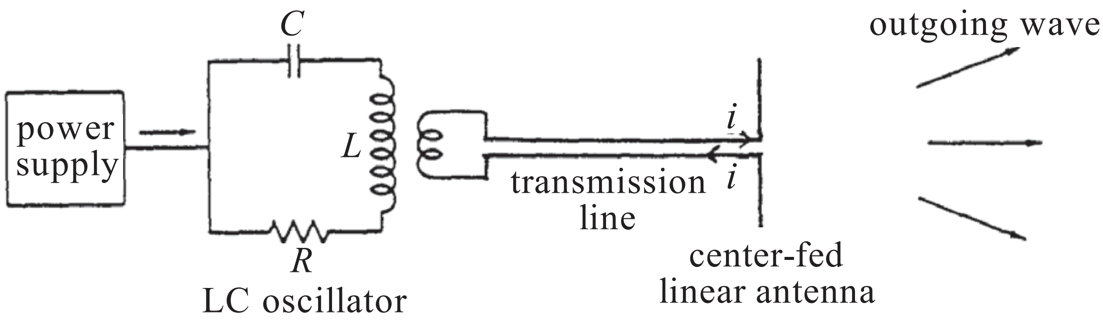


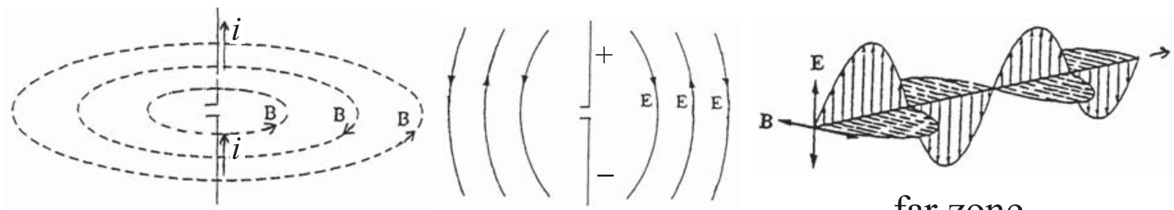
Comparison between Static and Time-dependent Cases

	relations between ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$	multipole expansion	definition of multipole moments	r-dependence of $\mathbf{E}$ and $\mathbf{B}$ (d : dimension of the source)
static case	$\rho(\mathbf{x}) \leftrightarrow \mathbf{E}(\mathbf{x})$ $\mathbf{J}(\mathbf{x}) \leftrightarrow \mathbf{B}(\mathbf{x})$	spherical harmonics expansion [(3.70)] or Taylor series [(4.10)] of $\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$q = \int \rho(\mathbf{x}')d^3x'$ $\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x'$ $Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')d^3x'$ $\mathbf{m} = \frac{1}{2}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'$	$\mathbf{E}$ or $\mathbf{B} \propto 1/r^{l+2}$ For $r \sim d$, all multipole fields can be significant. For $r \gg d$, multipole fields are dominated by the lowest-order nonvanishing term.
time-dependent case	$\begin{Bmatrix}\rho(\mathbf{x})\\ \updownarrow \\ \mathbf{J}(\mathbf{x})\end{Bmatrix} \leftrightarrow \begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \updownarrow \\ \mathbf{B}(\mathbf{x})\end{Bmatrix}$ $\Rightarrow$ EM waves	spherical harmonics expansion [(9.98)] of $\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	There is no time-dependent monopole for an isolated source (see p. 410). $\mathbf{p}$, Q_{ij}, and $\mathbf{m}$ have the same expressions as those of their static counterparts, but with the $e^{-i\omega t}$ time dependence. In time-dependent cases, electric multipoles can generate $\mathbf{B}$-fields and magnetic multipoles can generate $\mathbf{E}$-fields.	(a) near zone $\lambda \gg r \gg d$ $\mathbf{E}$ or $\mathbf{B} \propto e^{-i\omega t}/r^{l+2}$ Approx. the same field pattern and r-dependence as for the corresponding static multipole, but with $e^{-i\omega t}$ dependence (hence called *quasi-static* fields.) (b) far zone $r \gg \lambda \gg d$ $\mathbf{E}, \mathbf{B} \propto e^{ikr-i\omega t}/r$ [see (3)] (spherical EM waves) All multipole fields $\propto 1/r$, relative power levels unchanged with distance.

9.4 Center-Fed Linear Antenna

A Qualitative Look at the Center-Fed Linear Antenna :





near zone (static spatial fields) far zone

In the near zone, **E** and **B** are principally generated by ρ and **J**, respectively ($\Rightarrow$ largely static spatial field patterns). In the far zone, **E** and **B** are regenerative through $\frac{\partial}{\partial t}\mathbf{B}$ and $\frac{\partial}{\partial t}\mathbf{E}$ ($\Rightarrow$ EM waves).

Detailed Analysis: The center-fed linear antenna is a case of special interest, because it allows the solution of (9.3) in closed form for any value of kd, whereas in Secs. 9.2 and 9.3, we assume kd<<1.

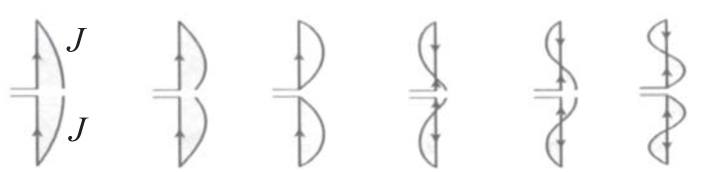
Rewrite $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$ (9.3)

Let $\mathbf{J}(\mathbf{x}) = I\sin(\frac{kd}{2} - k|z|)\delta(x)\delta(y)\mathbf{e}_z$ (9.53)

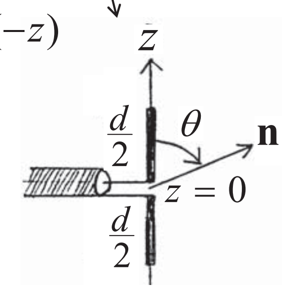
$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin(\frac{kd}{2} - k|z'|)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad [k = \frac{\omega}{c}]$$

a center-fed linear antenna

Note: (i) I = peak current only if $kd \geq \pi$. (ii) $\mathbf{J}(z) = \mathbf{J}(-z)$



$kd = \pi$	$3\pi/2$	2π	$5\pi/2$	3π	4π
$\lambda = 2d$	$4d/3$	d	$4d/5$	$2d/3$	$d/2$



Question: The antenna has open ends. How can there be **J** on it?

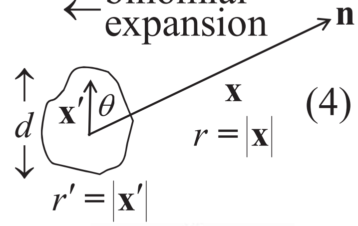
9.4 Center-fed Linear Antenna (*continued*)

$$|\mathbf{x}-\mathbf{x}'| = (r^2 - 2rr'\cos\theta + r'^2)^{\frac{1}{2}} = r[1-(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})]^{\frac{1}{2}}$$

$$= r[1 - \tfrac{1}{2}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2}) - \tfrac{1}{8}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})^2 + \cdots] \quad \leftarrow \text{binomial expansion}$$

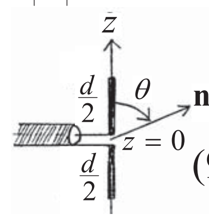
$$= r - \mathbf{n}\cdot\mathbf{x}' + \tfrac{1}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \cdots \qquad (4)$$

$$\Rightarrow |\mathbf{x}-\mathbf{x}'| \approx r - \mathbf{n}\cdot\mathbf{x}' = r - z'\cos\theta \quad \text{if } r \gg d$$



$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin(\frac{kd}{2} - k|z'|)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

$$\underset{\uparrow \boxed{r \gg d}}{\approx} \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin(\frac{kd}{2} - k|z'|)e^{-ikz'\cos\theta}}{\underbrace{r - z'\cos\theta}_{\approx r}} \qquad (9.54)$$

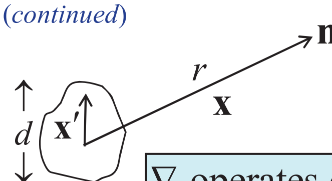


$$= \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})}{\sin^2\theta}\right] \quad \begin{bmatrix}\text{for } r \gg d \\ \text{and any } k\end{bmatrix} \qquad (9.55)$$

Note: $z'\cos\theta$ in $\frac{1}{r - z'\cos\theta}$ can be neglected if $r \gg d$. But $z'\cos\theta$ in $e^{ikz'\cos\theta}$ is an important part of the phase angle even if $r \gg d$.

9.4 Center-fed Linear Antenna (*continued*)

$$\text{Rewrite } \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \qquad (9.3)$$



$\boxed{\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}}$ $\quad$ $\boxed{\nabla \text{ operates on } \mathbf{x}}$

$$\nabla\times\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'[\nabla\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{J}(\mathbf{x}') + \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\cancel{\nabla\times\mathbf{J}(\mathbf{x}')}]$$

$$= \frac{\mu_0}{4\pi}\int d^3x'[e^{ik|\mathbf{x}-\mathbf{x}'|}(\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|})\times\mathbf{J}(\mathbf{x}') + \frac{1}{|\mathbf{x}-\mathbf{x}'|}(\nabla e^{ik|\mathbf{x}-\mathbf{x}'|})\times\mathbf{J}(\mathbf{x}')]$$

$$\boxed{\nabla|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \text{ [Eq. (1), Ch. 1]}}$$

$$= \frac{\mu_0}{4\pi}\int d^3x'[e^{ik|\mathbf{x}-\mathbf{x}'|}(-\frac{\mathbf{x}-\mathbf{x}'}{\underbrace{|\mathbf{x}-\mathbf{x}'|^3}_{\approx\frac{1}{r^2} \text{ if } r\gg d}})\times\mathbf{J}(\mathbf{x}') + \underbrace{\frac{ik}{|\mathbf{x}-\mathbf{x}'|}}_{\approx\frac{k}{r}=\frac{2\pi}{\lambda r} \text{ if } r\gg d} e^{ik|\mathbf{x}-\mathbf{x}'|}(\underbrace{\nabla|\mathbf{x}-\mathbf{x}'|}_{=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\underset{\uparrow r\gg d}{\approx}\mathbf{n}})\times\mathbf{J}(\mathbf{x}')]$$

Consider the far zone ($r \gg d, \lambda$) $\Rightarrow$ 1st term negligible

Note : k (or λ) can be any value provided $kr \gg 1$ (or $r \gg \lambda$).

$$\Rightarrow \nabla\times\mathbf{A}(\mathbf{x}) = ik\frac{\mu_0}{4\pi}\mathbf{n}\times\int d^3x'\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') = ik\mathbf{n}\times\mathbf{A}(\mathbf{x}) \quad \begin{bmatrix}\text{for any } k \\ \text{in far zone}\end{bmatrix}$$

$r \gg d, \lambda$ (far zone) $\qquad r \gg d, \lambda \qquad$ $\mathbf{A}$ is in z direction [(9.55)]

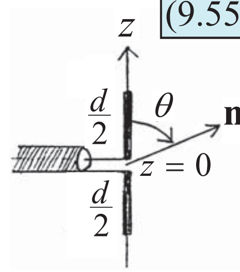
$$\mathbf{E} = Z_0 \mathbf{H}\times\mathbf{n}; \qquad \mathbf{H} = \frac{\nabla\times\mathbf{A}}{\mu_0} = \frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A} \Rightarrow |\mathbf{H}| = \frac{k\sin\theta}{\mu_0}|\mathbf{A}|$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle_t = \tfrac{1}{2}\mathrm{Re}\left[r^2 \mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{Z_0}{2} r^2 |\mathbf{H}|^2 = \frac{Z_0}{2\mu_0^2} k^2 r^2 \sin^2\theta \, |\mathbf{A}|^2 \quad (5)$$

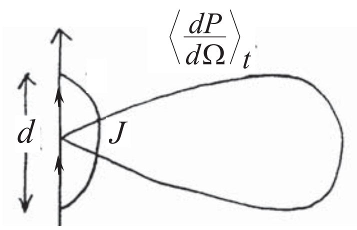
$\boxed{(9.55)}$

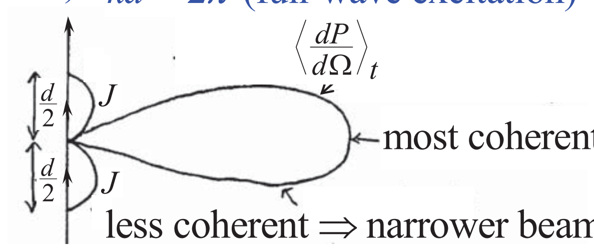
$$= \frac{Z_0 I^2}{8\pi^2} \frac{\left|\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})\right|^2}{\sin^2\theta} \quad \left[\begin{array}{l}\text{for any } k \text{ in far} \\ \text{zone } (r \gg d, \lambda)\end{array}\right] \quad (9.56)$$

$$= \frac{Z_0 I^2}{8\pi^2} \begin{cases} \cos^2(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = \pi \\ 4\cos^4(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = 2\pi \end{cases} \quad [k = \tfrac{\omega}{c}] \quad (9.57)$$



$kd = \pi$ (half-wave excitation) $\qquad kd = 2\pi$ (full-wave excitation)





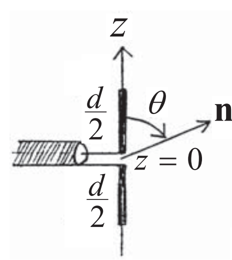
$\Rightarrow$ The smaller the $\frac{\lambda}{d}$ ratio, the narrower the radiation beamwidth.

Rewrite $\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I^2}{8\pi^2} \frac{\left|\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})\right|^2}{\sin^2\theta}$ $\left[\begin{array}{l}\text{for any } k \\ \text{in far zone}\end{array}\right]$ [(9.56)]

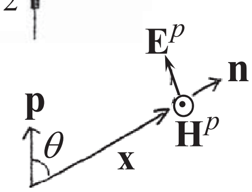
Limiting case in far zone: $kd \ll 1$ (i.e. $\lambda \gg d$)

Use $\cos x \simeq 1 - x^2/2$ $(x \ll 1)$

$$\Rightarrow \begin{cases} \cos\left(\frac{kd}{2}\cos\theta\right) \simeq 1 - \frac{k^2 d^2}{8}\cos^2\theta \\ \cos\left(\frac{kd}{2}\right) \simeq 1 - \frac{k^2 d^2}{8} \end{cases}$$



$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{8\pi^2} \frac{\left|1 - \frac{k^2 d^2}{8}\cos^2\theta - 1 + \frac{k^2 d^2}{8}\right|^2}{\sin^2\theta}$$

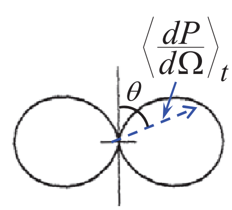


$$= \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta \quad \left[\begin{array}{l}\text{for } kd \ll 1 \\ \text{in far zone}\end{array}\right] \quad (6)$$

(6) has the same depedence on k and θ as

$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta \ [(9.23)],$$

which was derived for a diploe $\mathbf{p}$ under $kd \ll 1$.



dipole radiation

Radiation Resistance and Equivalent Circuit:

$$\mathbf{J}(\mathbf{x}) = I\sin(\tfrac{kd}{2} - k\,|\,z\,|)\delta(x)\delta(y)\mathbf{e}_z \underset{\boxed{kd \ll 1}}{\approx} \underbrace{\tfrac{kd}{2}I}_{I_0 \text{ (peak current, } \because |z| \le d)}(1 - \tfrac{2|z|}{d})\delta(x)\delta(y)\mathbf{e}_z$$

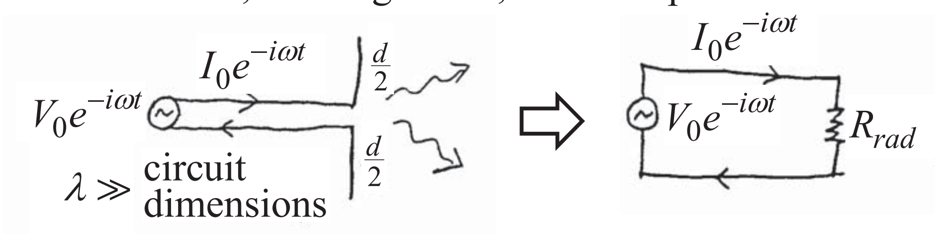
$$\Rightarrow \text{From (6), } \left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2 \sin^2\theta \quad (9.28)$$

$$\Rightarrow \langle P \rangle_t \approx \int \left\langle \frac{dP}{d\Omega} \right\rangle_t d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I_0^2}{48\pi}(kd)^2 \quad (9.29)$$

$$= \frac{I_0^2}{2} R_{rad}, \quad \left[\begin{array}{l} R_{rad}\text{: radiation resistance, part of the field} \\ \text{definition of Z [see 2nd term in (6.137)]} \end{array}\right]$$

where $R_{rad} \equiv \frac{Z_0}{24\pi}(kd)^2 \approx 5(kd)^2$ ohms [See pp. 412-3.]

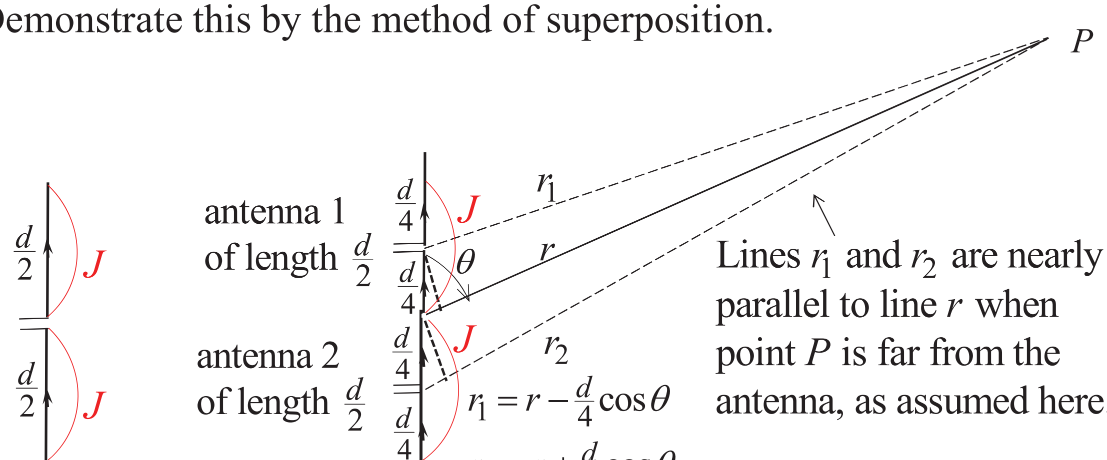
For $d \ll \lambda$, the longer d is, the more power it radiates/receives.



Equivalent circuit for a center-fed antenna

Problem 1 (*an exercise in wave superposition*):

A full-wave antenna of length d (left figure) should produce the same radiation as 2 half-wave antennas (right figure), each of length $d/2$, one above the other, and center-fed *in phase* by the same current. Demonstrate this by the method of superposition.

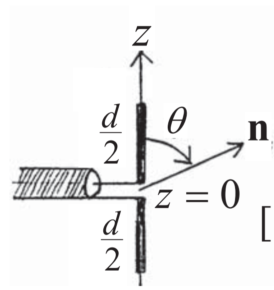


a full-wave antenna of length d

2 half-wave antennas, each of length $\frac{d}{2}$

Solution to Problem 1: Principle of superposition requires that we add the fields (not the powers) of the 2 antennas.

Rewrite $\mathbf{A}(\mathbf{x})$ [for $r \gg d$ and any k] for a single antenna of total length d (upper fig.)

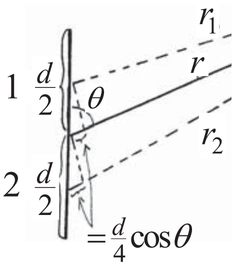


$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \quad [(9.55)]$$

$\Rightarrow \mathbf{A}(\mathbf{x})$ for the 2 antennas, each of total length $d/2$ (lower fig.), is given by (9.55) with d replaced with $d/2$ and r replaced with r_1 or r_2, where $\begin{cases} r_1 = r - \frac{d}{4}\cos\theta \\ r_2 = r + \frac{d}{4}\cos\theta \end{cases}$. Thus,

antenna 1 $\frac{d}{2}$

antenna 2 $\frac{d}{2}$



$$\mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi k \underbrace{r_{1,2}}_{\approx r}}\left[\frac{\cos(\frac{kd}{4}\cos\theta) - \cos(\frac{kd}{4})}{\sin^2\theta}\right], \quad (7)$$

Note: We may approximate $r_{1,2}$ in the denominator of (7) by r, but must use the exact $r_{1,2}$ in the exponential to get the correct phases.

Rewrite

$$\mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi kr}\left[\frac{\cos(\frac{kd}{4}\cos\theta) - \cos(\frac{kd}{4})}{\sin^2\theta}\right] \left[\begin{matrix}\text{total length} = \frac{d}{2}, \\ \text{any } kd\end{matrix}\right] \quad [(7)]$$

$$\left.\begin{cases}\text{For half-wave excitation, } k \cdot (\text{total length}) = \pi \\ \text{Total length of each antenna} = d/2\end{cases}\right\} \Rightarrow \text{Set } k\frac{d}{2} = \pi \text{ in } (7)$$

$$\Rightarrow \mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi kr}\left[\frac{\cos(\frac{\pi}{2}\cos\theta) - \cancel{\cos\frac{\pi}{2}}}{\sin^2\theta}\right] \left[\begin{matrix}\text{total length} = \frac{d}{2}, \\ \text{in half-wave excitation}\end{matrix}\right]$$

The 2 antennas are in phase and $r_1 = r - \frac{d}{4}\cos\theta$; $r_2 = r + \frac{d}{4}\cos\theta$

$$\Rightarrow \mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi}\frac{I}{kr}e^{ikr}\left[e^{-i\frac{\pi}{2}\cos\theta} + e^{i\frac{\pi}{2}\cos\theta}\right]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \quad (8)$$

$$= \mathbf{e}_z \frac{\mu_0}{\pi}\frac{I}{kr}e^{ikr}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

antenna 1 $\frac{d}{2}$ J

antenna 2 $\frac{d}{2}$ J

For $r \gg \lambda$, $\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0}{2\mu_0^2}k^2 r^2 \sin^2\theta |\mathbf{A}|^2$ [(5)]

$$\Rightarrow \left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0 I^2}{2\pi^2}\frac{\cos^4(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\left[\begin{matrix}\text{same as full-wave excitation of an} \\ \text{antenna of total length } d, \text{ see (9.57)}\end{matrix}\right]$$

Problem 2: Assume the 2 half-wave antennas in Problem 1 are 180^{o} out of phase, find $dP/d\Omega$ and compare with the $dP/d\Omega$ in problem 1.

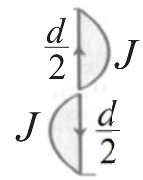
We simply replace the "+" sign in (8) with a "−" sign. Thus,

$$\mathbf{A}=\mathbf{A}_1-\mathbf{A}_2=\mathbf{e}_z\frac{\mu_0}{2\pi}\frac{1}{kr}e^{ikr}\left[e^{-i\frac{\pi}{2}\cos\theta}-e^{i\frac{\pi}{2}\cos\theta}\right]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$=-i\mathbf{e}_z\frac{\mu_0}{\pi}\frac{1}{kr}e^{ikr}\frac{\sin(\frac{\pi}{2}\cos\theta)\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

antenna 1

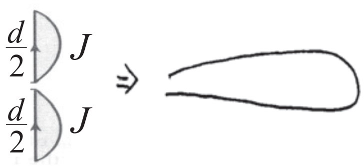
antenna 2

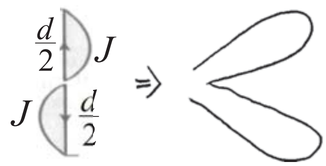


$$\Rightarrow\left\langle\frac{dP}{d\Omega}\right\rangle_t=\frac{Z_0}{2\mu_0^2}k^2r^2\sin^2\theta|\mathbf{A}|^2\quad[r\gg\lambda,\ \text{by}(5)]$$

$$=\frac{Z_0I^2}{2\pi^2}\frac{\sin^2(\frac{\pi}{2}\cos\theta)\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}=\frac{Z_0I^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

in phase $\Rightarrow$ dipole radiation $\quad 180^{\text{o}}$ out of phase $\Rightarrow$ quadrupole radiation





Examples of Linear Antennas

AM broadcast antenna

AM (amplitude modulation)

frequency : 535 − 1606 kHz

$\lambda_{free\ space}$: 186 − 560 m

The antenna length is usually 1/4 or 1/2 of $\lambda_{free\ space}$.

FM broadcast antenna

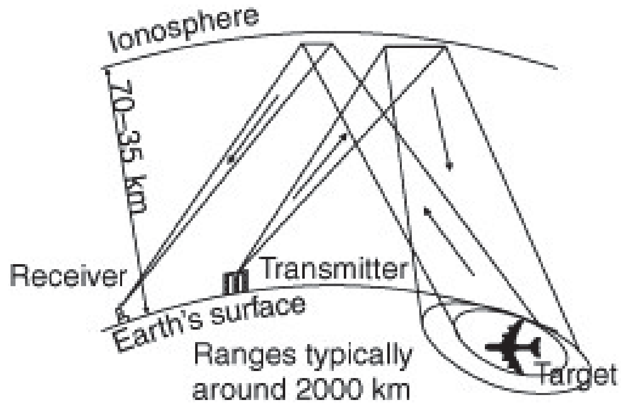
FM (frequency modulation)

frequency : 87.5 − 108 MHz

$\lambda_{free\ space}$: 2.8 − 3.4 m

Phased array antenna

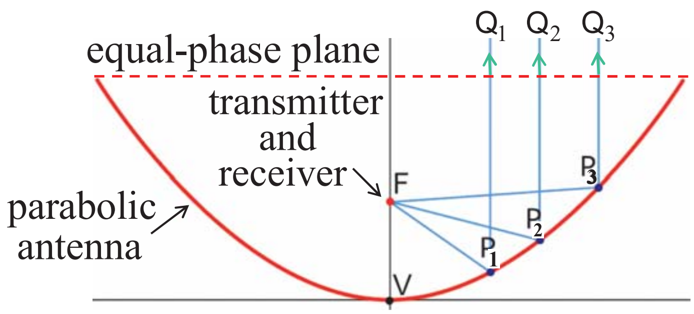
Over-the-horizon radar



Principles of over-the-horizon radar

- Transmitter frequency: f = 3–30 MHz (short waves)
- Adjusting relative phases of antennas →controlling wave direction
- Reflection from the ionosphere →1000's of km in range
- Use Doppler effect to distinguish scattered signals from the moving target and the background.

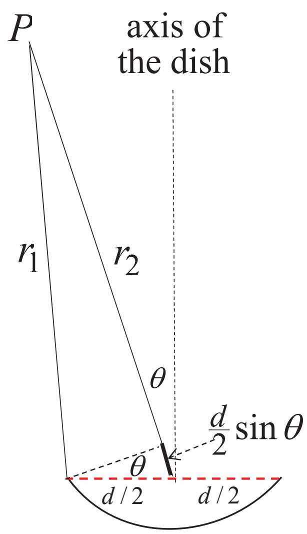
Parabolic (or Dish) Antennas : Assume $d \gg \lambda$ (opposite to Sec. 9.1). ⇒ Geometrical optics are valid to zero order (correction needed).



Parabolic antennas are based on the geometrical property of the paraboloid that the ray paths FP_1Q_1, FP_2Q_2, FP_3Q_3 are all of the same length, where F is the dish's focus. So wide-angle emissions from a transmitter at F will form an equal-phase plane, resulting in a plane wave travelling parallel to the dish's axis VF.

There is also a receiver at the focal point to detect reflected signals from the targets. So it functions as a radar (example 1 below). If there is only a receiver (no transmitter) at F, the antenna functions as a telescope (examples 2-4 below).

Angular width: As a correction to geometrical optics, the wave coming out from every pt. on the "equal-phase" plane actually has an angular spread (instead of vertical to the plane). Consider paths r_1 and r_2 which converge on a target point P located at an angle θ to the axis of the dish. If P is far away along r_2 (e.g. 1000 d), path r_1 will be nearly parallel to path r_2. Then, $\quad r_2 - r_1 \approx \frac{1}{2} d \sin\theta$



If $\frac{1}{2} d \sin\theta = \frac{1}{2}\lambda$, waves from all pts. superpose *destructively* at P. Usually, $d \gg \lambda$ (i.e. $\theta \ll 1$).

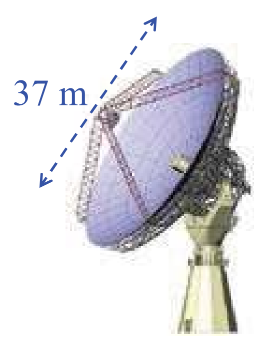
$$\Rightarrow \theta \approx \frac{\lambda}{d} \quad \left[\begin{array}{l}\underline{\text{angular width}} \text{ (or beamwidth)} \\ \text{of the antenna in radian}\end{array}\right]$$

Angular resolution: If the antenna is used as a receiver of waves emitted from the target point P, the same argument will show that, if $\theta \geq \frac{\lambda}{d}$, waves from P add destructively at the antenna receiver. Thus, the <u>angular resolution</u> of the antenna is also given by $\theta \approx \frac{\lambda}{d}$.

Example 1: Haystack Radar (upgraded to 92-100 GHz in 2014)

$$\text{For} \begin{cases} \text{imaging and tracking of space objects} \\ \text{radio astronomy} \end{cases}$$

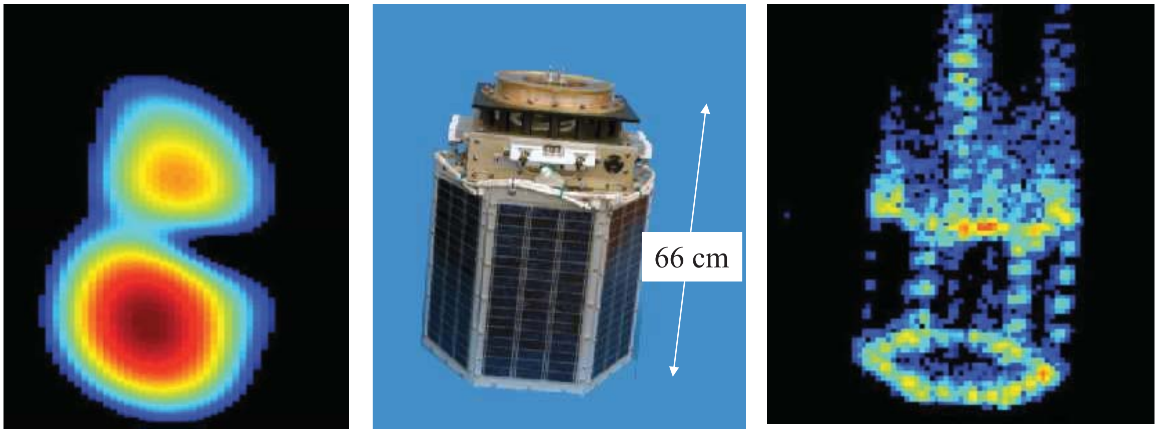
Largest radome-enclosed antenna in the world, operated by Lincoln Lab.



Haystack parabolic antenna
$d \approx 37$ m, $\lambda \approx 3.13$ mm
θ (beamwidth) $\approx 5\times10^{-3}$ degree
gain ≈ 90 dB
surface tolerance ≈ 0.1 mm

The <u>gain of an antenna</u> (expressed in dB) is the $dP/d\Omega$ (power per unit solid angle) in the direction of maximum radiation relative to that of a reference antenna emitting the same total power isotropically.

Satellite imaging by Haystack radar: X-band (old) vs W-band (new)
(simulated results)



9.5 – 10.5 GHz
25-cm resolution

Model of satellite

92 – 100 GHz
3-cm resolution

W. M. Brown and A. F. Pensa, Lincoln Lab Journal **21**, 4 (2014).
J. M. Usoff, M. T. Clarke, C. Liu, and M. J. Silver, ibid, p. 83.
M. G. Czerwinski and J. M. Usoff, ibid, p. 28.

Example 2: Five-hundred-meter Aperture Spherical Telescope (FAST)
(http://fast.bao.ac.cn/en/)

Located in a natural basin in Guizhou province and 500 m in diameter, FAST is the world's largest single-dish radio telescope.

It was completed in 2016 and officially operational in 2020. US$180 million was spent on the facility. US $270 million was spent to relocate ~8,000 people in a 5 km radius.

An antenna can emit and receive waves (as in a radar). FAST principally receives waves, at 70 MHz-3 GHz, from the outer space.

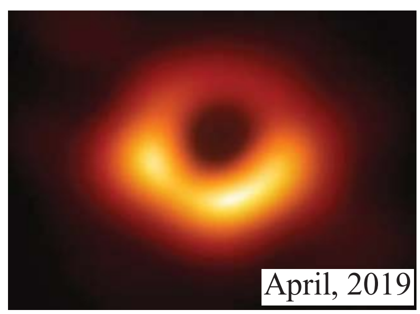
Scientific goals of FAST include detecting faint pulsars, mapping neutral hydrogen in galaxies, and listening to possible signals from other civilizations.

Example 3 : ALMA (Atacama Large Milli./Submilli. Array)

ALMA is designed to probe the universe through the millimeter and submillimeter wavelength. It consists of 66 movable antennas (12 m & 7 m in diameter). Signal synchronization makes the array a single giant antenna, with an angular resolution of $\sim 10^{-6}$ degree.

Located in Chile 5,000 meters above sea level and built at a cost of US$1.3 billion, ALMA is an international collaboration (including Taiwan) which has been operational since 2013.

Example 4 : EHT (The Event Horizon Telescope)
(https://eventhorizontelescope.org/)



The EHT (left fig.) is a global network of synchronized radio telescopes including ALMA. It has an angular resolution of 2.78×10^{-10} degree, about that of a single antenna of the size of the earth. In comparison, the human eye has a resolution 0.017 degree.

At a wavelength of 1.3 mm, the EHT captured the first ever image of a black hole surrounded by a halo of bright gas (plasma, right fig.), whose physical behavior reveals information relevant to the black hole.

Located 55 million light-years away, the black hole has a boundary 40 billion km across and a mass 6.5 billion times that of the Sun.

Exercise : The EHT has an angular resolution (θ) of

$$\theta = 20\ \mu\text{ac} = 9.7\times10^{-11}\ \text{radian}$$

[1 μac (micro-arc-second) $\approx 2.78\times10^{-10}$ degree $\approx 4.85\times10^{-12}$ radian]

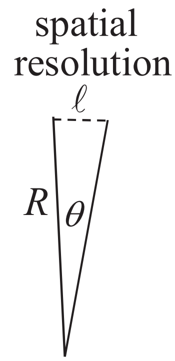
(a) What is its spatial resolution at a distance of $R = 3.8\times10^{5}$ km (earth-moon distance)?

Ans. The spatial resolution (ℓ) is

$$\ell = R\theta \quad [\theta \text{ in radian}]$$

Thus, $R = 3.8\times10^{5}\ \text{km} = 3.8\times10^{10}\ \text{cm}$

$$\Rightarrow \ell = R\theta = 3.8\times10^{10}\times9.7\times10^{-11} = 3.7\ \text{cm}$$

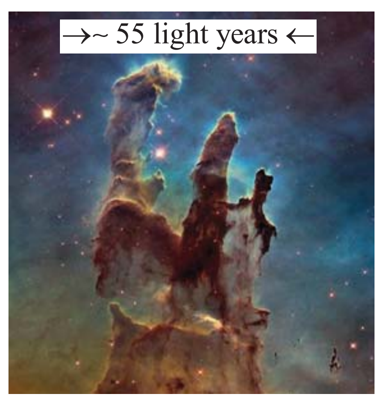


(b) What is its spatial resolution at a distance of 55 million light years (1 light year $= 9.46\times10^{12}$ km)?

Ans. R = 55 million light years

$$= 55\times10^{6}\times9.46\times10^{12}\ \text{km}$$
$$= 5.2\times10^{20}\ \text{km}$$
$$\Rightarrow \ell = R\theta = 5.2\times10^{20}\times9.7\times10^{-11}$$
$$= 5\times10^{10}\ \text{km} = 50\ \text{billion km}$$

A Famous Optical Telescope : Hubble Space Telescope (1990-)



Stars forming in the Eagle Nebula
(7000 light years away)