台大物理系 "古典電力學(二)" 任課老師:朱國瑞

"Classical Electrodynamics (II)"

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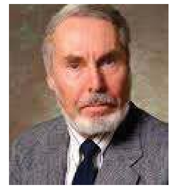
Spring Semester, 2024

1.**Textbook and Contents of the Course:**



J. D. Jackson
1925 - 2016

J. D. Jackson, "Classical Electrodynamics", 3rd edition, Chapters 7-11, 14.

The first seven chapters of the textbook [main subjects of Electrodynamics (I)] elegantly lay down the foundation of electrodynamics. This is done concurrently with a systematic exposition of some key topics in applied mathematics.

The remaining chapters address selected topics of deep physical insight as well as current interest, which will help students consolidate their knowledge of the fundamental principles.

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In Electrodynamics (II), we will cover selected topics in Chaps. 7-11 and 14 of the textbook, such as plane/spherical EM waves, reflection/refraction, waveguides/cavities, optical fibers, antennas, scattering/diffraction, special theory of relativity, and synchrotron radiation. We will start from p. 1, Ch. 7 of lecture notes.

2. **Reference:**

Other books on physics and mathematics will be referenced in the lecture notes when needed, e.g. M. A. Heald and J. B. Marion, "Classical Electromagnetic Radiation", 3rd edition. Chap. 14.

3. **Conduct of Class :**

Lecture notes will be projected sequentially on the screen during the class. Physical concepts will be emphasized, while algebraic details in the lecture notes will often be skipped. *Questions are encouraged*. It is assumed that students have at least gone through the algebra in the lecture notes before attending classes (*important*!).

4. **Grading Policy:** Midterm (40%); Final (50); Attendance (5%); Homework (5%). The final exam covers all materials taught in the class. The scores may be adjusted with no change on the order.
5. **Lecture Notes:** Starting from basic equations, the lecture notes follow Jackson closely with algebraic details filled in.

   Equations numbered in the format of (8.7), (8.9)... refer to Jackson. Supplementary equations derived in lecture notes, which will later be referenced, are numbered (1), (2)... [restarting from (1) in each chapter.] Equations in Appendices A, B…of each chapter are numbered (A.1), (A.2)…and (B.1), (B.2)…

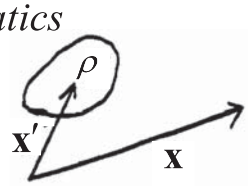
   Page numbers cited in the text (e.g. p. 395) refer to Jackson. Section numbers (e.g. Sec. 8.1) refer to Jackson (except for sections in Ch. 11). Main topics within each section are highlighted by **boldfaced** characters. Some words are typed in *italicized* characters for attention. Technical terms which are introduced for the first time are underlined.

1. Review of Static Laws and Equations

Electrostatics

Definition of **E**:

$\mathbf{F}_E = q\mathbf{E}$



Coulomb's law:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \qquad \nabla\times\mathbf{E} = 0$$

$$\Downarrow \qquad \Downarrow$$

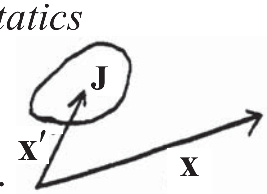
$$\oint \mathbf{E}\cdot d\mathbf{a} = q/\varepsilon_0 \qquad \oint \mathbf{E}\cdot d\ell = 0$$

Gauss's law of electrostatics

Magnetostatics

Definition of **B**:

$\mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$



Biot-Savart law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

$$\Downarrow \qquad \Downarrow$$

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0 \qquad \oint \mathbf{B}\cdot d\ell = \mu_0 I$$

Gauss's Law of magnetism — Ampere's law

Question: Which of the above eqs. still hold if $\frac{\partial}{\partial t} \neq 0$? Why?

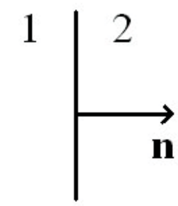
Helmholtz's Theorem (Griffiths, Sec. 1.6.1): If a vector field goes to 0 at infinity, it is uniquely determined by its divergence and curl.

Field energy: (for linear medium)
$$\begin{cases} W_E = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}d^3x & (4.89) \\ W_B = \frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}d^3x & (5.148) \end{cases} \quad \text{where} \quad \begin{cases} \mathbf{D} = \varepsilon_b \mathbf{E} & (4.37) \\ \mathbf{B} = \mu\mathbf{H} & (5.84) \end{cases}$$

$\boxed{\varepsilon_b \text{ is due to bound electrons.}}$

Forces:
$$\begin{cases} \mathbf{F}_E = \int \rho\mathbf{E}d^3x \\ \mathbf{F}_B = \int \mathbf{J}\times\mathbf{B}d^3x \end{cases}$$

Boundary conditions:
$$\begin{cases} (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = \sigma \\ (\mathbf{E}_2 - \mathbf{E}_1)\times\mathbf{n} = 0 \end{cases} \quad (4.40)$$
$$\begin{cases} (\mathbf{B}_2 - \mathbf{B}_1)\cdot\mathbf{n} = 0 & (5.86) \\ \mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} & (5.87) \end{cases}$$



Question 1: All of the above eqs. still hold if $\frac{\partial}{\partial t} \neq 0$. Why?

Question 2: Does **D** or **H** have a physical meaning?

Ans.: Jackson, p. 193: "The fundamental fields are **E** and **B**."; "The derived fields **D** and **H** are introduced as a matter of convenience."

2. Poynting's Theorem

All electromagnetic phenomea are governed by the four Maxewll Maxwell equations. In macroscopic form, they are

$$\begin{cases} \nabla\cdot\mathbf{D} = \rho & \nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D} \\ \nabla\cdot\mathbf{B} = 0 & \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \end{cases} \quad (6.6)$$

We begin with the derivation of a very useful theorem.

V $\mathbf{J}\cdot\mathbf{E}$

$\boxed{\begin{array}{l}\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} \\ \quad = \text{force/unit volume}\end{array}}$ $\boxed{= \nabla\times\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D}}$

$$\begin{bmatrix}\text{rate of work done by field} \\ \text{on charges in a volume } V\end{bmatrix} = \int_V \mathbf{f}\cdot\mathbf{v}d^3x = \int_V \rho\mathbf{v}\cdot\mathbf{E}d^3x = \underbrace{\int_V \mathbf{J}\cdot\mathbf{E}d^3x}$$

$\boxed{\text{rate of conversion of field energy into mechanical or thermal energy}}$

$$= \int_V (\underbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}} - \mathbf{E}\cdot\tfrac{\partial}{\partial t}\mathbf{D})d^3x$$

$\boxed{\begin{array}{l} = \mathbf{H}\cdot\nabla\times\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{H}) \\ = -\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} - \nabla\cdot(\mathbf{E}\times\mathbf{H}) \end{array}}$

$$= -\int_V \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\tfrac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\tfrac{\partial}{\partial t}\mathbf{B}\right]d^3x \quad (6.105)$$

Poynting's Theorem (*continued*)

Rewrite (6.105): $\int_V \mathbf{J}\cdot\mathbf{E}d^3x = -\int_V \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

The terms $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}$ and $\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$ in the integrand can be interpreted physically if we make the following assumptions:

Assumption 1: The medium is *linear* with *negligible dispersion* and *negligible losses*.

We can then write [reasons given in Ch. 7 of Electrodynamice (I) lecture notes]: $\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t)$

$$\Rightarrow \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}),\ \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{B}).$$

Assumption 2: The field energy density for static fields

$$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \quad (6.106)$$

represents the field energy density even for *time-dependent* fields.

From (6) and (6.106), we have

$$\frac{\partial u}{\partial t} = \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \begin{bmatrix}\text{rate of change of} \\ \text{field energy density}\end{bmatrix}$$

Poynting's Theorem (*continued*)

Rewrite (6.105): $\int_V \mathbf{J}\cdot\mathbf{E}d^3x = -\int_V \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

Sub. $\frac{\partial u}{\partial t}$ for $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$, we obtain

$$\int_V \mathbf{J}\cdot\mathbf{E}d^3x + \int_V \frac{\partial u}{\partial t}d^3x + \int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x = 0 \quad (6.107)$$

and, by divergence thm., $\underbrace{\int_V \mathbf{J}\cdot\mathbf{E}d^3x}_{\frac{d}{dt}E_{mech}} + \underbrace{\int_V \frac{\partial u}{\partial t}d^3x}_{\frac{d}{dt}E_{field}} + \underbrace{\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x}_{\oint_S \mathbf{S}\cdot\mathbf{n}da} = 0$

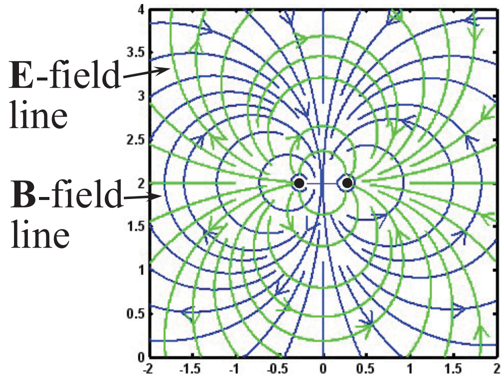
where $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ [Poynting vector] (6.109)

$\Rightarrow \frac{d}{dt}(E_{mech} + E_{field}) = -\oint_S \mathbf{S}\cdot\mathbf{n}da$ [Poynting's thm.] (6.111)

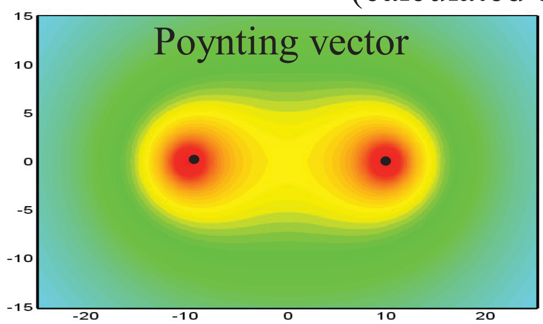
where E_{mech} is the total mechanical (including thermal) energy inside V (no particle moves into or out of V) and E_{field} is the total field energy inside V. By conservation of energy, $\oint_S \mathbf{S}\cdot\mathbf{n}da$ must be the total power flow into or out of V. We postulate $\mathbf{S}$ to be power/unit area at any point.

In differential form, we have [by (107)] $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$ (6.108)

Example 1: power lines



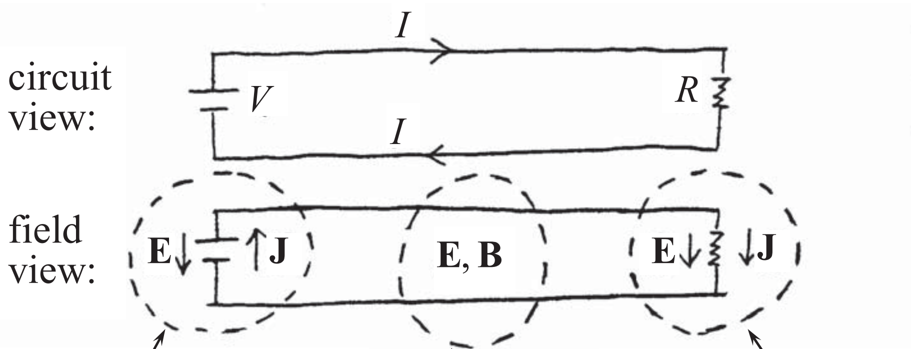
(calculated by C. Y. Kao)



Note: $\mathbf{S} = \mathbf{E} \times \mathbf{H} = (\sum_j \mathbf{E}_j) \times (\sum_j \mathbf{H}_j) [\neq \sum_j (\mathbf{E}_j \times \mathbf{H}_j)]$

Example 2: a DC circuit

$$\int_V \mathbf{J} \cdot \mathbf{E} d^3x + \int_V \frac{\partial u}{\partial t} d^3x + \oint_S \mathbf{S} \cdot \mathbf{n} da = 0 \underset{\boxed{\text{steady state}}}{\Rightarrow} \int_V \mathbf{J} \cdot \mathbf{E} d^3x + \oint_S \mathbf{S} \cdot \mathbf{n} da = 0$$



$\int_V \mathbf{J} \cdot \mathbf{E} d^3x < 0$

$\Rightarrow \oint_S \mathbf{S} \cdot \mathbf{n} da > 0$

$\Rightarrow$ Power flows out of battery.

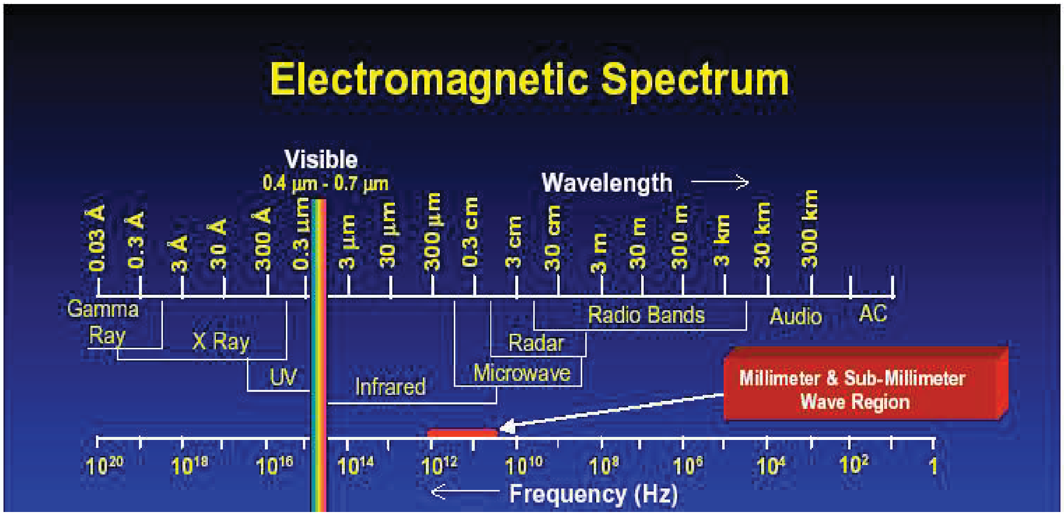
Power transmission by Poynting vector

$\int_V \mathbf{J} \cdot \mathbf{E} d^3x > 0$

$\Rightarrow \oint_S \mathbf{S} \cdot \mathbf{n} da < 0$

$\Rightarrow$ Power flows into resistor.

Chapter 7: Plane Electromagnetic Waves and Wave Propagation



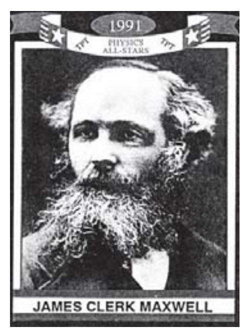
Radio wave: 3×10^{3}–3×10^{11} Hz
Microwave: 3×10^{8}–3×10^{11} Hz
THz wave: 10^{11}–10^{13} Hz
IR: 3×10^{11} Hz–4.3×10^{14} Hz

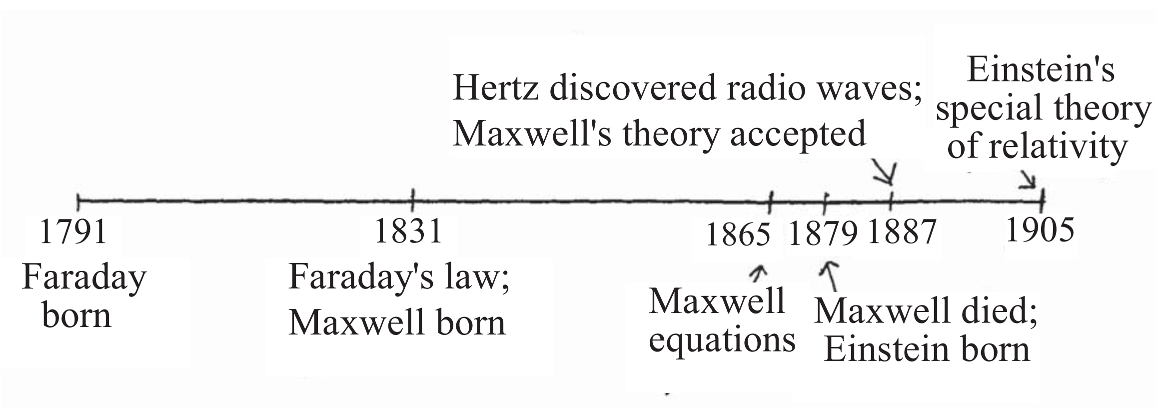
Visible: 4.3×10^{14}–7.9×10^{14} Hz
UV: 7.5×10^{14}–3×10^{16} Hz
X-ray: 3×10^{16}–3×10^{19} Hz
γ-ray: $>10^{19}$ Hz

An Historical Perspective:

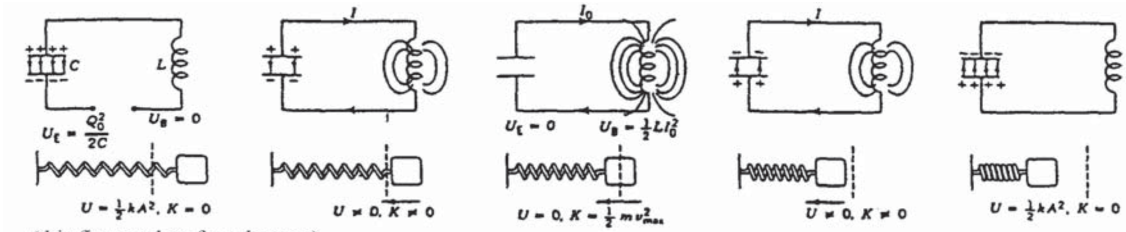
Faraday : Time-varying magnetic field generates electric field.

Maxwell : Time-varying electric field generates magnetic field.





A Note about Oscillatory Behavior:



(this figure taken from benson)

Common feature of oscillatory behavior: $\boxed{\text{energy type 1}} \Leftrightarrow \boxed{\text{energy type 2}}$

$$\Rightarrow \text{Oscillations require } \begin{cases} \text{energy storing mechanisms} \\ \text{energy exchange mechanism(s)} \end{cases}$$

example	energy storing mechanisms	energy exchange mechanism(s)	medium
mass-spring system	$\frac{1}{2}mv^2,\ \frac{1}{2}kx^2$	restoring force	mass & spring
LC oscillator	$\frac{B^2}{2\mu},\ \frac{\varepsilon E^2}{2}$	Q, I	L, C, & wire
EM wave	$\frac{B^2}{2\mu},\ \frac{\varepsilon E^2}{2}$	$\frac{dB}{dt},\ \frac{dE}{dt}$	not required

Organization of Lecture Notes on Ch. 7:

In Jackson, plane waves in dielectric media are treated in Secs. 7.1 and 7.2. Various special cases (plasma medium and high-frequency limit) are treated in Sec. 7.5. Plane waves in conductors are treated in Sec. 5.18 [e.g. Eqs. (5.163)-(5.169)] and Sec. 8.1 [e.g. Eqs. (8.9), (8.10), (8.12), (8.14), and (8.15)] by methods different from those in Secs. 7.1 and 7.2.

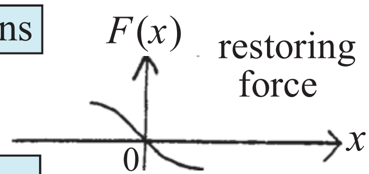
Here, we will cover these sections with a unified treatment of plane waves in both dielectrics and conductors, and at all frequencies. Equations in Jackson will be examined in detail, but in somewhat different order. So the first three sections of lecture notes are numbered Secs. I, II, and III rather than following Jackson's section numbers. However, Secs. 7.3, 7.4, 7.8, and 7.9 of Jackson will be followed closely in subsequent lecture notes.

We begin with a derivation of the generalized dielectric constant $\varepsilon/\varepsilon_0$, which is applicable to both dielectric and conducting media.

I. Derivation of the Generalized Dielectric Constant $\varepsilon/\varepsilon_0$ [Sec. 7.5 (part A)]

Dipole Moment of a Single Electron : The equation of motion for *both bound and free* electrons with mass m and charge $-e$ in an external E-field $\mathbf{E}(\mathbf{x},t)$ is [Neglect the small force due to $\mathbf{B}(\mathbf{x},t)$]

$$m\frac{d^2}{dt^2}\mathbf{x} = -e\mathbf{E}(\mathbf{x},t)\underbrace{-\gamma m\frac{d}{dt}\mathbf{x}}\overbrace{-m\omega_0^2\mathbf{x}}^{\text{restoring force; } \omega_0 = 0 \text{ for free electrons}} \qquad (7.49)$$



displacement of the electron from its equilibrium position $\mathbf{x} = 0$

γ: electron collision frequency
$-\gamma m\frac{d}{dt}\mathbf{x}$: damping force (rate of change of electron momentum due to collisions)

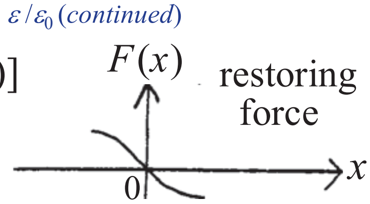
$F(x) = \underbrace{F(0)}_{0} + \underbrace{F'(0)}_{-m\omega_0^2}\, x + \cdots$
As in Sec. 4.6, we neglect higher-order terms.

Note: 1. The damping force ($-\gamma m\mathbf{v}$) is always opposite to $\mathbf{v}$. It converts the e^- energy into thermal energy (random kinetic energy).

2. ω_0 $\left(=\sqrt{\frac{|F'(0)|}{m}}\right)$ is the natural oscillation frequency of the e^-.

$$m\frac{d^2}{dt^2}\mathbf{x} = -e\mathbf{E}(\mathbf{x},t) - \gamma m\frac{d}{dt}\mathbf{x} - m\omega_0^2\mathbf{x} \quad [(7.49)]$$

$$\Rightarrow \; m\left(\frac{d^2}{dt^2}\mathbf{x} + \gamma\frac{d}{dt}\mathbf{x} + \omega_0^2\mathbf{x}\right) = -e\mathbf{E}(\mathbf{x},t)$$



Let* $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$ and expand $\mathbf{E}(\mathbf{x})$ about the equilibrium position $\mathbf{x} = 0$. $\Rightarrow$

$$\mathbf{E}(\mathbf{x}) = \mathbf{E}(0) + \underbrace{(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{x})\big|_{\mathbf{x}=0}}_{\sim \frac{x}{\lambda}\mathbf{E}(0)} + \cdots \underset{\text{if } \frac{x}{\lambda} \ll 1}{\approx} \mathbf{E}(0), \quad (1)$$

See Ch. 4 (Appendix A) for Taylor expansion of vector functions with vector argument.

where λ is the scale length of $\mathbf{E}(\mathbf{x})$. For example, if $\mathbf{E}(\mathbf{x})$ is a wave field, then $\lambda \approx$ wavelength. By neglecting $(\mathbf{x}\cdot\nabla)\mathbf{E}(\mathbf{x})\big|_{\mathbf{x}=0}$, we have assumed that the electron displacement is too small for the electron to see the spatial field variations. Thus, we assume that the electron is acted on by a *spatially uniform* field:

$$\mathbf{E}(\mathbf{x},t) \approx \mathbf{E}(0)e^{-i\omega t} \quad (\text{LHS} = \text{Re[RHS]}),$$

where $\mathbf{E}(0)$ is the field amplitude at the location of the electron.

*This is equivalent to a Fourier trnasformation to the ω space and $\mathbf{E}(\mathbf{x})$ is a complex quantity called the phasor [see Appendix A]

Let $\mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t}$ and substitute

$$\begin{cases}\mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t}\\ \mathbf{E}(\mathbf{x},t)=\mathbf{E}(0)e^{-i\omega t}\end{cases} \quad \text{into} \quad m(\frac{d^2}{dt^2}\mathbf{x}+\gamma\frac{d}{dt}\mathbf{x}+\omega_0^2\mathbf{x})=-e\mathbf{E}(\mathbf{x},t),$$

we obtain $m(-\omega^2-i\omega\gamma+\omega_0^2)\mathbf{x}_0=-e\mathbf{E}(0) \;\Rightarrow\; \mathbf{x}_0=-\frac{e}{m}\frac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma}$

$$\Rightarrow\; \mathbf{x}(t)=-\frac{e}{m}\frac{\mathbf{E}(0)e^{-i\omega t}}{\omega_0^2-\omega^2-i\omega\gamma} \quad (\text{LHS}=\text{Re[RHS]}),$$

which represents the *forced* oscillation of a simple harmonic oscillator. The natural oscillation frequency is ω_0 (see Exercise below).

The time-dependent $\mathbf{x}(t)$ results in a time-dependent dipole moment at $\mathbf{x}=0$ given by $\mathbf{p}(t)=\mathbf{p}_0 e^{-i\omega t}$,

where $\mathbf{p}_0=-e\mathbf{x}_0=\frac{e^2}{m}\frac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma}$ $\left[\Rightarrow \mathbf{p}_0=\frac{e^2\mathbf{E}(0)}{m\omega_0^2}\ [(4.72)] \text{ if } \omega=0\right]$

Questions: 1. If $\omega_0=0$, $\mathbf{p}_0\to\infty$ as $\omega\to 0$. What's the problem?

2. If $\mathbf{E}(0)=0$, under what condition can we still have a solution?

Exercise: Obtain the solution of $m(\frac{d^2}{dt^2}\mathbf{x}+\gamma\frac{d}{dt}\mathbf{x}+\omega_0^2\mathbf{x})=0.$

Let $\mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t} \;\Rightarrow\; m(-\omega^2-i\omega\gamma+\omega_0^2)\mathbf{x}_0=0$

For $\mathbf{x}_0\neq 0$, the only possibility is $\omega^2-\omega_0^2+i\omega\gamma=0$

Assume $\gamma\ll\omega$ and use the method of iteration:

$$\begin{cases}\text{To 0-th order, } \omega^2-\omega_0^2=0\Rightarrow\; \omega\approx\omega_0 \quad \text{(binding frequency)}\\ \text{To 1st order, } \omega^2-\omega_0^2+i\omega_0\gamma\approx 0\Rightarrow \omega\approx\omega_0(1-i\frac{\gamma}{\omega_0})^{1/2}\approx\omega_0(1-i\frac{\gamma}{2\omega_0})\end{cases}$$

$$\Rightarrow \omega\approx\omega_0-i\frac{\gamma}{2}\Rightarrow \mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t}\approx\mathbf{x}_0 e^{-\frac{1}{2}\gamma t}e^{-i\omega_0 t} \quad [\gamma/2: \text{damping rate}]$$

This is the *damped* oscillation of a simple harmonic oscillator with natural frequency ω_0 (see Halliday, Resnick, and Walker, Sec.16.8).

Discussion: This exercise shows:

(1) The method of iteration is a useful and systematic way to solve an equation containing a small term.

(2) ω can be a complex number.

(3) Collisions (γ) are responsible for damping.

Go back to $\begin{cases} \mathbf{E}(\mathbf{x},t) = \mathbf{E}(0)e^{-i\omega t} \\ \mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t} \\ \mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t} \end{cases}$ with $\begin{cases} \mathbf{x}_0 = -\dfrac{e}{m}\dfrac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma} \\ \mathbf{p}_0 = -e\mathbf{x}_0 = \dfrac{e^2}{m}\dfrac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma} \end{cases}$

In these equations, $\mathbf{E}(0)$, $\mathbf{x}_0$, and $\mathbf{p}_0$ are phasors containing the phase & amplitude information of $\mathbf{E}(0,t)$, $\mathbf{x}(t)$, & $\mathbf{p}(t)$, respectively.

The subscript "0" in $\mathbf{x}_0$ and $\mathbf{p}_0$ indicates that the oscillation is centered at $\mathbf{x}=0$, where $\mathbf{E}(\mathbf{x},t)$ is approximated by $\mathbf{E}(0)e^{-i\omega t}$ (its value at $\mathbf{x}=0$). If the oscillation is centered at an arbitrary point $\mathbf{x}$, the only difference is that the electron will see the field at point $\mathbf{x}$, i.e. $\mathbf{E}(\mathbf{x})e^{-i\omega t}$ instead of $\mathbf{E}(0)e^{-i\omega t}$. Thus, we generalize

$$\begin{cases} \mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t} \\ \mathbf{p}_0 = \dfrac{e^2}{m}\dfrac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma} \end{cases} \text{ to } \begin{cases} \mathbf{p}(t) = \mathbf{p} e^{-i\omega t} & \boxed{\mathbf{p}(t): \text{a } \textit{microscopic} \text{ quantity}} \\ \mathbf{p} = \dfrac{e^2}{m}\dfrac{\mathbf{E}(\mathbf{x})}{\omega_0^2-\omega^2-i\omega\gamma} \end{cases} \quad (7.50)$$

Note: In (7.50), $\mathbf{x}$ is a spatial coordinate (not the electron displacement), and $\mathbf{p}$ and $\mathbf{E}(\mathbf{x})$ are phasors (complex numbers).

The Generalized Dielectric Constant : Assume there are N molecules per unit volume and Z (bound + free) electrons for each molecule (= nuclear charge). Divide the electrons of a molecule into groups, each with electron number f_j ($\Sigma f_j = Z$), binding frequency ω_j ($\omega_j = 0$ for free electrons), and collision frequency γ_j. Then, the electric polarization (total dipole moment per unit volume) is

$$\underset{\uparrow}{\mathbf{P}}(\mathbf{x}) = N\sum_j f_j \mathbf{p}_j \overset{\nearrow}{=} \underbrace{\frac{Ne^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_0\chi_e}\mathbf{E}(\mathbf{x}) \overset{\uparrow}{=} \varepsilon_0\chi_e\mathbf{E}(\underset{\uparrow}{\mathbf{x}})$$

$\boxed{\text{a } \textit{macroscopic} \text{ quantity}}$ $\boxed{(7.50)}$ $\boxed{(4.36)}$ $\boxed{\text{a spatial variable}}$

Extending the definitions of the static electric displacement ($\mathbf{D}$) and permittivity (ε):

$$\begin{cases} \mathbf{D}(\mathbf{x}) \equiv \varepsilon_0\mathbf{E}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x}) & (4.34)\ (4.37) \\ \varepsilon = \varepsilon_0(1+\chi_e) & (4.38) \end{cases}$$

to fields with $\exp(-i\omega t)$ dependence, we obtain $\quad \mathbf{D}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x}) \quad (2)$

with $\dfrac{\varepsilon}{\varepsilon_0} = 1 + \dfrac{Ne^2}{\varepsilon_0 m}\displaystyle\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$ $\left[\begin{array}{l}\varepsilon: \text{generalized permittivity} \\ \quad \text{with free-electron effect}\end{array}\right]$ (7.51)

Divide the e's into $\begin{cases} \text{bound electrons: } \omega_j \neq 0 \\ \text{free electrons: } \omega_j = 0,\ f_j = f_0,\ \gamma_j = \gamma_0 \end{cases}$

$$(7.51) \Rightarrow \varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m} \sum_{j(\text{bound})} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\varepsilon_b} + i \underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\sigma/\omega} \qquad (7.56)$$

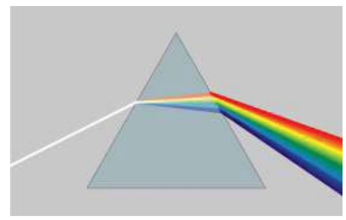
$$= \varepsilon_b + i\frac{\sigma}{\omega} \qquad \boxed{\sigma \text{ is due to free electrons.}} \qquad \begin{matrix}(7.56)\\ \text{short form}\end{matrix}$$

where $\sigma \equiv \dfrac{f_0 Ne^2}{m(\gamma_0 - i\omega)}$ [Drude model for the electrical conductivity. See Appendix B for a direct derivation of σ.] (7.58)

Note: $\gamma_0 \approx 4\times 10^{13}$/s for Cu (p. 312)

Questions: 1. $\varepsilon \to \infty$ as $\omega \to 0$, implying that the derivation breaks down. Why?

2. A medium is said to be dispersive if its ε is a function of ω (p. 323, top). What causes the medium to be dispersive? *Ans*: electron inertia



Discussion:

(i) The *linear* relations, $\mathbf{p} = \dfrac{e^2}{m}\dfrac{\mathbf{E}(\mathbf{x})}{\omega_0^2 - \omega^2 - i\omega\gamma}$ [(7.50)] & $\mathbf{D}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x})$ [(2)], result from the approx. that x (e displacement) is small so that $\begin{cases} F(\mathbf{x}) \propto x \ \ [(7.49)] \\ \mathbf{E}(\mathbf{x}) \approx \mathbf{E}(0) \ \ [(1)] \end{cases}$.

See Eq. (A.9) for the meaning of a *linear* relation in t-space.

(ii) When E is comparable to the interatomic E-field (10^5-10^8 V/m), "$F(\mathbf{x}) \propto x$" is no longer valid. Then, the **p**-**E** and **D**-**E** relations become *nonlinear* (Thus, laser $\Rightarrow$ the birth of nonlinear optics).

(iii) In the time-varying **E**-fields, free e's oscillate about an equilibrium position just like bound e's. Hence, both types of e's can be treated on equal footing. This makes it possible to obtain the *generalized* ε in (7.51) or (7.56), which is applicable to both dielectric and conducting materials.

The generalized ε allows a unified treatment of EM waves in both dielectric and conducting media (as will be shown later).

(iv) $\underbrace{\varepsilon = \varepsilon_0 + \frac{Ne^2}{m} \sum_{j(\text{bound})} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\varepsilon_b} + i \underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\text{due to free } e\text{'s}}$ [(7.56)]

Useful algebra: If $X = \frac{1}{a-ib}$ $\left(= \frac{a+ib}{a^2+b^2}\right)$, then $\frac{\text{Im}(X)}{\text{Re}(X)} = \frac{b}{a}$

Let $\varepsilon = \varepsilon' + i\varepsilon''$ [$\varepsilon' = \text{Re}(\varepsilon)$, $\varepsilon'' = \text{Im}(\varepsilon)$]. From (7.56), we see that ε'' is due to γ of (both bound and free) electrons. Since γ is the damping term in $m\frac{d^2}{dt^2}\mathbf{x} = -e\mathbf{E}(\mathbf{x},t) - \gamma m \frac{d}{dt}\mathbf{x} - m\omega_j^2\mathbf{x}$ [(7.49)], ε'' accounts for EM wave damping in dielectric and conducting media.

Bound e's are restrained by the binding force, which makes ω_j in the denominator of ε_b dominate over γ_j ($\because \omega_j \gg \gamma_j$, see p. 310). Thus, $\quad \varepsilon_b'' \ll \varepsilon_b'$

Free e's are not restrained by the binding force. There is no binding force term in the free e part of (7.56). Hence, $\varepsilon''(\text{due to free } e\text{'s}) \gg \varepsilon_b''$ $\Rightarrow$ EM waves damp much faster in a conductor than in a dielectric.

(v) In ω-space, only those quantities tranformed from t-space, such as $\mathbf{E}$, have a counterpart in t-space. Since ε is a *derived* quantity in ω-space, it has no conterpart in t-space. So, the <u>constitutive relation</u> $\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega)$ [(2)] is an ω-space relation. However, since $\mathbf{D}(\omega) \propto \mathbf{E}(\omega)$, $\mathbf{D}(\omega)$ has a counterpart in t-space, which may be obtained by an inverse Fourier transform:

$$\mathbf{D}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{D}(\omega) e^{-i\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\varepsilon(\omega)} \mathbf{E}(\omega) e^{-i\omega t} d\omega \qquad (3)$$

$$\boxed{\varepsilon(\omega) = \varepsilon_0 + \frac{Ne^2}{m} \sum_{j(\text{bound})} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} + i \frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)} \quad [(7.56)]}$$

$$\neq \varepsilon\mathbf{E}(t) \quad \left[\mathbf{E}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{-i\omega t} d\omega\right]$$

in general

However, there are 2 special cases where (3) gives $\mathbf{D}(t) = \varepsilon\mathbf{E}(t)$ in t-space for a dielectric medium [see items (vi) and (vii) below].

(vi) Rewrite $\mathbf{D}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{D}(\omega)e^{-i\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{-i\omega t}d\omega$ [(3)]

Consider a static $\mathbf{E}$ (i.e. $\omega=0,\ \mathbf{E}=const$) in a dielectric medium with no free electrons ($f_0=0$). We have

$$\begin{cases}\mathbf{E}(\omega)=\int_{-\infty}^{\infty}\mathbf{E}(t)e^{i\omega t}dt=\mathbf{E}\int_{-\infty}^{\infty}e^{i\omega t}dt=2\pi\mathbf{E}\delta(\omega)\\ \varepsilon(\omega)=\varepsilon_0+\frac{Ne^2}{m}\sum\limits_{j(\text{bound})}\frac{f_j}{\omega_j^2-\cancel{\omega}^2-i\cancel{\omega}\gamma_j}+\cancel{i\frac{Ne^2f_0}{m\omega(\gamma_0-i\omega)}}\quad[(7.56)]\\ \qquad=\varepsilon_0+\frac{Ne^2}{m}\sum\limits_{j(\text{bound})}\frac{f_j}{\omega_j^2}=\varepsilon_b\quad[Note:\varepsilon_b\text{ is real.}]\end{cases}$$

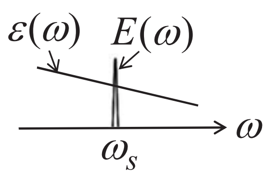
$\Rightarrow$ In t-space, we have a static $\mathbf{D}$ given by

$$\mathbf{D}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{-i\omega t}d\omega=\frac{\varepsilon_b}{2\pi}2\pi\mathbf{E}\int_{-\infty}^{\infty}\delta(\omega)e^{-i\omega t}d\omega=\varepsilon_b\mathbf{E}$$

So we have recovered the static relation $\mathbf{D}=\varepsilon_b\mathbf{E}$ [(4.37)] without making any approximation.

(vii) Let $E(\omega)$ be a signal at ω_s with bandwidth $\Delta\omega$ and assume

1. $\varepsilon(\omega)\approx const$ for ω within $\omega_s\pm\Delta\omega$. So the medium has *negligible dispersion* to the signal.
2. The medium has *negligible loss* (i.e. $\gamma_j\approx 0$).



Then, for this signal, we may let $\omega=\omega_s$ & $\gamma_j=0$ in (7.56).

$$\Rightarrow\varepsilon(\omega_s)=\varepsilon_0+\frac{Ne^2}{m}\sum_{j(\text{bound})}\frac{f_j}{\omega_j^2-\omega_s^2-i\omega_s\cancel{\gamma_j}}\quad\begin{bmatrix}\text{for dielectrics, }\gamma\text{ of free}\\ e\text{'s is non-negligible.}\end{bmatrix}$$

$$\approx\varepsilon_0+\frac{Ne^2}{m}\sum_{j(\text{bound})}\frac{f_j}{\omega_j^2-\omega_s^2}=\text{real and indep. of }\omega$$

$$\Rightarrow\mathbf{D}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{-i\omega t}d\omega\approx\frac{\varepsilon(\omega_s)}{2\pi}\int_{-\infty}^{\infty}\mathbf{E}(\omega)e^{-i\omega t}d\omega=\varepsilon(\omega_s)\mathbf{E}(t)$$

This explains the assumption made on p. 259: "The macroscopic medium is linear in its electrical properties, with *negligible dispersion* or *negligible losses*". Under this assumption, we have $\mathbf{D}(t)=\varepsilon\mathbf{E}(t)$, which is used in (6.105) to derive (6.107) in t-space.

Question: Why is the "negligible loss" assumption also required?

(viii) In general, we cannot neglect dispersion and losses. Such cases must be treated in ω-space (see Parts II and III below and Sec. 6.8).

A note about terminology: In general, the electric permitivity is a tensor (denote it by $\overleftrightarrow{\boldsymbol{\varepsilon}}$) and we may write

$$\mathbf{D} = \overleftrightarrow{\boldsymbol{\varepsilon}} \cdot \mathbf{E}, \quad \text{where } \overleftrightarrow{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

The electrical property of the medium is	if
uniform (or homogeneous)	$\overleftrightarrow{\boldsymbol{\varepsilon}}$ is indept. of $\mathbf{x}$
linear	$\overleftrightarrow{\boldsymbol{\varepsilon}}$ is indept. of $\mathbf{E}$
nondispersive	$\overleftrightarrow{\boldsymbol{\varepsilon}}$ is indept. of ω
isotropic	$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$, $\varepsilon_{ij} = 0$ if $i \neq j$

A note about notations:

The notation ε is commonly used to denote the permittivity of a dielectric medium, where *bound* electrons contribute to ε.

The notation σ is commonly used to denote the conductivity of a conducting medium, where *free* electrons contribute to σ.

For harmonic fields, we have derived a *generalized* ε [(7.51)] contributed by both bound & free electrons. For a clear distinction, we have denoted the commonly used permittivity (due to bound electrons only) by ε_b [as in (7.56)] and reserved the notation ε for the generalized permittivity. Thus, there are 2 definitions for $\mathbf{D}$:

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} = (\varepsilon_b + i\frac{\sigma}{\omega})\mathbf{E} \quad [(2)] \\ \mathbf{D} = \varepsilon_b \mathbf{E} \quad [(2) \text{ without the free-electron term } i\frac{\sigma}{\omega}] \end{cases} \tag{4}$$

These 2 definitions of $\mathbf{D}$ will be in no contradiction because $\mathbf{D}$ has no physical significance. We only need to choose a defintion suitable for the conditions under study [see (6) below].

Mazwell Equations for Harmonic Fields (i.e. $e^{-i\omega t}$ dependence) in Terms of the Generalized Dielectric Constant ε :

Bound and free electrons are combined in the generalized ε:

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad \left[\begin{array}{l}\text{including effects of both}\\ \text{bound and free electrons}\end{array}\right] \quad [(7.51)]$$

$$= \varepsilon_b + i\frac{\sigma}{\omega} \quad \text{[(7.56), in short form],}$$

If we characterize the (uniform, isotropic) medium by ε, the Maxwell eqs. no longer contain explicit ρ_{free} & $\mathbf{J}_{free}$. Hence,

$$\begin{cases}\nabla\cdot\mathbf{D}(\mathbf{x},t) = 0\\ \nabla\cdot\mathbf{B}(\mathbf{x},t) = 0\\ \nabla\times\mathbf{E}(\mathbf{x},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t)\\ \nabla\times\mathbf{H}(\mathbf{x},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t)\end{cases} \quad \overset{\substack{e^{-i\omega t}\\ \text{dependence}}}{\Rightarrow} \quad \begin{cases}\nabla\cdot[\varepsilon\mathbf{E}(\mathbf{x})] = 0\\ \nabla\cdot\mathbf{B}(\mathbf{x}) = 0\\ \nabla\times\mathbf{E}(\mathbf{x}) = i\omega\mathbf{B}(\mathbf{x})\\ \nabla\times\mathbf{H}(\mathbf{x}) = -i\omega\varepsilon\mathbf{E}(\mathbf{x})\end{cases} \quad (5)$$

In the following 4 pages, we will derive $\varepsilon = \varepsilon_b + i\frac{\sigma}{\omega}$ and (5) again (for a uniform, isotropic medium) using the usual form of Maxwell eqs. [(6.6)], in which bound and free electrons are treated separately.

II. Plane Wave Equations in Dielectrics and Conductors - A Unified Formalism

Basic Equations: We start from the macroscopic Maxwell eqs. in (6.6), in which $\mathbf{D}$, $\mathbf{H}$ contain effects of bound electrons, and ρ, $\mathbf{J}$ are due to free electrons [ρ, $\mathbf{J}$ in (6.6) are denoted by ρ_{free}, $\mathbf{J}_{free}$ here].

$$\begin{cases}\nabla\cdot\mathbf{D}(\mathbf{x},t) = \rho_{free}(\mathbf{x},t)\\ \nabla\cdot\mathbf{B}(\mathbf{x},t) = 0\\ \nabla\times\mathbf{E}(\mathbf{x},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t)\\ \nabla\times\mathbf{H}(\mathbf{x},t) = \mathbf{J}_{free}(\mathbf{x},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t)\end{cases} \quad (6.6)$$

> ρ_{free}, $\mathbf{J}_{free}$ here are neglected in (7.1).
> $\mathbf{E}(\mathbf{x},t)$, $\mathbf{D}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, and $\mathbf{H}(\mathbf{x},t)$ here are $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ in (7.1).

Equation of continuity (conservation of free charges):

$$\frac{\partial}{\partial t}\rho_{free}(\mathbf{x},t) + \nabla\cdot\mathbf{J}_{free}(\mathbf{x},t) = 0$$

As discussed earlier, $\mathbf{D} = \varepsilon_b\mathbf{E}$ (for bound electrons) and $\mathbf{D} = \varepsilon\mathbf{E}$ (for both bound and free electrons) are in general applicable only in the ω-space. Similarly, $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{J} = \sigma\mathbf{E}$ are also ω-space relations. To utilize these relation, we need to first go to the ω-space.

Assumption 1 : Harmonic time dependence (go to ω-space)

$$\text{Let } \underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x},t) \\ \mathbf{D}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) \\ \mathbf{H}(\mathbf{x},t) \\ \mathbf{J}_{free}(\mathbf{x},t) \\ \rho_{free}(\mathbf{x},t) \end{Bmatrix}}_{\text{real}} = \underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{D}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \\ \mathbf{J}_{free}(\mathbf{x}) \\ \rho_{free}(\mathbf{x}) \end{Bmatrix}}_{\text{complex phasors (ω-space quantities)}} e^{-i\omega t}$$

By convention, the LHS is the real part of the RHS.

$\mathbf{E}(\mathbf{x}), \mathbf{B}(\mathbf{x})$ here are $\mathbf{E}$, $\mathbf{B}$ in (7.2) and (7.3)

$\omega \neq 0$ for EM waves

$$\begin{cases} \nabla\cdot\mathbf{D}(\mathbf{x},t) = \rho_{free}(\mathbf{x},t) \\ \nabla\cdot\mathbf{B}(\mathbf{x},t) = 0 \\ \nabla\times\mathbf{E}(\mathbf{x},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t) = \mathbf{J}_{free}(\mathbf{x},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \quad (6.6) \quad \Rightarrow \begin{cases} \nabla\cdot\mathbf{D}(\mathbf{x}) = \rho_{free}(\mathbf{x}) \\ \nabla\cdot\mathbf{B}(\mathbf{x}) = 0 \\ \nabla\times\mathbf{E}(\mathbf{x}) = i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x}) = \mathbf{J}_{free}(\mathbf{x}) - i\omega\mathbf{D}(\mathbf{x}) \end{cases} \quad (6)$$

$$\frac{\partial}{\partial t}\rho_{free}(\mathbf{x},t) + \nabla\cdot\mathbf{J}_{free}(\mathbf{x},t) = 0 \;\Rightarrow\; -i\omega\rho_{free}(\mathbf{x}) + \nabla\cdot\mathbf{J}_{free}(\mathbf{x}) = 0$$

Question: What information have we lost in Assumption 1?

(Since the medium is linear, we may supperpose any multi-ω signal.)

II. Plane Wave Equations in Dielectrics and Conductors… (*continued*)

Assumption 2 : Linear and isotropic medium, i.e.

$$\mathbf{D}(\mathbf{x}) = \underbrace{\varepsilon_b}_{\text{due to bound } e\text{'s}} \mathbf{E}(\mathbf{x}),\; \mathbf{B}(\mathbf{x}) = \underbrace{\mu}_{\text{due to bound } e\text{'s}} \mathbf{H}(\mathbf{x}),\; \mathbf{J}_{free}(\mathbf{x}) = \underbrace{\sigma}_{\text{due to free } e\text{'s}} \mathbf{E}(\mathbf{x}) \quad (7)$$

Note: For (6), we must use the definition $\mathbf{D} = \varepsilon_b\mathbf{E}$, because free-$e$ effects are included in ρ_{free}, $\mathbf{J}_{free}$ [see discussion below (4)].

Rewrite $-i\omega\rho_{free}(\mathbf{x}) + \nabla\cdot\mathbf{J}_{free}(\mathbf{x}) = 0$ [conservation of charge]

$$\Rightarrow -i\omega\rho_{free}(\mathbf{x}) + \nabla\cdot\sigma\mathbf{E}(\mathbf{x}) = 0$$

$$\Rightarrow \rho_{free}(\mathbf{x}) = \frac{\nabla\cdot\sigma\mathbf{E}(\mathbf{x})}{i\omega} \quad \begin{bmatrix} \omega \neq 0 \text{ for} \\ \text{EM waves} \end{bmatrix}$$

Letting $\mathbf{J}_{free}(\mathbf{x}) = \sigma\mathbf{E}(\mathbf{x})$ implies the only $\mathbf{J}_{free}$ and ρ_{free} are due to $\mathbf{E}$ in (6), i.e. no external $\mathbf{J}_{free}$ and ρ_{free} (e.g. an electron beam).

Sub. ρ_{free} into $\nabla\cdot\mathbf{D}(\mathbf{x}) = \rho_{free}$

$$\Rightarrow \nabla\cdot\varepsilon_b\mathbf{E}(\mathbf{x}) = \frac{\nabla\cdot\sigma\mathbf{E}(\mathbf{x})}{i\omega} \Rightarrow \nabla\cdot(\varepsilon_b + i\tfrac{\sigma}{\omega})\mathbf{E}(\mathbf{x}) = 0 \Rightarrow \nabla\cdot\varepsilon\mathbf{E}(\mathbf{x}) = 0, \quad (8)$$

where $\varepsilon = \varepsilon_b + i\frac{\sigma}{\omega}$ [same form as (7.56)]

Question: What information have we lost in Assumption 2?

Similarly, $\nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}_{free}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x})$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{x})=\sigma\mathbf{E}(\mathbf{x})-i\omega\varepsilon_b\mathbf{E}(\mathbf{x})=-i\omega\overbrace{(\varepsilon_b+i\frac{\sigma}{\omega})}^{\varepsilon}\mathbf{E}(\mathbf{x}) \quad (7.57)$$

$$\Rightarrow \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x}) \quad (9)$$

Thus, ε_b and σ are again combined into $\varepsilon=\varepsilon_b+i\frac{\sigma}{\omega}$ as in (8).

Note: This is an alternative derivation of the generalized ε, by applying Maxwell eqs. [(6.6)] to *harmonic* fields (assumption 1) in a *linear* and *isotropic* medium (assumption 2). ε $(=\varepsilon_b+i\frac{\sigma}{\omega})$ in (8) has the same form as (7.56). However, (7.56) contains the explicit expressions for ε_b and σ, which will turn out to be very useful.

(8) and (9) give the macroscopic Maxwell equations in terms of phasor fields and the generalized ε in the same form as (5):

$$\begin{cases}\nabla\cdot[\varepsilon\mathbf{E}(\mathbf{x})]=0\\ \nabla\cdot\mathbf{B}(\mathbf{x})=0\\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x})\\ \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x})\end{cases}\quad\left[\begin{array}{l}\text{for } \textit{harmonic} \text{ fields in}\\ \text{a } \textit{linear} \text{ and } \textit{isotropic}\\ \text{medium}\end{array}\right]\quad [(5)]$$

Note: (i) Bound and free electrons are separated in (6.6) and (6):

$$\begin{cases}\nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho_{free}(\mathbf{x},t)\\ \nabla\cdot\mathbf{B}(\mathbf{x},t)=0\\ \nabla\times\mathbf{E}(\mathbf{x},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t)\\ \nabla\times\mathbf{H}(\mathbf{x},t)=\mathbf{J}_{free}(\mathbf{x},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t)\end{cases}\quad(6.6)\qquad \begin{cases}\nabla\cdot\mathbf{D}(\mathbf{x})=\rho_{free}(\mathbf{x})\\ \nabla\cdot\mathbf{B}(\mathbf{x})=0\\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x})\\ \nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}_{free}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x})\end{cases}\quad(6)$$

where $\mathbf{D}(\mathbf{x})=\varepsilon_b\mathbf{E}(\mathbf{x})$ and $\mathbf{J}_{free}(\mathbf{x})=\sigma\mathbf{E}(\mathbf{x})$. ε_b and σ account for the effects of bound and free electronics, respectively.

(ii) Bound and free electrons are combined in (5):

$$\begin{cases}\nabla\cdot[\varepsilon\mathbf{E}(\mathbf{x})]=0\\ \nabla\cdot\mathbf{B}(\mathbf{x})=0\\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x})\\ \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x})\end{cases}\quad [(5)]$$

where $\varepsilon(=\varepsilon_b+i\frac{\sigma}{\omega})$ includes the effects of both bound & free electrons.

Assumption 3 : Uniform medium (i.e. ε, μ independent of $\mathbf{x}$)

$$\begin{cases}\nabla\cdot[\varepsilon\mathbf{E}(\mathbf{x})]=0\\ \nabla\cdot\mathbf{B}(\mathbf{x})=0\\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x})\\ \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x})\end{cases}\quad [(5)]\ \Rightarrow\ \begin{cases}\nabla\cdot\mathbf{E}(\mathbf{x})=0 & (11)\\ \nabla\cdot\mathbf{B}(\mathbf{x})=0 & (12)\\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) & (13)\\ \nabla\times\mathbf{B}(\mathbf{x})=-i\omega\mu\varepsilon\mathbf{E}(\mathbf{x}) & (14)\end{cases}$$

$$\nabla\times\begin{Bmatrix}(13)\\(14)\end{Bmatrix}\ \Rightarrow\ \nabla^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\\mathbf{B}(\mathbf{x})\end{Bmatrix}+\mu\varepsilon\omega^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\\mathbf{B}(\mathbf{x})\end{Bmatrix}=0 \qquad (15)$$

(15) has the same form as (7.3), which is derived from the source-free Maxwell equations [(7.1)] for a non-conducting medium ($\sigma=0$). However, (15) is applicable to both dielectric and conducting media. In (7.3), $\varepsilon=\varepsilon_b$. In (15), $\varepsilon=\varepsilon_b+i\frac{\sigma}{\omega}$. The solution of (15) takes the same algebraic steps as (7.3). But with $\varepsilon=\varepsilon_b+i\frac{\sigma}{\omega}$, the solution of (15) will be applicable to both dielectric and conducting media.

Question: What information have we lost in Assumption 3?

Assumption 4 : $\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\\mathbf{B}(\mathbf{x})\end{Bmatrix}=\begin{Bmatrix}\mathbf{E}_0\\\mathbf{B}_0\end{Bmatrix}e^{i\mathbf{k}\cdot\mathbf{x}}$

$\mathbf{E}_0$, $\mathbf{B}_0$ here are $\mathfrak{E}$, $\mathfrak{B}$ in (7.8)-(7.12)

$$\nabla^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\\mathbf{B}(\mathbf{x})\end{Bmatrix}+\mu\varepsilon\omega^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\\mathbf{B}(\mathbf{x})\end{Bmatrix}=0\Rightarrow\left(-k^2+\mu\varepsilon\omega^2\right)\begin{Bmatrix}\mathbf{E}_0\\\mathbf{B}_0\end{Bmatrix}=0\Rightarrow k=\pm\sqrt{\mu\varepsilon}\,\omega.$$

The $\pm$ signs represent waves traveling in opposite directions. For an isotropic medium, we may choose the + sign without loss of generality.

Thus, $\quad k=\sqrt{\mu\varepsilon}\,\omega$ [dispersion relation, same as (7.4)] $\qquad$ (16)

Note: 1. $k^2\equiv\mathbf{k}\cdot\mathbf{k}$; $|\mathbf{k}|^2\equiv\mathbf{k}\cdot\mathbf{k}^*$; $k^2\neq|\mathbf{k}|^2$ and $k\neq|\mathbf{k}|$ unless $\mathbf{k}$ is real.

2. k can be complex, but $|\mathbf{k}|$ is always real and positive.

$$(11)\text{-}(13)\Rightarrow\begin{cases}\mathbf{k}\cdot\mathbf{E}_0=0 & (17)\\ \mathbf{k}\cdot\mathbf{B}_0=0 & (18)\\ \mathbf{B}_0=\frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0=\sqrt{\mu\varepsilon}\,\frac{\mathbf{k}\times\mathbf{E}_0}{k} & (19)\end{cases}$$

(14) gives $\mathbf{E}_0=-\frac{1}{\omega\mu\varepsilon}\mathbf{k}\times\mathbf{B}_0$, which is implicit in (17) and (19).

Question: What information have we lost in Assumption 4?

In Sec. 6.9 of lecture notes, we have derived the relation

$$\langle \mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\rangle_t = \tfrac{1}{2}\mathrm{Re}[\mathbf{E}^*(\mathbf{x})\times\mathbf{H}(\mathbf{x})] \quad \text{[valid for real } \omega\text{]}$$

$\Rightarrow \langle \mathbf{S}\rangle_t$ = time-averaged power flow per unit area (called intensity)

$$= \langle \underbrace{\mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)}_{\text{real quantities}}\rangle_t$$

$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{x}) &= \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}}\\ \mathbf{H}(\mathbf{x}) &= \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{\mathbf{B}_0}{\mu}e^{i\mathbf{k}\cdot\mathbf{x}}\\ &= \sqrt{\frac{\varepsilon}{\mu}}\frac{\mathbf{k}\times\mathbf{E}_0}{k}e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{[Use (19)]}\end{aligned}}$$

$$= \tfrac{1}{2}\mathrm{Re}[\underbrace{\mathbf{E}^*(\mathbf{x})\times\mathbf{H}(\mathbf{x})}_{\text{phasors}}]$$

$$= \tfrac{1}{2}\mathrm{Re}[\sqrt{\tfrac{\varepsilon}{\mu}}\tfrac{1}{k}\mathbf{E}_0^*\times(\mathbf{k}\times\mathbf{E}_0)e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}]$$

$$\boxed{|\mathbf{E}_0|^2 \equiv \mathbf{E}_0\cdot\mathbf{E}_0^*}$$

$$= \begin{cases} \frac{1}{2}\mathrm{Re}\left\{\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}[\mathbf{k}|\mathbf{E}_0|^2 - \mathbf{E}_0(\mathbf{k}\cdot\mathbf{E}_0^*)]e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}\right\} & \text{(20a)}\\ \frac{1}{2\omega}\mathrm{Re}\left\{\frac{1}{\mu}[\mathbf{k}|\mathbf{E}_0|^2 - \mathbf{E}_0(\mathbf{k}\cdot\mathbf{E}_0^*)]e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}\right\} & \text{(20b)}\end{cases}$$

Discussion: Rewrite
$$\begin{cases} k = \sqrt{\mu\varepsilon}\,\omega \quad \text{(dispersion relation)} & [(16)]\\ \mathbf{k}\cdot\mathbf{E}_0 = 0 & [(17)]\\ \mathbf{k}\cdot\mathbf{B}_0 = 0 & [(18)]\\ \mathbf{B}_0 = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0 = \sqrt{\mu\varepsilon}\frac{\mathbf{k}\times\mathbf{E}_0}{k} & [(19)]\end{cases}$$

(i) Assume μ, ε are given, so (16)-(19) are conditions imposed on ω, $\mathbf{k}$, $\mathbf{E}_0$, $\mathbf{B}_0$ by the Maxwell eqs.

(ii) The derivation of (16)-(19) only requires μ, ε, ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ to be constants, but not necessarily real. Thus, any set of complex μ, ε, ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ can be a valid solution of the Maxwell eqs. provided they satisfy (16)-(19).

(iii) ε is complex [as in (7.56)]. μ can also be complex. A complex ε or μ can lead to complex solutions for ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$. Even if ε and μ are real, boundary conditions (if any) can also lead to complex $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ [to be shown in Sec. 7.4].

(iv) Under Assumptions 1 & 4, we have a plane wave ($\sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$).

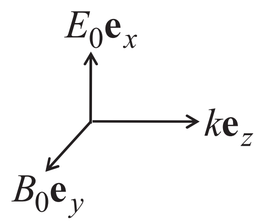
Question : What is a plane wave? *Ans* : The surface of constant *phase* (not constant *amplitude*) at the same t is a plane.

We may *construct* 2 types of plane waves satisfying (16)-(19).

Type 1. Homogeneous plane wave :

This is the most familiar plane wave

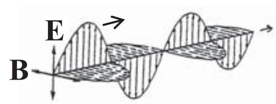
given by $\begin{cases} \mathbf{k} = k\mathbf{e}_z \\ \mathbf{E}_0 = E_0\mathbf{e}_x \\ \mathbf{B}_0 = B_0\mathbf{e}_y \end{cases}$ with $\begin{cases} B_0 = \sqrt{\mu\varepsilon}E_0 \\ k = \sqrt{\mu\varepsilon}\omega \end{cases}$ (21)



where $\mathbf{e}_x$, $\mathbf{e}_y$, & $\mathbf{e}_z$ are real unit vectors, but E_0, B_0, & k can all be complex due to complex μ or ε. Clearly, (21) satisfies (16)-(19).

Example : μ, ε are real $\Rightarrow E_0, B_0, k$ are also real. Then, in t-space,

$$\begin{cases} \mathbf{E}(\mathbf{x},t) = \mathrm{Re}[\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}] = E_0\cos(kz - \omega t)\mathbf{e}_x \\ \mathbf{B}(\mathbf{x},t) = \mathrm{Re}[\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}] = B_0\cos(kz - \omega t)\mathbf{e}_y \end{cases} \quad (22)$$



$\Rightarrow$ Any plane $\perp$ to the z-axis is a constant-phase plane.

Type 2. Inhomogeneous plane wave (surface wave) :

Construct another solu. satisfying (16)-(19): [derived in (85)-(88)]

$$\begin{cases} \mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z \\ \mathbf{E}_0 = E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z \\ \mathbf{B}_0 = iB_{0y}\mathbf{e}_y \end{cases} \text{with} \begin{cases} k^2 = \mathbf{k}\cdot\mathbf{k} = k_x^2 - k_z^2 = \mu\varepsilon\omega^2 \\ \mathbf{k}\cdot\mathbf{E}_0 = k_xE_{0x} - k_zE_{0z} = 0 \\ B_{0y} = (-k_xE_{0z} + k_zE_{0x})/\omega \end{cases} \quad (23)$$

where ω, k_x, k_z, E_{0x}, E_{0z}, and B_{0y} are all real constants.

$\mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z$ defined here can be converted to the form $\mathbf{k} = k\mathbf{n} = k(\mathbf{n}_R + i\mathbf{n}_I)$ as used on p. 298 of Jackson. Here, we reserve the notation $\mathbf{n}$ for later use as a *real* unit vector.

(23) is an ω-space solution. Its physical meaning becomes clear when we convert the E-field to t-space (see Appendix A).

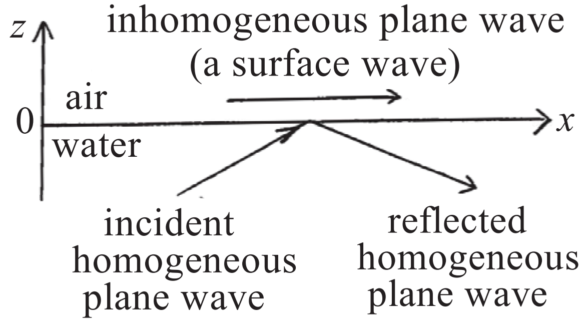
$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}[\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}] = \mathrm{Re}[(E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z)e^{-i\omega t + ik_x x - k_z z}]$$
$$= \left[E_{0x}\cos(\omega t - k_x x)\mathbf{e}_x + E_{0z}\sin(\omega t - k_x x)\mathbf{e}_z\right]e^{-k_z z} \quad (24)$$

Rewrite

$\mathbf{E}(\mathbf{x},t) = [E_{0x}\cos(\omega t - k_x x)\mathbf{e}_x + E_{0z}\sin(\omega t - k_x x)\mathbf{e}_z]e^{-k_z z}$ [(24)]

This represents a <u>surface wave</u> in the $z \geq 0$ half space, propagating along $\mathbf{e}_x$ with an amplitude decreasing exponentially along $+\mathbf{e}_z$. It is also called an <u>inhomogeneous plane wave</u> (p. 298) because any plane $\perp$ to the x-axis is a plane of constant phase. Note that the exponential decay is not due to any medium loss and it is in a direction $\perp$ to $\mathbf{e}_x$.

The surface wave discussed here is a general phenomenon, e.g. a plane wave incident from a dense to a tenuous medium (e.g. water to air) can be totally reflected. Due to b.c.'s at $z = 0$. Fields in the tenuous medium form a surface wave exactly as in (24).



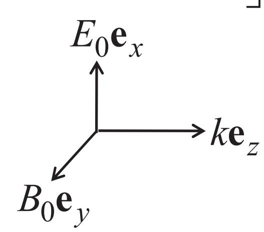
The surface wave also plays a key role in fiber optics (Sec. 8.11).

Question: Give an example of non-plane wave (*Ans.*: spherical wave).

(v) Orthogonality of vectors $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ in (17)-(19)

$$(17)\text{-}(19) \Rightarrow \left\{\begin{array}{l} \mathbf{k}\cdot\mathbf{E}_0 = 0 \\ \mathbf{k}\cdot\mathbf{B}_0 = 0 \\ \mathbf{E}_0\cdot\mathbf{B}_0 = 0 \end{array}\right\} \Rightarrow \left[\begin{array}{l} \mathbf{E}_0,\ \mathbf{B}_0, \text{ and } \mathbf{k} \text{ are } \textit{algebraically} \\ \text{orthogonal to one another.} \end{array}\right]$$

For the Type 1 (homogeneous) plane wave, $\mathbf{E}_0(= E_0\mathbf{e}_x)$, $\mathbf{B}_0(= B_0\mathbf{e}_y)$, and $\mathbf{k}(= k\mathbf{e}_z)$ are also *geometrically* orthogonal.



For the Type 2 (inhomogeneous) plane wave, the algebraic orthogonality of $\mathbf{k}(= k_x\mathbf{e}_x + ik_z\mathbf{e}_z)$, $\mathbf{E}_0(= E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z)$, & $\mathbf{B}_0(= iB_{0y}\mathbf{e}_y)$ does not imply geometric orthogonality because $\mathbf{k}$ and $\mathbf{E}_0$ do not have geometric directions (in ω-space). This is reflected in the t-space expression for the E-field just shown:

$\mathbf{E}(\mathbf{x},t) = \left[E_{0x}\cos(\omega t - k_x x)\mathbf{e}_x + E_{0z}\sin(\omega t - k_x x)\mathbf{e}_z\right]e^{-k_z z}$ [(24)],

which is a wave propagating along $\mathbf{e}_x$, but with $\mathbf{E}$ not $\perp$ to $\mathbf{e}_x$.

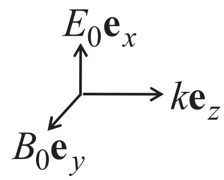
(vi) $\mathbf{k}\cdot\mathbf{E}_0 = 0$ does not necessarily imply $\mathbf{k}\cdot\mathbf{E}_0^* = 0$.

(A similar comment is made in Jackson, see footnote on p. 298.)

For the homogeneous plane wave: $\mathbf{k} = k\mathbf{e}_z$, $\mathbf{E}_0 = E_0\mathbf{e}_x$

$$\mathbf{k}\cdot\mathbf{E}_0 = 0$$

$$\Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = 0$$



But for the inhomogeneous plane wave:

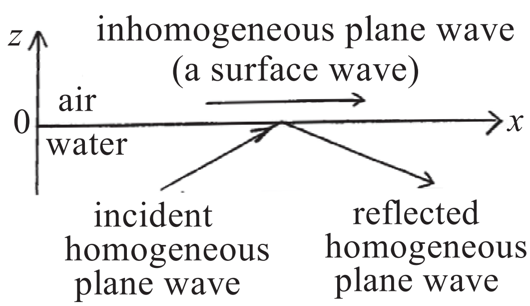
$$\begin{cases} \mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z \\ \mathbf{E}_0 = E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z \end{cases} \quad \text{[see (23)]}$$

$$\mathbf{k}\cdot\mathbf{E}_0 = 0 \Rightarrow k_xE_{0x} - k_zE_{0z} = 0$$

$$\Rightarrow k_xE_{0x} = k_zE_{0z}$$

$$\Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = k_xE_{0x} + k_zE_{0z}$$

$$= 2k_zE_{0z} \neq 0$$



Thus, at this point, the $\mathbf{k}\cdot\mathbf{E}_0^*$ term must be kept in (20a,b).

Assumption 5: $\mathbf{k} = k\mathbf{n} = (k_r + ik_i)\mathbf{n}$

k: complex constant
$\mathbf{n}$: real unit vector

Then, (17)-(19) can be written

$$\begin{cases} \mathbf{n}\cdot\mathbf{E}_0 = 0 & (25) \\ \mathbf{n}\cdot\mathbf{B}_0 = 0 & (26) \\ \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 & (27) \end{cases}$$

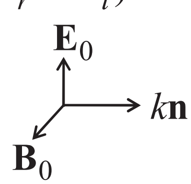
(16), (25)-(27) here are equivalent to (7.9)-(7.11) when $\mathbf{n}$ in (7.9)-(7.11) is a real unit vector and ε in (7.9)-(7.11) is interpreted as the generalized ε.

and $\mathbf{k}\cdot\mathbf{E}_0 = 0 \Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = 0. \Rightarrow$ The Poynting vector [(20)] reduces to

$$\langle\mathbf{S}\rangle_t = \tfrac{1}{2}\mathrm{Re}\sqrt{\tfrac{\varepsilon}{\mu}}\,|\mathbf{E}_0|^2\, e^{-2k_i\mathbf{n}\cdot\mathbf{x}}\,\mathbf{n} \qquad (28)$$

Exercise: Show $\mathbf{k} = k\mathbf{n} \Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = 0$.

$\mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z$ has no geometrical direction, but $\mathbf{k} = (k_r + ik_i)\mathbf{n}$ has the geometrical direction $\mathbf{n}$. $\Rightarrow$ (25)-(27) is a Type 1 plane wave with *geometrically* orthogonal $\mathbf{k}$, $\mathbf{E}_0$, & $\mathbf{B}_0$.



In $\mathbf{k} = (k_r + ik_i)\mathbf{n}$, $k_r\,(= \frac{2\pi}{\lambda})$ gives the wavelength, k_i gives the attenuation rate, and $\mathbf{n}$ gives the propagation direction.

Question: What information have we lost in Assumption 5?

Definition of impedance and admittance of the medium:

Rewrite $\mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0$ [(27)]

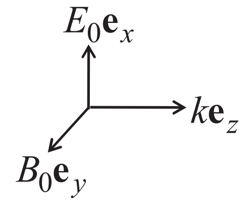
In engineering literature, this equation is often written

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{n}\times\mathbf{E}_0}{Z}, \tag{7.11}$$

where $Z \equiv \sqrt{\frac{\mu}{\varepsilon}}$ [impedance of the medium, Jackson, p. 297] (29)

The admittance of the medium is defined as $Y \equiv \frac{1}{Z} = \sqrt{\frac{\varepsilon}{\mu}}$.

Note: Z and Y are *intrinsic* properties of the medium.



Let $\mathbf{E}_0 = E_0\mathbf{e}_x$, $\mathbf{B}_0 = B_0\mathbf{e}_y$, & $\mathbf{n} = \mathbf{e}_z \Rightarrow Z = \dfrac{E_0}{H_0}$.

$\Rightarrow Z$ is the ratio of the wave amplitudes E_0 and H_0 in the medium. In general, Z is a complex number.

For the free space, $Z = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$ $\begin{bmatrix}\text{impedance}\\ \text{of free space}\end{bmatrix}$ (30)

A note about notations: Rewrite
$$\begin{cases} k = \sqrt{\mu\varepsilon}\,\omega & [(16)] \\ \mathbf{n}\cdot\mathbf{E}_0 = 0 & [(25)] \\ \mathbf{n}\cdot\mathbf{B}_0 = 0 & [(26)] \\ \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 & [(27)] \end{cases}$$

This set of equations is equivalent to (7.9)-(7.11) in Jackson, with ε in (7.9)-(7.11) interpreted as the generalized ε. The difference is in notation $\mathbf{k}$. Both here and in (7.9)-(7.11), $\mathbf{k} = k\mathbf{n}$. However, in (7.9)-(7.11), k is a real number and $\mathbf{n} = \mathbf{n}_R + i\mathbf{n}_I$ is a complex unit vector subject to the conditions: $\mathbf{n}\cdot\mathbf{n} = 1$, $n_R^2 - n_I^2 = 1$, & $\mathbf{n}_R\cdot\mathbf{n}_I = 0$ [(7.15)] But elsewhere in Jackson, $\mathbf{n}$ is treated as a real unit vector [e.g. Secs. 7.3, 7.4] while k as a complex number [e.g. (7.53)]. Here, for consistency, we always treat k $(= k_r + ik_i)$ as a complex number and $\mathbf{n}$ as real unit vector (with no additional condition on $\mathbf{n}$). Thus,

$$e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} = \begin{cases} e^{-k_i\mathbf{n}\cdot\mathbf{x}}e^{ik_r\mathbf{n}\cdot\mathbf{x} - i\omega t} & \text{[Lecture notes]} \\ e^{-k\mathbf{n}_I\cdot\mathbf{x}}e^{ik\mathbf{n}_R\cdot\mathbf{x} - i\omega t} & \text{[Jackson, p. 298]} \end{cases}$$

III. Properties of Plane Waves in Dielectrics and Conductors [A unified treatment of Secs. 5.18, 7.1, 7.2, 7.5, and 8.1 using the generalized ε in (7.51)]

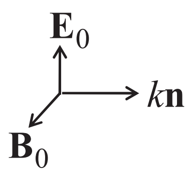
In Sec. II, under Assumptions 1-5, we have obtained the familiar plane-wave solution for a linear, uniform, and isotropic medium:

$$\begin{cases} \mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} & \text{with } \mathbf{k} = k\mathbf{n} \\ \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} & \\ \langle \mathbf{S} \rangle_t = \frac{1}{2}\mathrm{Re}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2 e^{-2k_i\mathbf{n}\cdot\mathbf{x}}\,\mathbf{n} & \text{[valid for real } \omega] \end{cases} \quad [(28)]$$

k: <u>wave vector</u> or <u>propagation vector</u>
n: direction of wave propagation

where the complex constants μ, ε, ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ must satisfy

$$\begin{cases} k = \sqrt{\mu\varepsilon}\,\omega \ [k: \text{\underline{wave number} or \underline{propagation constant}}] & [(16)] \\ \mathbf{n}\cdot\mathbf{E}_0 = 0 & [(25)] \\ \mathbf{n}\cdot\mathbf{B}_0 = 0 & [(26)] \\ \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 & [(27)] \end{cases}$$

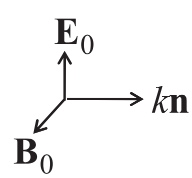


The medium property is implicit in the generalized ε:

$$\varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m}\sum_{j\,(\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_b} + i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\sigma/\omega} \quad [(7.56)]$$

On the basis of these eqs., there are 4 radically different cases:

Case 1. Waves in a dielectric medium
Case 2. Waves in a good conductor
Case 3. Waves at optical frequencies and beyond
Case 4. Waves in a plasma



Discussion: $\mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0$ with $\mathbf{E}_0, \mathbf{B}_0, \mathbf{n}$ mutually orthogonal.
$\Rightarrow$ We need $\mathbf{n}$ (e.g. $\mathbf{e}_z$) & only one amplitude (e.g. $E_0\mathbf{e}_x$) to specify a plane wave, e.g.

$$\begin{cases} \mathbf{E}(z,t) = E_0 e^{\pm ikz-i\omega t}\mathbf{e}_x \\ \mathbf{B}(z,t) = \pm\sqrt{\mu\varepsilon}E_0 e^{\pm ikz-i\omega t}\mathbf{e}_y \end{cases} \begin{bmatrix} \text{upper sign: } \mathbf{n} = \mathbf{e}_z \\ \text{lower sign: } \mathbf{n} = -\mathbf{e}_z \end{bmatrix}$$

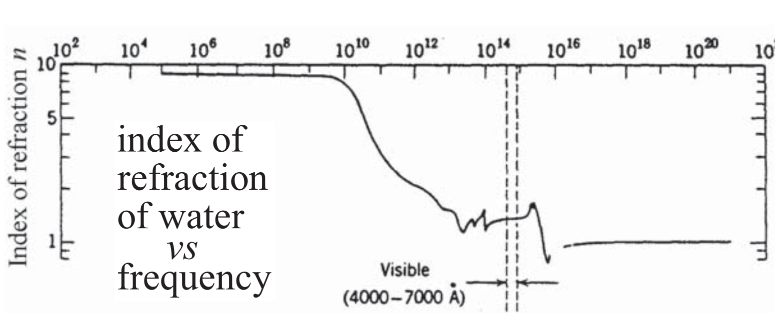
Since $\mathbf{n}$ is arbitrary in an isotropic medium and E_0 is arbitrary in a linear medium, the ω-k relation ($k = \sqrt{\mu\varepsilon}\,\omega$) distinguishes the 4 cases.

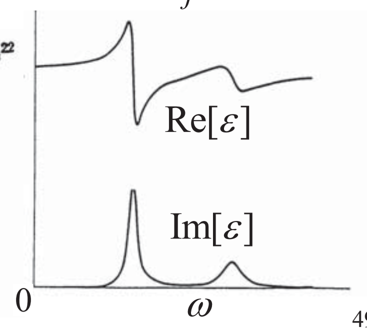
Case 1: Waves in a Dielectric Medium

$$\varepsilon = \varepsilon_0 + \frac{Ne^2}{m} \sum_{j\,(\text{bound})} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} + \underbrace{\cancel{i\,\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}}_{\text{negligible } (\because\, f_0 = 0 \text{ or } \approx 0)} \quad [(7.56)]$$

Properties of ε:

1. $\gamma_j \ll \omega_j$ (see p. 310) for most bound electrons. $\Rightarrow \text{Im}[\varepsilon] \ll \text{Re}[\varepsilon]$.
2. When ω is near each ω_j (binding frequency of the j-th group of electrons), ε exhibits resonant behavior in the form of anomalous dispersion and resonant absorption.
3. $\text{Re}[\varepsilon]$ decreases as ω increases above more and more ω_j's.





Case 1.1: Lossless dielectric (μ and ε are real. Secs. 7.1 and 7.2)

Plane wave propertities in a dielectric medium, governed by Eqs. (16), (25)-(28), are most clearly illustrated by the simple case of no medium loss (i.e. μ and ε are both real).

1. *Time-averaged quantities*:

$$(25) \Rightarrow \langle \mathbf{S} \rangle_t = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2 \mathbf{n} \quad \left[\underline{\text{intensity:}}\ \frac{\text{average power}}{\text{unit area}}\right] \quad (7.13)$$

The time-averaged energy density is given by

$$\langle u \rangle_t = \frac{1}{4}\underbrace{[\varepsilon \mathbf{E}(\mathbf{x}) \cdot \mathbf{E}^*(\mathbf{x}) + \frac{1}{\mu}\mathbf{B}(\mathbf{x}) \cdot \mathbf{B}^*(\mathbf{x})]} = \frac{\varepsilon}{2}|\mathbf{E}_0|^2 \quad (7.14)$$

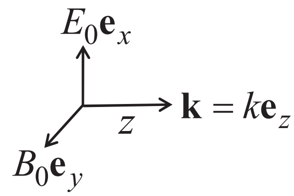
These 2 terms are equal [$\because \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n} \times \mathbf{E}_0$ (27)].
$\Rightarrow$ Equipartition of E-field and B-field energies

$$(7.13), (7.14) \Rightarrow \langle \mathbf{S} \rangle_t \cdot \mathbf{n} = \langle u \rangle_t \frac{1}{\sqrt{\mu\varepsilon}} \quad \left(\frac{1}{\sqrt{\mu\varepsilon}} = c \text{ in free space}\right)$$

The concept of group velocity will be considered in Sec. 7.8.

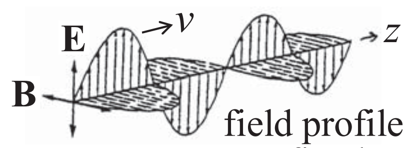
2. *Time-dependent quantities*: To be specific, we let $\mathbf{k} = k\mathbf{e}_z$ and write $\begin{cases} \mathbf{E}(\mathbf{x}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{B}(\mathbf{x}) = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} \end{cases}$ as $\begin{cases} \mathbf{E}(z) = E_0 e^{ikz}\mathbf{e}_x \\ \mathbf{B}(z) = \sqrt{\mu\varepsilon}\,E_0 e^{ikz}\mathbf{e}_y \end{cases}$ $\begin{bmatrix} \omega\text{-space} \\ \text{quantities} \end{bmatrix}$,

where $\mathbf{E}$ has a fixed direction (linearly polarized).



Let $E_0 = |E_0| e^{i\theta}$ and convert to t-space:

$$\begin{cases} \mathbf{E}(z,t) = \mathrm{Re}[\mathbf{E}_0 e^{ikz-i\omega t}] = |E_0|\cos(kz-\omega t+\theta)\,\mathbf{e}_x \\ \mathbf{B}(z,t) = \mathrm{Re}[\sqrt{\mu\varepsilon}\,E_0 e^{ikz-i\omega t}]\mathbf{e}_y \\ \qquad\quad = \sqrt{\mu\varepsilon}\,|E_0|\cos(kz-\omega t+\theta)\,\mathbf{e}_y \end{cases}$$



field profile at a fixed t

$$\Rightarrow \begin{cases} v_{ph} = \omega/k \text{ (speed of const. phase point)} \\ \mathbf{E}(z,t),\ \mathbf{B}(z,t) \text{ are } \textit{in phase } [\because \mu,\ \varepsilon \text{ are real}] \\ \mathbf{S}(z,t) = \mathbf{E}(z,t)\times\mathbf{H}(z,t) = \text{instantaneous Poynting vector [(6.109)]} \\ \qquad\quad = \sqrt{\frac{\varepsilon}{\mu}}|E_0|^2\cos^2(kz-\omega t+\theta)\,\mathbf{e}_z \\ \text{At a fixed } z,\ S \text{ varies between 0 and } \sqrt{\frac{\varepsilon}{\mu}}|E_0|^2 \text{ at frequency } 2\omega. \end{cases}$$

Two linearly polarized waves can be combined to give

$$\mathbf{E}(z,t) = \mathbf{E}_1(z,t) + \mathbf{E}_2(z,t) = (\mathbf{e}_x E_1 + \mathbf{e}_y E_2)e^{ikz-i\omega t} \qquad (7.19)$$

(7.19) consists of the following 3 cases:

1. (7.19) is a linearly polarized plane wave if E_1 and E_2 are in phase, i.e. if $E_1 = |E_1|e^{i\theta}$ and $E_2 = |E_2|e^{i\theta}$

2. (7.19) is an elliptically polarized plane wave if E_1 and E_2 are not in phase, i.e. if $E_1 = |E_1|e^{i\theta}$ and $E_2 = |E_2|e^{i(\theta+\varphi)}$.

3. (7.19) is a circularly polarized plane wave (a special case of elliptical polarization) if $|E_1| = |E_2| (= E_0)$ and $\varphi = \pm\frac{\pi}{2}$. Hence,

$$\mathbf{E}(z,t) = E_0(\mathbf{e}_x \pm i\mathbf{e}_y)e^{ikz-i\omega t} \qquad (7.20)$$

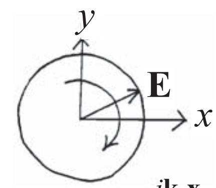
As is understood, in (7.20), LHS = Re[RHS], i.e.

$$\begin{aligned} \mathbf{E}(z,t) &= \mathrm{Re}[E_0(\mathbf{e}_x \pm i\mathbf{e}_y)e^{ikz-i\omega t}] \\ &= E_0[\cos(kz-\omega t)\mathbf{e}_x \pm \cos(kz-\omega t+\tfrac{\pi}{2})\mathbf{e}_y] \qquad (31) \end{aligned}$$

Field rotation of a circularly polarized wave: Rewrite (31):

$$\mathbf{E}(z,t) = E_0[\cos(kz-\omega t)\mathbf{e}_x \pm \cos(kz-\omega t+\tfrac{\pi}{2})\mathbf{e}_y]$$

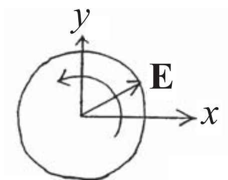
$$\Rightarrow \quad \begin{cases} E_x(z,t) = E_0\cos(kz-\omega t) \\ E_y(z,t) = \mp E_0 \sin(kz-\omega t) \end{cases}$$



$E_0(\mathbf{e}_x - i\mathbf{e}_y)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$

$\Rightarrow \mathbf{E}(z,t)$ rotates in t as shown to the right.

Exercise: Show that the instantaneous $\mathbf{S}$ of a circularly polarized plane wave is indep. of t.



$E_0(\mathbf{e}_x + i\mathbf{e}_y)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$

Medium property:

$k = \sqrt{\mu\varepsilon}\,\omega$ [(16)] gives the phase velocity:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}, \text{ where } n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad [\underline{\text{index of refraction}}] \qquad (7.5)$$

Next, consider plane waves in a lossy dielectric, where $\mathbf{E}, \mathbf{B}$ differ only slightly from those in a lossless dielectric (e.g. $\mathbf{E}, \mathbf{B}$ slightly out of phase). However, there is a qualitative difference: the medium can absorb the wave. So, our emphsis will be on the medium properties.

Case 1.2: Lossy dielectric [μ and/or ε = complex, Sec. 7.5 (Part B)]

$$\Rightarrow \quad k = \sqrt{\mu\varepsilon}\,\omega = \mathrm{Re}\sqrt{\mu\varepsilon}\,\omega + i\,\mathrm{Im}\sqrt{\mu\varepsilon}\,\omega = \beta + i\frac{\alpha}{2}, \qquad (7.53)$$

where β [$= \mathrm{Re}\sqrt{\mu\varepsilon}\,\omega$] gives (for arbitrary μ and ε) the

$$\begin{cases} \text{wavelength: } \lambda = \frac{2\pi}{\beta} \\ \text{phase velocity: } v = \frac{\omega}{\beta} = \frac{1}{\mathrm{Re}\sqrt{\mu\varepsilon}} \\ \text{index of refraction: } n = \frac{c}{v} = \mathrm{Re}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad [\text{used on p. 314}] \end{cases} \qquad (32)$$

See comment above (16) for the sign choice of the root of $\varepsilon\mu$.

To find the meaning of α, we set $k_i = \frac{\alpha}{2}$ and $\mathbf{n} = \mathbf{e}_z$ in

$$\langle \mathbf{S} \rangle_t = \tfrac{1}{2}\mathrm{Re}\sqrt{\tfrac{\varepsilon}{\mu}}|E_0|^2 e^{-2k_i\mathbf{n}\cdot\mathbf{x}}\,\mathbf{n} \quad [(28)]$$

$$\Rightarrow P = \langle \mathbf{S} \rangle_t \cdot \mathbf{n} = \tfrac{1}{2}\mathrm{Re}\sqrt{\tfrac{\varepsilon}{\mu}}|E_0|^2 e^{-\alpha z} \quad [\text{average power/unit area}] \; (33)$$

Hence, α is the *power* <u>attenuation constant</u> given by

$$\alpha = -\frac{1}{P}\frac{d}{dz}P \quad [\text{The } \textit{field} \text{ attenuation constant is } \tfrac{\alpha}{2}.] \qquad (34a)$$

From (7.53): $\alpha = 2\,\mathrm{Im}\sqrt{\mu\varepsilon}\,\omega \quad$ [used on p. 314] $\qquad$ (34b)

Problem: Assume $\mu = real$, calculate the time-averaged power loss per unit volume (p_{loss}) [relevant to microwave heating].

$$\begin{cases} \dfrac{\text{average wave power}}{\text{unit area}} : P = \frac{1}{2}\mathrm{Re}\sqrt{\frac{\varepsilon}{\mu}}|E_0|^2 e^{-\alpha z} \quad [(33)] \\ \text{power attenuation constant: } \alpha = 2\,\mathrm{Im}\sqrt{\mu\varepsilon}\omega \quad [(34b)] \end{cases}$$

$$\Rightarrow p_{loss} = -\frac{dP}{dz} = \alpha P = 2\,\mathrm{Im}\sqrt{\mu\varepsilon}\omega \tfrac{1}{2}\mathrm{Re}\sqrt{\frac{\varepsilon}{\mu}}|E_0|^2 e^{-\alpha z} \quad \begin{bmatrix} P: \mathrm{W/m^2} \\ p_{loss}: \mathrm{W/m^3} \end{bmatrix}$$

$$= \mathrm{Im}\sqrt{\varepsilon}\,\mathrm{Re}\sqrt{\varepsilon}\omega|E_0|^2 e^{-\alpha z} \quad [\text{for real } \mu \text{ \& arbitrary } \varepsilon]$$

Let $\varepsilon = (a+ib)^2 = \underbrace{a^2 - b^2}_{\varepsilon'} + i\underbrace{2ab}_{\varepsilon''}$ [a and b are real]

$$\Rightarrow \sqrt{\varepsilon} = \pm(a+ib) \Rightarrow \begin{cases} \mathrm{Re}\sqrt{\varepsilon} = \pm a \\ \mathrm{Im}\sqrt{\varepsilon} = \pm b \end{cases}$$

$$\Rightarrow p_{loss} = \mathrm{Im}\sqrt{\varepsilon}\,\mathrm{Re}\sqrt{\varepsilon}\omega|E_0|^2 e^{-\alpha z} = \overbrace{ab}^{\frac{1}{2}\varepsilon''}\omega|E_0|^2 e^{-\alpha z}$$

$$= \tfrac{1}{2}\varepsilon''\omega|E_0|^2 e^{-\alpha z} \quad \begin{bmatrix} \text{no restriction on relative} \\ \text{values of } \varepsilon' \text{ and } \varepsilon'' \end{bmatrix} \qquad (35)$$

Weak dielectric loss (for most dielectrics): Let $\mu = real$ and $\varepsilon = \varepsilon' + i\varepsilon''$ with $\varepsilon' \gg \varepsilon''$. Then, $\sqrt{\varepsilon} = \sqrt{\varepsilon'}(1 + i\frac{\varepsilon''}{\varepsilon'})^{\frac{1}{2}} \approx \sqrt{\varepsilon'}(1 + i\frac{\varepsilon''}{2\varepsilon'})$.

$$\Rightarrow k = \mathrm{Re}\sqrt{\mu\varepsilon}\omega + i\,\mathrm{Im}\sqrt{\mu\varepsilon}\omega \approx \sqrt{\mu\varepsilon'}\omega + \tfrac{i}{2}\sqrt{\tfrac{\mu}{\varepsilon'}}\varepsilon''\omega \quad [\text{for real } \mu \text{ \& } \varepsilon' \gg \varepsilon'']$$

$$\Rightarrow \begin{cases} \beta = k_r \approx \sqrt{\mu\varepsilon'}\omega = \sqrt{\dfrac{\mu\varepsilon'}{\mu_0\varepsilon_0}}\dfrac{\omega}{c} \quad \text{(propagation constant)} \\ v = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon'}} = \frac{c}{n} \quad \text{(phase velocity)} \\ n = \frac{c}{v} \approx \sqrt{\mu\varepsilon'}c = \sqrt{\dfrac{\mu\varepsilon'}{\mu_0\varepsilon_0}} \quad \text{(index of refraction)} \\ \alpha = 2k_i \approx \sqrt{\frac{\mu}{\varepsilon'}}\varepsilon''\omega = \frac{\varepsilon''}{\varepsilon'}\beta \quad \text{(power attenuation constant)} \qquad (7.55) \\ P = \frac{1}{2}\mathrm{Re}\sqrt{\frac{\varepsilon}{\mu}}|E_0|^2 e^{-\alpha z} \approx \frac{1}{2}\sqrt{\frac{\varepsilon'}{\mu}}|E_0|^2 e^{-\alpha z} \quad \text{(intensity)} \\ p_{loss} = \dfrac{\text{average power lost to the medium}}{\text{unit volume}} = -\dfrac{dP}{dz} \underset{\uparrow}{=} \frac{1}{2}\varepsilon''\omega|E_0|^2 e^{-\alpha z} \end{cases}$$

no restriction on relative values of ε' and ε'' [see (35)]

β reduces to the expression on p. 311 if $\mu = \mu_0$.

Loss tangent: In $\alpha = \frac{\varepsilon''}{\varepsilon'}\beta$ [(7.55)], the factor

$$\frac{\varepsilon''}{\varepsilon'}(\equiv \tan\delta_l) \tag{36}$$

is commonly referred to as the loss tangent.

Below is a table of ε' (Re[ε]) and loss tangent ($\tan\delta_l$ or $\frac{\varepsilon''}{\varepsilon'}$) of some dielectrics at different frequencies.

Material	ϵ'/ϵ_0			Loss tangent, $10^4\,\epsilon''/\epsilon'$		
	$f = 10^6$	$f = 10^8$	$f = 10^{10}$	$f = 10^6$	$f = 10^8$	$f = 10^{10}$
Glass, Corning 707	4.00	4.00	4.00	8	12	21
Fused quartz	3.78	3.78	3.78	2	1	1
Ruby mica	5.4	5.4	—	3	2	—
Ceramic Alsimag 393	4.95	4.95	4.95	10	10	9.7
Titania	100	100	—	3	2.5	—
Polystyrene	2.56	2.55	2.54	0.7	1	4.3
Neoprene	5.7	3.4	—	950	1600	—

from Ramo, Whinnery, and Van Duzer, p.334.

A miraculous property of water

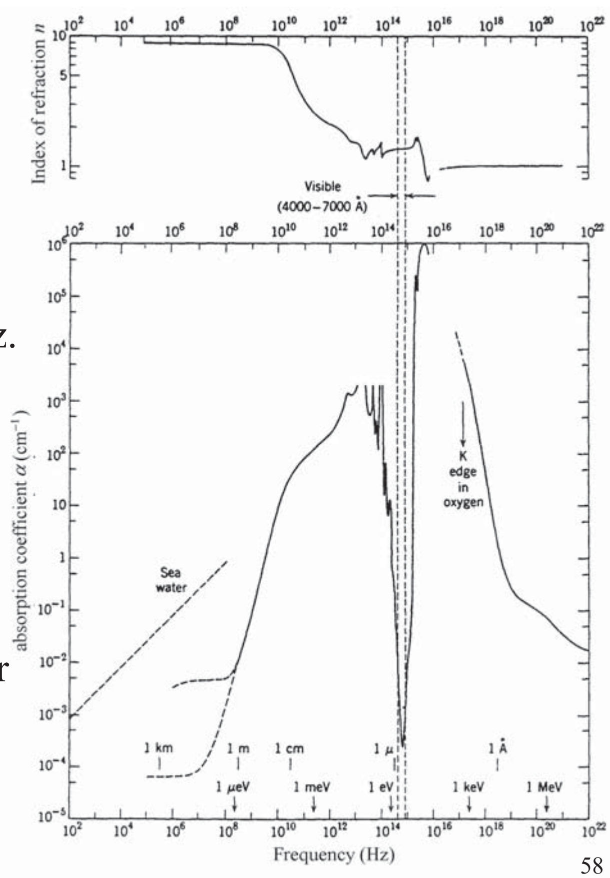
Upper figure: n (index of refraction) of liquid water vs f (wave frequency in Hz).

Question: No data for $f < 10^5$ Hz. Why? *Ans.*: Water is a good conductor at low f (no longer a dielectric). See next 2 pages.

Lower figure: α (absorption coefficient) of liquid water vs f.

α falls precipitously by a factor of $10^{7\text{-}8}$ in the visible light region!

"Mother Nature has certainly exploited her window!" (p.315).



Case 2 : Waves in a Good Conductor [Secs. 5.18 and 8.1]

Applicable to waves in metals with $f < 10^{12}$ Hz

Criterion of a good conductor: Rewrite

$$\varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m} \sum_{j(\text{bound})} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\varepsilon_b} + i \underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\sigma/\omega} = \varepsilon_b + i\frac{\sigma}{\omega} \quad [(7.56)]$$

For bound e's, we have $\gamma_j \ll \omega_j$ (p. 310) $\Rightarrow$ $\text{Re}[\varepsilon_b] \gg \text{Im}[\varepsilon_b]$

$\Rightarrow$ $\varepsilon_b \approx$ real.

For free e's, assume $\omega \ll \gamma_0$ $(\overbrace{\sim 4\times 10^{13}/\text{s}}^{\text{For Cu (p.312)}})$ or $f\,(=\frac{\omega}{2\pi}) < \overbrace{10^{12}}^{1\text{ THz}}$ Hz

$\Rightarrow \sigma = \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)} \approx \frac{n_0 e^2}{\gamma_0 m}$ [$n_0 = Nf_0$ = free electron density]

i.e. $\sigma \approx$ real and indep. of ω for $f\,(=\frac{\omega}{2\pi}) < 10^{12}$ Hz

We define the good-conductor criterion as $\quad \frac{\sigma}{\omega\varepsilon_b} \gg 1 \quad$ (37)

$\Rightarrow \varepsilon = \varepsilon_b + i\frac{\sigma}{\omega} \approx i\frac{\sigma}{\omega}$, i.e. free e's dominate over bound e's.

Rewrite $\frac{\sigma}{\omega\varepsilon_b} \gg 1$ [good-conductor criterion, (37)]

Examples (assume* $\varepsilon_b = \varepsilon_0 = 8.85\times 10^{-12}$ farad/m):

*This assumption may be quite off in some cases (e.g. Fig. 7.9 shows $\varepsilon_b \approx 80\varepsilon_0$ for liquid water at 10^5 Hz $< f < 10^9$ Hz)

Material	σ $(/\Omega\text{m})$	$\frac{\sigma}{\omega\varepsilon_b}$ (60 Hz)	$\frac{\sigma}{\omega\varepsilon_b}$ (10^{10} Hz)
Copper	5.9×10^7	1.8×10^{16}	1.1×10^8
Graphite	6×10^4	1.8×10^{13}	1.1×10^5
Sea water	~ 10	$\sim 3\times 10^9$	~ 18
Ground	$\sim 10^{-2}$	$\sim 3\times 10^6$	$\sim 1.8\times 10^{-2}$
Pure water	$\sim 10^{-5}$	$\sim 3\times 10^3$	$\sim 1.8\times 10^{-5}$
Glass	$\sim 10^{-12}$	$\sim 3\times 10^{-4}$	$\sim 1.8\times 10^{-12}$
Air	$\sim 5\times 10^{-15}$	$\sim 6\times 10^{-6}$	$\sim 9\times 10^{-15}$
Teflon	$\sim 10^{-23}$	$\sim 3\times 10^{-15}$	$\sim 1.8\times 10^{-23}$

Note: Even insulators contain some free e's.

Question: Why is it dangerous if an electrical heater falls into your bath tub?

Skin depth in a good conductor :

For a good conductor ($\frac{\sigma}{\omega\varepsilon_b} \gg 1$), we have

$$\sqrt{\varepsilon} = (\varepsilon_b + i\tfrac{\sigma}{\omega})^{\frac{1}{2}} \approx (i\tfrac{\sigma}{\omega})^{\frac{1}{2}} = \sqrt{\tfrac{\sigma}{2\omega}}(1+i) \quad \boxed{i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \tfrac{1}{\sqrt{2}}(1+i)} \qquad (39)$$

$$\Rightarrow k = \sqrt{\mu\varepsilon}\,\omega = \sqrt{\tfrac{\mu\sigma\omega}{2}}(1+i) = \tfrac{1+i}{\delta} \quad \text{[for forward wave]} \qquad (5.164) \text{ and } (40)$$

$$\text{where } \delta \equiv \sqrt{\tfrac{2}{\mu\sigma\omega}} \quad \begin{bmatrix} \delta: \ \text{skin depth} \\ \mu \text{ is real by assumption.} \end{bmatrix} \qquad (5.165), (8.8), \text{ and } (41)$$

Thus, $e^{ikz} = e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}} \Rightarrow E, B$ damp by a factor of e^{-1} in a distance of $\delta \Rightarrow \delta$ is the field penetration depth into a good conductor.

Examples :

$$\text{For copper, } \delta_{cu} \approx \begin{cases} 0.85 \text{ cm at } f = 60 \text{ Hz (household current)} & (42a) \\ 6.5\times10^{-5} \text{ cm at } f = 10^{10} \text{ Hz (microwave)} & (42b) \\ 6.5\times10^{-6} \text{ cm at } f = 10^{12} \text{ Hz (THz wave)} & (42c) \end{cases} \qquad \boxed{\mu_{cu} \approx \mu_0}$$

Fields in a good conductor : Assume a plane wave is incident normally ($\mathbf{n} = \mathbf{e}_z$) from free space on the conductor. Most of it will be reflected, but a small fraction will be transmitted (and dissipated) into the conductor ($\because \sigma \neq \infty$).

Wave dynamics in the conductor is of physical and practical importance. Here, we focus on properties of the transmitted wave (Reflection/transmission will be treated in Sec. 7.3).

Using $\mathbf{H}_0 = \sqrt{\varepsilon/\mu}\ \mathbf{n}\times\mathbf{E}_0$ [(27)], we specify [$\mathbf{E}(z)$, $\mathbf{H}(z)$] of the incident ($\mathbf{n} = \mathbf{e}_z$), reflected ($\mathbf{n} = -\mathbf{e}_z$), and transmitted ($\mathbf{n} = \mathbf{e}_z$) waves:

Free space : $k_0 = \sqrt{\mu_0\varepsilon_0}\,\omega$

incident wave $\mathbf{e}_z \longrightarrow$

$(E_0^i e^{ik_0z}\mathbf{e}_x,\ \sqrt{\varepsilon_0/\mu_0}\,E_0^i e^{ik_0z}\mathbf{e}_y)$

reflected wave $-\mathbf{e}_z \longleftarrow$

$(E_0^r e^{-ik_0z}\mathbf{e}_x,\ -\sqrt{\varepsilon_0/\mu_0}\,E_0^r e^{-ik_0z}\mathbf{e}_y)$

x

Conductor : $k = \sqrt{\mu\varepsilon}\,\omega$

(ω satisfies $\frac{\sigma}{\omega\varepsilon_b} \gg 1$)

$\rightarrow \mathbf{e}_z$ transmitted wave

$(E_0 e^{ikz}\mathbf{e}_x,\ \sqrt{\varepsilon/\mu}\,E_0 e^{ikz}\mathbf{e}_y)$

0 $\qquad z$

Free space: $k_0 = \sqrt{\mu_0\varepsilon_0}\,\omega$

incident wave $\mathbf{e}_z \longrightarrow$

$(E_0^i e^{ik_0 z}\mathbf{e}_x,\ \sqrt{\varepsilon_0/\mu_0}\,E_0^i e^{ik_0 z}\mathbf{e}_y)$

reflected wave $-\mathbf{e}_z \longleftarrow$

$(E_0^r e^{-ik_0 z}\mathbf{e}_x,\ -\sqrt{\varepsilon_0/\mu_0}\,E_0^r e^{-ik_0 z}\mathbf{e}_y)$

Conductor: $k = \sqrt{\mu\varepsilon}\,\omega$

(ω satisfies $\frac{\sigma}{\omega\varepsilon_b} \gg 1$)

$\rightarrow \mathbf{e}_z$ transmitted wave

$(E_0 e^{ikz}\mathbf{e}_x,\ \sqrt{\varepsilon/\mu}\,E_0 e^{ikz}\mathbf{e}_y)$

x, z, 0

Reflection/transmission amplitudes (E_0^r & E_0) can be expressed in terms of E_0^i by the continuity of $E_{\|}$ & $H_{\|}$ at $z=0$ [see (76a,b) below]. However, basic properties of the transmitted wave in the conductor are indep. of its amplitude E_0. We only need its spatial field profiles:

$$\begin{cases} \mathbf{E}(\mathbf{x}) = \mathbf{E}(z) = E_0 e^{ikz}\mathbf{e}_x = E_0 e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}}\mathbf{e}_x & \text{(43a)} \\ \mathbf{H}(\mathbf{x}) = \mathbf{H}(z) = \sqrt{\frac{\varepsilon}{\mu}}E_0\mathbf{e}_y e^{ikz} = \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0 e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}}\mathbf{e}_y & \text{(43b)} \end{cases}$$

where we have used $k = \frac{1+i}{\delta}$ [(40] and $\sqrt{\varepsilon} = \sqrt{\frac{\sigma}{2\omega}}(1+i)$ [(39)].

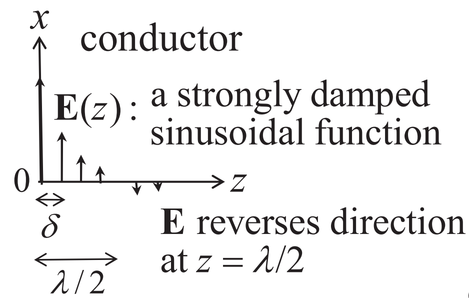
Rewrite
$$\begin{cases} \mathbf{E}(z) = E_0 e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}}\mathbf{e}_x \ \text{[(43a)]} \\ \mathbf{H}(z) = \sqrt{\frac{\sigma}{2\mu\omega}}\underbrace{(1+i)}_{=\sqrt{2}\exp(i\pi/4)}E_0 e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}}\mathbf{e}_y \ \text{[(43b)]} \end{cases}$$

[Let E_0 = real without loss of generality]

$$\Rightarrow \begin{cases} \mathbf{E}(z,t) = \mathrm{Re}[\mathbf{E}(z)e^{-i\omega t}] = E_0 e^{-\frac{z}{\delta}}\cos(\tfrac{z}{\delta}-\omega t)\mathbf{e}_x & \text{(44a)} \\ \mathbf{H}(z,t) = \mathrm{Re}[\mathbf{H}(z)e^{-i\omega t}] = \sqrt{\frac{\sigma}{\mu\omega}}E_0 e^{-\frac{z}{\delta}}\cos(\tfrac{z}{\delta}-\omega t+\tfrac{\pi}{4})\mathbf{e}_y & \text{(44b)} \end{cases}$$

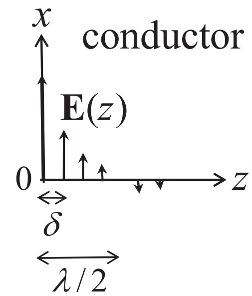
$\Rightarrow$ It has a wavelength $\lambda = 2\pi\delta$. $\mathbf{E}(z,t)$ & $\mathbf{H}(z,t)$ are 45° out of phase and damp by $\frac{1}{e}$ in length δ. Though markedly different from the EM wave in a dielectric, it is also an EM wave with the characteristic properties:

1. $\nabla\cdot\mathbf{E}(z) = 0$ (no ρ)
2. $\mathbf{k}$, $\mathbf{E}$, $\mathbf{H}$ mutually orthogonal
3. Propagating and transporting energy; expenential damping



Rewrite

$$\begin{cases} \mathbf{E}(z,t) = E_0 e^{-\frac{z}{\delta}} \cos(\frac{z}{\delta} - \omega t)\mathbf{e}_x & [(44a)] \\ \mathbf{H}(z,t) = \sqrt{\frac{\sigma}{\mu\omega}} E_0 e^{-\frac{z}{\delta}} \cos(\frac{z}{\delta} - \omega t + \frac{\pi}{4})\mathbf{e}_y & [(44b)] \end{cases}$$



Examples: Let $f = \frac{\omega}{2\pi} = 10^{10}$ Hz

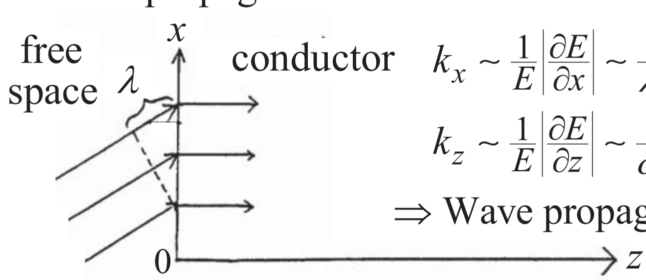
(typical microwave frequency, $\lambda_{free\ space} = 3$ cm)

glass ($\frac{\mu}{\mu_0} \approx 1,\ \frac{\varepsilon'}{\varepsilon_0} \approx 4,\ \frac{\varepsilon''}{\varepsilon'} \approx 2.1\times10^{-3}$)	copper ($\delta \approx 7\times10^{-5}$ cm)
$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\mu\varepsilon'}\omega} \approx 1.5$ cm (Case 1.2) $\alpha = \frac{2\pi}{\lambda}\frac{\varepsilon''}{\varepsilon'} \approx 8.8\times10^{-3}\ \text{cm}^{-1}$ (7.55)	$\lambda = 2\pi\delta \approx 4.4\times10^{-4}$ cm $\alpha = -\frac{1}{P}\frac{dP}{dz} = \frac{2}{\delta} \approx 4.5\times10^{3}\ \text{cm}^{-1}$

Question 1: $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \Rightarrow \delta_{\text{sea water}}$ (at $f = 10^{10}$ Hz) ≈ 0.16 cm. The sunlight has $f \approx 5\times10^{14}$ Hz. Why can it penetrate deep into sea water?

Question 2: $\mathbf{E}(z)$ & $\mathbf{H}(z)$ in (43a,b) are || to the conductor surface. Is this still true if the wave is incident obliquely on the conductor?

The wave propagates in the direction of $\mathbf{k}$ [$\sim \nabla E(\mathbf{x}) = E_0 \nabla e^{i\mathbf{k}\cdot\mathbf{x}}$].



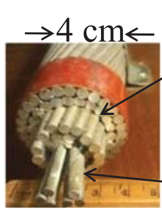
$$k_x \sim \frac{1}{E}\left|\frac{\partial E}{\partial x}\right| \sim \frac{1}{\lambda(\text{free space wavelength})}$$

$$k_z \sim \frac{1}{E}\left|\frac{\partial E}{\partial z}\right| \sim \frac{1}{\delta(\text{skin depth})} \gg k_x$$

$\Rightarrow$ Wave propagates essentially along $\mathbf{e}_z$.

Since $E(\mathbf{x})$ in the conductor varies rapidly along $\mathbf{e}_z$, $\mathbf{k}$ inside the conductor is essentially along $\mathbf{e}_z$ for a wave incident at any angle. $\Rightarrow \mathbf{E}(z)$ & $\mathbf{H}(z)$ are essentially || to the conductor surface. Hence, (43a,b) is still valid even if the wave is incident at an oblique angle.

Structure of a power cable:



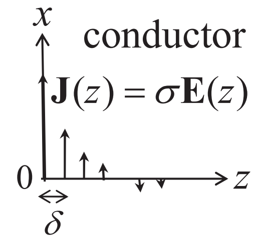
aluminium ($\sigma_{Al} \approx 0.59\sigma_{cu}$)

($\delta_{Al} \approx 1.1$ cm at 60 Hz)

steel ($\sigma_{steel} \approx 0.11\sigma_{cu}$)

Surface current $\mathbf{K}_{eff}$ *on a good conductor*:

If $\delta \neq 0$, the "surface" current $\mathbf{K}_{eff}$ is not exactly on the surface. $\mathbf{J}(z)$ [$= \sigma \mathbf{E}(z)$] penetrates a depth of $\sim \delta$. Hence, $\mathbf{K}_{eff}$ (unit: A/m) is an integrated value of $\mathbf{J}(z)$ (unit: A/m^2) over z.



$$\mathbf{K}_{eff} = \int_0^\infty \mathbf{J}(z)dz = \sigma \int_0^\infty \mathbf{E}(z)dz = \sigma E_0 \underbrace{\int_0^\infty e^{\frac{-1+i}{\delta}z} dz}_{= \frac{\delta}{1-i} = \frac{\delta(1+i)}{2} = \frac{(1+i)}{\sqrt{2\mu\sigma\omega}}}\, \mathbf{e}_x$$

$$\boxed{\mathbf{E}(z) = E_0 e^{-\frac{z}{\delta}} e^{i\frac{z}{\delta}} \mathbf{e}_x \ (43a)}$$

$$= \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0 \mathbf{e}_x \overset{(43b)}{=} -\mathbf{e}_z \times \mathbf{H}(0) \quad \begin{bmatrix} \text{"}\mathbf{n}\text{" in (8.14) is} \\ \text{"}-\mathbf{e}_z\text{" here.} \end{bmatrix} \quad \text{(8.14) and (45)}$$

Note: $\mathbf{e}_z \times \mathbf{H} = \mathbf{e}_z \times \mathbf{H}_{||}$ ($\mathbf{H}_{||}$: the component of $\mathbf{H}$ parallel to x-y plane)

Question: In (45), $\mathbf{K}_{eff}$ depends only on $\mathbf{H}$, but not on ω, σ, μ. Why?

Answer (important physics): By Faraday's law, $\mathbf{K}_{eff}$ is the conductor's response to shield its inside from a time-varying $\mathbf{H}$ on its surface. $\mathbf{K}_{eff}$ is hence determined entirely by the surface $\mathbf{H}$. This explains why ω, σ, and μ are cancelled out in the derivation of $\mathbf{K}_{eff} = -\mathbf{e}_z \times \mathbf{H}(0)$ [(45)].

Time-averaged power loss on the surface of a good conductor:

$$\frac{dP_{loss}}{da} = \frac{\text{average power going into conductor}}{\text{unit area of conductor surface}} = \langle \mathbf{S}(z=0) \rangle_t \cdot \mathbf{e}_z$$

$$= \tfrac{1}{2}\mathrm{Re}\left[\mathbf{E}(0) \times \mathbf{H}^*(0)\right] \cdot \mathbf{e}_z$$

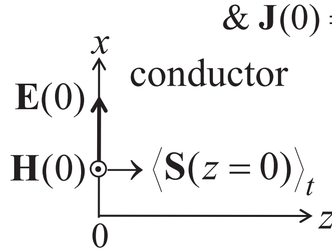
$$\boxed{\mathbf{E}(0^+) = \mathbf{E}(0^-);\ \mathbf{H}(0^+) = \mathbf{H}(0^-)}$$

$\mathbf{E}$ || surface; $\mathbf{H}$ || surface & $\mathbf{J}(0) \neq \infty$

$$\boxed{\mathbf{H}(0) = \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0 \mathbf{e}_y \ [(43b)]}$$

$$= \tfrac{1}{2}\mathrm{Re}\left[E_0 \mathbf{e}_x \times \sqrt{\frac{\sigma}{2\mu\omega}}(1-i)E_0^* \mathbf{e}_y\right] \cdot \mathbf{e}_z$$

$$= \frac{1}{2}\sqrt{\frac{\sigma}{2\mu\omega}}\left|E_0\right|^2 \quad (46)$$



$$(43b) \Rightarrow \left|E_0\right| = \sqrt{\frac{\mu\omega}{\sigma}}\left|H_0\right| \quad (47)$$

$$\text{Sub. (47) int (46)} \Rightarrow \frac{dP_{loss}}{da} = \frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}}\left|H_0\right|^2 \quad \begin{bmatrix} \Rightarrow \text{Explanation of} \\ \text{induction heating} \end{bmatrix} \quad (48)$$

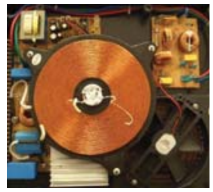
$$\boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}}}$$

$$= \frac{1}{4}\mu\omega\delta\left|H_0\right|^2 \quad \text{(8.12) and (49)}$$

$$= \frac{1}{2}\frac{1}{\sigma\delta}\left|H_0\right|^2 \overset{(45)}{=} \frac{1}{2}\frac{1}{\sigma\delta}\left|K_{eff}\right|^2 \quad \text{(8.15) and (50)}$$

Note: $\left|H_0\right|^2 = \left|\mathbf{H}_0^i(\text{incident}) + \mathbf{H}_0^r(\text{reflected})\right|^2$

Question: How does an induction cooker work?



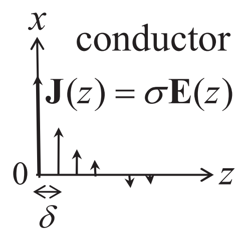
Inside view of an induction cooker: An AC current (e.g. at 24 kHz) is passed through a large copper coil to form the magnetic field.

Note: dP_{loss}/da in (8.12) is obtained by the Poynting vector method. It is in fact exactly the Ohmic power dissipated inside the conductor.

$$P_{resistive}(z) = \frac{\text{ohmic power deposited inside the conductor}}{\text{unit volume}}$$

$$= \tfrac{1}{2}\mathrm{Re}[\mathbf{J}(z)\cdot\mathbf{E}^*(z)] = \tfrac{1}{2}\sigma|\mathbf{E}(z)|^2$$

$$\underset{\boxed{(43a)}}{=} \tfrac{1}{2}\sigma|E_0|^2 e^{-\frac{2z}{\delta}} \underset{\boxed{(47)}}{=} \tfrac{1}{2}\mu\omega|H_0|^2 e^{-\frac{2z}{\delta}}$$



(5.169) and (51)

$$\frac{dP_{loss}}{da} = \int_0^\infty P_{resistive}\,dz$$

$$= \tfrac{1}{2}\mu\omega|H_0|^2\int_0^\infty e^{-\frac{2z}{\delta}}dz = \tfrac{1}{4}\mu\omega\delta|H_0|^2 \quad \text{[exactly as (8.12) and (49)]}$$

Definitions: surface impedance Z_s, surface resistance R_s, and surface reactance X_s of metal

$$(45) \Rightarrow \mathbf{K}_{eff} = \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)\mathbf{E}(0) \underset{(41)}{=} \frac{\sigma\delta}{1-i}\mathbf{E}(0) = \frac{\mathbf{E}(0)}{Z_s} \quad \left[\begin{array}{c} Z_s\text{: ratio of } E(0) \\ \text{to } K_{eff}\end{array}\right]$$

where $Z_s \equiv \frac{1-i}{\sigma\delta}$ [Jackson p. 356, bottom] is called the surface impedance. We may write

$$\begin{cases} Z_s = R_s - iX_s, \\ \text{where } \underset{\boxed{\text{surface resistance}}}{R_s} = \underset{\boxed{\text{surface reactance}}}{X_s} = \dfrac{1}{\sigma\delta} \end{cases} \qquad (52)$$

Examples:
- R_s of copper $\approx 0.026\ \Omega$ at 10^{10} Hz [microwave] (53a)
- R_s of copper $\approx 0.26\ \Omega$ at 10^{12} Hz [THz wave] (53b)

Z_s is an *intrinsic* (rather than surface) property of metal. It is in fact the impedance Z [$\equiv\sqrt{\mu/\varepsilon}$, see (29)] of a good conductor:

$$Z_s = \sqrt{\frac{\mu}{\varepsilon(\text{metal})}} = \frac{\sqrt{\mu}}{\sqrt{\frac{\sigma}{\omega}}\,i^{1/2}} = \frac{\sqrt{\mu}}{\sqrt{\frac{\sigma}{2\omega}}(1+i)} = \frac{1-i}{\sqrt{\frac{2\sigma}{\mu\omega}}} = \frac{1-i}{\sigma\delta} \qquad (54)$$

Case 3: Waves above Optical Frequencies [Sec. 7.5, Part D]

Case 3.1: $\omega >> \gamma_0$ but $\omega < \omega_j$ for some of the bound electrons, applicable to visible light and ultraviolet frequencies

$$\varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m}\sum_{j(\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_b} + i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\substack{\approx -\frac{Ne^2 f_0}{m\omega^2} \\ (\because\ \omega\gg\gamma_0)}} \quad [(7.56)]$$

$$\approx \varepsilon_b - \frac{Ne^2 f_0}{m\omega^2} \qquad (55)$$

In Case 2 ($\omega <<\gamma_0$), the free electron term is predominantly imaginary. Here, we have $\omega >>\gamma_0$ and the free electron term becomes predominantly real. This is a qualitative departure from Case 2, which will radically change the metal's response to EM waves. See examples below and in Case 3.2.

Light reflection off mirrors & ultraviolet transparency of metals

Rewrite $\varepsilon = \varepsilon_b - \dfrac{Ne^2 f_0}{m\omega^2}$ [(55)]

Let $n_0 = Nf_0$ be the free electron density in the conductor ($f_0 \sim 1$, i.e. each atom in the conductor contains on average approximately one free electron, see p. 312), we obtain

$$\varepsilon = \varepsilon_b - \frac{\omega_p^2}{\omega^2}\varepsilon_0 \qquad (56)$$

where ω_p is the plasma frequency of the conduction electrons

$$\omega_p^2 = \frac{n_0 e^2}{m^* \varepsilon_0} \qquad \text{[See bottom of p. 313.]} \qquad (57)$$

and we have replaced m in (7.51) with the effective mass m^* of the conduction electrons to account for the effects of binding. For simplicity, we assume ε_b to be real by neglecting the weak damping effects of bound electrons.

Sub. $\varepsilon = \varepsilon_b - \frac{\omega_p^2}{\omega^2}\varepsilon_0$ [(56)] into $k = \sqrt{\mu\varepsilon}\,\omega$ [(16)], we obtain

$$k = \sqrt{\mu(\varepsilon_b - \frac{\omega_p^2 \varepsilon_0}{\omega^2})}\ \omega \tag{58}$$

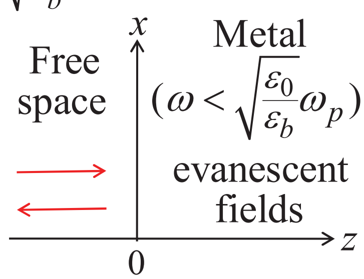
Hence, depending on the value of ω, k is either real (propagatng freely) or purely imaginary (cutoff to the wave, total reflection).

1. Cutoff regime (light reflection off mirrors) : $\varepsilon_b < \frac{\omega_p^2 \varepsilon_0}{\omega^2}$

$$(58) \Rightarrow k = i\sqrt{\mu(\frac{\omega_p^2 \varepsilon_0}{\omega^2} - \varepsilon_b)}\,\omega = i|k|, \text{ if } \omega < \sqrt{\frac{\varepsilon_0}{\varepsilon_b}}\,\omega_p \quad \text{[cutoff regime]}$$

$$\Rightarrow \begin{cases} \mathbf{E}(z) = E_0 e^{ikz}\mathbf{e}_x = E_0 e^{-|k|z}\mathbf{e}_x & (59a) \\ \mathbf{H}(z) \underset{\uparrow}{=} \sqrt{\frac{\varepsilon}{\mu}}\mathbf{e}_z \times \mathbf{E}(z) \underset{\uparrow}{=} i\sqrt{\frac{|\varepsilon|}{\mu}} E_0 e^{-|k|z}\mathbf{e}_y & (59b) \end{cases}$$

$\boxed{(27)}$ $\quad$ $\boxed{\sqrt{\varepsilon} = \sqrt{-|\varepsilon|} = i\sqrt{|\varepsilon|}}$



Question: The fields in (59) no longer represent a wave. Why?

Rewrite $\mathbf{E}(z) = E_0 e^{-|k|z}\mathbf{e}_x$; $\mathbf{H}(z) = i\sqrt{\frac{|\varepsilon|}{\mu}} E_0 e^{-|k|z}\mathbf{e}_y$ [(59a,b)]

E(z), **H**(z) are evanescent "fringe fields" in the metal left behind by a totally-reflected incident wave. It carries no power because Re[**E**(z)×**H***(z)] = 0. This explains the "light reflection off mirrors".

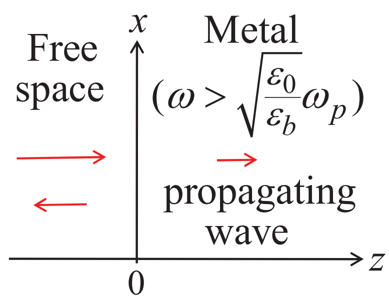
In comparison, in microwave reflection off a good conductor (Case 2), **E**(z) and **H**(z) are "wave fields" 45° out of phase. Hence, Re[**E**(z)×**H***(z)] ≠ 0 ⇒ There is power flowing into the conductor.

2. Propagating regime (UV transparency of metals) : $\varepsilon_b > \frac{\omega_p^2 \varepsilon_0}{\omega^2}$

Rewrite $k = \sqrt{\mu(\varepsilon_b - \frac{\omega_p^2 \varepsilon_0}{\omega^2})}\ \omega$ [(58)]

$\Rightarrow k = \text{real, if } \ \omega > \sqrt{\frac{\varepsilon_0}{\varepsilon_b}}\,\omega_p$ $\left[\begin{array}{l}\text{propagating} \\ \text{regime}\end{array}\right]$

Thus, the wave can propagate without attenuation inside the metal. This explains the "ultraviolet transparency of metals".



Case 3.2: $\omega >> \gamma_j$ and $\omega >> \omega_j$ for all electrons in the medium, applicable to X-ray frequencies and beyond

Under the conditions $\omega >> \gamma_j$ (including γ_0) and $\omega >> \omega_j$, we may neglect γ_j and ω_j in (7.51),

$$\varepsilon = \varepsilon_0 + \underbrace{\frac{Ne^2}{m} \sum_{j(\text{bound})} \frac{f_j}{\cancel{\omega_j^2} - \omega^2 - i\omega \cancel{\gamma_j}} + i \frac{Ne^2 f_0}{m\omega(\cancel{\gamma_0} - i\omega)}}_{\approx -\frac{NZe^2}{m\omega^2} \ (\sum_{j(all)} f_j = Z)} \qquad (7.56)$$

$$\Rightarrow \ \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}, \qquad (7.59)$$

where $\omega_p^2 \equiv \dfrac{NZe^2}{m\varepsilon_0}$ $\left[\begin{array}{l} NZ \text{ is the density of all electrons} \\ \text{(bound and free) in the medium.} \end{array}\right]$ (7.60)

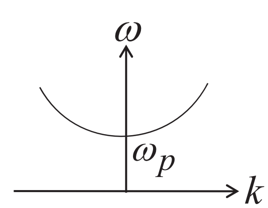
(7.59) is valid for ultra high ω ($\omega \gg \gamma_j$ & ω_j, such as gamma ray). In this regime, we have in general $\omega \gg \omega_p$ (see p. 313).

Sub. $\dfrac{\varepsilon}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$ [(7.59)] into $k = \sqrt{\mu\varepsilon}\,\omega$ and assume $\mu = \mu_0$,

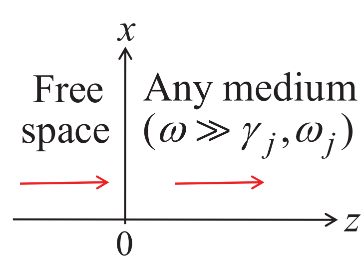
we obtain $k^2 = \mu\varepsilon\omega^2 = \underbrace{\mu_0\varepsilon_0}_{1/c^2}(1 - \dfrac{\omega_p^2}{\omega^2})\omega^2$

$$\Rightarrow \quad \omega^2 = k^2c^2 + \omega_p^2 \qquad (7.61)$$

Although (7.61) predicts evanescent fields for $\omega < \omega_p$, the validity of (7.61) requires $\omega \gg \gamma_j$, ω_j for all electrons in the medium. This in general results in $\omega \gg \omega_p$, hence $\varepsilon \approx \varepsilon_0$ and $\omega \approx kc$.



Thus, in this ultra-high-frequency regime, the wave can not only propagate freely, but also be ~100% transmitted into the metal due to $\varepsilon \approx \varepsilon_0$ (good impedance match).



Case 3.2 above applies to both dielectric and conducting media.

Discussion :

(i) To see the physical reason why we may neglect collisions and binding forces in (7.51) under the conditions $\omega \gg \gamma_j$ and $\omega \gg \omega_j$, we go back to the eq. of motion for the electrons.

Rewrite $m\frac{d^2}{dt^2}\mathbf{x}(t) = -e\mathbf{E}(\mathbf{x},t) - \gamma_j m\frac{d}{dt}\mathbf{x}(t) - m\omega_j^2\mathbf{x}(t)$ [(7.49)]

Let $\mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t}$ and $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(0)e^{-i\omega t}$:

$$\Rightarrow \mathbf{x}(t) = -\frac{e}{m}\frac{\mathbf{E}(0)e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad \Rightarrow \mathbf{v}(t) = \frac{d}{dt}\mathbf{x}(t) = \frac{e}{m}\frac{i\omega\mathbf{E}(0)e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad (60a)$$

$\Rightarrow$ If $\omega \gg \gamma_j$ & ω_j, then $\mathbf{x} \propto \frac{1}{\omega^2}$ & $\mathbf{v} \propto \frac{1}{\omega}$ [bound and free e's] (60b)

$\Rightarrow$ For sufficiently large ω, the displacement ($\mathbf{x}$) and velocity ($\mathbf{v}$) are so small that the electrons are essentially *stationary* with negligible binding force ($m\omega_j^2\mathbf{x}$) and damping force ($-\gamma_j m\mathbf{v}$).

(ii) In deriving the generalized ε, we have assumed $x \ll \lambda$ [see (1)]. (60b) shows "$x \ll \lambda$" is even more valid for smaller λ (or higher ω).

(iii) Unattenuated propagation of X- and γ-rays in materials in Case 3.2 is a good approximation based on classical electrodynamics. However, this is not quite true if we consider complicated non-classical effects, such as absorption and scattering of energetic photons by electrons in different quantum states. These effects are in fact responsible for the detection of metal objects by X-ray scanners.

Case 4 : Waves in Plasmas [Sec. 7.5, Part D]

The plasma is a partially ionized gas (e.g. ionosphere) or fully ionized gas (e.g. fusion plasmas). It has a much lower density than solid. So, we may in general neglect collisions. There is an equal number of electrons and ions. At sufficiently high frequencies (as here), we may neglect ion motion and consider only the *e*'s. Thus,

$$\varepsilon = \varepsilon_0 + \underbrace{\cancel{\frac{Ne^2}{m}\sum_{j(\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}}_{\text{negligible } (\because \text{low gas density})} + \underbrace{i\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\approx -\frac{Ne^2 f_0}{m\omega^2}\ (\omega\gg\gamma_0)} \qquad [(7.56)]$$

$$\Rightarrow \frac{\varepsilon}{\varepsilon_0} = 1-\frac{\omega_p^2}{\omega^2} \quad \left[\begin{array}{l}\text{same form as (7.59) but its vilidity}\\ \text{doesn't require } \omega\gg\omega_p \text{ as in (7.59)}\end{array}\right] \qquad (61)$$

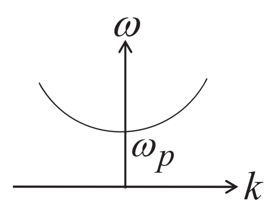
where ω_p is the plasma frequency defined as

$$\omega_p^2 \equiv \frac{ne^2}{\varepsilon_0 m} \quad \left[\begin{array}{l}n = Nf_0 = \text{plasma electron density, normally}\\ \text{much smaller than the density of solids.}\end{array}\right] \qquad (62)$$

Sub. $\frac{\varepsilon}{\varepsilon_0} = 1-\frac{\omega_p^2}{\omega^2}$ [(61)] into $k=\sqrt{\mu\varepsilon}\omega$, we obtain

$$k^2 = \mu\varepsilon\omega^2 = \overbrace{\mu_0\varepsilon_0}^{1/c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)\omega^2 \quad (\mu=\mu_0 \text{ for plasmas})$$

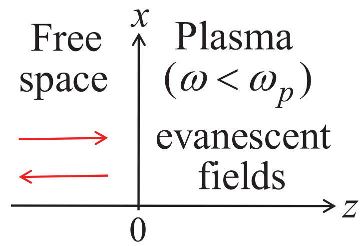
$$\Rightarrow \omega^2 = k^2c^2+\omega_p^2 \quad \left[\begin{array}{l}\text{same form as (7.61) but valid}\\ \text{for arbitrary relative values}\\ \text{of } \omega \text{ and } \omega_p \text{ (see p. 313)}\end{array}\right] \qquad (63)$$



(63) is the well known dispersion relation for electromagnetic waves in a plasma in the absence of an externally applied static magnetic field (Sec. 7.6 considers the dispersion relation for a magnetized plasma). When ω is extremely large (such as the gamma ray), all materials have a dispersion relation given by (63) (Case 3.2). But for the plasma, (63) is valid for all frequencies (e.g. MHz).

Rewrite $\omega^2 = k^2 c^2 + \omega_p^2$ [(63)]

1. Cutoff regime : $\omega < \omega_p$

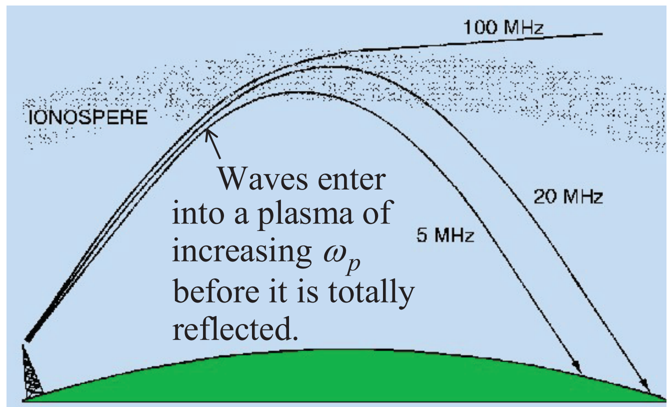


$\Rightarrow k = i\,|k|$ (purely imaginary) $\Rightarrow \mathbf{E}(z), \mathbf{H}(z)$ are evanescent fields (not a wave) given by

$$\mathbf{E}(z) = E_0 e^{-|k|z}\mathbf{e}_x;\ \mathbf{H}(z) = i\sqrt{\frac{|\varepsilon|}{\mu}} E_0 e^{-|k|z}\mathbf{e}_y \ \text{[as in (59a,b)]} \qquad (64)$$

$\Rightarrow$ An incident wave with $\omega < \omega_p$ will be totally reflected.

Reflections of "short waves" (approx. 3-30 MHz) off the ionospheric plasma (F-layer, ~200 km above the earth surface) are exploited for long-distance communications. This is an example of waves traveling in a *non-uniform* medium.

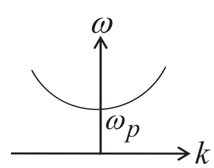


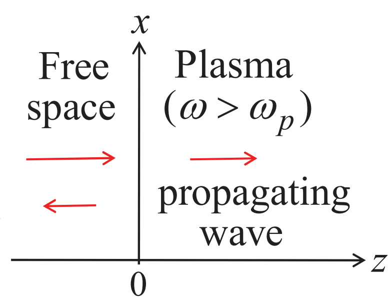
2. Propagating regime : $\omega > \omega_p$

Rewrite $\omega^2 = k^2 c^2 + \omega_p^2$ [(63)]

$\omega > \omega_p \Rightarrow k = \text{real}.$

$\Rightarrow$ The wave can propagate freely with $v_{ph} = \dfrac{\omega}{k} > c$ [see (63)].

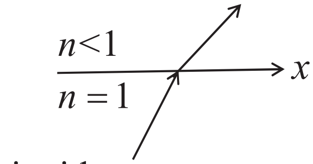




Rewrite $\dfrac{\varepsilon}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$ [(61)]

For $\omega > \omega_p$ and $\mu = \mu_0$, the plasma index of refraction is

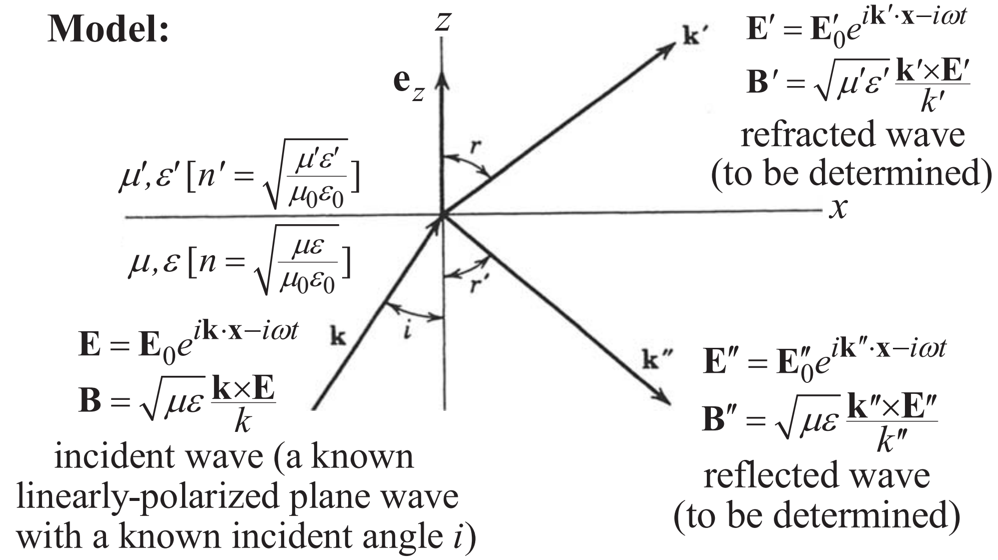
$$n = \mathrm{Re}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} < 1 \qquad (65)$$



This explains the gradual bending of the (refracted) short wave in the ionosphere before it is totally reflected.

7.3 Reflection and Refraction of Electromagnetic Waves at a Plane Interface Between Dielectrics

Model:



$$
Note: \begin{cases} \text{In (32), we have } n \equiv \frac{c}{v} = \mathrm{Re}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \; [\text{index of refraction}] \\ \text{Here, } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \begin{bmatrix} \text{complex index of refraction (see p. 306,} \\ \text{top). It is useful for algebraic analysis} \end{bmatrix} \end{cases}
$$

Kinematic Properties : i.e. relations between directions (but not phase, amp., etc) of incident, reflected, and refracted waves

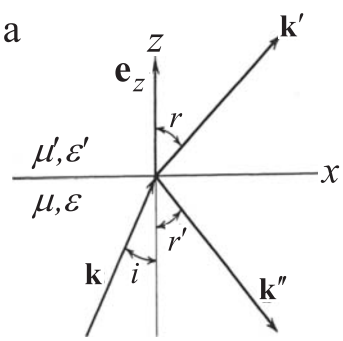
On the $z = 0$ plane, i.e. the interface between the two media,

$$
\text{we have} \quad \begin{cases} \text{incident E-field: } \mathbf{E}_0 e^{ik_x x + ik_y y} \\ \text{reflected E-field: } \mathbf{E}_0'' e^{ik_x'' x + ik_y'' y} \\ \text{refracted E-field: } \mathbf{E}_0' e^{ik_x' x + ik_y' y} \end{cases}
$$

To satisfy the boundary conditions at any x and y, we must have

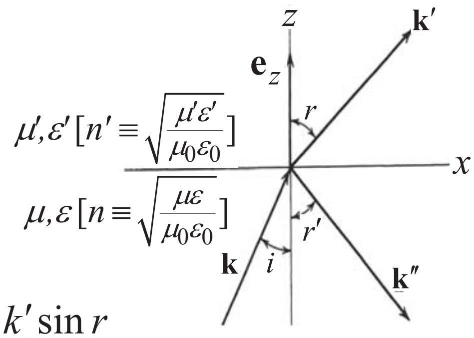
$$k_x = k_x'' = k_x' \text{ and } k_y = k_y'' = k_y'$$

Without loss of generality, we may choose a coordinate system in which $k_y = k_y'' = k_y' = 0$. Then, $\mathbf{k}$, $\mathbf{k}''$, and $\mathbf{k}'$ all lie in the x-z plane, which is called the plane of incidence.



The incident and reflected waves are in the same medium. Hence, $k = k'' = \sqrt{\mu\varepsilon}\,\omega$, but $\mathbf{k} \neq \mathbf{k}''$ ($\because$ their directions are different).

ε, ε', μ, μ', n, & n' are in general complex (p. 306, top). For kinematic properties, we assume they are all real numbers.



We have just shown that b.c.'s at $z=0$ require $\begin{cases} k_x = k_x'' = k_x' \\ \text{or } \; k\sin i = k''\sin r' = k'\sin r \end{cases}$

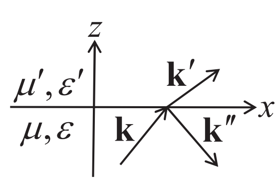
Then, $k = k'' \Rightarrow i = r'$ (angle of incidence = angle of reflection)

With $\begin{cases} k = \sqrt{\mu\varepsilon}\,\omega = \frac{\omega}{c} n \\ k' = \sqrt{\mu'\varepsilon'}\,\omega = \frac{\omega}{c} n' \end{cases}$ $\left[c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \right]$,

we have $\frac{\sin i}{\sin r} = \frac{k'}{k} = \frac{n'}{n}$ (Snell's law) (7.36)

Dynamic Properties : i.e. magnitude, phase, and polarization of $\mathbf{E}_0'$ & $\mathbf{E}''$ relative to the $\mathbf{E}_0$ (polarization: driection of $\mathbf{E}$-feild).

For the study of dynamic properties, we assume ε, ε', μ, and μ' (hence n and n') to be *complex* numbers and ε and ε' to be the generalized electric permittivity. Hence, the results derived below apply to any medium (including conductor). It's impractical to assume the lower (incident) medium to be a conductor, but a conducting upper medium is a case of general interest.



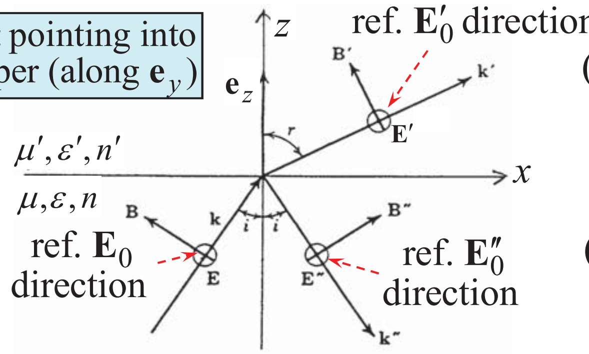
Dynamic properties are contained in the field amplitudes: $\mathbf{E}_0$, $\mathbf{E}_0'$, & $\mathbf{E}_0''$, which obey the following b.c.'s at $z=0$ [same as (7.37)]:

$$\begin{cases} D_\perp \text{ continuous} \Rightarrow [\varepsilon\left(\mathbf{E}_0 + \mathbf{E}_0''\right) - \varepsilon'\mathbf{E}_0'] \cdot \mathbf{e}_z = 0 & (66) \\ B_\perp \text{ continuous} \Rightarrow [\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'' - \mathbf{k}'\times\mathbf{E}_0'] \cdot \mathbf{e}_z = 0 & (67) \\ E_{||} \text{ continuous} \Rightarrow [\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0'] \times \mathbf{e}_z = 0 & (68) \\ H_{||} \text{ continuous} \Rightarrow [\frac{1}{\mu}(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'') - \frac{1}{\mu'}(\mathbf{k}'\times\mathbf{E}_0')] \times \mathbf{e}_z = 0 & (69) \end{cases}$$

Case 1: $\mathbf{E}_0 \perp$ plane of incidence (x-z plane)

$$\begin{cases} \mathbf{k} = k_x \mathbf{e}_x + k_z \mathbf{e}_z \\ \mathbf{k}' = k_x \mathbf{e}_x + k'_z \mathbf{e}_z \\ \mathbf{k}'' = k_x \mathbf{e}_x - k_z \mathbf{e}_z \end{cases} \qquad (70)$$

$\otimes$: pointing into paper (along $\mathbf{e}_y$)

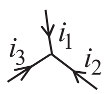


$$\begin{cases} \mathbf{E}_0 = E_0 \mathbf{e}_y \\ \mathbf{E}'_0 = E'_0 \mathbf{e}_y \\ \mathbf{E}''_0 = E''_0 \mathbf{e}_y \end{cases} \quad \begin{bmatrix} \mathbf{E}_0,\ \mathbf{E}'_0,\ \mathbf{E}''_0 \\ \text{are E-field} \\ \text{amplitudes.} \end{bmatrix} \qquad (71)$$

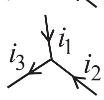
Note: 1. In (70), (71), $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$, $\mathbf{E}_0$, $\mathbf{E}'_0$, and $\mathbf{E}''_0$ are based on their "reference directions" specified in the above figure.

2. If a quantity is + (or −), it is in the + (or −) reference direction.

Example:



$\Rightarrow i_1 + i_2 + i_3 = 0 \Rightarrow$ If $i_1 = i_2 = 1$ A, then $i_3 = -2$ A



$\Rightarrow i_1 + i_2 - i_3 = 0 \Rightarrow$ If $i_1 = i_2 = 1$ A, then $i_3 = 2$ A

In these 2 figures, reference directions of i_3 are specified oppositely; hence, the charge conservation law results in different signs for i_3.

$[\varepsilon(\mathbf{E}_0 + \mathbf{E}''_0) - \varepsilon' \mathbf{E}'_0] \cdot \mathbf{e}_z = 0$ [(66)] is satisfied ($\because \mathbf{E}_0, \mathbf{E}''_0, \mathbf{E}'_0 \perp \mathbf{e}_z$).

Rewrite the b.c. $[\mathbf{E}_0 + \mathbf{E}''_0 - \mathbf{E}'_0] \times \mathbf{e}_z = 0$ [(68)]

$$\Rightarrow E_0 + E''_0 - E'_0 = 0 \quad \text{[also given by b.c. (67)]} \qquad (72)$$

Rewrite the b.c. $[\frac{1}{\mu}(\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}''_0) - \frac{1}{\mu'}(\mathbf{k}' \times \mathbf{E}'_0)] \times \mathbf{e}_z = 0$ [(69)]

$$\Rightarrow \frac{1}{\mu}(k_x E_0 \mathbf{e}_z - k_z E_0 \mathbf{e}_x) \times \mathbf{e}_z + \frac{1}{\mu}(k_x E''_0 \mathbf{e}_z + k_z E''_0 \mathbf{e}_x) \times \mathbf{e}_z$$

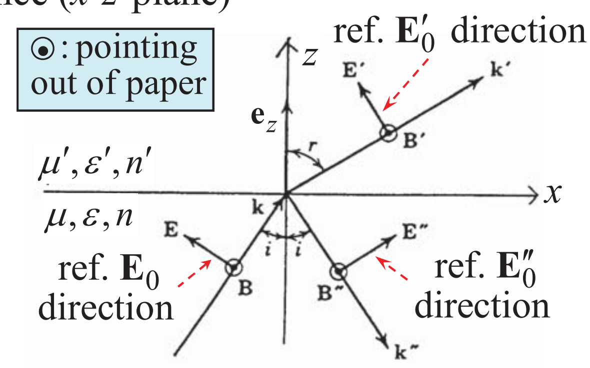
$$- \frac{1}{\mu'}(k_x E'_0 \mathbf{e}_z - k'_z E'_0 \mathbf{e}_x) \times \mathbf{e}_z = 0 \Rightarrow \frac{1}{\mu} k_z (E_0 - E''_0) - \frac{1}{\mu'} k'_z E'_0 = 0$$

$$\Rightarrow \frac{n}{\mu}(E_0 - E''_0)\cos i - \frac{n'}{\mu'} E'_0 \cos r = 0 \longleftarrow \boxed{\begin{aligned} k_z &= k\cos i = \tfrac{\omega}{c} n \cos i \\ k'_z &= k' \cos r = \tfrac{\omega}{c} n' \cos r \end{aligned}} \qquad (73)$$

$$\left.\begin{aligned}(72)\\(73)\end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{E'_0}{E_0} = \dfrac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \\[2ex] \dfrac{E''_0}{E_0} = \dfrac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \end{cases} \quad \begin{bmatrix} \text{For } \mathbf{E}_0 \perp \text{plane} \\ \text{of incidence} \end{bmatrix} \qquad (7.39)$$

Case 2: $\mathbf{E}_0 \parallel$ plane of incidence (x-z plane)

$$\begin{cases} \mathbf{k} = k\left(\sin i\mathbf{e}_x + \cos i\mathbf{e}_z\right) \\ \mathbf{k}' = k'\left(\sin r\mathbf{e}_x + \cos r\mathbf{e}_z\right) \\ \mathbf{k}'' = k''\left(\sin i\mathbf{e}_x - \cos i\mathbf{e}_z\right) \end{cases} \quad (74)$$



$$\begin{cases} \mathbf{E}_0 = E_0\left(-\cos i\mathbf{e}_x + \sin i\mathbf{e}_z\right) \\ \mathbf{E}_0' = E_0'\left(-\cos r\mathbf{e}_x + \sin r\mathbf{e}_z\right) \\ \mathbf{E}_0'' = E_0''\left(\cos i\mathbf{e}_x + \sin i\mathbf{e}_z\right) \end{cases} \quad (75)$$

Similarly, sub. (74)-(75) into b.c.'s (66)-(69) yields

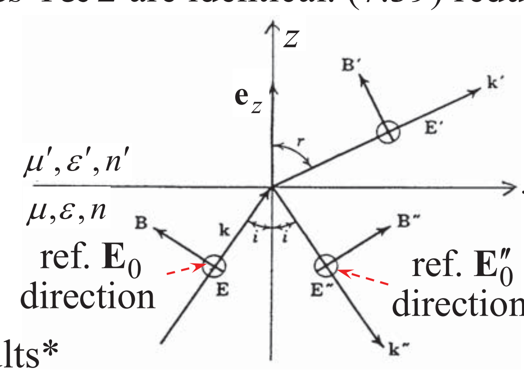
$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \\ \dfrac{E_0''}{E_0} = \dfrac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \end{cases} \quad (7.41)$$

Note: Components of $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{k}''$, $\mathbf{E}_0$, $\mathbf{E}_0'$, & $\mathbf{E}_0''$ in (74),(75) are specified by their reference directions in the above figure.

Note: (7.39) and (7.41) apply to complex ε, μ, and n (p. 306, top)

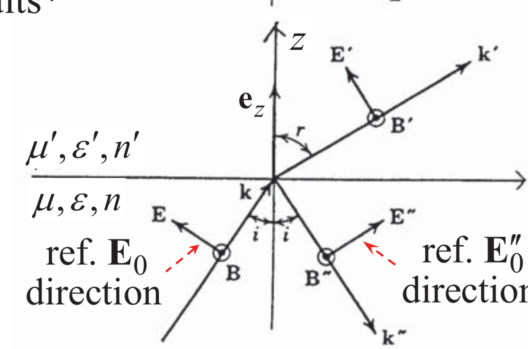
For $\perp$ *incidence* ($i = 0$): *Cases* 1 & 2 are identical. (7.39) reduces to

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} \xrightarrow[\mu=\mu']{} \dfrac{2n}{n+n'} & (76a) \\ \dfrac{E_0''}{E_0} = \dfrac{1-\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} \xrightarrow[\mu=\mu']{} \dfrac{\overbrace{n-n'}^{\boxed{(7.42),\,+\,\text{sign}}}}{n+n'} & (76b) \end{cases}$$



and (7.41) reduces to same results*

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \xrightarrow[\mu=\mu']{} \dfrac{2n}{n'+n} & (77a) \\ \dfrac{E_0''}{E_0} = \dfrac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \xrightarrow[\mu=\mu']{} \dfrac{\overbrace{n'-n}^{\boxed{(7.42),\,-\,\text{sign}}}}{n'+n} & (77b) \end{cases}$$



*For $i = 0$, reference directions of $\mathbf{E}_0$ & $\mathbf{E}_0''$ are the same for (76b) and opposite for (77b). Hence, (76b) & (77b) differ by a sign.

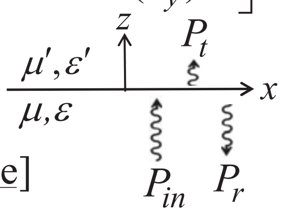
Reflectance and transmittance [for $\perp$ incidence ($i = 0$)]:

Either *Case 1* or *Case 2* applies. Define a reflection coefficient (Γ):

$$\Gamma = \frac{E_0''}{E_0} = \frac{1-\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} \tag{78}$$

[*Note*: For $\perp$ incidence, (76b) & (77b) differ only by a sign due to opposite ref. directions. Here, for simpliciy, we use (76b) for which $\mathbf{E}_0$ & $\mathbf{E}_0''$ have the same ref. direction ($\mathbf{e}_y$).]

$$\Rightarrow \begin{cases} \dfrac{\text{reflected power } (P_r)}{\text{incident power } (P_{in})} = |\Gamma|^2 \ \ [\underline{\text{reflectance}}] \\[2ex] \dfrac{\text{transmitted power } (P_t)}{\text{incident power } (P_{in})} = 1-|\Gamma|^2 \ \ [\underline{\text{transmittance}}] \end{cases} \tag{79}$$



Γ is applicable to all materials if we use the generalized ε (or ε'):

$$\varepsilon = \varepsilon_b + i\frac{\sigma}{\omega} \ \ [(7.56)] \text{ with } \sigma = \frac{n_0 e^2}{m(\gamma_0 - i\omega)} \ \ \left[\begin{array}{l}(7.58)\\ n_0 : \text{free electron density}\end{array}\right]$$

Define $\sigma_{DC} = \dfrac{n_0 e^2}{m\gamma_0}$ [DC conductivity, values given in handbooks]

$$\Rightarrow \varepsilon = \varepsilon_b + i\frac{\sigma_{DC}}{\omega(1-i\frac{\omega}{\gamma_0})} \ \ [\text{applicable to all materials at any } \omega]$$

Example: Normal incidence of a wave from free space to copper

$$\text{For copper, } \begin{cases} \varepsilon_b \approx \varepsilon_0;\ \mu \approx \mu_0;\ \gamma_0 \approx 4\times 10^{13}/\text{s} \ \ [\text{p. 312}] \\ \sigma_{DC} = 5.9\times 10^7 \ \ (\Omega\text{m})^{-1} \ [\text{handbook value}] \end{cases}$$

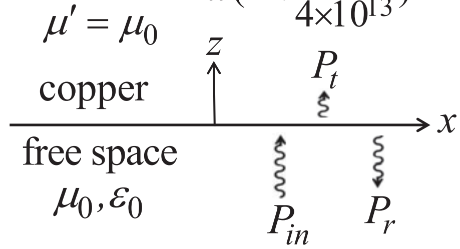
$$\Rightarrow \varepsilon' = \varepsilon_b + i\frac{\sigma_{DC}}{\omega(1-i\frac{\omega}{\gamma_0})} \approx \varepsilon_0 + i\frac{5.9\times 10^7}{\omega(1-i\frac{\omega}{4\times 10^{13}})} \ \ [\text{for copper at any } \omega] \tag{80}$$

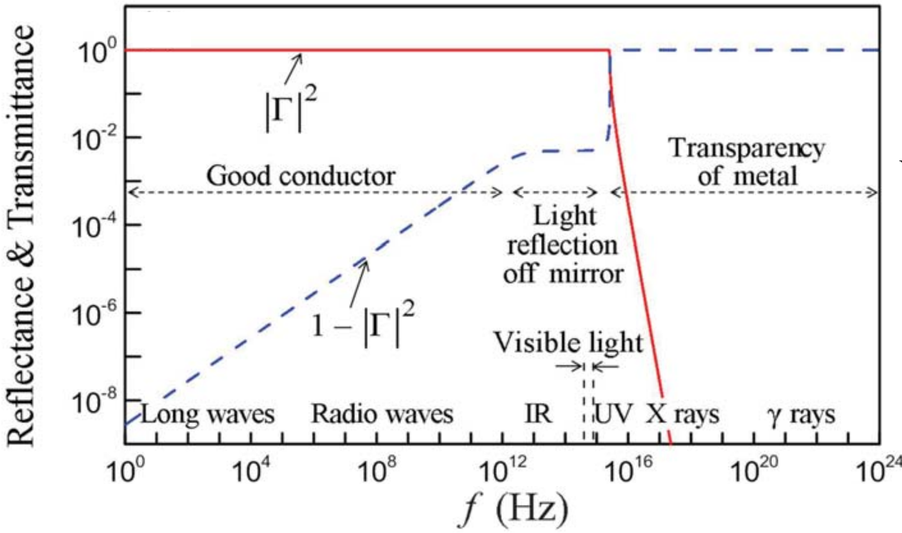
$$\Rightarrow \Gamma = \frac{E_0''}{E_0} \approx \frac{1-\sqrt{\frac{\varepsilon'}{\varepsilon_0}}}{1+\sqrt{\frac{\varepsilon'}{\varepsilon_0}}} = \frac{1-\sqrt{1+i\frac{5.9\times 10^7}{\varepsilon_0\omega(1-i\frac{\omega}{4\times 10^{13}})}}}{1+\sqrt{1+i\frac{5.9\times 10^7}{\varepsilon_0\omega(1-i\frac{\omega}{4\times 10^{13}})}}}$$

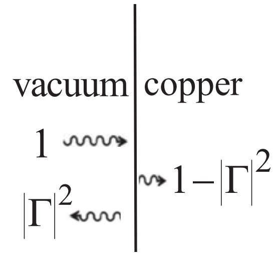
[for a wave incident along $\mathbf{e}_z$ from free space to copper at any ω]

Below we plot and interpret the reflectance ($|\Gamma|^2 = P_r / P_{in}$) and the transmittance ($1-|\Gamma|^2 = P_t / P_{in}$) as a function of f ($=\omega/2\pi$) over a broad range of f (1 Hz $\rightarrow 10^{24}$ Hz).

$$\varepsilon' = \varepsilon_0 + i\frac{5.9\times 10^7}{\omega(1-i\frac{\omega}{4\times 10^{13}})}$$



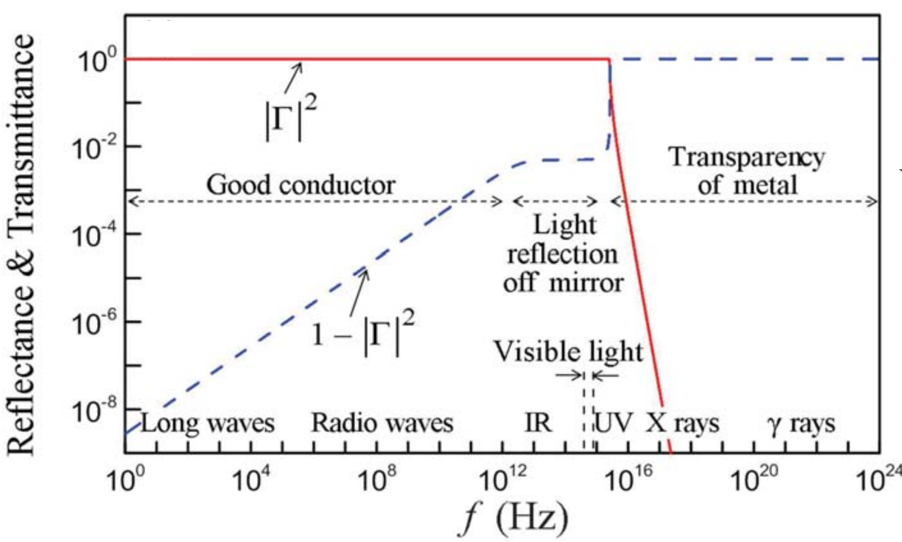


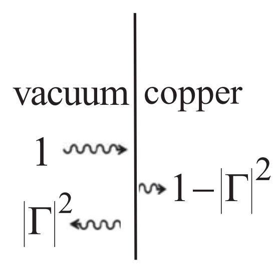


Taken from M. K. Shen and K. R. Chu, Am. J. Phys. **82**, 110 (2014)
[(77b) is used for Γ in this paper, which gives the same $|\Gamma|^2$.]

In the "good conductor" regime (Sec. 3, Case 2), $|\Gamma|^2 \approx 1$, but there is still some absorption (given by $1-|\Gamma|^2$). Copper absorbs ~ 50 times more IR radiation ($10^{12\text{-}14}$ Hz) than 2.45 GHz μ-waves. This is why walls of a μ-wave oven are much cooler than a conventional oven.

7.3 Reflection and Refraction… (*continued*)

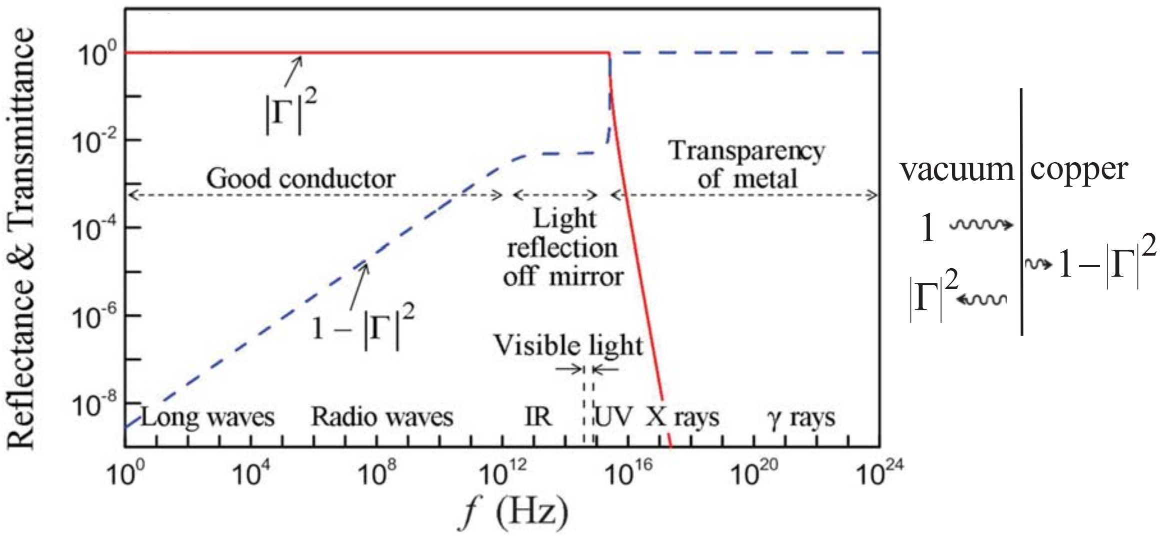




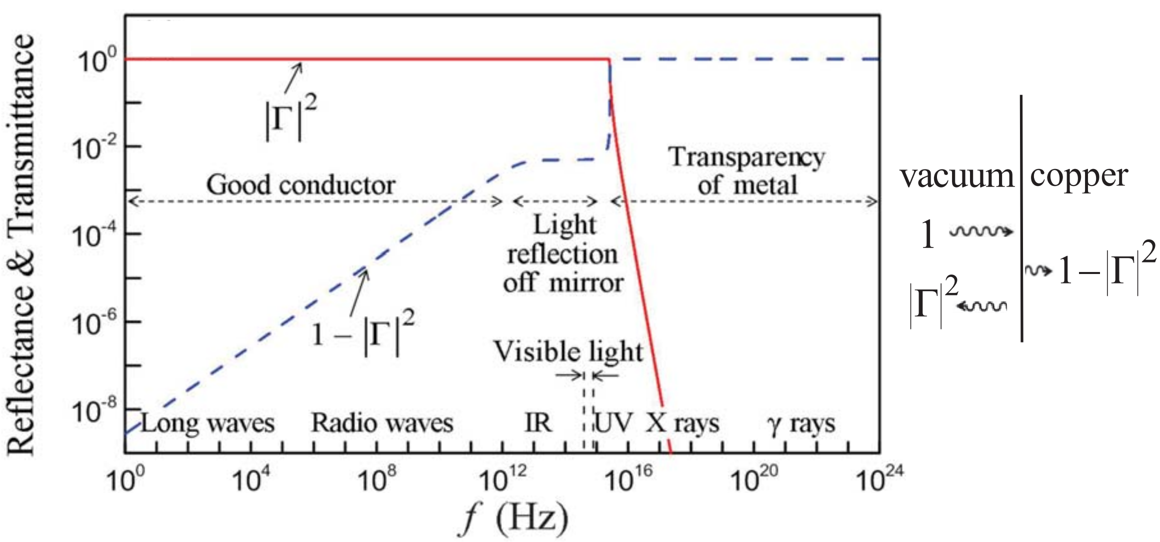
Rewrite $\varepsilon' = \varepsilon_0 + i\dfrac{5.9\times10^7}{\omega(1-i\omega/4\times10^{13})}$ [(80)]

$$\omega \gg 4\times10^{13}/\text{sec}\ (=\gamma_0 \text{ for Cu}) \Rightarrow \varepsilon \approx \varepsilon_0 - \frac{2.36\times10^{21}}{\omega^2} = \varepsilon_0\left(1-\frac{6.8\times10^{30}}{f^2}\right)$$

$$\Rightarrow \begin{cases} \varepsilon < 0, \text{ if } f < 2.6\times10^{15} \text{ Hz} \Rightarrow \text{"light-reflection-off-mirror" regime} \\ \varepsilon > 0, \text{ if } f > 2.6\times10^{15} \text{ Hz} \Rightarrow \text{"transparency-of-metal" regime} \end{cases}$$



In the "light-reflection-off-mirror" regime (Sec. 3, Case 3.1), theory predicts total reflection if copper had no loss, and there is *evanescent* fields inside copper. In reality, copper has a very small resistivity. This results in $\sim 0.5\%$ absorption of the evanescent fields by copper.



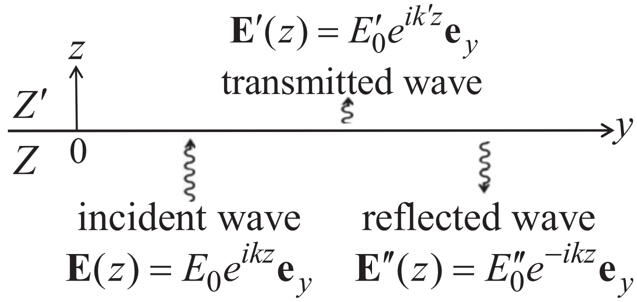
In the "transparency-of-metal" regime (Sec. 3, Cases 3.1 and 3.2) applicable to ultraviolet waves, X-rays, and γ-rays, theory predicts almost total transmission. However, the small difference in ε between the two media results in a negligibly small reflection.

Problem 1: Show that the *high* reflectance off a good conductor is due to a *poor* impedance match.

In terms of the impedance : $Z = \sqrt{\frac{\mu}{\varepsilon}}$ [(29)], we obtain from (76b)

[*Note*: For (76b), $\mathbf{E}_0$ & $\mathbf{E}_0''$ have the same ref. direction : $\mathbf{e}_y$]

$$\Gamma = \frac{E_0''}{E_0} = \frac{1-\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} = \frac{\sqrt{\frac{\mu'}{\varepsilon'}}-\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{\frac{\mu'}{\varepsilon'}}+\sqrt{\frac{\mu}{\varepsilon}}} = \frac{Z'-Z}{Z'+Z} \qquad (81)$$

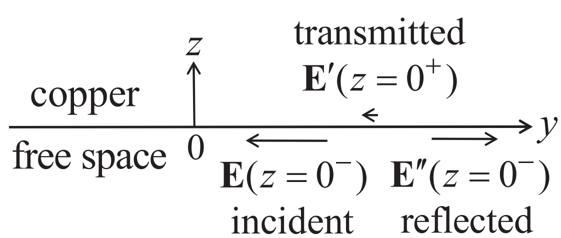


Impedance of $\begin{cases} \text{Cu at 10 GHz}: Z' = Z_s = (0.026 - i0.026)\ \Omega \text{ [(53a)]} \\ \text{free space}: Z = Z_0 = 376.7\ \Omega \text{ [(30)]} \end{cases}$

$\Rightarrow$ a poor impedance match between copper and free space.

$\Rightarrow \Gamma \approx -1$ (or $E_0'' \approx -E_0$) $\Rightarrow$ Most of the incident $\mathbf{E}$ is reflected with a *phase reversal* of $\mathbf{E}''$ at $z=0$, i.e. $\quad \mathbf{E}''(z=0^-) \approx -\mathbf{E}(z=0^-) \qquad (82)$

Problem 2: We find in *Problem 1* that, due to a poor impedance match, almost all of the incident wave (from free space to copper) is reflected at $z=0$ with a phase reversal of the reflected wave, i.e. $\mathbf{E}''(z=0^-) \approx -\mathbf{E}(z=0^-)$. Give a physical interpretation in terms of Faraday's law.



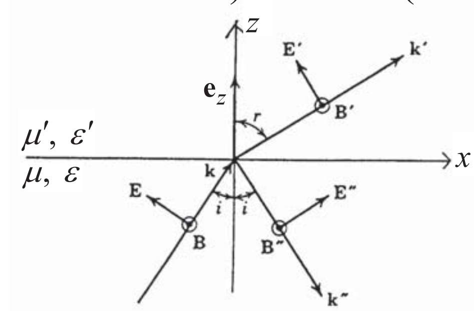
Rewrite (45): $\mathbf{K}_{eff} = \int_0^\infty \mathbf{J}(z)dz = -\mathbf{e}_z \times \mathbf{H}(0)$ [$\mathbf{J}(z) = \sigma\mathbf{E}'(z) \propto e^{-\frac{z}{\delta}}$]

By Faraday's law, $\mathbf{K}_{eff}$ is to shield the conductor from a time-varying $\mathbf{H}$ by causing an exponential decay of $\mathbf{H}$ at $z \geq 0$ [discussed below (45)]. Since $\sigma(z \geq 0)$ is very large, only a small $\mathbf{E}'(z \geq 0)$ is required to drive $\mathbf{K}_{eff}$ to the value in (45). The result in *Problem 1* ($\mathbf{E}'' \approx -\mathbf{E}$ at $z=0^-$) implies a small net $\mathbf{E}$-field at $z=0^-$. Since the net $\mathbf{E}$ at $z=0^-$ is || the $z=0$ plane, it is continuous across $z=0$. Thus, $\mathbf{E}'(z=0^+)$ is also small, as required.

7.4. Polarization by Reflection and Total Internal Reflection

Brewster's Angle i_B (for $\mathbf{E}_0 \parallel$ plane of incidence): Rewrite (7.41):

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \\[2ex] \dfrac{E_0''}{E_0} = \dfrac{\frac{\mu}{\mu'} n'^2 \cos i - n\sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \end{cases}$$



Assume $\varepsilon,\ \varepsilon',\ \mu,\ \mu',\ n,$ and n' are all real and $\mu = \mu'$. (7.41) gives $E_0'' = 0$ (no reflected wave) if $i = i_B$, where i_B satisfies

$$n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B} \qquad (83)$$

Hence, for incident waves with mixed polarization (e.g. sunlight), only the portion with $\mathbf{E}_0 \perp x\text{-}z$ plane is reflected and the reflected $\mathbf{E}_0''$ is $\perp x\text{-}z$ plane. This is a way to produce linearly-polarized waves.

Calculation of i_B:

Rewrite $n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$ [(83)]

$$\Rightarrow n'^4 \cos^2 i_B = n^2(n'^2 - n^2 \sin^2 i_B)$$

$$\Rightarrow n'^4(1 - \sin^2 i_B) = n^2 n'^2 - n^4 \sin^2 i_B$$

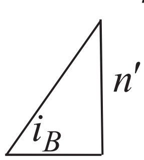
$$\Rightarrow (n^4 - n'^4)\sin^2 i_B = n'^2(n^2 - n'^2)$$

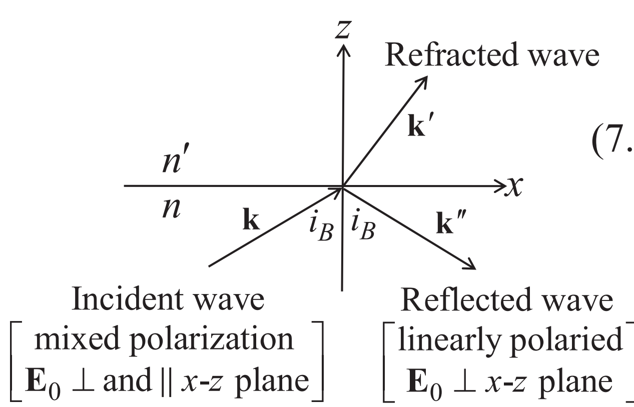
$$\Rightarrow \sin^2 i_B = \frac{n'^2}{n^2 + n'^2}$$

$$\Rightarrow \tan i_B = \frac{n'}{n}$$

Typical example:

$i_B = 56^\circ$ if $\frac{n'}{n} = 1.5$





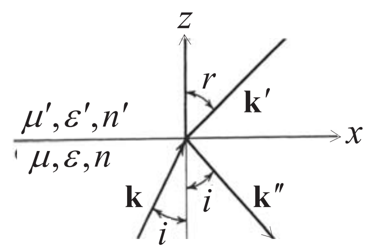
(7.43)

Total Internal Reflection:

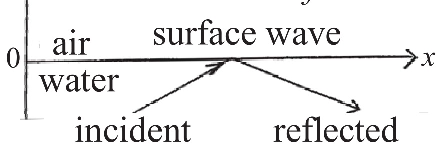
Assume *real* $\varepsilon, \varepsilon', \mu, \mu', n,$ & n' with $n > n'$

Let i_0 (critical angle) $\equiv \sin^{-1}\frac{n'}{n}$ $(< \frac{\pi}{2})$

Then, $\frac{\sin i}{\sin r} = \frac{n'}{n}$ [(7.36)] $\Rightarrow \sin r = \frac{\sin i}{\sin i_0}$



Total internal reflection

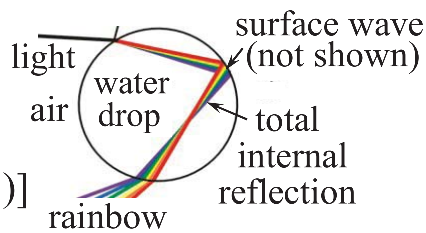


$\Rightarrow$ If $i > i_0$, we have $\sin r = \frac{\sin i}{\sin i_0} > 1$

$\Rightarrow \cos r = [1 - \sin^2 r]^{1/2} = i[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2}$

$\Rightarrow \mathbf{k}' = k' \sin r \mathbf{e}_x + k' \cos r \mathbf{e}_z$

$$= \underbrace{k' \frac{\sin i}{\sin i_0}}_{k'_x\ (>0)} \mathbf{e}_x + i \underbrace{k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2}}_{k'_z\ (>0)} \mathbf{e}_z \quad (84)$$



$\Rightarrow \mathbf{k}' = k'_x \mathbf{e}_x + i k'_z \mathbf{e}_z$ [same form as in (23)] (85)

$\Rightarrow e^{i\mathbf{k}' \cdot \mathbf{x}} = e^{ik'_x x} e^{-k'_z z}$

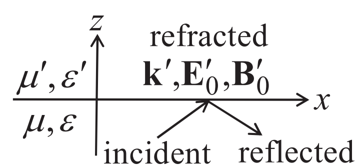
$$= e^{ik' \frac{\sin i}{\sin i_0} x} e^{-k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2} z}$$

[This is a surface wave [see (24)]. Power flows along x, but not along z.] (7.46)

Exercise: Prove $\mathbf{E}'_0$, $\mathbf{B}'_0$ of the refracted surface wave are given by (23).

Rewrite $\mathbf{k}' = k'_x \mathbf{e}_x + i k'_z \mathbf{e}_z$ [(85)]

where $k'_x = k' \frac{\sin i}{\sin i_0}$ and $k'_z = k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2}$



are real and positive numbers.

Note: $\mathbf{k}'$ in (85) satisfies $\mathbf{k}' \cdot \mathbf{k}' = k'^2_x - k'^2_z = k'^2 = \mu' \varepsilon' \omega^2$ [(16)]

To determine $\mathbf{E}'_0$ and $\mathbf{B}'_0$, we consider the case of $\mathbf{E}'_0 \parallel x\text{-}z$ plane.

Let $\mathbf{E}'_0 = E'_{0x} \mathbf{e}_x + i E'_{0z} \mathbf{e}_z$ (86)

Then, $\mathbf{k}' \cdot \mathbf{E}'_0 = 0$ [(17)] $\Rightarrow k'_x E'_{0x} - E'_{0z} k'_z = 0 \Rightarrow \frac{E'_{0x}}{E'_{0z}} = \frac{k'_z}{k'_x}$ (87)

Since k'_x and k'_z in (87) are real and positive, we may let E'_{0x} and E'_{0z} be real and positive without loss of generality (field amplitudes are relative).

$$\mathbf{B}'_0 = \sqrt{\mu' \varepsilon'} \frac{\mathbf{k}' \times \mathbf{E}'_0}{k'} \text{ [(19)]} \Rightarrow \mathbf{B}'_0 = i \frac{-k'_x E'_{0z} + k'_z E'_{0x}}{\omega} \mathbf{e}_y \quad (88)$$

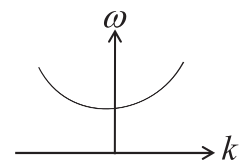
(85), (86), (88) are in exactly the same form as constructed in (23).

7.8 Superposition of Waves in One Dimension; Group Velocity

A Single-Frequency Wave :

Consider a single-ω (hence single-k) wave, $\cos(\omega t - kx)$, in a dispersive medium obeying $\omega = \omega(k)$ (see figs. below).

$\cos(\omega t - kx) \rightarrow$ 



A wave pulse follows $\Delta k \Delta x \geq \frac{1}{2}$ [(7.82)] $\Rightarrow$ a single-k wave ($\Delta k = 0$) is of infinite length ($\Delta x = \infty$). Any constant-phase point ($\omega t - kx = const$), e.g. any peak, can be seen to move at the phase velocity $v_{ph} (= \omega / k)$.

On the other hand, the speed at which the energy propagates is called the group velocity (v_g). $v_g \leq c$ and it is in general not v_{ph}. For example, for the above dispersion relation, we have $v_{ph} \rightarrow \infty$ as $k \rightarrow 0$. Thus, v_{ph} is clearly not v_g. The wave energy does propagate, but we can't see from an infinite wave how fast it is. To determine v_g, we need to make use of the dispersive property of the medium: $\omega = \omega(k)$

Superposition of 2 Waves : Consider 2 waves (Fig. 1), $\cos(\omega_1 t - k_1 x)$ and $\cos(\omega_2 t - k_2 x)$, in a dispersive medium characterized by $\omega = \omega(k)$. Assume $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$, then $\frac{\omega_1}{k_1} \approx \frac{\omega_2}{k_2}$ is approximately the phase velocity (v_{ph}) of the superposed wave (see Fig.2). The difference in wavelengths results in alternating regions of constructive/destructive interferences, or spatial modulations of the superposed wave (Fig. 2).

The regions of constructive interference represent packets of field energy. Their speed gives the group velocity, which can be calculated (next page) and observed.

$\cos(\omega_1 t - k_1 x) \rightarrow$

$\cos(\omega_2 t - k_2 x) \rightarrow$

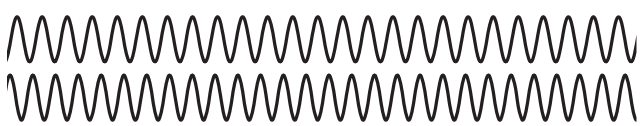


Fig. 1

$\begin{cases} \cos(\omega_1 t - k_1 x) \\ + \cos(\omega_2 t - k_2 x) \end{cases} \rightarrow$

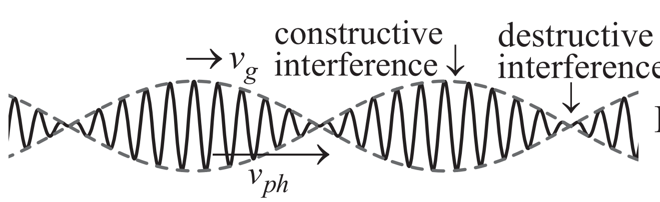


Fig. 2

The above qualitative picture can be analyzed as follows.

$$\cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$
$$= 2\cos\left(\tfrac{\omega_1-\omega_2}{2}t - \tfrac{k_1-k_2}{2}x\right)\cos\left(\tfrac{\omega_1+\omega_2}{2}t - \tfrac{k_1+k_2}{2}x\right)$$
$$\approx 2\underbrace{\cos\left(\tfrac{\omega_1-\omega_2}{2}t - \tfrac{k_1-k_2}{2}x\right)}_{\text{(A)}}\underbrace{\cos(\omega t - kx)}_{\text{(B)}},$$

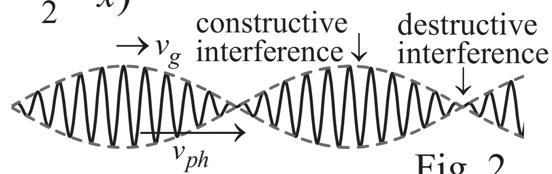


Fig. 2

where $\omega = \frac{\omega_1+\omega_2}{2}(\approx \omega_1 \approx \omega_2)$ and $k = \frac{k_1+k_2}{2}(\approx k_1 \approx k_2)$.

Factor (A) is the envelope function of the modulated wave (Fig. 2), which divides the wave into packets, each propagating at the speed:

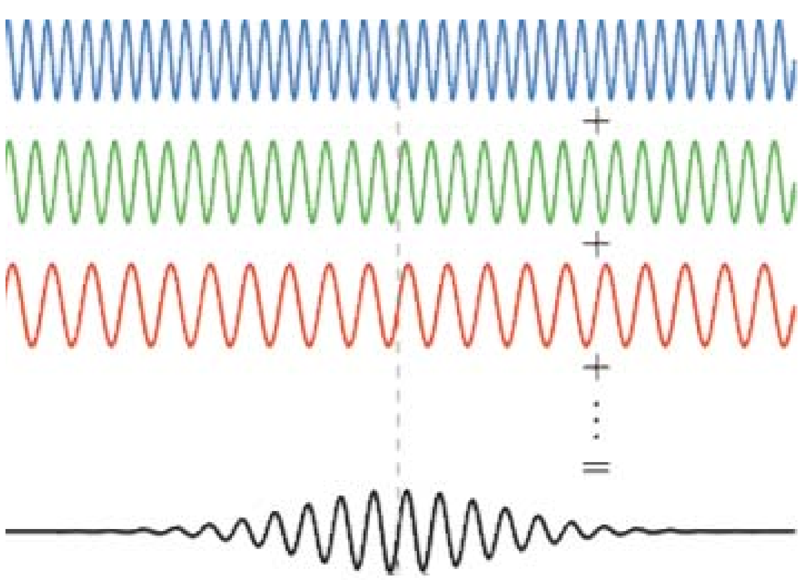
$$v_g = \frac{\frac{\omega_1-\omega_2}{2}}{\frac{k_1-k_2}{2}} = \frac{\omega_1-\omega_2}{k_1-k_2} \approx \frac{d\omega}{dk} \quad \left[\begin{array}{l}\text{Simplest derivation of the group}\\ \text{velocity : speed of energy transport}\end{array}\right]$$

Factor (B) gives the phase velocity of the wave within each packet:

$$v_{ph} = \frac{\omega}{k} \quad \text{[phase velocity]}$$

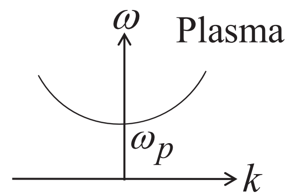
Superposition of an Infinite Number of Waves :

When an infinite number of waves are superposed, interferences can result in cancellation everywhere except for a region of finite length called a wave packet, a pulse, or a wave train.



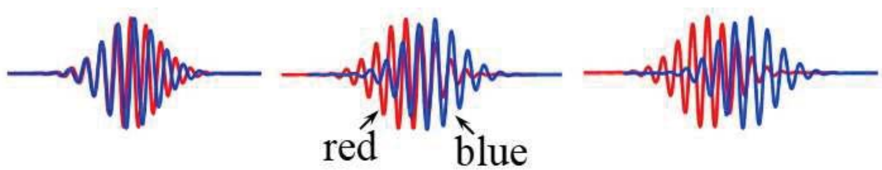
Pulse Broadening - a Qualitative Explanation

Consider a light pulse which consists a range of freqencies (from red to blue). Each frequency has a different v_g $(= d\omega / dk)$. Thus, $d^2\omega / dk^2 = dv_g / dk \neq 0$ and each frequency travel at a different v_g. As a result, the pulse width will broaden with time.



Example: A light pulse consists of a continuous distribution of frequencies. When it propagates in a plasma, we have $v_g(\text{blue}) > v_g(\text{red})$

$\Rightarrow$



[from Del Mar Photonics]

Since $\Delta k \Delta x \geq \frac{1}{2}$ [(7.82)], a shorter pulse has a greater spread in k (and v_g). Hence, it broadens faster than a longer pulse.

However, a pulse in vacuum remains undistorted $\left(\omega = kc \Rightarrow \frac{d^2\omega}{dk^2} = 0\right)$.

A More Detailed Analysis

Consider an infinite number of waves in k-space centered around k_0 with a spread of Δk (see Fig. 4). Assume they are destructively superposed in x-space everywhere except for a region of length Δx. A wave packet is thus formed in the region (see Fig. 3).

Assume that k and $\omega(k)$ are both real (no dissipation). We can construct a wave packet by a linear superposition of waves of different k.

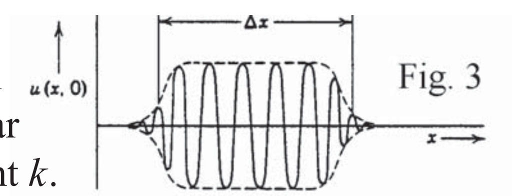


Fig. 3

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} dk \tag{7.80}$$

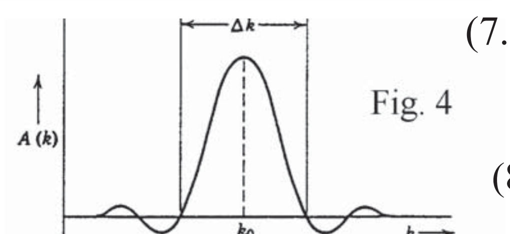


Fig. 4

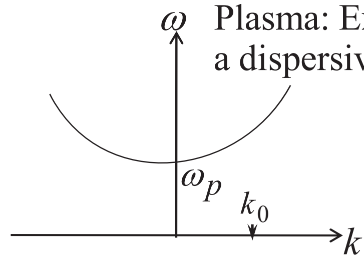
If the field profile at $t = 0$ is

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk. \tag{89}$$

Thus, $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx,$ (7.81)

In general, $\Delta x \Delta k \geq \frac{1}{2}$ [see example in (94)] (7.82)

Taylor expand $\omega(k)$ about k_0



$$\omega(k) = \underbrace{\omega(k_0)}_{\omega_0} + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0) + \ldots$$

$$\approx \omega_0 + v_g(k-k_0), \quad (7.83)$$

where $v_g \equiv \left.\frac{d\omega}{dk}\right|_{k_0}$ $\left[\begin{array}{l} Note: \text{As } k_0 \to 0, \\ v_g \to 0,\ v_{ph} \to \infty \end{array}\right]$ $\quad (7.86)$

Sub. (7.83) into $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}dk$ $\quad (7.80)$

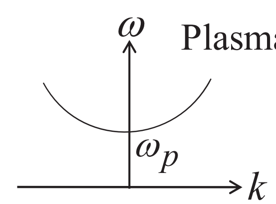
$$\Rightarrow u(x,t) \approx e^{i(k_0 v_g - \omega_0)t} \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ik(x-v_g t)}dk}_{\boxed{= u(x-v_g t,0) \left[\begin{array}{l} \text{In } u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk\ [(89)], \\ \text{change } x \text{ to } x - v_g t \end{array}\right]}} \quad (7.84)$$

$$= u(x-v_g t,0)e^{i(k_0 v_g - \omega_0)t} \quad (7.85)$$

(7.85) represents a wave packet propagating at v_g with its shape unchanged in time [$\because$ high order terms in (7.83) are neglected].

Rewrite $u(x,t) \approx u(x-v_g t,0)e^{i(k_0 v_g - \omega_0)t}$ [(7.85)]

The pulse shape give by (7.85) is undistorted in time. However, if high order terms (e.g. $\frac{d^2\omega}{dk^2}$) are included in the expansion of $\omega(k)$ in (7.83), the pulse will broaden with time.

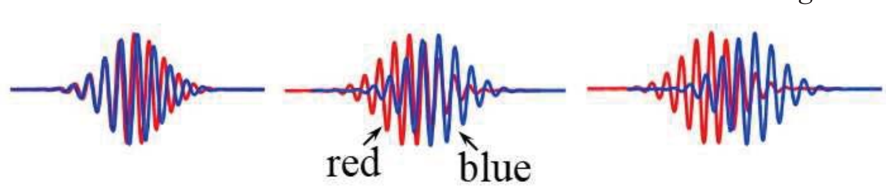


v_g(blue)>v_g(red)

Reason : As discussed earlier,

$$\frac{d^2\omega}{dk^2} \neq 0 \Rightarrow \frac{d}{dk}v_g \neq 0 \Rightarrow \text{Waves with}$$

different k travel at different v_g.



Pulse broadening will be analyzed exactly in the following example.

7.9 Illustration of the Spreading of a Pulse as It Propagates in a Dispersive Medium

Here, we give an analytical treatment of the wave packet including the effect of pulse broadening.

Rigorously, the *real* quantity $u(x,t)$, which we expressed in (7.80) as $u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk$, should be written:

$$u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk+c.c. \quad \boxed{c.c.\Rightarrow \text{complex conjugate}} \tag{7.90}$$

$$\Rightarrow\begin{cases} u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk+\frac{1}{2}\frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}A^*(k)e^{-ikx+i\omega(k)t}dk} \\ \frac{\partial}{\partial t}u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-i\omega(k)A(k)e^{ikx-i\omega(k)t}dk \\ \qquad +\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}i\omega(k)A^*(k)e^{-ikx+i\omega(k)t}dk \end{cases}$$

Assume $\omega^*=\omega$ and $k^*=k$, i.e. no dissipation

Note: $A(k)$ is not the Fourier transform of $u(x,t)$ because, in (7.90), $e^{ikx-i\omega(k)t}$ is not of the simple e^{ikx} form. Hence, the "realty condition" $A(k)=A^*(-k)$ [see Sec. 2.8 of lecture notes] is not applicable here.

$$\text{At } t=0:\begin{cases} u(x,0)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[A(k)e^{ikx}+A^*(k)e^{-ikx}]dk \\ \frac{\partial}{\partial t}u(x,0)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[-i\omega(k)A(k)e^{ikx}+i\omega(k)A^*(k)e^{-ikx}]dk \end{cases}$$

where $\frac{\partial}{\partial t}u(x,0)=\frac{\partial}{\partial t}u(x,t)\Big|_{t=0}$. We may evaluate $A(k)$ as follows:

$$\int_{-\infty}^{\infty}e^{-ik'x}u(x,0)dx=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[A(k)e^{i(k-k')x}+A^*(k)e^{-i(k+k')x}]dkdx$$

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k-k')x}dx=\delta(k-k');\ \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i(k+k')x}dx=\delta(k+k')\ [\text{Ch. 2}]}$$

$$=\frac{\sqrt{2\pi}}{2}[A(k')+A^*(-k')] \tag{92}$$

Similarly,

$$\int_{-\infty}^{\infty}e^{-ik'x}\frac{\partial}{\partial t}u(x,0)dx=\frac{\sqrt{2\pi}}{2}[-i\omega(k')A(k')+i\,\underbrace{\omega(-k')}\,A^*(-k')] \tag{93}$$

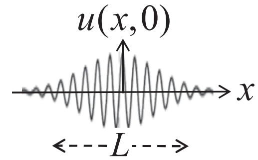
$(92)-\frac{(93)}{i\omega(k')}$ to eliminate $A^*(-k')$

Assume isotropic medium $\Rightarrow \omega(-k')=\omega(k')$

$$\Rightarrow A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}[u(x,0)+\frac{i}{\omega(k)}\frac{\partial}{\partial t}u(x,0)]dx \tag{7.91}$$

where we have changed the notation k' to k.

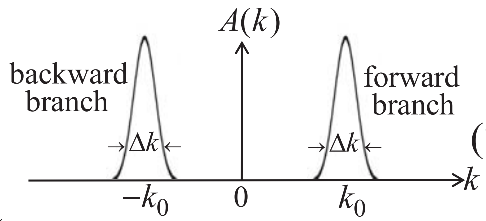
$$Example:\begin{cases} u(x,0)=\exp(-\frac{x^2}{2L^2})\cos k_0 x & (7.92)\\ \frac{\partial}{\partial t}u(x,0)[=\frac{\partial}{\partial t}u(x,t)\Big|_{t=0}]=0 & (7.93)\end{cases}$$



$$\Rightarrow A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\Big[u(x,0)+\frac{i}{\omega(k)}\cancel{\frac{\partial}{\partial t}u(x,0)}\Big]dx \quad [(7.91)]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}e^{-x^2/2L^2}\cos k_0 x dx$$

$$=\frac{L}{2}\Big[e^{-\frac{L^2}{2}(k-k_0)^2}+e^{-\frac{L^2}{2}(k+k_0)^2}\Big] \quad (7.94)$$



$\Rightarrow A(k)$ has 2 branches, each with a <u>Gaussian</u> profile of width Δk given by $\Delta k\cdot L\sim 1$. Since the pulse width $\Delta x\sim L$, we have $\Delta k\Delta x\sim 1$ [in agreement with (7.82)] (94)

Assume $\omega(k)=\nu[1+\frac{a^2k^2}{2}]$ $\left[\begin{array}{l}\text{dispersion relation}\\ \text{of the medium}\end{array}\right]$ (7.95)

$\Rightarrow \frac{d}{dk}v_g=\frac{d^2\omega}{dk^2}=\nu a^2\neq 0\Rightarrow v_g$ is a function of k.

$\Rightarrow$ Pulse broadening is expected, as shown on next page.

Rewrite $u(x,t)=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk+c.c.$ [(7.90)]

$$\Rightarrow u(x,t)=\frac{1}{\sqrt{2\pi}}\mathrm{Re}\int_{-\infty}^{\infty}A(k)e^{ikx-i\omega(k)t}dk$$

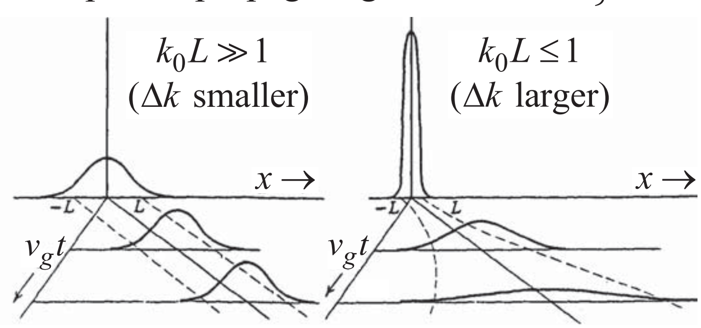
$$\boxed{\begin{array}{c}(7.94)\\(7.95)\end{array}}\rightarrow=\frac{L}{2\sqrt{2\pi}}\mathrm{Re}\int_{-\infty}^{\infty}\Big[e^{-\frac{L^2}{2}(k-k_0)^2}+e^{-\frac{L^2}{2}(k+k_0)^2}\Big]e^{ikx-i\nu t(1+\frac{a^2k^2}{2})}dk$$

$$=\frac{1}{2}\mathrm{Re}\left\{\begin{array}{l}\frac{1}{(1+\frac{ia^2\nu t}{L^2})^{\frac{1}{2}}}\exp\Big[-\frac{(x-\nu a^2k_0t)^2}{2L^2(1+\frac{ia^2\nu t}{L^2})}\Big]\cdot\underset{\substack{\uparrow\\ \text{a wave packet propagating forward}}}{\exp\Big[ik_0x-i\nu(1+\frac{a^2k_0^2}{2})t\Big]}\\ +\ (k_0\rightarrow -k_0)\leftarrow\text{a wave packet propagating backward}\end{array}\right\} \quad (7.98)$$

It can be shown that the spatial width of the pulse is given by [see (7.99)]:

$L(t)=[L^2+(\frac{a^2\nu t}{L})^2]^{\frac{1}{2}}$,

which increases with t.



Appendix A. *t*-space and ω-space

$$\begin{cases} \mathbf{E}(\omega) = \int_{-\infty}^{\infty} \mathbf{E}(t) e^{i\omega t} dt & \text{(A.1)} \\ \mathbf{E}(t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{-i\omega t} d\omega & \\ \quad\;\; = \dfrac{1}{2\pi} \int_{0}^{\infty} \left[\mathbf{E}(\omega) e^{-i\omega t} + \mathbf{E}(-\omega) e^{i\omega t} \right] d\omega & \text{(A.2)} \end{cases}$$

Note: If $\mathbf{E}(t)$ is real (a physical quantity in t-space must be a real quantity), then,

$$\mathbf{E}(-\omega) = \mathbf{E}^*(\omega) \qquad \text{(A.3)}$$

Appendix A. *t*-space and ω-space (*continued*)

phase constant

Example 1: $\mathbf{E}(t) = \mathbf{E}_0 \cos(\underbrace{\omega t + \theta}_{\text{phase angle}})$ (A.4)

a single-frequency real quantity

amplitude (real vector)

Sub. (A.4) into $\mathbf{E}(\omega) = \int_{-\infty}^{\infty} \mathbf{E}(t) e^{i\omega t} dt$

$$\begin{aligned} \mathbf{E}(\omega') &= \int_{-\infty}^{\infty} \mathbf{E}_0 \cos(\omega t + \theta) e^{i\omega' t} dt \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} [\mathbf{E}_0 e^{i\theta} e^{i(\omega' + \omega)t} + \mathbf{E}_0 e^{-i\theta} e^{i(\omega' - \omega)t}] dt \\ &= \pi [\mathbf{E}_\omega \delta(\omega' - \omega) + \mathbf{E}_\omega^* \delta(\omega' + \omega)] \end{aligned}$$

where $\mathbf{E}_\omega \equiv \mathbf{E}_0 e^{-i\theta}$ (A.5)

$$\begin{aligned} \text{(A.2)} \Rightarrow \mathbf{E}(t) &= \frac{1}{2\pi} \pi \int_{-\infty}^{\infty} [\mathbf{E}_\omega \delta(\omega' - \omega) + \mathbf{E}_\omega^* \delta(\omega' + \omega)] e^{-i\omega' t} d\omega' \\ &= \tfrac{1}{2} [\mathbf{E}_\omega e^{-i\omega t} + \mathbf{E}_\omega^* e^{i\omega t}] = \mathrm{Re}[\mathbf{E}_\omega e^{-i\omega t}] \qquad \text{(A.6)} \end{aligned}$$

In linear equations, we may omit the "Re" sign and write (A.6) as

$$\mathbf{E}(t) = \mathbf{E}_\omega e^{-i\omega t} \quad (\Rightarrow \text{LHS} = \text{Re[RHS]}) \qquad \text{(A.7)}$$

Rewrite $\mathbf{E}(t) = \mathbf{E}_\omega e^{-i\omega t}$ $(\Rightarrow \text{LHS} = \text{Re[RHS]})$ (A.7)

Thus, by writing $\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t + \theta)$ as $\mathbf{E}(t) = \mathbf{E}_\omega e^{-i\omega t}$. We have entered from the t-space into the ω-space. Equations derived by using (A.7) are thus ω-space equations, e.g.

$$\mathbf{D}_\omega = \varepsilon \mathbf{E}_\omega \quad \text{[derived in Sec. I of the main text]} \qquad \text{(A.8)}$$

$\mathbf{E}_0$ in (A.4) is a *real* t-space quantity. $\mathbf{E}_\omega (= \mathbf{E}_0 e^{-i\theta})$ in (A.4) is a *complex* ω-space quantity and is called a phasor.

To convert a phasor back into the t-space, we multiply it by $e^{-i\omega t}$ and take the real part [as shown in (A.6)]. Thus

$$\mathbf{D}(t) = \text{Re}[\mathbf{D}_\omega e^{-i\omega t}] = \text{Re}[\varepsilon \mathbf{E}_\omega e^{-i\omega t}]$$
$$= \text{Re}[|\varepsilon| e^{i\varphi} \mathbf{E}_0 e^{-i\theta} e^{-i\omega t}] = |\varepsilon| \mathbf{E}_0 \cos(\omega t + \theta - \varphi) \qquad \text{(A.9)}$$

Note: The "linear" relation in (A.8) and (A.9) means the relative amplitude, phase, and direction between **D** and **E** are indep. of the absolute amplitudes of **D** and **E**, either in ω-space or t-space.

Discussion:

(i) A complex number carries twice the information as a real number, e.g. $\mathbf{E}_0$ in (A.4) gives the amplitude of $\mathbf{E}(t)$, whereas $\mathbf{E}_\omega (= \mathbf{E}_0 e^{-i\theta})$ in (A.7) gives both the amplitude and phase angle of $\mathbf{E}(t)$. Hence, the algebra is simpler in the ω-space. This is the reason why we often work in the ω-space.

(ii) In (A.8), $\mathbf{D}_\omega$ and $\mathbf{E}_\omega$ are phasors. But ε [derived in (7.51)] is a complex number derived in the ω-space. It is not a phasor. Hence, $\text{Re}[\varepsilon e^{-i\omega t}]$ is not a corresponding t-space quantity.

(iii) The same mathematics can be found in circuit theory:

$$V = IZ \text{ in circuit theory } \Leftrightarrow \mathbf{D} = \varepsilon \mathbf{E} \text{ here} \begin{bmatrix} V, I \Leftrightarrow \mathbf{D}, \mathbf{E} \\ Z \Leftrightarrow \varepsilon \end{bmatrix}$$

Example 2: a rotating vector

$$\mathbf{E}(t) = E_0\left(\cos\omega t\mathbf{e}_x + \sin\omega t\mathbf{e}_y\right) \tag{A.10}$$

Following the same procedure leading to (A.6), we obtain

$$\mathbf{E}(t) = \mathrm{Re}[(E_0\mathbf{e}_x + E_0 e^{i\frac{\pi}{2}}\mathbf{e}_y)e^{-i\omega t}]$$

$$= \mathrm{Re}[\mathbf{E}_\omega e^{-i\omega t}] \tag{A.11}$$

where $\mathbf{E}_\omega \equiv \mathrm{E}_0(\mathbf{e}_x + i\mathbf{e}_y)$ (A.12)

Discussion:

Examining the phasors $\mathbf{E}_\omega \equiv \mathbf{E}_0 e^{-i\theta}$ (A.5) and $\mathbf{E}_\omega \equiv \mathrm{E}_0(\mathbf{e}_x + i\mathbf{e}_y)$ (A.12), we find that the phasor, an ω-space quantity, may or may not have a clear geometric direction. For example, $\mathbf{E}_\omega$ in (A.5) has the same geometric direction as $\mathbf{E}_0$, but $\mathbf{E}_\omega$ in (A.12) does not have a clear geometric direction. The reason is that, in the time space, $\mathbf{E}(t) = E_0(\cos\omega t\mathbf{e}_x + \sin\omega t\mathbf{e}_y)$ has a geometric direction which rotates with time. When $\mathbf{E}(t)$ is transformed into the ω-space, in which t is no longer a variable, we obtain a phasor $\mathbf{E}_\omega$ without a clear geometric direction.

Appendix B. Electrical Conductivity of Metals

The current density $\mathbf{J}$ at any point in a conductor is given by

$$\mathbf{J}(t) = -n_0 e \mathbf{v}(t),$$

where n_0 is the *free* electron density, and their velocity $\mathbf{v}$ obeys

$$m\frac{d\mathbf{v}(t)}{dt} = -e\mathbf{E}(\mathbf{x},t) - m\gamma_0 \mathbf{v}(t) \;\; [\gamma_0\text{: average collision frequency}]$$

Assume the electron displacement is too small see the $\mathbf{x}$-variaton of $\mathbf{E}(\mathbf{x},t)$. Hence, $\mathbf{E}(\mathbf{x},t) \approx \mathbf{E}_0 e^{-i\omega t}$. Let

$$\begin{Bmatrix} \mathbf{E}(t) \\ \mathbf{v}(t) \\ \mathbf{J}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0 \\ \mathbf{v}_0 \\ \mathbf{J}_0 \end{Bmatrix} e^{-i\omega t} \Rightarrow -im\omega \mathbf{v}_0 = -e\mathbf{E}_0 - m\gamma_0 \mathbf{v}_0$$

$$\Rightarrow \;\; \mathbf{v}_0 = \frac{-e}{m(\gamma_0 - i\omega)}\mathbf{E}_0 \;\; \Rightarrow \;\; \mathbf{J}_0 = -n_0 e \mathbf{v}_0 = \frac{n_0 e^2}{m(\gamma_0 - i\omega)}\mathbf{E}_0$$

$$\Rightarrow \qquad \mathbf{J}_0 = \sigma \mathbf{E}_0$$

where $\sigma \equiv \dfrac{n_0 e^2}{m(\gamma_0 - i\omega)}$ [electrical conductivity]

With n_0 replaced with $f_0 N$, this is the same σ as in (7.58).