台大物理系 “古典電力學(一)” 任課老師:朱國瑞

“Classical Electrodynamics (I)”

Department of Physics, National Taiwan University

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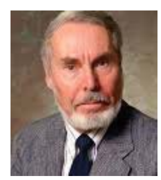
Fall Semester, 2020

1. Textbook and Contents of the Course:

Textbook: J. D. Jackson, “Classical Electrodynamics”, 3rd ed. Other books will be referenced when needed.

Prerequisite: Vector calculus (e.g., Griffiths, pp. 1-45). In general, this is all you need.



J. D. Jackson
1925 - 2016

In this course [Classical Electrodynamics (I)] , we will cover Chs. 1-6 and most of Ch. 7, which elegantly lay down the foundation of electrodynamics with a systematic exposition of some key topics in applied mathematics.

Next semester, in Classical Electrodynamics (II), we will cover selected topics of deep physical insight as well as current interest. These topics (including microwaves, radiating systems, scattering, diffraction, special theory of relativity, and synchrotron radiation) will help students consolidate the fundamental principles in Classical Electrodynamics (I).

2. **Conduct of Class :** Lecture notes will be projected sequentially on the screen during the class. Physical concepts will be emphasized. Straightforward algebraic details (provided in the lecture notes) may be skipped. *Questions are highly encouraged*. It is assumed that students have at least gone through the algebra in the lecture notes before attending classes (*important*!).
3. **Homework:** Approximately 3 problems per chapter will be assigned by the TAs. Students are encouraged to do as many unassigned problems as time allows. Find the problems that appeal to you. Always do them by yourself. If unsure or unsuccessful, solutions (not always correct!) can be found by, for example, a Google search under "Jackson electrodynamics problem solutions".
4. **Teaching Assistants:**
劉立中, f06222048@ntu.edu.tw, 0919–399–468, Rm. 103
林明勳, ck920274@yahoo.com.tw, 0936–565–863, Rm. 616
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5. **Grading Policy:** Midterm (40%); Final (50); Attendance (5%); Homework (5%). The final exam covers all materials taught in the class. The scores may be adjusted with no change on the order.
6. **Lecture Notes:**

Starting from basic equations, the lecture notes follow Jackson closely with algebraic details filled in.

Equations numbered in the format of (1.1), (1.2)… refer to Jackson. Supplementary equations derived in lecture notes, which will later be referenced, are numbered (1), (2)... [restarting from (1) in each chapter.] Equations in Appendices A, B…of each chapter are numbered (A.1), (A.2)…and (B.1), (B.2)…

Page numbers cited in the text (e.g. p. 120) refer to Jackson.

Section numbers (e.g. Sec. 1.1) refer to Jackson. Main topics within each section are highlighted by **boldfaced** characters. Some words are typed in *italicized* characters for attention. Technical terms which are introduced for the first time are <u>underlined</u>.

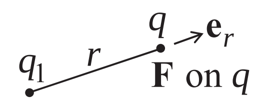
Chapter 1: Introduction to Electrostatics

1.1 Coulomb's Law

Coulomb's law, discovered experimentally in 1785, is a fundamental law governing all electrostatic phenomena. It states :

1. The force $\mathbf{F}$ exerted on point charge q by point charge q_1 obeys

$$\mathbf{F} = \frac{qq_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \Rightarrow \begin{cases} F \propto q,\ q_1, \text{ and } \frac{1}{r^2}. \\ F \text{ is along } \pm\mathbf{e}_r \text{ (central force).} \\ F \text{ is attractive if } q \text{ and } q_1 \text{ have opposite signs.} \\ F \text{ is repulsive if } q \text{ and } q_1 \text{ have the same sign.} \end{cases}$$



2. For multiple charges, the total force on q is the vector sum of the 2-body Coulomb forces between q and each of its surrounding charges.

Note: The exponent "2" in r^2 has later been determined by precision measuement to be accurate up to at least 16 decimal points.

Questions: (1) What is the principle of linear superposition?

(2) What force holds our body from falling apart?

1.2 Electric Field

The electric field at point $\mathbf{x}$ due to one or more charges is defined by

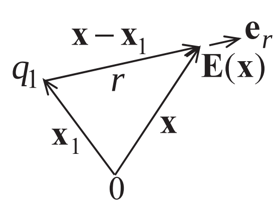
$$\mathbf{E}(\mathbf{x}) \equiv \frac{\mathbf{F}}{q} \quad [q : \text{a test charge at } \mathbf{x};\ \mathbf{F} : \text{total Coulomb force on } q] \quad (1.1)$$

Note: 1. $q \to 0$ so it will not displace the charges producing $\mathbf{E}$.

2. The word "field" was first used by Faraday in 1845.

$$\Rightarrow \mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2}\mathbf{e}_r = \frac{q_1}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} \quad [\mathbf{E} \text{ at } \mathbf{x} \text{ due to } q_1 \text{ at } \mathbf{x}_1] \quad (1.3)$$

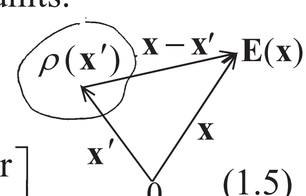
$$\boxed{\begin{aligned} r\mathbf{e}_r &= \mathbf{x}-\mathbf{x}_1 = (x-x_1)\mathbf{e}_x + (y-y_1)\mathbf{e}_y + (z-z_1)\mathbf{e}_z \\ r &= |\mathbf{x}-\mathbf{x}_1| \equiv \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} \end{aligned}}$$



Question: Why write "$r\mathbf{e}_r$" as "$\mathbf{x}-\mathbf{x}_1$"?

Note: In Jackson (1.2) and (1.3), $k = \frac{1}{4\pi\varepsilon_0}$ in SI units.

For a continuous charge distribution, let $\rho(\mathbf{x})$ be its volume density (charge/unit volume). Then,



$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \left[\begin{matrix}\text{principle of linear} \\ \text{superposition}\end{matrix}\right] \quad (1.5)$$

Questions:

(1) $\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} \to \infty$ as $\mathbf{x} \to \mathbf{x}_1$. Is this physical?

Answer: All charged particles have a finite size (even the electron has a non-zero, but unknown, size). Hence, they do not produce an infinite electric field. The particle size is so small that we are always far away (relative to particle size) from the particle in electrodynamics problems. Thus, the particle can be treated as a point charge.

(2) Does a point charge experience a force in its own electric field?

Answer: Imagine the charge is distributed in a particle of radius R. The Coulomb forces within the particle are *internal* forces, with a zero vector sum by Newton's third law. This is true even as $R \to 0$. Thus, a point charge experiences no net force in its own electrostatic field.

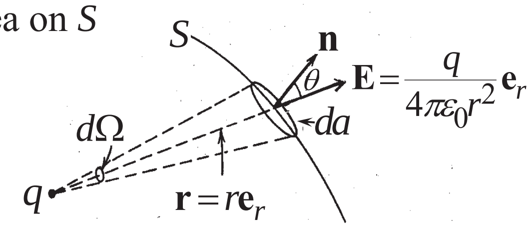
(3) Any theory on what holds the electron together?

This is out of the scope of this course. See Feynman Lectures II, Sec. 28-4 if you are interested.

1.3 Gauss's Law

Consider a point charge q inside or outside of a *closed* surface S.

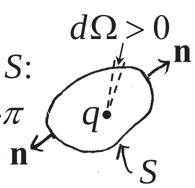
Let
- da: a differential surface area on S
- $d\Omega$: solid angle from q to da
- $\mathbf{n}$: unit vector normal to da and pointing *out of* S
- $\mathbf{e}_r$: unit vector along $\mathbf{r}$
- θ: angle between $\mathbf{n}$ and $\mathbf{e}_r$



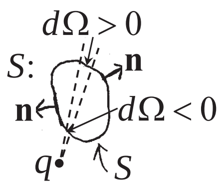
$$\mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r\cdot\mathbf{n}da = \frac{q}{4\pi\varepsilon_0 r^2}\overbrace{\cos\theta da}^{r^2 d\Omega} = \frac{q}{4\pi\varepsilon_0}d\Omega \quad \left[d\Omega = \frac{\overbrace{\mathbf{e}_r\cdot\mathbf{n}}^{\cos\theta}da}{r^2}\right]$$

$$\begin{cases} d\Omega \begin{pmatrix}\text{leaving } S \\ 0 \le \theta < 90^\circ\end{pmatrix} > 0 \\ d\Omega \begin{pmatrix}\text{entering } S \\ 90^\circ < \theta \le 180^\circ\end{pmatrix} < 0 \end{cases} \Rightarrow$$

q inside S: $\int d\Omega = 4\pi$



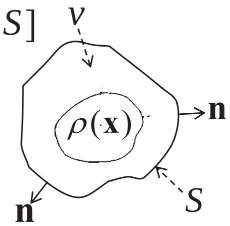
q outside S: $\int d\Omega = 0$



$$\Rightarrow \oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \begin{cases} q/\varepsilon_0, & q \text{ inside } S \\ 0, & q \text{ outside } S \end{cases} \quad \left[\begin{matrix}\underline{\text{Gauss's law}} \text{ for} \\ \text{a single charge}\end{matrix}\right] \quad (1.9)$$

Exercise: Compare with Griffiths's derivation of Gauss's law.

Principle of linear superposition $\Rightarrow$ Gauss's law for a discrete set of charges: $\oint_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_0}\sum_i q_i$ [all q_i inside S] (1.10)



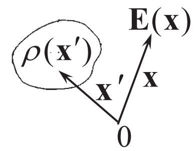
and for a distribution of charges:

$$\oint_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x}) d^3x \quad [\rho \text{ inside } S] \tag{1.11}$$

Discussion:

1. In deriving Gauss's law, $\mathbf{n}$ ($\perp$ to S) is assumed to point *out of* v.

2. (1.11) is the *integral* form of Gauss's law. It is a powerful law derived from Coulomb's law:



$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \tag{1.5}$$

3. In (1.11), $\int_v \cdots d^3x$ is over the ρ inside (not on) an arbitrary S.

4. In (1.5), $\int_v \cdots d^3x'$ is over all the ρ, which contributes to $\mathbf{E}(\mathbf{x})$.

A note on notation: $\rho(\mathbf{x})$ is used when $\mathbf{x}$ is the only variable [as in (1.11)]. $\rho(\mathbf{x}')$ is used to distinguish 2 variables, $\mathbf{x}'$ and $\mathbf{x}$ [as in (1.5)].

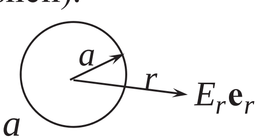
Exercise 1: Prove the 2 shell theorems.

Halliday, Resnick, and Walker, "Fundamentals of Physics":

Thorem 1: A uniform spherical shell of charges behaves, for external points, as if all its charges were concentrated at its center.

Theorem 2: A uniform spherical shell of charges exerts no force on a charge placed inside the shell (i.e. $\mathbf{E} = 0$ inside the shell).

Proof: Symmetry consideration $\Rightarrow \mathbf{E} = E_r \mathbf{e}_r$



$$\left\{\begin{array}{l}\text{Gauss's law:}\\ \oint_S \mathbf{E}\cdot\mathbf{n}\, da = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x}) d^3x\end{array}\right. \Rightarrow 4\pi r^2 E_r = \begin{cases}\frac{Q}{\varepsilon_0}, & r > a\\ 0, & r < a\end{cases}$$

a: radius of shell; Q: total charge on shell

$$\Rightarrow E_r = \begin{cases}\frac{Q}{4\pi\varepsilon_0 r^2}, & r > a \quad (\text{as if } Q \text{ were at } r = 0)\\ 0, & r < a \quad (Q \text{ produces no } \mathbf{E})\end{cases}$$

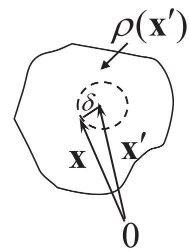
Note: E_r is discontinuous at $r = a$ [see (1.22), Sec. 1.6].

Exercise 2: Coulomb's law states $\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$.

Assume $\rho(\mathbf{x}')$ is a finite function of $\mathbf{x}'$. Use the integral form of Gauss's law and order-of-magnitude considerations to explain why $\mathbf{E}(\mathbf{x}) \neq \infty$ as $\mathbf{x} \to \mathbf{x}'$ [$\mathbf{x}'$ is any point in $\rho(\mathbf{x}')$].

Solution: If $\mathbf{E}(\mathbf{x}) \to \infty$, it must be due to charge(s) closest to $\mathbf{x}$.

Apply Gauss's law: $\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x})d^3x$ to a sphere of infinistesimal radius δ around $\mathbf{x}'$, so $S \propto \delta^2$ and $v \propto \delta^3$. Let $\mathbf{x}$ be a point on the sphere. Then, $\mathbf{x} \to \mathbf{x}'$ as $\delta \to 0$.



Gauss's law $\Rightarrow$ $\underbrace{E(\mathbf{x})}\delta^2 \sim \frac{1}{\varepsilon_0}\rho\delta^3$

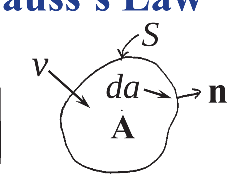
E on the sphere of radius δ around $\mathbf{x}'$ due to charges inside the sphere

$\Rightarrow E(\mathbf{x}) \sim \delta \to 0$ as $\delta \to 0$

Note: $\begin{cases}\text{Point charge } q \text{ is a } \textit{microscopic} \text{ quantity.} \\ \rho \text{ (charge/unit volume) is a } \textit{macroscopic} \text{ quantity.}\end{cases}$

1.4 Differential Form of Gauss's Law

Using the divergence theorem:



$\int_v \nabla\cdot\mathbf{A}d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}da$ $\begin{bmatrix}\text{see Jackson,} \\ \text{front cover}\end{bmatrix}$

on Gauss's law: $\underbrace{\oint_S \mathbf{E}\cdot\mathbf{n}\, da}_{\int_v \nabla\cdot\mathbf{E}d^3x} = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x})d^3x$ [(1.11)], we obtain

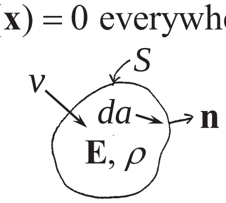
In both the divergence thm. and Gauss's law, $\mathbf{n}$ is required to be a unit normal pointing *out of* v.

$\int_v \nabla\cdot\mathbf{E}d^3x = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x})d^3x$

$\Rightarrow \int_v (\nabla\cdot\mathbf{E} - \frac{\rho}{\varepsilon_0})d^3x = 0$ [for any v] (1.12)

If $\int_v f(\mathbf{x})d^3x = 0$ for any volume v, then $f(\mathbf{x}) = 0$ everywhere.

$\Rightarrow \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$ $\begin{bmatrix}\text{differential form} \\ \text{of Gauss's law}\end{bmatrix}$ (1.13)



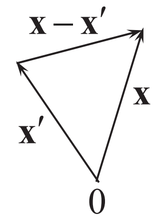
Question: If $\oint_S \mathbf{A}\cdot\mathbf{n}da = 0$ for any *closed* surface S, does it imply $\mathbf{A} = 0$ everywhere?

Ans.: No, e.g. $\oint_S \mathbf{B}(\text{magnetic field})\cdot\mathbf{n}\, da = 0$ for any S $\not\Rightarrow$ $\mathbf{B} = 0$

1.5 Another Equation of Electrostatics and the Scalar Potential

$$\nabla|\mathbf{x}-\mathbf{x}'|^n$$
$$=\frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_x$$
$$+\frac{\partial}{\partial y}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_y$$
$$+\frac{\partial}{\partial z}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_z$$
$$=\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(x-x')\mathbf{e}_x$$
$$+\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(y-y')\mathbf{e}_y$$
$$+\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(z-z')\mathbf{e}_z$$
$$=n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \qquad (1)$$

∇ operates on $\mathbf{x}$.
∇' operates on $\mathbf{x}'$.
$\nabla'|\mathbf{x}-\mathbf{x}'|^n=-\nabla|\mathbf{x}-\mathbf{x}'|^n$



Examples: (frequently used later)

(1) $\nabla|\mathbf{x}-\mathbf{x}'|=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$; (2) $\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$; (3) $\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|^3}=-3\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^5}$

$$\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'=-\frac{1}{4\pi\varepsilon_0}\int\nabla\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

∇ operates on $\mathbf{x}$, hence can be moved out of the d^3x'-integral.

$\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$

$$=-\frac{1}{4\pi\varepsilon_0}\nabla\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-\nabla\phi(\mathbf{x})\ [\phi:\text{electrostatic potential}]$$

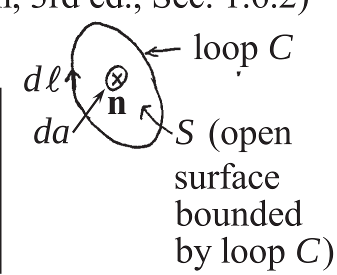
where $\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ $\left[\begin{array}{l}\text{no } \rho \text{ at } \infty \text{ (by assumption)} \Rightarrow\\ \mathbf{x}' \text{ is finite.} \Rightarrow \phi\to 0 \text{ as } \mathbf{x}\to\infty.\end{array}\right]$ (1.17)

Question: What and where is the reference value of ϕ in (1.17)?

$\mathbf{E}=-\nabla\phi \ \Rightarrow\ \nabla\times\mathbf{E}=0$ [by the identity: $\nabla\times\nabla f=0$] (1.14)

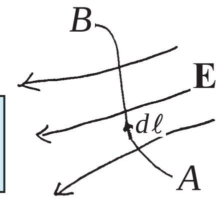
Note: $\nabla\times\mathbf{E}=0$ also implies $\mathbf{E}=-\nabla\phi$ (Griffith, 3rd ed., Sec. 1.6.2)

Stokes's theorem: $\oint_C\mathbf{A}\cdot d\ell=\int_S(\nabla\times\mathbf{A})\cdot\mathbf{n}\,da$



$d\ell$: a differential length on a closed loop C
S: arbitrary *open* surface bounded by loop C
$\mathbf{n}$: unit vector normal to surface element da in the direction given by the *right-hand rule*

Work done to bring charge q from position A to position B along any path:



$$
\begin{aligned}
W &= -\int_A^B \mathbf{F}\cdot d\ell \\
&= -q\int_A^B \mathbf{E}\cdot d\ell \\
&= q\int_A^B \nabla\phi\cdot d\ell \\
&= q\int_A^B d\phi \\
&= q(\phi_B-\phi_A)
\end{aligned}
$$

Question: Is W the work done by $\mathbf{E}$ or by an external force?

$\nabla\phi=\frac{\partial\phi}{\partial x}\mathbf{e}_x+\frac{\partial\phi}{\partial y}\mathbf{e}_y+\frac{\partial\phi}{\partial z}\mathbf{e}_z;\ d\ell=dx\mathbf{e}_x+dy\mathbf{e}_y+dz\mathbf{e}_z$

$\Rightarrow \nabla\phi\cdot d\ell=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi$

$\Rightarrow d\phi$ is an infinitesimal change of ϕ due to an infinitesimal displacement $d\ell$.

Thus, W depends only on the values of ϕ at A and B, and it is indep. of the charge's path from A to B, which justifies the concept of potential energy. In other words, the total work done on q along *any* closed path C is 0, i.e. $\oint_C \mathbf{E}\cdot d\ell=0$ (1.21)

Stokes's thm.: $\oint_C \mathbf{E}\cdot d\ell=\int_S(\nabla\times\mathbf{E})\cdot\mathbf{n}\,da \Rightarrow \int_S(\nabla\times\mathbf{E})\cdot\mathbf{n}\,da=0.$

S is an arbitrary *open* surface $\Rightarrow$ We again obtain $\nabla\times\mathbf{E}=0$ [(1.14)].

1.7 Poisson and Laplace Equations

(Sec. 1.6 will be covered later)

Rewrite $\begin{cases}\nabla\cdot\mathbf{E}=\dfrac{\rho}{\varepsilon_0} & (1.13)\\ \mathbf{E}=-\nabla\phi & (1.16)\end{cases}$

$\begin{cases}(1.13)\\(1.16)\end{cases}\Rightarrow$ D.E. (differential eq.): $\nabla^2\phi=-\dfrac{\rho}{\varepsilon_0}$ [Poisson eq.] (1.28)

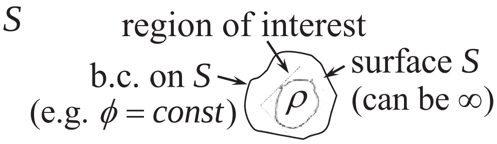
In a charge-free region, (1.28) reduces to

$$\nabla^2\phi=0 \quad \text{[Laplace euqation]} \qquad (1.29)$$

Questions:

1. What needs to be specified to solve the D.E. $\nabla^2\phi=-\rho/\varepsilon_0$?

$\begin{cases}\text{1. a region of interest enclosed by } S\\ \text{2. } \rho \text{ in the region of interest}\\ \text{3. b.c. (boundary condition) on } S\end{cases}$



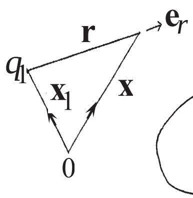
2. Do we need to know the charges on or outside S? (*Ans.*: No).
3. If $\rho=0$ but $\phi\neq 0$ in the region of interest, where are the charges that produce ϕ? (*Ans.*: Charges on and/or outside S).

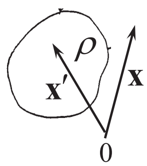
Summary of Secs. 1-5 and 7:

$$\boxed{\mathbf{F}=\frac{qq_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r} \longrightarrow$$

$\uparrow$

$$\mathbf{E}\equiv\lim_{q\to 0}\frac{\mathbf{F}}{q}$$





$$\mathbf{E}=\frac{q_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \qquad \leftarrow r\mathbf{e}_r=\mathbf{x}-\mathbf{x}_1,\quad r=|\mathbf{x}-\mathbf{x}_1|$$

$$=\frac{q_1(\mathbf{x}-\mathbf{x}_1)}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_1|^3} \qquad \leftarrow \text{principle of linear superposition}$$

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x' \qquad \leftarrow \nabla|\mathbf{x}-\mathbf{x}'|^n$$

$$=-\frac{1}{4\pi\varepsilon_0}\nabla\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \qquad =n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}')$$

$$=-\nabla\phi \qquad \leftarrow \phi\equiv\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$\nabla\times\nabla f=0\rightarrow$ $\boxed{\nabla\times\mathbf{E}=0}$

$\leftarrow$ derived in Sec.1.3 using $\mathbf{E}=\dfrac{q\mathbf{e}_r}{4\pi\varepsilon_0 r^2}$

$$\boxed{\oint_S \mathbf{E}\cdot\mathbf{n}da=\frac{1}{\varepsilon_0}\int_V\rho(\mathbf{x})d^3x}$$

$\leftarrow$ divergence thm.

$$\boxed{\nabla\cdot\mathbf{E}=\rho/\varepsilon_0}$$

$$\boxed{\nabla^2\phi=-\rho/\varepsilon_0}$$

Questions on *Secs. 1-5 and 7*:

1. Can one calculate $\mathbf{E}$ by using $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$ alone? *Ans*.: No
   Helmholtz's Theorem (Griffiths, "Intro. to Electrodynamics", Sec. 1.6.1): If a vector field goes to 0 at infinity, it is uniquely determined by its divergence and curl.
2. Coulomb's law gives $\nabla\times\mathbf{E}=0$ and $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$. Can it give any other independent relation for $\mathbf{E}$? No (by Helmholtz's Theorem)
3. Why break one equation [$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$] into 2 equations:
   $\nabla\times\mathbf{E}=0$ and $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$? *Ans*.: For example, these 2 equations give the D.E. $\nabla^2\phi=-\rho/\varepsilon_0$, which is often much easier to solve.
4. To solve $\nabla^2\phi=-\rho/\varepsilon_0$, we need to know the "region of interest" and "boundary condition". Do we need the same information to solve $\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$? *Ans*.: The region is ∞ and $\phi(\infty)=0$, but all ρ of interest must be integrated (from easy to impossible).

5. Is $\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x$ (integral form of Gauss's law) *mathematically* equivalent to its differential form $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$?

Answer: Yes. To prove the "equivalence", we need to show that $\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x$ is both a *sufficient* and *necessary* condition for $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$. This can be done as follows:

$$\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x \tag{1.11}$$

$$\Uparrow \;\; \Downarrow \quad \leftarrow \text{divergence thm.}$$

$$\int_V(\nabla\cdot\mathbf{E} - \rho/\varepsilon_0)d^3x = 0 \quad \text{(for arbitrary volume } v)$$

$$\Uparrow \;\; \Downarrow$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.13}$$

Downward manipulation ($\Downarrow$) shows that (1.11) is a sufficient condition for (1.13). Upward manipulation ($\Uparrow$) shows that (1.11) is a necessary condition for (1.13). Thus, the two forms of Gauss's law are mathematically equivalent (hence always *physically* equivalent). 19

6. Is $\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x$ (Gauss's law) *mathematically* equivalent to $\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$ (Coulomb's law)?

Answer: No, because Coulomb's law is a sufficient but not a necessary condition for Gauss's law. That is, we may derive Gauss's law from Coulomb's law, but not the reverse.

Coulomb's law completely determines **E**, but Gauss's law cannot completely determine **E**. This becomes clear if we write Gauss's law in its differential form, $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$. By Helmholtz's Theorem, we also need $\nabla\times\mathbf{E}$ to uniquely determine **E**. In electrostatics, we have $\nabla\times\mathbf{E} = 0$. In general, $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ [Faraday's law, Ch. 5].

7. Is $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$ (or its integral form) valid for time-varying **E**?

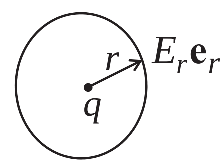
Answer: Yes, by law of nature. In fact, it is one of the 4 Maxwell eqs. (Ch. 6). Thus, if $\partial/\partial t \neq 0$, $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$ and $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ give an **E** different from $\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$, i.e. the latter is valid only in electrostatics.

8. Is Gauss's law *physically* equivalent to Coulomb's law?

Answer: Yes, but only in *static* cases, in which $\mathbf{E}$ is spherically symmetric about point charge q, so that $\mathbf{E} = E_r \mathbf{e}_r$.

Note: $\mathbf{E}$ is *asymmetric* about a *moving* charge (See Fig. 11.9).

On a spherical surface of radius r centered at a static charge q, we have $\mathbf{E} = E_r \mathbf{e}_r$ by symmetry.



Thus, $\oint_s \mathbf{E} \cdot \mathbf{n} da = \frac{q}{\varepsilon_0}$ gives $E_r 4\pi r^2 = \frac{q}{\varepsilon_0}$

$$\Rightarrow E_r = \frac{q}{4\pi\varepsilon_0 r^2} \Rightarrow \mathbf{E} = E_r \mathbf{e}_r = \frac{q}{4\pi\varepsilon_0 r^2} \mathbf{e}_r$$

In the above, we have derived Coulomb's law from Gauss's law. In Sec. 1.3, we have also derived Gauss's law from Coulomb's law. Hence, the two laws are equivalent in electrostatics. The fact they are *conditionally* equivalent (i.e. only in electrostatics) also indicates that they are not mathematically equivalent.

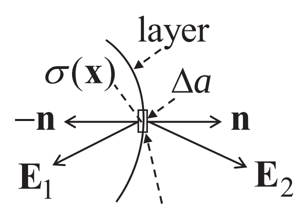
Gauss's law alone cannot determine $\mathbf{E}$ unless under special conditions (p.31), e.g. the symmetry above or Prob. 1 in Sec.1.6.

1.6 Surface Distributions of Charges and Dipoles and Discontinuities in the Electric Field and Potential

Single Layer : A layer of surface charges with zero thickness

Define $\sigma(\mathbf{x}) \equiv \lim_{\Delta a \to 0} \frac{\Delta q}{\Delta a}$ [charge/unit area]

Draw a pillbox of area Δa with thickness $\ll$ radius of Δa. Let its size be infinitesimal so it will enclose an essentially flat area of the layer at point $\mathbf{x}$ (area direction $\mathbf{n} \perp$ layer).



$$\oint \mathbf{E} \cdot \mathbf{n} da = \frac{q}{\varepsilon_0} \Rightarrow (-\mathbf{E}_1 \cdot \mathbf{n} + \mathbf{E}_2 \cdot \mathbf{n}) \Delta a = \frac{\sigma \Delta a}{\varepsilon_0}$$

$$\Rightarrow (\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \frac{\sigma}{\varepsilon_0} \quad \left[\begin{array}{l} \Rightarrow \mathbf{E}_\perp \text{ discontinuous across } \sigma; \\ \text{will be used frequently later.} \end{array}\right] \qquad (1.22)$$

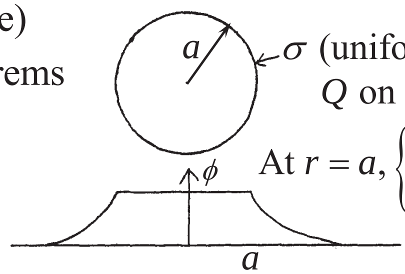
In Gauss's law, $\mathbf{n}$, $\mathbf{E}_1$, & $\mathbf{E}_2$ refer to values on the pillbox. We take them to be the values at $\mathbf{x}$ since the pillbox thickness $\to 0$.

Treat the rectangle above as a "loop" and apply $\oint_C \mathbf{E} \cdot d\ell = 0$. It can be easily seen that $\mathbf{E}_{||}$ is continuous across the layer.

Example: (see figure)

Apply the shell theorems

$$\Rightarrow \phi = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 a}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r}, & r > a \end{cases}$$



σ (uniform distribution of Q on a layer of radius a)

At $r = a$, $\begin{cases} \phi \text{ is continuous.} \\ E_r \text{ is discontinuous.} \end{cases}$

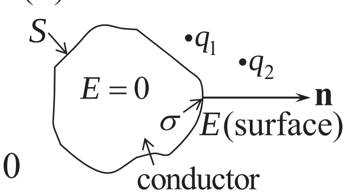
Questions:

1. E and ϕ of a *point* charge diverge as one moves infinistesimally close to the charge. Explain why E and ϕ due to *surface* charges do not diverge as one moves infinistesimally close to the surface.

Answer: A point charge is finite amount of charge at a point (zero volume). σ is finite everywhere on the surface (no point charges). We must integrate σ over a finite area to obtain a finite amount of charge. Hence, there is only an infinistesimal amount of charge at any surface point ($\Rightarrow$ no divergence).

2. Why is ϕ continuous across the layer?

Problem 1: A conductor has a non-uniform $\sigma(\mathbf{x})$ on its surface. Find $\mathbf{E}$ *on* (not *away from*) the surface S.



Solution: In electrostatics, charges are always on the surface of a conductor and $\sigma(\mathbf{x})$ is always self-adjusted so that: (1) E (inside) = 0 and (2) $\mathbf{E}$ is $\perp$ to the surface of the conductor (Otherwise, there will be current inside or on the surface, hence non-static).

Apply $(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n} = \frac{\sigma}{\varepsilon_0}$ [(1.22)] to any point on the surface,

where $\begin{cases} \mathbf{n} = \text{unit normal to surface, pointing outward} \\ \mathbf{E}_2 = E(\text{surface})\mathbf{n}; \quad \mathbf{E}_1 = \mathbf{E}(\text{inside}) = 0 \end{cases}$

$\Rightarrow \mathbf{E}(\text{surface}) = \frac{\sigma}{\varepsilon_0}\mathbf{n}$ [Even if S is *curved* or σ is *non-uniform*]

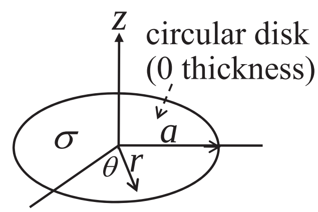
Question: Why does $\mathbf{E}$(surface) depend on σ at only one point?

[In deriving (1.22), the pillbox area $\to 0$. See also Prob. 2]

Note: 1. We cannot use (1.22) to find $\mathbf{E}$ *away from* the surface, since (1.22) relates $\mathbf{E}$ across a layer of zero thickness.

2. External charges (q_1, q_2, etc.) affect E (surface) by changing σ.

Problem 2: A disk of radius a has a uniform σ. Find ϕ & $\mathbf{E}$ on its axis ($r = 0,\ \mathbf{E} = E\mathbf{e}_z$) in cylindrical coordinates.



Solution: $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad \begin{bmatrix} \mathbf{x} = z\mathbf{e}_z \\ \mathbf{x}' = r'\mathbf{e}_r \end{bmatrix}$

$\Rightarrow \phi(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^a \frac{2\pi r' dr'}{(z^2+r'^2)^{1/2}} \quad [\phi(\mathbf{x}) \text{ at } r = 0]$

$$\Rightarrow \begin{cases} \phi(z \ge 0) = \frac{\sigma}{2\varepsilon_0}[z(1+\frac{a^2}{z^2})^{1/2} - z] & \begin{bmatrix} \text{Valid only for } z \ge 0. \text{ By} \\ \text{symmetry, } \phi(z \le 0) = \phi(z \ge 0) \end{bmatrix} \\ \mathbf{E}(z \ge 0) = -\frac{\partial}{\partial z}\phi(z \ge 0)\mathbf{e}_z = \frac{\sigma}{2\varepsilon_0}[1 - \frac{z}{(z^2+a^2)^{1/2}}]\mathbf{e}_z & \begin{bmatrix} \mathbf{E}: \text{discontinuous} \\ \text{across } \sigma \text{ by (1.22)} \end{bmatrix} \end{cases}$$

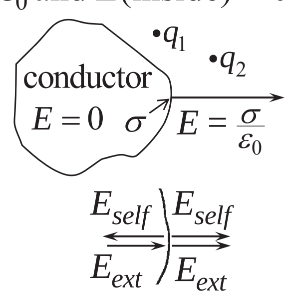
$$\text{Checks:} \begin{cases} \phi(z \gg a) \underset{\uparrow}{\approx} \frac{\sigma}{2\varepsilon_0}[z(1+\frac{a^2}{2z^2}) - z] = \frac{\pi a^2 \sigma}{4\pi\varepsilon_0}\frac{1}{z} = \underbrace{\frac{Q}{4\pi\varepsilon_0}\frac{1}{z}}_{} \\ \boxed{(1+\varepsilon)^n \approx 1+n\varepsilon, \text{ if } \varepsilon \ll 1, \text{ see (2) below}} \quad \text{far field of } Q, \text{ as expected} \\ \mathbf{E}(a \to \infty) = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \ \&\ \mathbf{E}(a = 0) = 0, \text{ both as expected} \end{cases}$$

An interesting observation: $\mathbf{E}(z = 0,\ a \ne 0) = \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z \quad [z = 0 \Rightarrow z = 0^+]$

$\Rightarrow$ $\mathbf{E}$ *on* the surface is indep. of a as long as $a \ne 0$.

$\Rightarrow$ $\mathbf{E}$ *on* the surface is due to an infinistesimal area of σ around $r = 0$.

Problem 3: Consider the conductor in Prob. 1 again (upper figure). What are the charges which make E (surface) = σ/ε_0 and E(inside) = 0?



Ans.: In Prob. 2, we find that charges on an infinitesimal area of σ will produce equal and opposite fields (call it E_{self}, lower figure) on both sides of σ and $E_{self} \perp \sigma$. However, $E_{self} = \sigma/(2\varepsilon_0)$, which is only half of E (surface) on a conductor.

Since E (inside) = 0, all other charges (on and off the conductor) not in the infinitesimal area of σ must have produced a field (call it E_{ext}) and $E_{ext} = E_{self} = \sigma/(2\varepsilon_0)$, so that E_{ext} cancels E_{self} (inside) and hence doubles E_{self} (outside), which makes E(inside)=0 and E (surface)=σ/ε_0.

Problem 4: Find the force on the σ of the conductor surface.

Ans.: σ can only experience a force due to E_{ext} (not E_{self}). Thus,

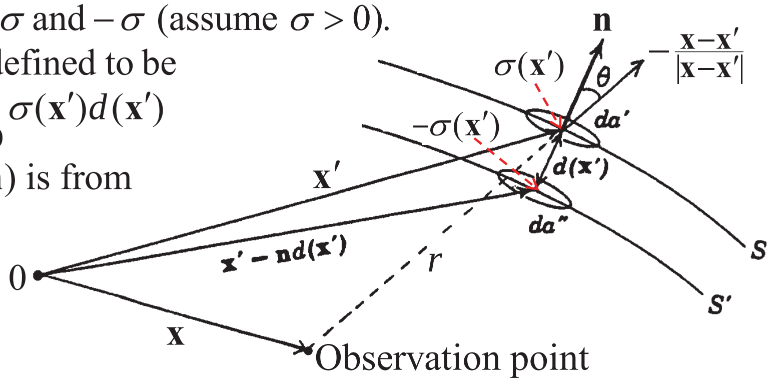
$$\frac{\text{Force on the surface}}{\text{unit area}} = \sigma E_{ext} = \frac{\sigma^2}{2\varepsilon_0} \quad \begin{bmatrix} \text{Griffiths, Sec. 2.5.3} \\ \text{1st line, Jackson, p. 43} \end{bmatrix}$$

Dipole Layer : A dipole layer is formed of 2 closely-spaced single layers with σ and $-\sigma$ (assume $\sigma > 0$).

Its strength is defined to be

$$D(\mathbf{x}') \equiv \lim_{d(\mathbf{x}')\to 0} \sigma(\mathbf{x}')d(\mathbf{x}')$$

Its direction ($\mathbf{n}$) is from $-\sigma$ to σ.



$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da' - \int_{S'} \frac{\sigma(\mathbf{x}'-\mathbf{n}d)}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|}da''\right]$$

Since $d \to 0$, we may let $\sigma(\mathbf{x}'-\mathbf{n}d) = \sigma(\mathbf{x}')$ and $da' = da''$. However, the small separation (d) of the 2 layers cannot be neglected because it is the dominant cause for ϕ. Thus,

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \sigma(\mathbf{x}')\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|}\right]da'$$

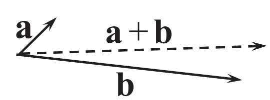
Using the binomial expansion: (This expansion will be used many times later. It's most useful if only the first 2 or 3 terms are important.)

$\boxed{n \text{ can be negative and/or a non-integer.}}$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots, \quad (2)$$

we obtain

$$\frac{1}{|\mathbf{b}+\mathbf{a}|} = \frac{1}{(b^2+a^2+2\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}} = \frac{1}{b}(\overset{x}{1} + \overbrace{\frac{a^2}{b^2} + 2\frac{\mathbf{a}\cdot\mathbf{b}}{b^2}}^{y})^{\overbrace{-\frac{1}{2}}^{n}}$$



$$= \frac{1}{b}\left(1 - \frac{a^2}{2b^2} - \frac{\mathbf{a}\cdot\mathbf{b}}{b^2} + \cdots\right) \underset{\uparrow \boxed{\frac{a}{b}\to 0}}{\approx} \frac{1}{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{b^3} \quad [\text{valid for } a \ll b]$$

$$\text{Let} \begin{cases} \mathbf{b} \to \mathbf{x}-\mathbf{x}' \\ \mathbf{a} \to \mathbf{n}d \end{cases}$$

$$\Rightarrow \frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|} \approx \frac{1}{|\mathbf{x}-\mathbf{x}'|} - d\mathbf{n}\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \quad [\text{valid for } d \ll |\mathbf{x}-\mathbf{x}'|]$$

Sub. $\dfrac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|} \approx \dfrac{1}{|\mathbf{x}-\mathbf{x}'|} - d\mathbf{n}\cdot\dfrac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$

into $\phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\int_S \sigma(\mathbf{x}')\left[\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} - \dfrac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|}\right]da'$, we obtain

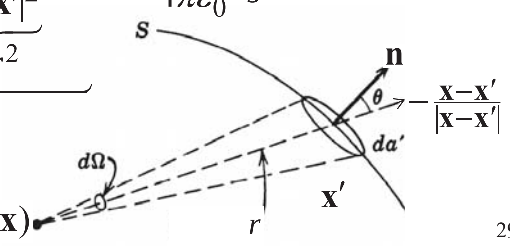
$$\boxed{= -\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \text{[See (1)]}}$$

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \underbrace{\sigma(\mathbf{x}')d(\mathbf{x}')}_{D(\mathbf{x}')}\mathbf{n}\cdot\overbrace{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}\,da' = \frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\mathbf{n}\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}da' \quad (1.24)$$

σ and *d* appear as a product here, so it's meaningful to define their product as the dipole layer strength.

$$\text{or } \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\underbrace{\underbrace{\mathbf{n}\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}}_{-\cos\theta}\underbrace{\frac{1}{|\mathbf{x}-\mathbf{x}'|^2}}_{1/r^2}}_{-d\Omega}da' = -\frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega \quad (1.26)$$

$$\begin{cases} \mathbf{n} \text{ is from } -\sigma \text{ to } \sigma \\ d\Omega > 0, \text{ if } \cos\theta > 0 \\ d\Omega < 0, \text{ if } \cos\theta < 0 \end{cases}$$

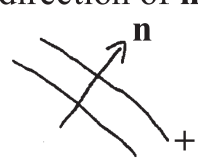


Observation point ($\mathbf{x}$)

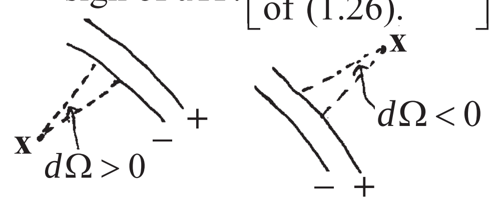
$$\text{Rewrite}: \phi(\mathbf{x}) = \begin{cases} -\dfrac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega \quad [D > 0 \text{ by definition}] & (1.26) \\ \dfrac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\mathbf{n}\cdot\nabla'\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}da' & (1.24) \end{cases}$$

Note: (1) The direction of **n** and sign of $d\Omega$ are shown below with respect to the polarity of the dipole layer:

direction of **n**:



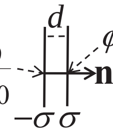
sign of $d\Omega$: [See derivation of (1.26).]



(2) The RHS of (1.24) is an explicit function of **x** (point of observation). The RHS of (1.26) is an implicit function of **x**, because the total solid angle depends on **x**.

Questions: (1) Under what condition are (1.24) and (1.26) invalid?
(2) What is the reference point for ϕ in these 2 eqs.?

Special case: A double layer with $D = const.$ ($D > 0$ by definition)

$\phi(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega$ [(1.26)] $\rightarrow$ $\phi_- = -\frac{D}{2\varepsilon_0}$  $\phi_+ = \frac{D}{2\varepsilon_0}$

$$\phi_+ - \phi_- = \frac{D}{2\varepsilon_0} - \left(-\frac{D}{2\varepsilon_0}\right) = \frac{D}{\varepsilon_0} \quad (1.27)$$

$$\Rightarrow \begin{cases} \text{electric field between layers: } E_\perp = \frac{D}{\varepsilon_0 d}. \\ \phi \text{ is discontinuous across the dipole layer. } \Delta\phi \text{ occurs inside the} \\ \text{dipole layer (i.e. between the 2 single layers of opposite } \sigma\text{).} \end{cases}$$

Question: **E** is continuous across the dipole layer (i.e. **E** on both side of the layer are equal). Why? (Since $d \to 0$, we may show this by drawing a pillbox of infinitesimal thickness to enclose the dipole layer and use Gauss law as in the case of a single layer.)

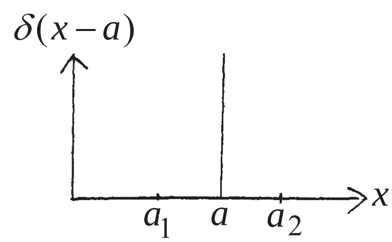
$$\text{Thus, across } \begin{cases} \text{a single layer, } \phi \text{ is continuous; } \mathbf{E} \text{ is discontinuous.} \\ \text{a dipole layer, } \phi \text{ is discontinuous; } \mathbf{E} \text{ is continuous.} \end{cases}$$

The case of a point dipole [(1.25)] will be considered in Sec. 4.1.

Delta Functions (pp. 26 - 27)

Definition of <u>*delta function*</u>:

$$\begin{cases} \delta(x-a) = 0\ , & \text{if } x \neq a \\ \int_{a_1}^{a_2} \delta(x-a)dx = 1\ , & \text{if } a_1 < a < a_2 \end{cases}$$



Note: Since the delta function is defined in terms of an integral, it takes an integration to bring out its full meaning.

Question: What is the value of $\int_a^{a_2} \delta(x-a)dx$? (undeterminable; Problem 4 in Sec. 3.10 shows a physical way of handling it.).

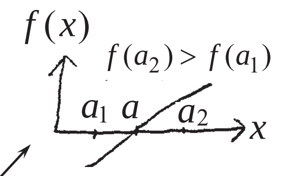
Properties of delta functions:

$$\text{(i) } \int_{a_1}^{a_2} f(x)\delta(x-a)dx = f(a) \quad (3)$$

$$\text{(ii) } \int_{a_1}^{a_2} f(x)\delta'(x-a)dx = \overbrace{f(x)\delta(x-a)\Big|_{a_1}^{a_2}}^{0} - \int_{a_1}^{a_2} f'(x)\delta(x-a)dx$$

$$= -f'(a) \quad (4)$$

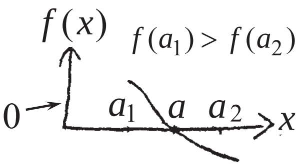
(iii) Let $x=a$ be the root of $f(x)=0$, then



$$\int_{a_1}^{a_2}\delta[f(x)]dx=\underset{\uparrow}{\int_{f(a_1)}^{f(a_2)}}\delta[f(x)]\frac{1}{\frac{d}{dx}f(x)}df(x)$$

Convert x-integration to f-integration

$$=\begin{cases}\int_{f(a_1)}^{f(a_2)}\frac{1}{f'}\delta(f)df=\frac{1}{f'(a)}=\frac{1}{|f'(a)|}, & \text{if } f'(a)>0\\ -\int_{f(a_2)}^{f(a_1)}\frac{1}{f'}\delta(f)df=-\frac{1}{f'(a)}=\frac{1}{|f'(a)|}, & \text{if } f'(a)<0\end{cases}$$



Note: In both expressions above, the integration over f is from a samller to a larger f value, as in the definition of the delta function.

$$\int_{a_1}^{a_2}\delta[f(x)]dx=\frac{1}{|f'(a)|}\Rightarrow\delta[f(x)]=\frac{\delta(x-a)}{|f'(a)|}\ \left[=\frac{\delta(x-a)}{|f'(x)|}\right]\quad(5a)$$

If $f(x)$ has multiple roots x_i $[f(x_i)=0,\ i=1,2,\cdots]$, then

$$\delta[f(x)]=\sum_i\frac{1}{|f'(x_i)|}\delta(x-x_i)\ \left[=\sum_i\frac{1}{|f'(x)|}\delta(x-x_i)\right]\quad(5b)$$

Exercise: Use (5a) to show $\delta(cx)=\frac{\delta(x)}{|c|}$ & $\delta(a-x)=\delta(x-a)$ (5c)

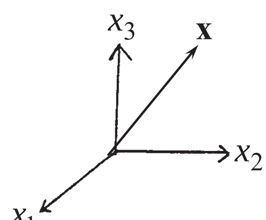
Extension to 3 dimensions:

1. Cartesian coordinates: $\mathbf{x}=(x_1,x_2,x_3)$

$$\delta(\mathbf{x}-\mathbf{x}')\equiv\delta(x_1-x_1')\delta(x_2-x_2')\delta(x_3-x_3')\quad(6a)$$

$$\Rightarrow\int_V\delta(\mathbf{x}-\mathbf{x}')d^3x=\int\delta(x_1-x_1')dx_1\int\delta(x_2-x_2')dx_2\int\delta(x_3-x_3')dx_3$$

$$=\begin{cases}0, & \text{if } \mathbf{x}' \text{ lies outside } V\\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V\end{cases}$$



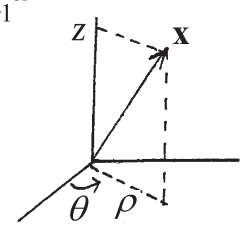
2. Cylindrical coordinates: $\mathbf{x}=(\rho,\theta,z)$

$$\delta(\mathbf{x}-\mathbf{x}')\equiv\frac{1}{\rho}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z')\quad(6b)$$

$$\Rightarrow\int_V\delta(\mathbf{x}-\mathbf{x}')d^3x=\int_V\delta(\mathbf{x}-\mathbf{x}')\rho d\rho d\theta dz$$

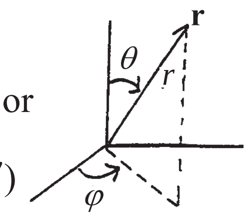
$$=\int\delta(\rho-\rho')d\rho\int\delta(\theta-\theta')d\theta\int\delta(z-z')dz$$

$$=\begin{cases}0, & \text{if } \mathbf{x}' \text{ lies outside } V\\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V\end{cases}$$



Question: If x and $\mathbf{x}$ both have the unit of cm, what are the units of $\delta(x)$ and $\delta(\mathbf{x})$? [*Ans.* $\delta(x)$: 1/cm; $\delta(\mathbf{x})$: 1/cm^3]

3. Spherical coordinates: $\mathbf{r} = (r,\ \theta,\ \varphi)$



$$\delta(\mathbf{r}-\mathbf{r}') \equiv \begin{cases} \dfrac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi'), \text{ or} \\ \dfrac{1}{r^2}\delta(r-r')\underbrace{\delta(\cos\theta-\cos\theta')}\delta(\varphi-\varphi') \end{cases} \qquad (6c)$$

By (5a), $\delta(\cos\theta-\cos\theta') = \dfrac{1}{|\sin\theta|}\delta(\theta-\theta') = \dfrac{1}{\sin\theta}\delta(\theta-\theta'),\ 0 \le \theta \le \pi$

$$\int_V \delta(\mathbf{r}-\mathbf{r}')d^3x = \int_V \frac{\delta(r-r')}{r^2}\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi')\underbrace{r^2 dr\, d(\cos\theta)d\varphi}_{\substack{d^3x \\ \text{[see (7b) below]}}}$$

$$= \begin{cases} 0, \text{ if } \mathbf{r}' \text{ lies outside } V \\ 1, \text{ if } \mathbf{r}' \text{ lies inside } V \end{cases}$$

Note: There are 2 ways to write d^3x in spherical coordinates.

$$\int_0^\infty dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi = \int_0^\infty r^2 dr \underbrace{\int_0^\pi \sin\theta d\theta}_{\boxed{-\int_1^{-1} d(\cos\theta)}} \int_0^{2\pi} d\varphi$$

$$= \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi \qquad (7a)$$

$$\Rightarrow d^3x = \begin{cases} r^2 \sin\theta dr d\theta d\varphi, \text{ or} \\ r^2 dr\, d(\cos\theta) d\varphi \end{cases} \qquad (7b)$$

$d(\cos\theta)$ is to be integrated from a smaller $\cos\theta$ to a larger $\cos\theta$.

Approximate representations of the delta function:

The delta function, $\delta(x)$, can be represented analytically by the following functions because they satisfy the definition of the delta function in the limit $\gamma \to 0$ $(\gamma > 0)$.

$$\delta(x) = \lim_{\gamma\to 0} \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$$

$$\delta(x) = \lim_{\gamma\to 0} \frac{1}{\sqrt{2\pi}\gamma} \mathrm{e}^{-\frac{x^2}{2\gamma^2}}$$

$$\delta(x) = \lim_{\gamma\to 0} \begin{cases} \dfrac{1}{\gamma}, & \text{for } -\dfrac{\gamma}{2} < x < \dfrac{\gamma}{2} \\ 0, & \text{otherwise} \end{cases}$$

Delta Functions (*continued*)

Representation of volume charge density $\rho(\mathbf{x})$ *by delta functions*:

$$\rho(\mathbf{x})\begin{cases}\text{of a point charge } q \text{ at the origin} \\ \text{of a uniform line charge } \lambda \text{ on the } z\text{-axis (from } -\infty \text{ to } \infty) \\ \text{of a uniform surface charge } \sigma \text{ on the entire } x\text{-}y \text{ plane}\end{cases}$$

$$=\begin{cases} q\delta(x)\delta(y)\delta(z), \text{ triple infinity} \\ \lambda\delta(x)\delta(y), \text{ double infinity} \\ \sigma\delta(z), \text{ single infinity}\end{cases}$$

$\Rightarrow$ As one approaches the charge,

$$E\begin{cases}\text{due to } q \text{ at the origin is most divergent } (\propto \frac{1}{r^2}) \\ \text{due to } \lambda \text{ on the } z\text{-axis is less divergent } (\propto \frac{1}{r}) \\ \text{due to } \sigma \text{ on the } x\text{-}y \text{ plane is a finite value } (\frac{\sigma}{2\varepsilon_0})\end{cases}$$

$\Rightarrow$ Although ρ of point, line, & surface charges are all infinite (the the charge is in 0 volume), their "degree of infinity" is different. The conceptual difficulty of infinite ρ can be resolved by giving the point, line, or surface a finite radius a or thickness t and let a or $t \to 0$.

Delta Functions (*continued*)

Problem 1: A total charge Q is uniformly distributed around a circular ring of radius a and infinitesimal thickness. Write the voulme charge density $\rho(\mathbf{x})$ in cylindrical coordinates.

Solution:

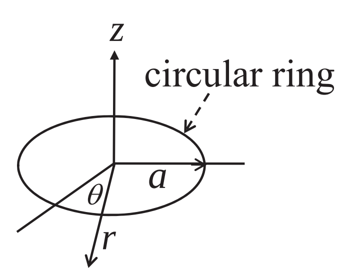
Let $\rho(\mathbf{x}) = K\delta(r-a)\delta(z)$ and find K as follows.

$$\int \rho(\mathbf{x})d^3x = K \int \delta(r-a)\delta(z)rdrd\theta dz$$

$$= 2\pi K a = Q$$

$$\Rightarrow K = \frac{Q}{2\pi a}$$

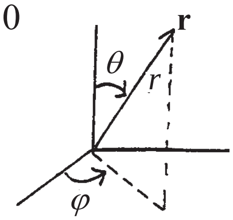
$$\Rightarrow \rho(\mathbf{x}) = \frac{Q}{2\pi a}\delta(r-a)\delta(z)$$



Note: ρ has the dimension of "charge/volume" as expected.

Problem 2: Prove $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$ (8)

Solution: Definition of $\delta(\mathbf{r}):\begin{cases}\delta(\mathbf{r}) = 0, \text{ if } r \neq 0 \\ \int \delta(\mathbf{r}) d^3x = 1\end{cases}$



Hence, we need to prove

(i) $\nabla^2 \frac{1}{r} = 0, \text{ if } r \neq 0$

(ii) $\int \nabla^2 \frac{1}{r} d^3x = -4\pi \int \delta(\mathbf{r}) d^3x = -4\pi$

It is convenient to use the spherical coordinates. To prove (i), we we write ∇^2 as (see back cover of Jackson)

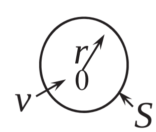
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d}{dr}\frac{1}{r}) = -\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{r^2}) = 0 \quad \text{if} \quad r \neq 0$$

Note: $\frac{r^2}{r^2}$ is undetermined at $r = 0$. However, here we are only concerned with the region $r > 0$.

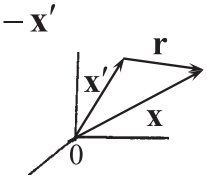
To prove (ii), we integrate $\nabla^2 \frac{1}{r}$ over a spherical volume v

$$\int_v \nabla^2 \frac{1}{r} d^3x = \int_v \nabla \cdot \nabla \frac{1}{r} d^3x \underset{\boxed{\text{divergence thm.}}}{=} \oint_S \mathbf{e}_r \cdot \underbrace{\nabla \frac{1}{r}}_{-\frac{1}{r^2}\mathbf{e}_r} \underbrace{da}_{\boxed{r^2 d\Omega}} = -\oint_S \frac{r^2}{r^2} d\Omega = -4\pi \quad (9)$$



Note: $r > 0$ on $S \Rightarrow$ No need to evaluate r^2 / r^2 at $r = 0$.

Change to a coordinate system in which $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $r = |\mathbf{x} - \mathbf{x}'|$. We obatin from $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$



$$\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad \begin{bmatrix}\text{A very useful} \\ \text{relation}\end{bmatrix} \quad (1.31)$$

Note: Jackson outlined the same method to prove (1.31) on p. 35.

Problem 4: Derive $\nabla^2\phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\varepsilon_0}$ from $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

Solution: $\nabla^2\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')[-4\pi\delta(\mathbf{x} - \mathbf{x}')] d^3x' = -\frac{\rho(\mathbf{x})}{\varepsilon_0} \quad \text{[same as (1.28)]}$$

Note: Jackson did this problem with a different method (p. 35).

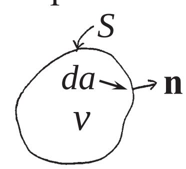
1.8 Green's Theorem

Green's theorem - a powerful theorem for treating electrostatic boundary value problems. It is derived from the divergence thm.:

$$\int_v \nabla\cdot\mathbf{A}\, d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}\, da \quad \text{[divergence thm., } \mathbf{n}\text{: outward normal]}$$

Let $\mathbf{A} = \phi\nabla\psi$, where ϕ and ψ are arbitrary functions of position.

$$\Rightarrow \begin{cases} \nabla\cdot\mathbf{A} = \nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi \\ \mathbf{A}\cdot\mathbf{n} = \phi\nabla\psi\cdot\mathbf{n} = \phi\frac{\partial\psi}{\partial n} \end{cases}$$



$\mathbf{n}$: unit normal pointing out of v

Sub. the above $\nabla\cdot\mathbf{A}$ and $\mathbf{A}\cdot\mathbf{n}$ into the divergence thm., we obtain Green's 1st identity:

$$\int_v (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}da \qquad (1.34)$$

Interchange ϕ and ψ in (1.34).

$\frac{\partial}{\partial n}$ is a derivative along (outward) $\mathbf{n}$

$$\Rightarrow \int_v (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)d^3x = \oint_S \psi\frac{\partial\phi}{\partial n}da$$

Substracting these 2 eqs., we obtain Green's 2nd identity:

$$\int_v (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})da \qquad (1.35)$$

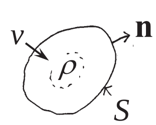
1.8 Green's Theorem *(continued)*

Green's thm. relates a volume integral to a surface integral and it contains ∇^2 operating on arbitrary functions ϕ, ψ. These features are useful for the manipulation of Poisson eq. in bounded space. For, example, we may convert the Poisson eq. into an integral relation by

Green's 2nd identity: $\int_v (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})da$ [(1.35)]

In (1.35), let $\psi = \frac{1}{R}(=\frac{1}{|\mathbf{x}-\mathbf{x}'|})$, $\phi =$ the electrostatic potential (thus, $\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$), and change the integration variable from $\mathbf{x}$ to $\mathbf{x}'$

$$\Rightarrow \int_v [-4\pi\phi\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{\varepsilon_0 R}\rho(\mathbf{x}')]d^3x' = \oint_S [\phi\frac{\partial}{\partial n'}(\tfrac{1}{R}) - \tfrac{1}{R}\frac{\partial\phi}{\partial n'}]da$$



$$\Rightarrow \phi(\mathbf{x}) \overset{\boxed{\mathbf{x}\text{ inside }v}}{=} \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{4\pi}\oint_S [\tfrac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}(\tfrac{1}{R})]da' \qquad (1.36)$$

Note: (1.36) is an integral relation (not a solu.) for ϕ. However, in infinite space, $\phi(R\to\infty) \propto \frac{1}{R}$. $\Rightarrow$ The the surface integral (2nd term) vanishes and (1.36) reduces to Coulomb's law: $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$

This again indicates that we must include all ρ to apply Coulomb's law.

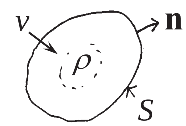
1.9 Uniqueness of Solution with Dirichlet or Neumann Boundary Conditions

$$\begin{cases} \underline{\text{Dirichlet boundary condition}}\text{: } \phi_s \text{ specified} \\ \underline{\text{Neumann boundary condition}}\text{: } \frac{\partial}{\partial n}\phi_s \text{ specified} \\ \underline{\text{Cauchy boundary condition}}\text{: } \phi_s \text{ and } \frac{\partial}{\partial n}\phi_s \text{ both specified} \end{cases}$$

As another application of Green's theorem, we use it to prove the <u>uniqueness theorem</u> for the solution of the Poisson equation.

Let there be two solutions, ϕ_1 and ϕ_2, both satisfying the D.E.

$$\nabla^2\phi = -\rho/\varepsilon_0 \text{ with } \begin{cases} \text{either } \phi = \phi_n \text{ on } S \text{ (Dirichlet b.c.)} \\ \text{or } \frac{\partial}{\partial n}\phi = \phi_n' \text{ on } S \text{ (Neumann b.c.)} \end{cases}$$



$$\text{i.e. } \begin{cases} \nabla^2\phi_1 = -\rho/\varepsilon_0 \\ \nabla^2\phi_2 = -\rho/\varepsilon_0 \end{cases} \text{ with } \begin{cases} \text{either } \phi_{1,2} = \phi_n \text{ on } S \\ \text{or } \frac{\partial}{\partial n}\phi_{1,2} = \phi_n' \text{ on } S \end{cases}$$

$$\text{Define } U \equiv \phi_1 - \phi_2 \Rightarrow \nabla^2 U = 0 \text{ with } \begin{cases} \text{either } U = \phi_n - \phi_n = 0 \text{ on } S \\ \text{or } \frac{\partial}{\partial n}U = \phi_n' - \phi_n' = 0 \text{ on } S \end{cases}$$

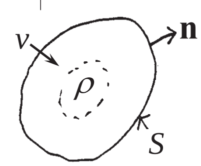
Rewrite Green's 1st identity: $\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}da$

Let $\phi = \psi = U (\equiv \phi_1 - \phi_2)$

b.c.: either $U = 0$ or $\partial U/\partial n = 0$ on S

$$\Rightarrow \int_V (U\underbrace{\nabla^2 U}_{0} + \nabla U\cdot\nabla U)d^3x = \oint_S U\frac{\partial U}{\partial n}da = 0 \Rightarrow \int_V |\nabla U|^2 d^3x = 0$$

$\Rightarrow \nabla U = 0$ everywhere within V



$\Rightarrow U = \phi_1 - \phi_2$

$$= \begin{cases} 0, & \text{if } U = 0 \text{ on } S \quad \Rightarrow \text{Both } \phi \text{ and } \mathbf{E} \text{ are uuique.} \\ const, & \text{if } \frac{\partial U}{\partial n} = 0 \text{ on } S \quad \Rightarrow \mathbf{E} \text{ is uuique (} \phi \text{ differ by a constant)} \end{cases}$$

Question: What is a "solution"? *Ans.*: It satisfies both D.E. and b.c.

An electrostatic problem is solved by specifying either ϕ or $\partial\phi/\partial n$ on a closed surface, e.g. if ϕ on S determines ϕ everywhere (including $\partial\phi/\partial n$ on S), we no longer have the freedom to specify $\partial\phi/\partial n$ on S.

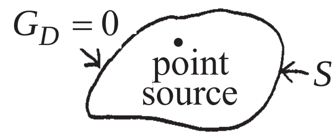
Problem: Prove that there cannot be any static $\mathbf{E}$ inside a closed, hollow conductor if there is no charge in the hollow region.

1.10 Formal Solution of Electrostatic Boundary-Value Problem with Green Function

The <u>Green function</u> is the solution of a boundary-value problem with a *point* source. In Jackson (Sec. 1.10), 3 notations appear:

$$\begin{cases} G(\mathbf{x},\mathbf{x}')\text{: Green function under either Dirichlet or Neumann b.c.} \\ G_D(\mathbf{x},\mathbf{x}')\text{: Green function under Dirichlet b.c.} \\ G_N(\mathbf{x},\mathbf{x}')\text{: Green function under Neumann b.c.} \end{cases}$$

These all refer to Green functions in electrostatics. $G_N(\mathbf{x},\mathbf{x}')$ is rarely used. We will consider only $G_D(\mathbf{x},\mathbf{x}')$.



Green Function under Dirichlet B.C.

$G_D(\mathbf{x},\mathbf{x}')$ is the solution of (see figure)

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ for } \mathbf{x} \text{ on } S, \quad (10)$$

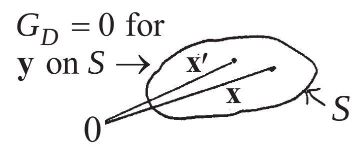
where $\mathbf{x}$ is treated as a variable and $\mathbf{x}'$ (the *arbitrary* position of a *point source* within S) is treated as a const. (10) is a pure mathematical eq.

Example: The solution of (10) in ∞ space is $G_D(\mathbf{x},\mathbf{x}') = 1/|\mathbf{x}-\mathbf{x}'|$

If we let $4\pi \to q/\varepsilon_0$ in (10), $G_D(\mathbf{x},\mathbf{x}')$ is the potential at $\mathbf{x}$ due to a point charge q at $\mathbf{x}'$ subject to the b.c. that the potential vanishes on S.

Exercise: Prove $G_D(\mathbf{x}',\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}')$

Consider 2 eqs. in $\mathbf{y}$-space: one with a point source at $\mathbf{x}$, the other with a point source at $\mathbf{x}'$.



$$\begin{cases} \nabla_y^2 G_D(\mathbf{y},\mathbf{x}) = -4\pi\delta(\mathbf{y}-\mathbf{x}), & \text{b.c. } G_D(\mathbf{y},\mathbf{x}) = 0 \text{ for } \mathbf{y} \text{ on } S \\ \nabla_y^2 G_D(\mathbf{y},\mathbf{x}') = -4\pi\delta(\mathbf{y}-\mathbf{x}'), & \text{b.c. } G_D(\mathbf{y},\mathbf{x}') = 0 \text{ for } \mathbf{y} \text{ on } S \end{cases}$$

Rewrite: $\int_V (\phi\nabla_y^2\psi - \psi\nabla_y^2\phi)d^3y = \oint_S [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}]da \quad (1.35)$

Let $\phi = G_D(\mathbf{y},\mathbf{x})$ and $\psi = G_D(\mathbf{y},\mathbf{x}')$, where $\mathbf{y}$ is the variable.

$$\Rightarrow \int_V [G_D(\mathbf{y},\mathbf{x})\overbrace{\nabla_y^2 G_D(\mathbf{y},\mathbf{x}')}^{-4\pi\delta(\mathbf{y}-\mathbf{x}')} - G_D(\mathbf{y},\mathbf{x}')\overbrace{\nabla_y^2 G_D(\mathbf{y},\mathbf{x})}^{-4\pi\delta(\mathbf{y}-\mathbf{x})}]d^3y = \overbrace{0}^{\substack{\text{RHS of (1.35)} = 0 \\ \because\, \phi=\psi=0 \text{ on } S}}$$

$$\Rightarrow G_D(\mathbf{x}',\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}') \text{ [symmetry property, e.g. } \underset{\uparrow \atop \text{solu. of (10) in } \infty \text{ space}}{G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}}]$$

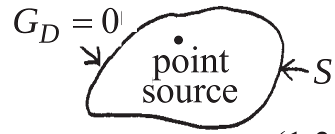
By $G_D(\mathbf{x}',\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}')$, the b.c. "$G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}$ on S" means "$G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}'$ on S". So we can simply say "$G_D(\mathbf{x},\mathbf{x}') = 0$ on S".

Problem: Give 2 physical examples to show $G_D(\mathbf{x}',\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}')$.

Formal Solution of Electrostatic Boundary-Value Problem :

Rewrite $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'),\ G_D(\mathbf{x},\mathbf{x}') = 0$ on S (10)

and Green's 2nd identity:



$$\int_V [\phi(\mathbf{x}')\nabla'^2\psi(\mathbf{x}') - \psi(\mathbf{x}')\nabla'^2\phi(\mathbf{x}')]d^3x'$$
$$= \oint_S [\phi(\mathbf{x}')\frac{\partial}{\partial n'}\psi(\mathbf{x}') - \psi(\mathbf{x}')\frac{\partial}{\partial n'}\phi(\mathbf{x}')]da' \quad (1.35)$$

Consider a general electrostatic problem with Dirichlet b.c.:

$$\nabla^2\phi(\mathbf{x}) = -\rho(\mathbf{x})/\varepsilon_0 \ \text{ with } \phi(\mathbf{x}) = \phi_s(\mathbf{x}) \text{ for } \mathbf{x} \text{ on } S \quad (11)$$

In (1.35), let $\phi(\mathbf{x}')$ be the solution of (11) and let $\psi(\mathbf{x}')$ [$= \mathrm{G_D}(\mathbf{x},\mathbf{x}')$] be the solution of (10) in the same region enclosed by S. Then,

$$\int_V [\phi(\mathbf{x}')\overbrace{\nabla'^2 G_D(\mathbf{x},\mathbf{x}')}^{-4\pi\delta(\mathbf{x}-\mathbf{x}')} - G_D(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\phi(\mathbf{x}')}^{-\rho(\mathbf{x}')/\varepsilon_0}]d^3x'$$
$$= \oint_S [\phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}') - \underbrace{G_D(\mathbf{x},\mathbf{x}')}_{=\,0 \text{ on } S}\frac{\partial}{\partial n'}\phi(\mathbf{x}')]da'$$
$$\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}da' \quad (1.44)$$

1.10 Formal Solution of Electrostatic Boundary-Value Problem…(*continued*)

Rewrite

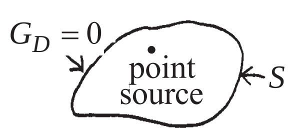
$$\begin{cases} \nabla^2\phi(\mathbf{x}) = -\rho(\mathbf{x})/\varepsilon_0 \ \text{ with } \phi(\mathbf{x}) = \phi_s(\mathbf{x}) \text{ on } S & (11) \\ \nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'), \text{ with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ on } S & (10) \\ \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}da' & (1.44) \end{cases}$$

Note: 1. $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ is applicable only to infinite space.

2. (1.44) is a solution applicable to any region with Dirichlet b.c. ρ in (1.44) does not include charges on or outside surface S.

3. (1.44) is derived without using (1.36) and (1.40)-(1.42).

Steps to evaluate (1.44):



1. Solve (10) for $G_D(\mathbf{x},\mathbf{x}')$.

2. Substitute $\rho(\mathbf{x}'),\ G_D(\mathbf{x},\mathbf{x}'),\ \phi_s(\mathbf{x})$, and $\partial G_D(\mathbf{x},\mathbf{x}')/\partial n'$ into (1.44).

It is often much simpler to solve $G_D(\mathbf{x},\mathbf{x}')$ from (10) than solving ϕ directly from (11), since (10) has the simple b.c. of $G_D(\mathbf{x},\mathbf{x}') = 0$ on S.

1.11 Electrostatic Potential Energy and Energy Density; Capacitance

Electric Field Energy: Let ϕ be due to ρ. The work done to add $\delta\rho$ is

$$\delta W = \int_V \delta\rho(\mathbf{x})\phi(\mathbf{x})d^3x$$

$\boxed{\text{Change } \delta\rho \text{ to } \delta\mathbf{E} \text{ by } \delta\rho = \varepsilon_0\nabla\cdot\delta\mathbf{E}}$

$$= \varepsilon_0\int_V \phi(\mathbf{x})\nabla\cdot\delta\mathbf{E}(\mathbf{x})d^3x$$

Using $\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$ we obtain

$$\phi\nabla\cdot\delta\mathbf{E} = \nabla\cdot(\phi\delta\mathbf{E}) - \delta\mathbf{E}\cdot\nabla\phi = \nabla\cdot(\phi\delta\mathbf{E}) + \mathbf{E}\cdot\delta\mathbf{E}$$

$$= \varepsilon_0\underbrace{\int_V \nabla\cdot(\phi\delta\mathbf{E})d^3x}_{} + \varepsilon_0\int_V \mathbf{E}\cdot\delta\mathbf{E}d^3x = \varepsilon_0\int_V \mathbf{E}\cdot\delta\mathbf{E}d^3x$$

$$= \oint_S \underbrace{\phi}_{\frac{1}{r}}\,\underbrace{\delta\mathbf{E}}_{\frac{1}{r^2}}\cdot\underbrace{d\mathbf{a}}_{r^2} = 0 \quad \uparrow \; r\to\infty$$

Note: For this integral to vanish, the integration must be over all space.

By conservation of energy, we postulate W to be the total $\mathbf{E}$-field energy.

$$\Rightarrow W = \varepsilon_0\int_V d^3x\int_0^{E(\mathbf{x})}\overbrace{\mathbf{E}(\mathbf{x})\cdot d\mathbf{E}(\mathbf{x})}^{\frac{1}{2}dE^2(\mathbf{x})} = \frac{\varepsilon_0}{2}\int_V \overbrace{|\mathbf{E}(\mathbf{x})|^2}^{E^2(\mathbf{x})} d^3x \quad [v\to\infty] \qquad (1.54)$$

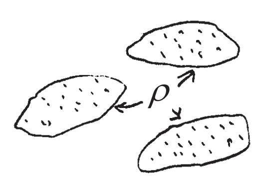
$$|\mathbf{E}|^2 = \mathbf{E}\cdot\mathbf{E} = -\mathbf{E}\cdot\nabla\phi = -\nabla\cdot(\phi\mathbf{E}) + \phi\nabla\cdot\mathbf{E} = -\nabla\cdot(\phi\mathbf{E}) + \frac{\rho\phi}{\varepsilon_0}$$

$$\Rightarrow W = \frac{1}{2}\int_V \rho(\mathbf{x})\phi(\mathbf{x})d^3x - \frac{\varepsilon_0}{2}\overbrace{\oint_S \phi\mathbf{E}\cdot d\mathbf{a}}^{\to 0 \text{ as } r\to\infty} = \frac{1}{2}\int_V \rho(\mathbf{x})\phi(\mathbf{x})d^3x \qquad (1.53)$$

Exercise: An alternative derivation of (1.53) and (1.54):

Consider a state in which a charge density $\rho(\mathbf{x})$ has produced an electrostatic potential $\phi(\mathbf{x})$, i.e.

$\rho(\mathbf{x}) \to \phi(\mathbf{x})$.



Then, by the principle of superposition, $\varepsilon\rho(\mathbf{x}) \to \varepsilon\phi(\mathbf{x})$, where $0 \le \varepsilon \le 1$.

To find the electric field energy, we consider the energy needed to build up $\phi(\mathbf{x})$ from $\varepsilon = 0$ (no charge and no potential) to $\varepsilon = 1$ (the present state). At any stage in the build-up process, the relative charge density (hence the relative potential) remains the same; namely, the intermediate state is characterized by the charge density $\varepsilon\rho(\mathbf{x})$ and potential $\varepsilon\phi(\mathbf{x})$.

In such a build-up process, when the potential is $\varepsilon\phi(\mathbf{x})$, the work done by adding an incremental charge $\rho(\mathbf{x})d\varepsilon$ is

$$dW = \int_V d^3x\varepsilon\phi(\mathbf{x})\rho(\mathbf{x})d\varepsilon$$

Hence, the total work done from $\varepsilon = 0$ to $\varepsilon = 1$ is

$$W = \int_{\varepsilon=0}^{\varepsilon=1} dW = \int_V d^3x \rho(\mathbf{x})\phi(\mathbf{x}) \int_0^1 \varepsilon d\varepsilon = \frac{1}{2}\int_V d^3x \overbrace{\rho(\mathbf{x})}^{\boxed{-\varepsilon_0 \nabla^2 \phi}} \phi(\mathbf{x}) \quad (1.53)$$

$$= -\frac{\varepsilon_0}{2}\int_V \phi \nabla^2 \phi d^3x \overset{\boxed{\text{Green's 1st identity}}}{=} \frac{\varepsilon_0}{2}\Big[\int_V \nabla\phi \cdot \nabla\phi d^3x - \overbrace{\oint_S \phi\Big(\frac{\partial\phi}{\partial n}\Big)da}^{\boxed{\text{This integral} \to 0 \text{ only if } r \to \infty}}\Big]$$

$$= \frac{\varepsilon_0}{2}\int_V |\mathbf{E}|^2 d^3x \quad [v = \infty] \qquad \boxed{\sim \frac{1}{r}} \boxed{\sim \frac{1}{r^2}} \boxed{-r^2} \quad (1.54)$$

We postulate $w_E = \frac{\varepsilon_0}{2}|\mathbf{E}|^2$ $\begin{bmatrix}\text{E-field energy density} \\ \text{"intuitively reasonable" (p.41)}\end{bmatrix}$ (1.55)

Questions: 1. If we bring q and $-q$ toward each other, the work done is negative. Why is then $W = \frac{\varepsilon_0}{2}\int_V |\mathbf{E}|^2 d^3x$ always positive?

2. Give one example to show that the **E**-field carries energy.

3. If we do work to bring q from ϕ_1 to ϕ_2, where does the work end in?

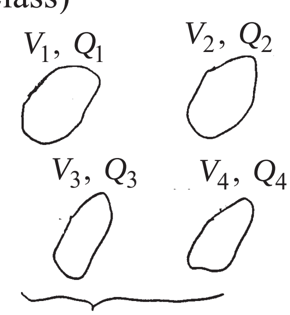
4. Can the field energy density of multiple charges be separately calculated, then linearly summed?

Answer: No, because $w_E = \frac{\varepsilon_0}{2}|\mathbf{E}|^2 = \frac{\varepsilon_0}{2}(\Sigma \mathbf{E}_j)\cdot(\Sigma \mathbf{E}_j) \neq \frac{\varepsilon_0}{2}\Sigma |\mathbf{E}_j|^2$

(This page will not be covered in class)

Capacitance: Refer to the figure

$$\left\{\begin{aligned} V_1 &= \sum_{j=1}^{n} P_{1j}Q_j \\ V_2 &= \sum_{j=1}^{n} P_{2j}Q_j \\ &\vdots \\ V_n &= \sum_{j=1}^{n} P_{nj}Q_j \end{aligned}\right. \quad \underset{\boxed{\text{Invert the equations}}}{\Rightarrow} \quad \left\{\begin{aligned} Q_1 &= \sum_{j=1}^{n} C_{1j}V_j \\ Q_2 &= \sum_{j=1}^{n} C_{2j}V_j \\ &\vdots \\ Q_n &= \sum_{j=1}^{n} C_{nj}V_j \end{aligned}\right.$$



A system of n conductors

$V_i \propto Q_j$, principle of linear superposition

C_{ii}: capacitance

C_{ij} $(i \neq j)$: coefficient of induction

P_{ij} and C_{ij} depend on the geometrical shape and position of the conductors. Potential energy of the i-th conductor is [using (1.53)]

$$W_i = \frac{1}{2}\int \rho_i(\mathbf{x})\phi_i(\mathbf{x})d^3x = \frac{1}{2}Q_iV_i \quad \left[\phi_i(\mathbf{x}) = V_i;\ \int \rho_i(\mathbf{x})d^3x = Q_i\right]$$

$$\Rightarrow \begin{bmatrix}\text{potential energy} \\ \text{of the system}\end{bmatrix} = \frac{1}{2}\sum_{i=1}^{n} Q_iV_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}V_iV_j \quad (1.62)$$

Appendix A: Unit Systems and Dimensions

Unit Systems:

Two systems of electromagnetic units are in common use today: the SI and Gaussian systems. Regardless of one's personal preference, it is important to be familiar with both systems and, in particular, the conversion from one system to the other. Conversion formulae can be divided into two categories: "symbol/equation conversion [such as E and $E = q/(4\pi\varepsilon_0 r^2)$]" and "unit conversion (such as coulomb)".

Conversion formulae for symbols and equations are listed in Table 3 on p. 782 of Jackson and conversion formulae for units in Table 4 on p. 783 (both tables attached on next page). These two tables are all we need to convert between SI and Gaussian systems. Correct use of the tables requires practices.

Appendix A: Unit Systems and Dimensions (*continued*)

Table 3 Conversion Table for Symbols and Formulas

The symbols for mass, length, time, force, and other not specifically electromagnetic quantities are unchanged. To convert any equation in SI variables to the corresponding equation in Gaussian quantities, on both sides of the equation replace the relevant symbols listed below under "SI" by the corresponding "Gaussian" symbols listed on the left. The reverse transformation is also allowed. Residual powers of $\mu_0\epsilon_0$ should be eliminated in favor of the speed of light ($c^2\mu_0\epsilon_0 = 1$). Since the length and time symbols are unchanged, quantities that differ dimensionally from one another only by powers of length and/or time are grouped together where possible.

Quantity	Gaussian	SI
Velocity of light	c	$(\mu_0\epsilon_0)^{-1/2}$
Electric field (potential, voltage)	$\mathbf{E}(\Phi, V)/\sqrt{4\pi\epsilon_0}$	$\mathbf{E}(\Phi, V)$
Displacement	$\sqrt{\epsilon_0/4\pi}\,\mathbf{D}$	$\mathbf{D}$
Charge density (charge, current density, current, polarization)	$\sqrt{4\pi\epsilon_0}\,\rho(q, \mathbf{J}, I, \mathbf{P})$	$\rho(q, \mathbf{J}, I, \mathbf{P})$
Magnetic induction	$\sqrt{\mu_0/4\pi}\,\mathbf{B}$	$\mathbf{B}$
Magnetic field	$\mathbf{H}/\sqrt{4\pi\mu_0}$	$\mathbf{H}$
Magnetization	$\sqrt{4\pi/\mu_0}\,\mathbf{M}$	$\mathbf{M}$
Conductivity	$4\pi\epsilon_0\sigma$	σ
Dielectric constant	$\epsilon_0\epsilon$	ϵ
Magnetic permeability	$\mu_0\mu$	μ
Resistance (impedance)	$R(Z)/4\pi\epsilon_0$	$R(Z)$
Inductance	$L/4\pi\epsilon_0$	L
Capacitance	$4\pi\epsilon_0 C$	C

$$c = 2.997\,924\,58 \times 10^8 \text{ m/s}$$
$$\epsilon_0 = 8.854\,187\,8 \ldots \times 10^{-12} \text{ F/m}$$
$$\mu_0 = 1.256\,637\,0 \ldots \times 10^{-6} \text{ H/m}$$
$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.730\,3 \ldots\ \Omega$$

Jackson, p. 782, Table 3

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16} c^2 = 8.987\,55 \ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

Physical Quantity	Symbol	SI		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work Energy	W U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

Jackson, p. 783, Table 4

Conversion of symbols and equations:

Consider, for example, the conversion of the SI equation

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{(A.1)}$$

into the Gaussian system.

This involves the conversion of symbols and equations. So we use Table 3. First, we note from Table 3 (top) that mechanical symbols (e.g. time, length, mass, force, energy, and frequency) are unchanged in the conversion. Thus, we only need to deal with electromagnetic symbols on *both* sides of (A.1).

From Table 3, we find $E^{SI} \to \frac{E^G}{\sqrt{4\pi\varepsilon_0}}$ and $q^{SI} \to \sqrt{4\pi\varepsilon_0}\, q^G$ (A.2)

Sub. $E^G/\sqrt{4\pi\varepsilon_0}$ and $\sqrt{4\pi\varepsilon_0}\, q^G$, respectively, for E and q in (A.1), we obtain the corresponding equation in the Gaussian system:

$$\frac{E^G}{\sqrt{4\pi\varepsilon_0}} = \frac{\sqrt{4\pi\varepsilon_0}\, q^G}{4\pi\varepsilon_0 r^2} \Rightarrow E^G = \frac{q^G}{r^2} \qquad \text{(A.3)}$$

Conversion of units and evaluation of physical quantities:

Consider again the SI equation : $E = \frac{q}{4\pi\varepsilon_0 r^2}$ (A.1)

Given $r = 0.01$ m, $q = 1$ statcoulomb, we may evaluate E in 3 steps:

Step 1: Express r, q, and ε_0 in SI units. From Table 3 (bottom) and Table 4, we find

$$\begin{cases} \varepsilon_0 = 8.854\times10^{-12} \text{ Farad/m} = \frac{1}{36\pi\times10^9} \text{ Farad/m} \\ r = 0.01 \text{ m (same as given)} \\ q(=1 \text{ statcoulomb}) = \frac{1}{3\times10^9} \text{ coulomb} \end{cases} \qquad \text{(A.4)}$$

Step 2: Sub. the numbers (*but not the units*) from (A.4) into (A.1).

This gives $E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{\frac{1}{3\times10^9}}{4\pi\times\frac{1}{36\pi\times10^9}\times(0.01)^2} = 3\times10^4$

Step 3: Look up Table 4 for the SI unit of E. As shown in Table 4, the SI unit of E is V/m. Thus, $E = 3\times10^4$ V/m (A.5)

As another exercise, we write (A.1) in the Gaussian system :

$$E = \frac{q}{r^2} \tag{A.3}$$

and evaluate E for the same r ($=0.01$ m) and q ($=1$ statcoulomb).

Step 1: Express r and q in Gaussian units. From Table 4, we find

$$\begin{cases} r(=0.01 \text{ m}) = 1 \text{ cm} \\ q = 1 \text{ statcoulomb (same as given)} \end{cases} \tag{A.6}$$

Step 2: Sub. the numbers (*but not the units*) from (A.6) into (A.3). This gives $E = \frac{q}{r^2} = \frac{1}{1} = 1$

Step 3: Look up Table 4 for the Gaussian unit of E. We find the unit to be statvolt/cm. Thus, $E = 1$ statvolt/cm (A.7)

Table 4 shows 1 statvolt/cm $= 3\times10^4$ V/m. Hence, the 2 results in (A.5) and (A.7): $\left\{\begin{array}{l} E = 3\times10^4 \text{ V/m} \\ E = 1 \text{ statvolt/cm} \end{array}\right\}$ are identical as expected.

Units and Dimensions :

In the Gaussian system, the basic units are length (ℓ), mass (m), and time (t). In the SI system, they are the above plus the current (I). [See Table 1 (top) on p. 779 of Jackson.] All other units are derived units.

If a physical quantity is expressed in term of the basic units, we have the dimension of this quantity.

A mechanical quantity has the same dimension in both systems. For example, the acceleration a $(= d^2x/dt^2)$ has the dimension of ℓt^{-2}. From $f = ma$, we obtain the dimension of force : $m\ell t^{-2}$, which in turn gives the dimension of work ($f \cdot \ell$) or energy: $m\ell^2 t^{-2}$.

An electromagnetic quantity has different dimensions in different systems. For example, the charge q has the SI dimension of It. From the Gaussian equation $f = q_1 q_2 / r^2$ and the dimensions of force and length, we find the Gaussian dimension of q to be $m^{1/2}\ell^{3/2}t^{-1}$. Since $q\phi$ has the dimension of energy ($m\ell^2 t^{-2}$), the potential ϕ has the SI dimension of $m\ell^2 t^{-3} I^{-1}$ and the Gaussian dimension of $m^{1/2}\ell^{1/2}t^{-1}$.

All physical quantities in an equation must be expressed in the same unit system and all terms must have the same dimension. For example, by Stokes's theorem, we have

$$\oint_C \mathbf{E} \cdot d\ell = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\ da \tag{A.8}$$

where both terms have the dimension of $\ell \cdot$(the dimension of E).

In the definition of the delta function:

$$\int_{a_1}^{a_2} \delta(x-a)dx = 1, \tag{A.9}$$

the RHS is dimensionless. Thus, if x has the dimension of ℓ, $\delta(x-a)$ must have the dimension of ℓ^{-1}. However, "0" is *not* to be regarded as a dimensionless quantity. This is clear if we write (A.8) as

$$\oint_C \mathbf{E} \cdot d\ell - \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\ da = 0.$$

Well-known equations need not be checked for dimensional consistency. However, for newly derived equations, a dimensional check can be a convenient way to find mistakes.