CHAPTER 2: Boundary-Value Problems in Electrostatics: I

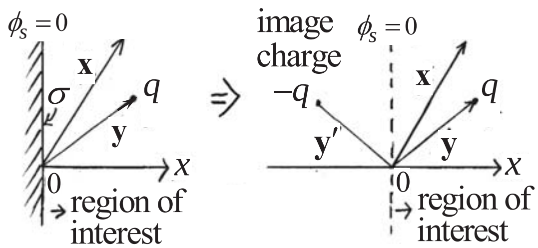
2.1 Method of Images

The method of images works only for a limited no of problems. Consider a point charge q in front of an infinite and grounded plane conductor (left figure).

The *region of interest* is $x \geq 0$, where $\phi(\mathbf{x})$ obeys

$$\nabla^2 \phi(\mathbf{x}) = -\frac{q}{\varepsilon_0} \delta(\mathbf{x} - \mathbf{y})$$

and b.c. $\phi_s = \phi(x = 0) = 0$.



(1)

To keep $\phi_s = 0$ on $x = 0$, σ will be induced (by q) on the conductor (left figure). We simulate the effects of σ with a *hypothetical* "image charge", $-q$, located symmetrically *inside* the conductor (right figure).

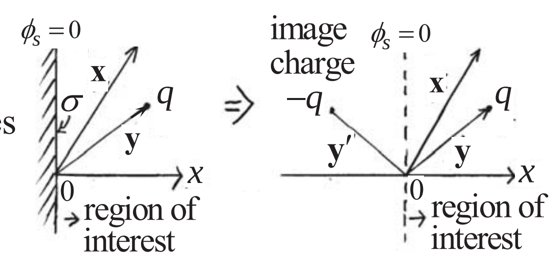
$$\Rightarrow \phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}'|}\right].$$

Question: How to determine this is a valid solution? See next page.

$$\phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}'|}\right]$$

1. By symmetry, $\phi(\mathbf{x})$ satisfies the b. c. $\phi_s = \phi(x = 0) = 0$.

2. Operate $\phi(\mathbf{x})$ with ∇^2



$$\Rightarrow \nabla^2 \phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}[\delta(\mathbf{x} - \mathbf{y}) - \delta(\mathbf{x} - \mathbf{y}')]$$

$\mathbf{y}'$ is outside the region of interest. Thus, in the region of interest ($x \geq 0$), we have $\delta(\mathbf{x} - \mathbf{y}') = 0$. $\Rightarrow \phi(\mathbf{x})$ obeys the original Poisson eq.

$$\nabla^2 \phi(\mathbf{x}) = -\frac{q}{\varepsilon_0} \delta(\mathbf{x} - \mathbf{y}) \qquad (1)$$

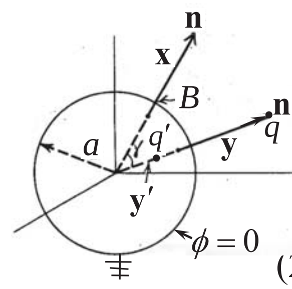
Since $\phi(\mathbf{x})$ satisfies both the D.E. & b.c. in the region of interest, it is a solution. By the uniqueness theorem, it is the only solution.

Note: 1. The image charge must be put outside the region of interest.

2. The solution $\phi(\mathbf{x})$ outside the region of interest is irrelevant.

3. Boundary charges are not required to solve a Poisson eq.

2.2 Point Charge in the Presence of a Grounded Conducting Sphere



Consider the grounded conducting sphere of radius a shown in the figure. A point charge q is at $r = y$ ($> a$). Find $\phi(\mathbf{x})$ in the region $r \geq a$.

Put an image charge q' at $r = y'$ ($< a$). Then,

$$\phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} + \frac{q'/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}'|} = \frac{q/4\pi\varepsilon_0}{|x\mathbf{n}-y\mathbf{n}'|} + \frac{q'/4\pi\varepsilon_0}{|x\mathbf{n}-y'\mathbf{n}'|} \quad (2.2)$$

Boundary condition requires

$$\phi(r=a) = \frac{q/4\pi\varepsilon_0}{a\left|\mathbf{n}-\frac{y}{a}\mathbf{n}'\right|} + \frac{q'/4\pi\varepsilon_0}{y'\left|\frac{a}{y'}\mathbf{n}-\mathbf{n}'\right|} = 0$$

$$\Rightarrow \phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}$$

First, set $\frac{y}{a} = \frac{a}{y'}$ (i.e. $y' = \frac{a^2}{y}$) so that $\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right| = \left|\frac{a}{y'}\mathbf{n} - \mathbf{n}'\right|$

[Note: $y' < a$; hence, q' lies outside the region of interest.]

Next, set $\frac{q}{a} = -\frac{q'}{y'}$ so that RHS = 0.

This gives $q' = -\frac{y'}{a} q = -\frac{a}{y} q$.

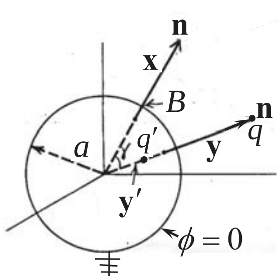
Note: If $y \to a$, then $y' \to a$, i.e. q' and q are so close that their attractive force can approach ∞.

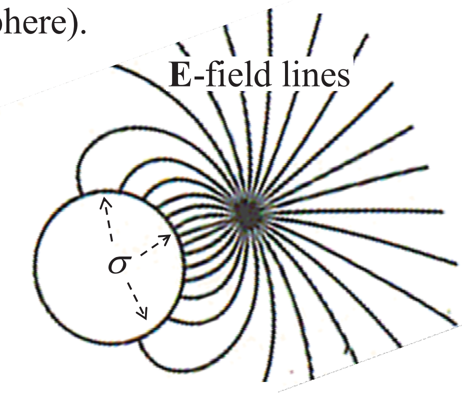
Rewrite $\phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{a}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}\right]$ [This is equivalent to (2.1) & (2.4).]

$$\Rightarrow \phi(\mathbf{x}) \text{ satisfies } \begin{cases} \nabla^2\phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y}) \text{ in the region of interest } (r \geq a) \\ \text{b.c. } \phi(r=a) = 0 \end{cases}$$

$\Rightarrow \phi(\mathbf{x})$ is the only solution.

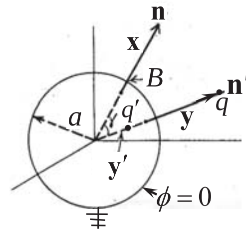
$\Rightarrow q'$ produces the same ϕ at $r \geq a$ as that produced by the actual charge (i.e. σ on the sphere).





E on the conductor surface is always ⊥ to the conductor to keep the charges in static equilibrium, i.e. $\phi(r = a) = 0$. (Prob. 1, Sec. 1.6)

σ on the sphere:



Rewrite $\phi(\mathbf{x}) = \dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{1}{|\mathbf{x}-\mathbf{y}|} - \dfrac{a}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}\right]$

Let γ be the angle between $\mathbf{x}$ and $\mathbf{y}$. Then

$$\phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2+y^2-2xy\cos\gamma)^{1/2}} - \frac{a}{y(x^2+\frac{a^4}{y^2}-2\frac{xa^2}{y}\cos\gamma)^{1/2}}\right]$$

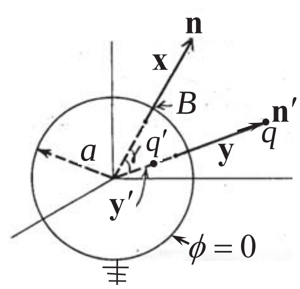
$\mathbf{E}(r<a)=0. \Rightarrow$ By Gauss's law, σ at point B is [see (1.22)]

$$\sigma = \varepsilon_0 E_r(x=a) = -\varepsilon_0 \frac{\partial\phi}{\partial x}\bigg|_{x=a} \quad \left[\begin{array}{l}\partial\phi/\partial x \text{ is a derivative normal}\\ \text{to the surface at point B.}\end{array}\right]$$

$$= \frac{q}{8\pi}\left[\frac{2a-2y\cos\gamma}{(a^2+y^2-2ay\cos\gamma)^{3/2}} - \frac{a(2a-2\frac{a^2}{y}\cos\gamma)}{y(a^2+\frac{a^4}{y^2}-2\frac{a^3}{y}\cos\gamma)^{3/2}}\right]$$

$$= \frac{-q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma)^{3/2}} \quad \left[\begin{array}{l}\text{This is the actual}\\ \text{charge producing the}\\ \text{2nd term of } \phi(\mathbf{x}).\end{array}\right] \quad (2.5)$$

Total charge on the sphere:



The total σ can be obtained by integrating σ over the spherical surface. However, it can be deduced from a simple argument: In the region $r \geq a$, $\mathbf{E}$ due to the total σ is exactly the $\mathbf{E}$ due to the image charge q'. Hence, by Gauss's law, the total σ must be $q'(=-\frac{a}{y}q)$.

Force on q:

At the position of q, $\mathbf{E}$ due to q' is the $\mathbf{E}$ due to σ. Hence, the force on q is the Coulomb force between q' & q.

n, **n**′ point out of the sphere enclosing $q' \Rightarrow$ convenient to apply Gauss's law and Coulomb's law to q'.

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(y-y')^2}\mathbf{n}' = \frac{-1}{4\pi\varepsilon_0}\frac{q(\frac{a}{y}q)}{(y-\frac{a^2}{y})^2}\mathbf{n}' = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\frac{1}{(1-\frac{a^2}{y^2})^2}\frac{\mathbf{y}}{\mathrm{y}} \quad (2.6)$$

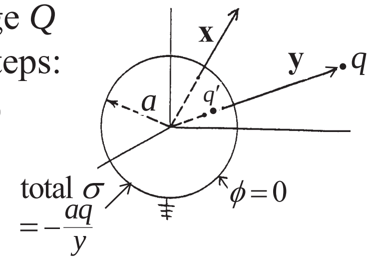
These 2 examples make the image charge a useful concept. They also show the merits of problem solving by physical arguments (more examples are given in the following Section).

2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere (with Total Charge Q)

If the sphere is insulated with total charge Q on its surface, we may obtain $\phi(\mathbf{x})$ in two steps:

Step 1: Ground the sphere (upper fugure)

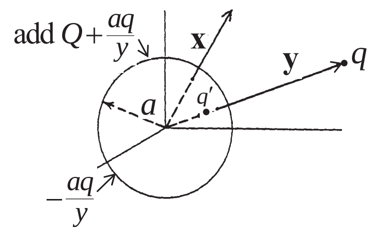
$\Rightarrow$ Same problem as in Sec. 2.2



$$\Rightarrow \phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\,|\mathbf{x}-a^2\mathbf{y}/y^2|}$$

with a total σ given by $q' = -aq/y$.

Step 2: Disconnect the ground wire. Add $Q + aq/y$ to the sphere (lower figure) so that the total charge on the sphere is Q.



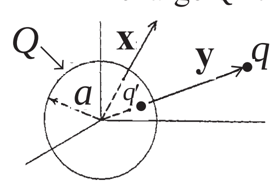
To keep $\phi(r=a)$ at a constant value, the added charge $Q + aq/y$ must be distributed uniformly on the surface. By the shell theorem,

$\phi(r \ge a)$ due to added charge $Q + aq/y$ is $\phi(\mathbf{x}) = \dfrac{Q + aq/y}{4\pi\varepsilon_0 |\mathbf{x}|}$

$\Rightarrow$ The the total ϕ is $\phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{|\mathbf{x}-\mathbf{y}|} - \dfrac{aq}{y\,|\mathbf{x}-a^2\mathbf{y}/y^2|} + \dfrac{Q+aq/y}{|\mathbf{x}|}\right]$ (2.8)

$$\Rightarrow \text{The force on } q \text{ is } \mathbf{F} = \underbrace{\frac{-1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3 \frac{1}{\left(1-\frac{a^2}{y^2}\right)^2}\frac{\mathbf{y}}{y}}_{(2.6)} + \underbrace{\frac{q(Q+aq/y)}{4\pi\varepsilon_0}\frac{\mathbf{y}}{y^3}}_{\text{due to added charge } Q+aq/y}$$

$$\Rightarrow \mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]\frac{\mathbf{y}}{y} \qquad (2.9)$$



$\Rightarrow \begin{cases} \text{As } y \to \infty,\ F \to \dfrac{qQ}{4\pi\varepsilon_0 y^2} \text{ (Coulomb force between point charges)} \\ \text{As } y \to a,\ F \text{ is always } \textit{attractive} \text{ even if } q \text{ and } Q \text{ have the same sign.} \end{cases}$

Question: If there is an excess of electrons on the surface, why don't they leave the surface due to mutual repulsion?

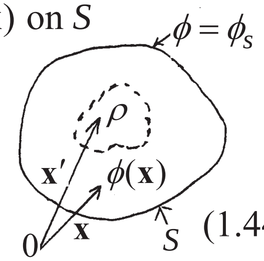
(See p. 61 for a discussion on the work function of a metal.)

2.6 Green Function for the Sphere; General Solution for the Potential

(We will skip Sec. 2.4 and treat Sec. 2.5 in Sec. 3.3)

Consider again the general electrostatic problem with Dirichlet b.c. (upper figure): $\nabla^2\phi(\mathbf{x}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{x})$ with $\phi(\mathbf{x}) = \phi_S(\mathbf{x})$ on S

In Sec. 1.10, we show it has the formal solution:

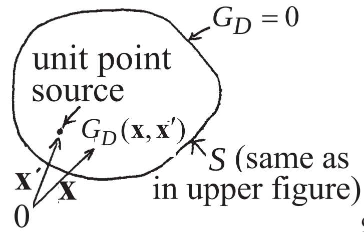


$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da', \quad (1.44)$$

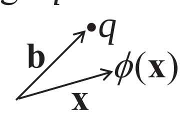
where the Green function $G_D(\mathbf{x},\mathbf{x}')$ is the solution of (lower figure)

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ on the same } S$$

Physically, if $4\pi \to q/\varepsilon_0$, $G_D(\mathbf{x},\mathbf{x}')$ is the ϕ at $\mathbf{x}$ due to a point charge q at $\mathbf{x}'$ under the b. c. $G_D(\mathbf{x},\mathbf{x}') = 0$ on S [i.e. for either $\mathbf{x}$ or $\mathbf{x}'$ on S since $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}',\mathbf{x})$]



Example 1: Use (1.44) to find ϕ due to a point charge q at $\mathbf{x} = \mathbf{b}$ in infinite space.



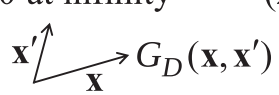
$$\nabla^2\phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{b}) \text{ with } \phi = 0 \text{ at infinity}$$

The solution is obviously given by $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{x}-\mathbf{b}|}$. We will solve the problem here as a simple exercise of (1.44).

First, obtain the Green function from

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ at infinity} \quad (2)$$

The solution of (2) is $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$



Next, sub. $\rho(\mathbf{x}') = q\delta(\mathbf{x}'-\mathbf{b})$ and $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ into (1.44)

$$\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{q\delta(\mathbf{x}'-\mathbf{b})}\ \overbrace{G_D(\mathbf{x},\mathbf{x}')}^{\frac{1}{|\mathbf{x}-\mathbf{x}'|}}d^3x' - \frac{1}{4\pi}\oint_S \overbrace{\phi(\mathbf{x}')}^{0}\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}da'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{x}-\mathbf{b}|}$$

Example 2: $\nabla^2\phi(\mathbf{x}) = 0$ with b.c. $\phi(r=a) = \phi(a,\theta,\varphi)$

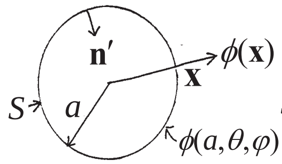
Find $\phi(\mathbf{x})$ in the region $r \ge a$ (see left figure).

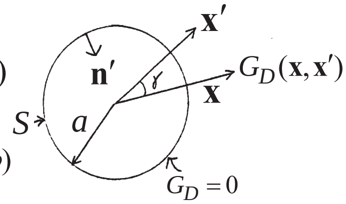
First, find $G_D(\mathbf{x},\mathbf{x}')$ from

$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$

with $G_D = 0$ on S (right fig.)

Then, use (1.44) on left fig.





Note: (1.44) is derived from Green's thm., which requires $\mathbf{n}'$ to point *outward* from the region of interest (i.e. the $r \ge a$ region).

By the method of images, we have shown (let $q \to 4\pi\varepsilon_0$, $\mathbf{y} \to \mathbf{x}'$):

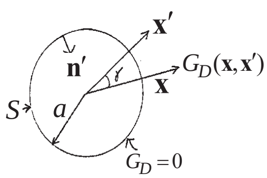
$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \qquad (2.16)$$

$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ in region of interest $(r \ge a)$

$$= \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}} \qquad (2.17)$$

γ: angle between $\mathbf{x}$ and $\mathbf{x}'$

Note: (2.17) shows $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}',\mathbf{x})$.

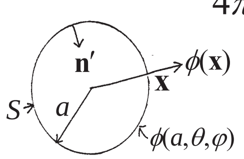


$\mathbf{n}'$ is a unit normal out of the $r \ge a$ region $\Rightarrow \frac{\partial}{\partial n'} = -\frac{\partial}{\partial x'}$

$$\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{x'=a} = -\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial x'}\right|_{x'=a} = -\frac{(x^2-a^2)}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

By (1.44),

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{=0} G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\overbrace{\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}} da'$$



$$= \frac{1}{4\pi}\oint_S \phi(a,\theta',\varphi')\frac{a(x^2-a^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}d\Omega' \qquad (2.19)$$

Question: In (2.16), we have $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|}$ as a solution of $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. But $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ also satisfies the same equation in the region of interest $(r \ge a)$. Does this violate the uniqueness theorem?

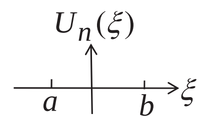
2.7 Conducting Spheres with Hemisphere…

(to be covered in Sec. 3.3 by a different method)

2.8 Orthogonal Functions and Expansions

Definition of Orthogonal Functions :

Consider a set of real or complex functions $U_n(\xi)$ $(n = 1, 2, \cdots)$ which are square integrable on the interval $a \le \xi \le b$.



$$U_n(\xi)\text{'s are } \begin{cases} \text{orthogonal, if } \overbrace{\int_a^b U_n^*(\xi) U_m(\xi) d\xi}^{\text{inner product}} \begin{cases} = 0, & m \ne n \\ \ne 0, & m = n \end{cases} \\ \text{orthonormal, if } \int_a^b U_n^*(\xi) U_m(\xi) d\xi = \delta_{mn} = \begin{cases} 0, & m \ne n \\ 1, & m = n \end{cases} \end{cases}$$

Geometrical analogy: $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are an orthonormal set of unit vectors, i.e. $\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}$. By comparison, the dot product $\mathbf{e}_m \cdot \mathbf{e}_n$ is similar to the inner product . But the algebraic set $U_n(\xi)$ can be infinite in number.

Linearly Independent Functions :

$$\begin{cases} \text{The set of } U_n(\xi)\text{'s are said to be linearly independent} \\ \text{if the only solution of} \\ \qquad \sum_n a_n U_n(\xi) = 0 \quad \begin{bmatrix} \text{for all values of } \xi \text{ on} \\ \text{the interval } a \le \xi \le b \end{bmatrix} \\ \text{is } a_n = 0 \text{ for any } n. \end{cases} \tag{3a}$$

$$\begin{cases} \text{If a set of functions are orthogonal, they are also} \\ \text{linearly independent.} \end{cases} \tag{3b}$$

Proof: Let $\sum_n a_n U_n(\xi) = 0$ for all ξ.

Multiply both sides by $U_m^*(\xi)$ and integrate from a to b.

$$\Rightarrow \int_a^b \sum_n a_n U_n(\xi) U_m^*(\xi) d\xi = \sum_n a_n \overbrace{\int_a^b U_n(\xi) U_m^*(\xi) d\xi}^{=0,\text{ unless } m=n}$$

$$= a_n \int_a^b |U_n(\xi)|^2 \, d\xi = 0$$

$$\Rightarrow a_n = 0 \text{ for any } n$$

Gram - Schmidt Orthogonalization Procedure:

Although "orthogonality" always implies "linear independence", "linear independence" does not ensure "orthogonality". However, a set of linearly independent functions, if not orthogonal, can be reconstructed into an orthogonal set by the Gram-Schmidt orthogonalization procedure. A simple example is given below.

Consider two vectors: $\mathbf{e}_x$ and $(\mathbf{e}_x + \mathbf{e}_y)$. These two vectors are linearly independent since $a\mathbf{e}_x + b(\mathbf{e}_x + \mathbf{e}_y) = 0 \;\Rightarrow a = b = 0,$ but they are not orthogonal, since $\mathbf{e}_x \cdot (\mathbf{e}_x + \mathbf{e}_y) \neq 0.$

We may form two new vectors $(\mathbf{e}_1, \mathbf{e}_2)$ as linear combinations of of the old vectors as follows. Let $\mathbf{e}_1 = \mathbf{e}_x$ and $\mathbf{e}_2 = \mathbf{e}_x + \mathbf{e}_y + \alpha \mathbf{e}_x$, and demand $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$. Then, $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \Rightarrow 1 + \alpha = 0 \Rightarrow \alpha = -1 \Rightarrow \mathbf{e}_2 = \mathbf{e}_y$

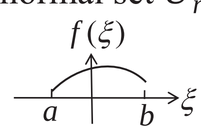
The new set, $\mathbf{e}_1 (= \mathbf{e}_x)$ and $\mathbf{e}_2 (= \mathbf{e}_y)$, are thus orthogonal (as well as linearly independent).

The same procedure can be applied to algebraic functions.

Completeness of a Set of Functions :

Expand an arbitrary, square-integrable function $f(\xi)$ in terms of a finite number (N) of functions in the orthonormal set $U_n(\xi)$,

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n U_n(\xi) \tag{2.30}$$



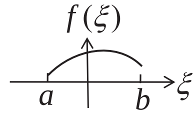
and let $M_N \equiv \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2 d\xi$ [mean square error], (2.31)

where
$$a_n = \int_a^b U_n^*(\xi) f(\xi) d\xi \tag{2.32}$$

If there exists a finite number N_0 such that, for $N > N_0$, M_N can be made smaller than any arbitrarily small positive quantity, then the set $U_n(\xi)$ is said to be complete and the series representation in

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) \quad [U_n(\xi): \text{orthonormal set}] \tag{2.33}$$

is said to converge in the mean to $f(\xi)$. Here, a limiting concept is used to define "=", i.e. the difference between the 2 sides of (2.33) is *arbitrarily* close to 0 (instead of exactly 0).

Rewrite (2.33): $f(\xi)=\sum\limits_{n=1}^{\infty} a_n U_n(\xi)$,  (2.33)

Note : $\sum\limits_{n=1}^{\infty} \Rightarrow$ a sum over the entire set, e.g. in (4a) below, it is $\sum\limits_{n=-\infty}^{\infty}$.

Multiply both sides by $U_n^*(\xi)$, integrate from a to b, and apply the orthonormal property of $U_n(\xi)$, we obtain (2.32) again

$$a_n=\int_a^b U_n^*(\xi)f(\xi)d\xi \tag{2.32}$$

Change ξ in (2.32) to ξ': $\quad a_n=\int_a^b U_n^*(\xi')f(\xi')d\xi' \qquad (2.32')$

Sub. a_n in (2.32′) into (2.33):

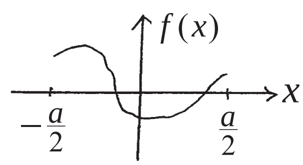
$$f(\xi)=\int_a^b\Big[\sum_{n=1}^{\infty} U_n^*(\xi')U_n(\xi)\Big]f(\xi')d\xi' \tag{2.34}$$

$f(\xi)$ in (2.34) is arbitrary $\Rightarrow \underbrace{\sum\limits_{n=1}^{\infty} U_n^*(\xi')U_n(\xi)=\delta(\xi-\xi')}_{\text{completeness or closure relation}} \qquad (2.35)$

Fourier Series : Example of complete set of orthogonal functions

Exponential representation of $f(x)$ *on the interval* $-\frac{a}{2}\le x\le\frac{a}{2}$:

$$\begin{cases} f(x)=\sum\limits_{n=-\infty}^{\infty} a_n e^{ik_n x} \quad [k_n=\frac{2\pi n}{a}], \\ \text{where } a_n=\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)e^{-ik_n x}dx \end{cases} \tag{4a}$$



Question: Why "$n=-\infty$ to ∞" instead of "$n=0$ to ∞"? *Ans* :

$\because e^{ik_n x}$ ($n=-\infty$ to ∞) are orthogonal, hence linearly indep. [see (3b)].

In (4a), $f(x)$ is in general a complex function and, even when $f(x)$ is real, a_n is in general a complex constant. However, if $f(x)$ is *real*, we have the realty condition: $\quad a_n=a_{-n}^*$ [for real $f(x)$] $\qquad$ (4b)

Proof : $f(x)=real \Rightarrow f(x)=f^*(x)$ $\boxed{n\to -n}$

$$\Rightarrow \sum_{n=-\infty}^{\infty} a_n e^{ik_n x}=\sum_{n=-\infty}^{\infty} a_n^* e^{-ik_n x}=\sum_{n=-\infty}^{\infty} a_{-n}^* e^{ik_n x}$$

Each $e^{ik_n x}$ is linearly independent $\Rightarrow a_n=a_{-n}^*$

Trignometric representation of $f(x)$ *on the interval* $-\frac{a}{2} \le x \le \frac{a}{2}$:

From (4a): $f(x) = \sum\limits_{n=-\infty}^{\infty} a_n e^{ik_n x} = a_0 + \sum\limits_{n=1}^{\infty} \left(a_n e^{ik_n x} + a_{-n} e^{-ik_n x}\right)$

$$= a_0 + \sum_{n=1}^{\infty} \left[\left(a_n \cos k_n x + a_{-n} \cos k_n x\right) + i\left(a_n \sin k_n x - a_{-n} \sin k_n x\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n + a_{-n}\right) \cos k_n x + \sum_{n=1}^{\infty} i\left(a_n - a_{-n}\right) \sin k_n x$$

$$\Rightarrow f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos k_n x + B_n \sin k_n x\right], \quad k_n = \frac{2\pi n}{a} \tag{5}$$

where $\boxed{\text{same as (2.36) and (2.37)}}$

$$\begin{cases} \underset{(n=0\to\infty)}{A_n} = a_n + a_{-n} = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\underbrace{\left(e^{-ik_n x} + e^{ik_n x}\right)}_{2\cos k_n x} dx = \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\cos k_n x dx \\ \underset{(n=1\to\infty)}{B_n} = i\left(a_n - a_{-n}\right) = \frac{i}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\underbrace{\left(e^{-ik_n x} - e^{ik_n x}\right)}_{-2i\sin k_n x} dx = \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\sin k_n x dx \end{cases}$$

Discussion: It is often more convenient to represent a physical quantity (a real number) by exponential rather than trigonometric functions, because the complex coefficient (a_n) of an exponential term carries twice the information of the real coefficients (A_n or B_n) of trigonometric functions. For example, if

$$x(t) = ae^{i\omega t} \quad \text{[By convention, LHS = real part of RHS]}$$

is the displacement of a simple harmonic oscillator, the complex constant a $(= |a| e^{i\varphi})$ contains both the magnitude ($|a|$) and phase (φ) of the displacement. In terms of trigonometric functions, the same information is expressed by 2 real constants in

$$x(t) = |a| \cos(\omega t + \varphi) \quad \text{or} \quad x(t) = A\cos\omega t + B\sin\omega t.$$

Exponential terms are also easier to manipulate (such as multiplication and differentiation). This point will be further discussed in Ch. 7.

Fourier Transform :

In (4), $k_n = \frac{2\pi n}{a}$ $(n = 1, 2, \cdots)$. Thus, $a \to \infty \Rightarrow k_n \to$ a continuum.

$\Rightarrow$ The series becomes an integral. This leads to the Fourier transform

(see p. 69): $\begin{cases} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$

Question: Does $A(k)$ contain any more or less information than $f(x)$?

Change x to x' in (2.45) and sub. (2.45) into (2.44)

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk}_{\delta(x-x')}$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x') \quad \text{[completeness relation]} \qquad (2.47)$$

This is an extension of $\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi')$ [(2.35)] to continuous index k.

Rewrite (2.47): $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x')$

Interchange notations x and k

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta(k-k') \quad \text{[orthogonality condition]} \qquad (2.46)$$

Let $y = k - k'$ and substitute it into (2.46)

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy} dx = \delta(y)$$

$\delta(y) = \delta(-y)$ [see (5c), Ch. 1] $\Rightarrow$ $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} dx = \delta(y)$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm ixy} dx = \delta(y) \quad \text{[most general expression]} \qquad (6)$$

A note on unit: Rewrite $\begin{cases} f(x) = \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} & [(4)] \\ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk & [(2.44)] \end{cases}$

If $f(x)$ is dimensionless and x is in unit of "m", then (1) k and k_n are in unit of $\frac{1}{\text{m}}$; (2) a_n is dimensionless, and (3) $A(k)$ is in unit of m.

There are two useful theorems involving the Fourier transform.

(1) Parseval's theorem:

The Parseval's theorem states $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |A(k)|^2 dk$ (7)

Proof:

Rewrite
$$\begin{cases} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x) f^*(x) dx$$
$$= \int_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^*(k') e^{-ik'x} dk' \right]$$
$$= \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dk' A^*(k') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x}}_{\delta(k-k')} = \int_{-\infty}^{\infty} |A(k)|^2 dk$$

2.8 Orthogonal Functions and Expansions (*continued*)

(2) Convolution theorem: Mathews and Walker (M&W), "Math. Meth. of Phys.", 2nd ed. (our main ref. on math.), p. 113.

The convolution of $f_1(x)$ and $f_2(x)$ is defined as

$$g(x) \equiv \int_{-\infty}^{\infty} f_1(x-\xi) f_2(\xi) d\xi$$

The convolution theorem states that the Fourier transform of $g(x)$ is given by $A_1(k) A_2(k) \times const$. For the convention of the Fourier transform in (2.44) and (2.45), the *const* is $\sqrt{2\pi}$, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overbrace{\int_{-\infty}^{\infty} f_1(x-\xi) f_2(\xi) d\xi}^{g(x)} \, e^{-ikx} dx = \sqrt{2\pi} A_1(k) A_2(k) \qquad (8)$$

Proof: LHS of (8) $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(\xi) d\xi \underbrace{\int_{-\infty}^{\infty} f_1(x-\xi) e^{-ikx} dx}_{\text{Let } \eta = x-\xi \ (\Rightarrow dx = d\eta)}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(\xi) d\xi \int_{-\infty}^{\infty} f_1(\eta) e^{-ik(\xi+\eta)} d\eta$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(\xi) e^{-ik\xi} d\xi \int_{-\infty}^{\infty} f_1(\eta) e^{-ik\eta} d\eta = \sqrt{2\pi} A_1(k) A_2(k)$$

2.9 Separation of Variables, Laplace Equation in Rectangular Coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \begin{bmatrix}\text{Laplace equation in} \\ \text{Cartesian coordinates}\end{bmatrix} \tag{2.48}$$

We may use the method of separation of variables to solve this partial D.E., i.e. let $\phi(x, y, z) = X(x)Y(y)Z(z)$ (2.49)

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{2.50}$$

Since each term is a function of only one variable, each of the 3 terms must be separately constant. We express them as follows

$$\frac{d^2X}{dx^2} = -\alpha^2 X;\ \frac{d^2Y}{dy^2} = -\beta^2 Y;\ \frac{d^2Z}{dz^2} = \gamma^2 Z \text{ subject to } \gamma^2 = \alpha^2 + \beta^2$$

$$\Rightarrow X(x) = \begin{cases} e^{i\alpha x} \\ e^{-i\alpha x}\end{cases};\ Y(y) = \begin{cases} e^{i\beta y} \\ e^{-i\beta y}\end{cases};\ Z(z) = \begin{cases} e^{\gamma z} \\ e^{-\gamma z}\end{cases} \text{ with } \gamma = \sqrt{\alpha^2 + \beta^2}$$

So far we have solved a D.E. [(2.48)]. However, a physics problem contains a D.E., a region of interest, and b.c.'s, as shown below.

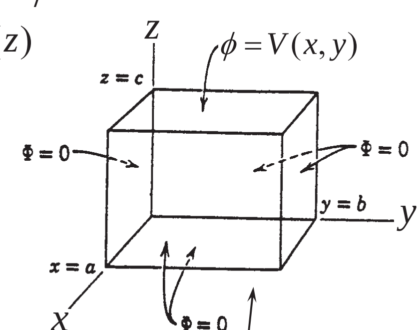
Problem 1: Find ϕ inside a charge-free rectangular box (see figure) with b.c.'s: $\phi\ (x,\ y,\ z = c) = V(x,\ y)$ and $\phi = 0$ on other 5 sides.

$$\nabla^2\phi(x, y, z) = 0,\ \phi(x, y, z) = X(x)Y(y)Z(z)$$

$$X(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$$

$$\begin{cases} X(0) = 0 \Rightarrow B = -A \\ \qquad \Rightarrow X = A(e^{i\alpha x} - e^{-i\alpha x}) = A'\sin\alpha x \\ X(a) = 0 \Rightarrow \alpha = \alpha_n = \frac{\pi n}{a},\ \ n = 1, 2, \ldots \end{cases}$$

$$\Rightarrow X(x) = A'\sin\alpha_n x,\ \ \alpha_n = \frac{\pi n}{a},\ n = 1, 2, \ldots$$



Note: a Cartesian coordinate system usually follow the right-hand convention.

Similarly, $Y(y) = Ae^{i\beta y} + Be^{-i\beta y}$.

$Y(0) = 0$ and $Y(b) = 0$ give

$$Y(y) = A''\sin\beta_m y,\ \beta_m = \frac{\pi m}{b},\ m = 1, 2, \ldots$$

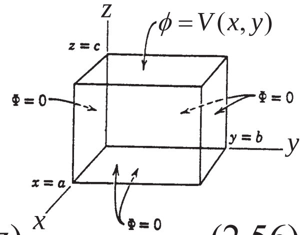
Solution for Z: $Z(z) = Ae^{\gamma z} + Be^{-\gamma z}$

$$\gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2}$$

$$Z(0) = 0 \Rightarrow B = -A \Rightarrow Z(z) = A(e^{\gamma z} - e^{-\gamma z}) = A'''\sinh\gamma_{nm} z$$

Next, we use the method of expansion in orthogonal functions

Rewrite $\begin{cases} X(x) = A' \sin\alpha_n x, \ \alpha_n = \dfrac{\pi n}{a}, \ n = 1, 2, \ldots \\ Y(y) = A'' \sin\beta_m y, \ \beta_m = \dfrac{\pi m}{b}, \ m = 1, 2, \ldots \\ Z(z) = A''' \sinh\gamma_{nm} z, \ \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} \end{cases}$



$$\Rightarrow \phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \quad (2.56)$$

b.c. at $z = c$: $V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$ (2.57)

Operate both sides of (2.57) by $\int_0^a dx \int_0^b dy \sin(\alpha_n x) \sin(\beta_m y)$, then apply the orthogonal property of each of the $\sin(\alpha_n x)$ & $\sin(\beta_m y)$ sets.

$$\Rightarrow A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y) \quad (2.58)$$

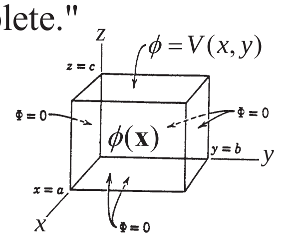
Questions: 1. $\rho = 0$ in the region of interest, what has generated ϕ? *Ans*: ρ on and/or outside the boundary (their effect is implicit in b.c.)

2. Why use the method of expansion? *Ans.*: (a) The base functions satisfy the D.E. & b.c. (b) Can use their orthogonality to determine A_{nm}.

3. Expansion in orthogonal functions is a general method. Why?

P. 68: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete."

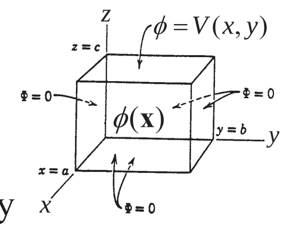
$\sin\alpha_n x$ & $\sin\beta_m y$ occurring in this problem are 2 sets of such functions, so they can represent any physical function or any reasonable mathematical function of x & y (see M&W, p.173, for the meaning of "reasonable"). Thus, any b.c. $V(x, y)$ at $z = c$ can be written as $V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$ [(2.57)]



Note: In (5), $k_n = \dfrac{2\pi n}{a}$. Here, $\alpha_n = \dfrac{\pi n}{a}$. $\Rightarrow$ The $\sin\alpha_n x$ series has the same number as the $(\sin k_n x, \cos k_n x)$ series in (5) (M&W, p. 100).

4. The method of images is not a general method. Why?

$\phi_i \left(= \dfrac{1}{4\pi\varepsilon_0} \dfrac{q_i}{|\mathbf{x} - \mathbf{x}_i|}, \ i = 1, 2, \cdots\right)$ of *hypothetical* image charges (q_i) do not form a complete set (e.g. ϕ_i's are not even orthogonal).



5. Can we find σ on the boundary?

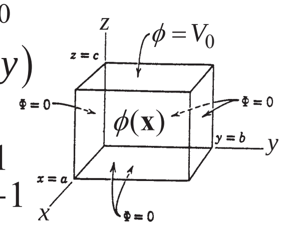
In general, we cannot find σ on the boundary. To find ϕ (hence **E**) inside the box, all we need is ϕ on all 6 sides of the box (outside ϕ not needed). However, we need **E** on *both* sides of the boundary in order to find σ by Gauss's law [see (1.22)]. Since the outside **E** is not in the region of interest, we have no way of knowing it.

For the special case that the boundary is the inner surface of a conductor, we have the extra information that $\mathbf{E}=0$ immediately outside the boundary. σ can thus be determined by Gauss's law.

For this problem, the side on $z=c$ can be the inner surface of a dielectric or no material at all (an imaginary boundary). In the case of an imaginary boundary, all we can tell about the outside is the continuity of **E** across $z=c$ [$\because \sigma = 0$ in (1.22)], but nothing more.

6. $\sin\alpha_n x$ and $\sin\beta_m y$ are *complete* sets, but they both vanish at the ends (i.e. $x=y=0$; $x=a$; $y=b$). Can they express a b.c. [e.g. $V(x,y)=V_0$] which does not vanish at the ends? See next page.

Special case: $V(x,y)=V_0$. In (2.58), let $V(x,y)=V_0$



$$\Rightarrow A_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy V_0 \sin(\alpha_n x)\sin(\beta_m y)$$

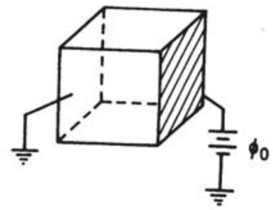
$$= \frac{4V_0}{ab\sinh(\gamma_{nm}c)}\begin{cases}\dfrac{4ab}{nm\pi^2}, & \text{odd } n,m\\ 0, & \text{even } n,m\end{cases} \Rightarrow \text{Let } \begin{cases} n = 2i-1\\ m = 2j-1\end{cases}$$

$$\Rightarrow A_{ij} = \frac{16V_0}{\sinh(\gamma_{ij}c)\pi^2(2i-1)(2j-1)},\ \gamma_{ij} = \pi\sqrt{(\tfrac{2i-1}{a})^2 + (\tfrac{2j-1}{b})^2},\ i,j=1,2,\cdots$$

$$(2.56) \Rightarrow \phi(x,y,z) = \frac{16V_0}{\pi^2}\sum_{i,j=1}^{\infty}\frac{\sin[\frac{\pi(2i-1)x}{a}]\sin[\frac{\pi(2j-1)y}{b}]\sinh(\gamma_{ij}z)}{(2i-1)(2j-1)\sinh(\gamma_{ij}c)} \qquad (9)$$

$$\Rightarrow \phi(x,y,z=c) = \frac{16V_0}{\pi^2}\underbrace{\sum_{i,j=1}^{\infty}\frac{\sin[\frac{\pi(2i-1)x}{a}]}{2i-1}\frac{\sin[\frac{\pi(2j-1)y}{b}]}{2j-1}}_{\pi^2/16 \text{ (see formula below)}} = V_0,\ \text{if } \begin{cases}0<x<a\\ 0<y<b\end{cases}$$

$$\sum_{k=1}^{\infty}\frac{\sin(2k-1)\theta}{2k-1} = \frac{\pi}{4},\ \text{for } 0<\theta<\pi \quad \left[\begin{array}{l}\text{Gradshteyn \& Ryzhik, "Table}\\ \text{of Integrals,}\cdots\text{", Eq. (1.442.1)}\end{array}\right]$$



Problem 2: A hollow metal cube (see figure) has six square sides. There is no charge inside. Five sides are grounded. The 6th side, insulated from the others, is held at a constant potential ϕ_0. Show that ϕ at the center of the cube is $\phi_0/6$.

We use the linear superposition property of the D.E. (see p. 72).

Consider 6 separate solutions as in (9): $\phi_1, \phi_2, \cdots \phi_6$, each equal to ϕ_0 on a different side and equal to 0 on the other 5 sides. Then, $\phi(\mathbf{x}) = \phi_1 + \phi_2 + \cdots + \phi_6$ satisfies the D.E. $\nabla^2\phi(\mathbf{x}) = 0$ and b.c. $\phi = \phi_0$ on all 6 sides. The solution of $\nabla^2\phi(\mathbf{x}) = 0$ with $\phi = \phi_0$ on all 6 sides is clearly $\phi(\mathbf{x}) = \phi_0$ everywhere. Thus, by the uniqueness theorem, $\phi(\mathbf{x}) = \phi_1 + \phi_2 + \cdots + \phi_6 = \phi_0$ everywhere. By symmetry, all ϕ_i's have the same value at the center; hence, $\phi_i = \phi_0/6$ (for all i) at the center.

Note: The prob. is solved without obtaining the 6 solus.: $\phi_1, \cdots \phi_6$. This is another example of prob. solving by math./phys. arguments.

Question: Will σ on the inner surface of the $\phi = \phi_0$ side change if the other 5 sides are brought to $\phi = \phi_0$? *Ans.*: σ changes to 0 (Why?).