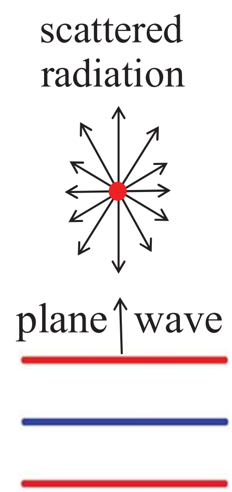
Chapter 10: Scattering and Diffraction

10.1 Scattering at Long Wavelength

Scattering is a commonly-encountered phenomenon in physics. Particle-particle collisions are well-known examples. Scattering of alpha particles by gold nuclei (Rutherford scattering) has led to the discovery of the nucleus.

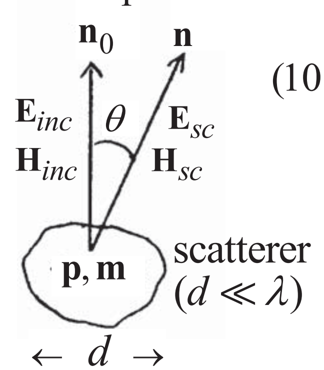
Scattering also occurs between EM waves and objects (called scatterers) of any size. This section deals with the scattering of a plane wave by a small object (size << λ), which results in spherical waves radiated by the object in all directions.

Wave scattering by small objects is often referred to as Rayleigh scattering.



Model : A plane wave propagating along $\mathbf{n}_0$ in free space:

$$\begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{H}_{inc} = \mathbf{n}_0 \times \mathbf{E}_{inc} / Z_0 \end{cases} \left[Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \right] \qquad (10.1)$$



is incident on an object of dimension $d \ll \lambda$, where $\boldsymbol{\varepsilon}_0$ (unit polarization vector) can be real (e.g. $\boldsymbol{\varepsilon}_x$, linearly polarized) or complex [e.g. $\boldsymbol{\varepsilon}_{0\pm} = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_x \pm i\boldsymbol{\varepsilon}_y)$, circularly polarized].

Formalism : $\mathbf{E}_{inc}$, $\mathbf{H}_{inc}$ will induce multipoles on the object, which generate scattered radiation $\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$. For $\lambda \gg d$, the induced $\mathbf{p}$ & $\mathbf{m}$ are dominant (See Exercises 1 & 2 below). Thus, from Ch. 9, we have

$$\begin{cases} \mathbf{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c] \\ \mathbf{H}_{sc} = \mathbf{n}\times\mathbf{E}_{sc}/Z_0 \quad \left[\begin{array}{l}\mathbf{n}: \text{direction of} \\ \text{scattered wave}\end{array}\right] \end{cases} \left[\begin{array}{l}(9.19)+(9.36) \\ \text{with near fields} \\ \text{neglected.}\end{array}\right] \qquad (10.2)$$

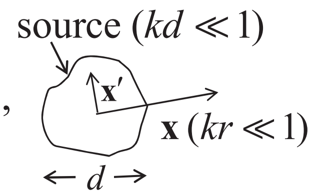
where the induced $\mathbf{p}$ and $\mathbf{m}$ are derived in Exercises 1 and 2 below.

Review of near-field equations: (Lecture Notes, Sec. 9.1)

$$\begin{cases}\mathbf{J}(\mathbf{x},t)=\mathbf{J}(\mathbf{x})e^{-i\omega t}\\ \rho(\mathbf{x},t)=\rho(\mathbf{x})e^{-i\omega t}\end{cases}\Rightarrow\begin{Bmatrix}\mathbf{A}(\mathbf{x})\\ \Phi(\mathbf{x})\end{Bmatrix}=\begin{Bmatrix}\frac{\mu_0}{4\pi}\\ \frac{1}{4\pi\varepsilon_0}\end{Bmatrix}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\begin{Bmatrix}\mathbf{J}(\mathbf{x}')\\ \rho(\mathbf{x}')\end{Bmatrix}\quad [(9.3)]$$

With r & d (source dimension) $\ll\lambda$ (or kr & $kr'\ll 1$), we have

$$e^{ik|\mathbf{x}-\mathbf{x}'|}\approx 1\quad\Rightarrow\quad\begin{cases}\mathbf{A}(\mathbf{x})\approx\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ \Phi(\mathbf{x})\approx\frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\end{cases},$$

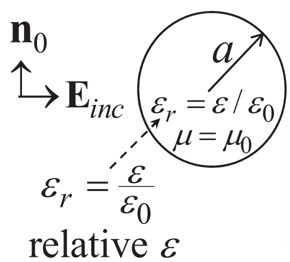


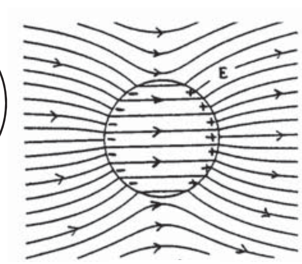
which are the *static* fields given by (1.17), (5.32). Thus, the *spatial* profiles of the near-zone fields approximately obey the static eqs.:

$$\begin{cases}\nabla\cdot\mathbf{B}(\mathbf{x})=0; & \nabla\times\mathbf{H}(\mathbf{x})\approx\mathbf{J}_{free}(\mathbf{x})\\ \nabla\cdot\mathbf{D}(\mathbf{x})=\rho_{free}(\mathbf{x}); & \nabla\times\mathbf{E}(\mathbf{x})\approx 0\end{cases},$$

where $\mathbf{D}(\mathbf{x})=\varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x}),\ \mathbf{B}(\mathbf{x})=\mu(\mathbf{x})\mathbf{H}(\mathbf{x}),$ and $\varepsilon(\mathbf{x}),\ \mu(\mathbf{x})$ are in general complex constants. *Note*: k in a conducting scatterer is much greater. Static spatial fields may not apply (see *Exercise* 2 below).

Exercise 1: Find the dipole moments $\mathbf{p}$ and $\mathbf{m}$ induced by the plane wave in (10.1) on a uniform dielectric sphere (radius $a\ll\lambda$, $\mu=\mu_0$, arbitrary ε).





total $\mathbf{E}$

Solution: (1) $\mathbf{p}$ due to $\mathbf{E}_{inc}$

$a\ll\lambda\Rightarrow$ We may assume a uniform $\mathbf{E}_{inc}$ in the region of interest $(0\le r\ll\lambda)$.

$r,a\ll\lambda\Rightarrow$ We are in the near zone.

$\Rightarrow\mathbf{E}(\mathbf{x})$ obeys static laws: $\nabla\times\mathbf{E}(\mathbf{x})\approx 0\Rightarrow\mathbf{E}(\mathbf{x})=-\nabla\phi.$

ε is uniform. $\Rightarrow\ \nabla\cdot\mathbf{D}(\mathbf{x})=\varepsilon\nabla\cdot\mathbf{E}(\mathbf{x})=\cancel{\rho_{free}}(\mathbf{x})\Rightarrow\nabla^2\phi=0.$

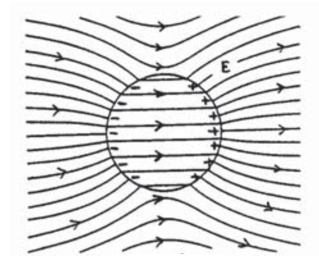
Following the same steps as in Sec. 4.4, we obtain the electric dipole moment $\mathbf{p}$ induced on the sphere by a uniform $\mathbf{E}_{inc}$

$$\mathbf{p}=4\pi\varepsilon_0\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)a^3\mathbf{E}_{inc}\qquad (4.56)\ \&\ (10.5)$$

Note: If $\mathbf{E}_{inc}$ (hence $\mathbf{p}$) is linearly polarized, so will be $\mathbf{E}_{sc}$.

Discussion:

1. For a uniform sphere as in the present case, polarization charges lie on the surface with a surface charge density given by (4.58).



total $\mathbf{E}$

2. $\mu = \mu_0$ implies that the material is non-permeable, i.e. in the presence $\mathbf{H}_{inc}$, the atomic currents remain randomly oriented, with a zero net $\mathbf{m}$. However, there is still the possibility of a net $\mathbf{m}$ due to an induced macroscopic $\mathbf{J}$ by $\mathbf{E}_{inc}$, as considered below.

(2) $\mathbf{m}$ due to $\mathbf{E}_{inc}$

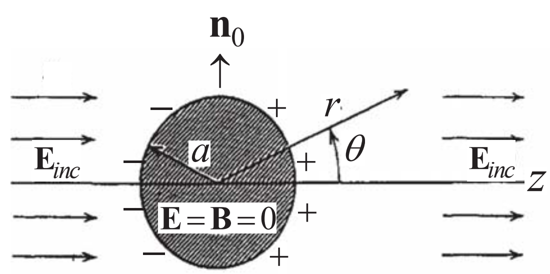
A time varying $\mathbf{P}$ [$\mathbf{P}$: electric polarization, Eq. (4.28)] gives rise to a polarization current $\mathbf{J}_{pol} = \frac{\partial}{\partial t}\mathbf{P}$ (lecture notes, Ch. 4, Appendix B), which can produce a magnetic dipole moment:

$$\mathbf{m} = \tfrac{1}{2}\int (\mathbf{x}\times\mathbf{J}_{pol})d^3x \quad [(5.54)]$$

For a uniform dielectric sphere, $\mathbf{P}$ & $\mathbf{J}_{pol}$ are uniform. So, in the integrand, every $\mathbf{x}\times\mathbf{J}_{pol}d^3x$ is cancelled by $(-\mathbf{x})\times\mathbf{J}_{pol}d^3x \Rightarrow \mathbf{m} = 0.$

Exercise 2: Find $\mathbf{p}$ and $\mathbf{m}$ induced by the plane wave in (10.1) on a *perfectly conducting* sphere (radius $a \ll \lambda$, $\mu = \mu_0$).

Solution : (1) $\mathbf{p}$ due to $\mathbf{E}_{inc}$



$\mathbf{E} = \mathbf{B} = 0$ inside the sphere of radius a and $a \ll \lambda \Rightarrow$ The region of interest is $a < r \ll \lambda$.

$a \ll \lambda \Rightarrow$ The incident $\mathbf{E}_{inc}$ is uniform in the region of interest.

$r, a \ll \lambda \Rightarrow \mathbf{E}(\mathbf{x})$ obeys static laws : $\mathbf{E} = -\nabla\phi;\ \nabla^2\phi = 0,\ a < r \ll \lambda$

In Prob. 2 of Sec. 3.3, we have shown a uniform $\mathbf{E}_{inc}$ will induce surface charges on the a conducting sphere, producing ϕ given by

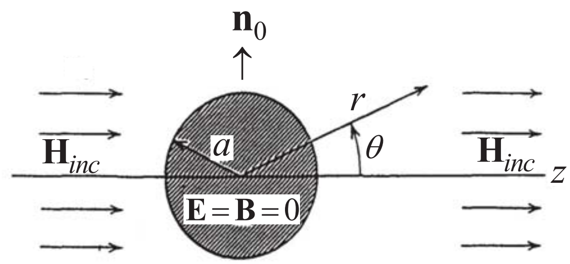
$$\phi(r,\theta) = E_{inc}\frac{a^3}{r^2}\cos\theta \quad \text{[lecture notes, Ch. 3, Eq. (3)]}$$

Form (4.10) : ϕ due to $\mathbf{p} = p\mathbf{e}_z$ is $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$

The 2 eqs. above imply that the induced $\mathbf{p}$ on the sphere is

$$\mathbf{p} = 4\pi\varepsilon_0 a^3\mathbf{E}_{inc} \qquad (10.12)$$

(2) $\mathbf{m}$ due to $\mathbf{H}_{inc}$



For the same reasons as in Case (1), the region of interest is again $a < r \ll \lambda$.

$a \ll \lambda \Rightarrow$ The incident $\mathbf{H}_{inc}$ is uniform in the region of interest.

$r, a \ll \lambda \Rightarrow \mathbf{H}(\mathbf{x})$ obeys static laws: $\nabla \times \mathbf{H} \approx \mathbf{J}_{free}$

To shield the inside from $\mathbf{H}_{inc}$, $\mathbf{K}_{eff}$ is induced at $r = a$. However, in the region $r > a$, $\mathbf{J}_{free} = 0 \Rightarrow \nabla \times \mathbf{H} = 0 \Rightarrow \nabla \times \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \phi$

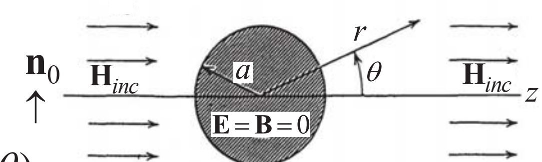
$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \nabla^2 \phi = 0$ with the solution: [Sec. 3.1 of lecture notes]

$$\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix} \begin{bmatrix} \text{near-zone solutions} \\ \text{for } r > a \end{bmatrix} \text{ subject to}$$

$$\text{b.c.'s}: \begin{cases} \mathbf{B}(r \to \infty) = \mu_0 H_{inc} \mathbf{e}_z \Rightarrow \phi(r \to \infty) = \mu_0 H_{inc} z = \mu_0 H_{inc} r \cos\theta \\ B_\perp(r = a) = 0 \Rightarrow \frac{\partial}{\partial r}\phi\Big|_{r=a} = 0 \end{cases}$$

$$\text{Rewrite } \phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

$$\text{b.c.'s}: \begin{cases} \phi \text{ is independent of } \varphi \\ \phi \text{ is finite at } \cos\theta = \pm 1 \end{cases}$$



$$\Rightarrow \phi = \sum_{l=0}^{\infty} \left[A_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta)$$

$P_1(\cos\theta) = \cos\theta$

b.c. $\phi(r \to \infty) = \mu_0 H_{inc} r \cos\theta$

$\mu_0 H_{inc} r \cos\theta$ gives a uniform $\mathbf{H}_{inc}$ at all $\mathbf{x}$. It's a good approx. in the region of interest.

$\Rightarrow A_1 = \mu_0 H_{inc}$ & $A_l = 0$ if $\ell \neq 1$

$$\Rightarrow \phi = \mu_0 H_{inc} r \cos\theta + \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta)$$

b.c. $\frac{\partial}{\partial r}\phi\Big|_{r=a} = 0$

$$\Rightarrow (\mu_0 H_{inc} - \tfrac{2}{a^3} C_1)\cos\theta - \sum_{l=2}^{\infty} \tfrac{l+1}{a^{l+2}} C_l P_l(\cos\theta) = 0$$

$\Rightarrow C_1 = \frac{1}{2}\mu_0 a^3 H_{inc}$ & $C_l = 0$ if $\ell \neq 1$

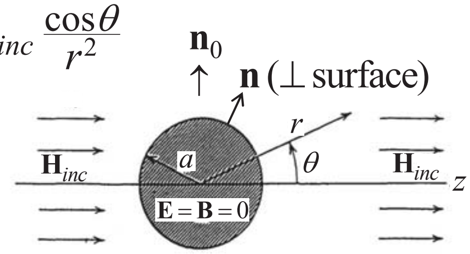
$$\Rightarrow \phi = \mu_0 H_{inc} r \cos\theta + \tfrac{1}{2}\mu_0 a^3 H_{inc} \frac{\cos\theta}{r^2}$$

The 2nd term on th RHS is due to the sphere.

Rewrite $\phi = \mu_0 H_{inc} r\cos\theta + \frac{1}{2}\mu_0 a^3 H_{inc}\dfrac{\cos\theta}{r^2}$

$\Rightarrow$ **B** (due to sphere) $= \nabla\phi$ (2nd term)

$$= -\frac{\mu_0 a^3}{2} H_{inc} \frac{2\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta}{r^3}$$



Compare with (5.41):

$$\mathbf{B}\,(\text{due to } \mathbf{m} = I\pi a^2 \mathbf{e}_z) = \frac{\mu_0}{4\pi}\overbrace{I\pi a^2}^{m}\frac{2\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta}{r^3} \quad [(5.41)]$$

$\Rightarrow$ **B** (due to sphere) has the form of a static dipole magnetic field due to a magnetic dipole moment of $\quad \mathbf{m} = -2\pi a^3 \mathbf{H}_{inc}$ (10.13)

Discussion:

1. **m** is due to the surface current $\mathbf{K}_{free}(a) = \mathbf{n}\times\mathbf{H}$ [see (5.87)] to prevent $\mathbf{H}_{inc}$ from entering the perfectly conducting sphere.

$\Rightarrow \int \mathbf{J}(\mathrm{x}')d^3x' = 0$ (by symmetry considerations).

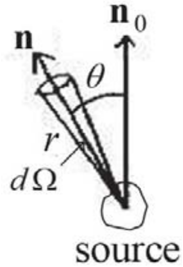
$\Rightarrow$ **m** is independent of the point of reference (see Prob. 2, Sec. 5.6).

2. In Exercises 1 & 2, when $\lambda \gg d$, the dipole moments **p** & **m** are almost exact solutions, hence dominant over higher moments.

Differential Scattering Cross Section :

For scattering problems, a useful quantity is the scattered power ralative to the incident power. Also, it is often desirable to divide the scattered power into two orthogonal polarization states. Thus, we define a <u>differential scattering cross section</u> (m^2/unit solid angle) as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0) \equiv \frac{\dfrac{\text{radiated power in } \mathbf{n}\text{-direction with } \boldsymbol{\varepsilon}\text{-polarization}}{\text{unit solid angle}}}{\dfrac{\text{incident power in } \mathbf{n}_0\text{-direction with } \boldsymbol{\varepsilon}_0\text{-polarization}}{\text{unit area}}}$$



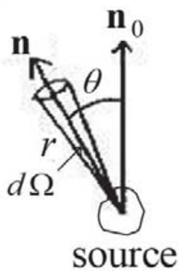
$$= \frac{r^2 \frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^* \cdot \mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^* \cdot \mathbf{E}_{inc}\right|^2} \quad (10.3)$$

[σ has the dimension of area. Its meaning will become clear in (10.11).]

$\boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}$ are polarization vectors. They can both be complex. $\boldsymbol{\varepsilon}_0$ is in the direction of $\mathbf{E}_{inc}$ [see (10.1)]. $\boldsymbol{\varepsilon}$ is not necessarily in the direction of $\mathbf{E}_{sc}$. We pick the desired component of $\mathbf{E}_{sc}$ by dotting $\mathbf{E}_{sc}$ with $\boldsymbol{\varepsilon}^*$.

Rewrite

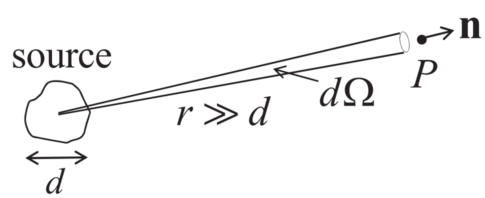
$$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon},\mathbf{n}_0,\boldsymbol{\varepsilon}_0)\equiv\frac{\dfrac{\text{radiated power in } \mathbf{n}\text{-direction with } \boldsymbol{\varepsilon}\text{-polarization}}{\text{unit solid angle}}}{\dfrac{\text{incident power in } \mathbf{n}_0\text{-direction with } \boldsymbol{\varepsilon}_0\text{-polarization}}{\text{unit area}}}$$



$$=\frac{r^2\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0{}^*\cdot\mathbf{E}_{inc}\right|^2}\quad [(10.3)]$$

where $\begin{cases}\boldsymbol{\varepsilon}_0\ \&\ \boldsymbol{\varepsilon}_0{}^*\perp\mathbf{n}_0\\ \boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0{}^*=1\end{cases}$; $\begin{cases}\boldsymbol{\varepsilon}\ \&\ \boldsymbol{\varepsilon}^*\perp\mathbf{n}\\ \boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^*=1\end{cases}$

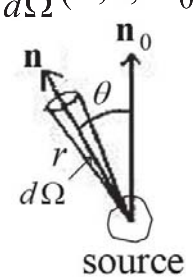
Let P be the observation pt. The greater the source-to-P distance, the more equal the source-to-P directions ($\mathbf{n}$). Since $r\gg d$, we have assumed $\mathbf{n}$ to be the same from all source points to P. Similarly, we have assumed $d\Omega$ from all source points to P to be along the same $\mathbf{n}$.



Rewrite $\quad\dfrac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0)=\dfrac{r^2\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0{}^*\cdot\mathbf{E}_{inc}\right|^2}\quad [(10.3)]$

Sub. $\begin{cases}\mathbf{E}_{inc}=\boldsymbol{\varepsilon}_0E_0e^{ik\mathbf{n}_0\cdot\mathbf{x}}\quad [(10.1)]\\ \mathbf{E}_{sc}=\dfrac{k^2}{4\pi\varepsilon_0}\dfrac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}-\mathbf{n}\times\mathbf{m}/c\right]\ [(10.2)]\end{cases}$ into (10.3)

$$\Rightarrow\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0)=\frac{k^4}{(4\pi\varepsilon_0E_0)^2}\left|\underbrace{\boldsymbol{\varepsilon}^*\cdot[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}]}-\boldsymbol{\varepsilon}^*\cdot\frac{\mathbf{n}\times\mathbf{m}}{c}\right|^2$$



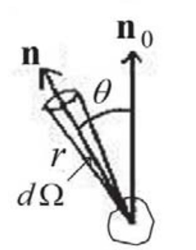
$$\boxed{=\boldsymbol{\varepsilon}^*\cdot[\mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})]=\boldsymbol{\varepsilon}^*\cdot\mathbf{p}-(\cancel{\boldsymbol{\varepsilon}^*\cdot\mathbf{n}})(\mathbf{n}\cdot\mathbf{p})=\boldsymbol{\varepsilon}^*\cdot\mathbf{p}}$$

$$=\frac{k^4}{(4\pi\varepsilon_0E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p}+\frac{(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}}{c}\right|^2\qquad(10.4)$$

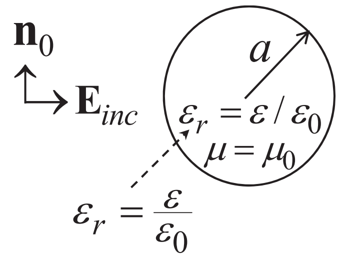
Note: (i) $\left|\boldsymbol{\varepsilon}_0{}^*\cdot\mathbf{E}_{inc}\right|^2=\left|E_0\right|^2$. Let E_0 be real $\Rightarrow\left|E_0\right|^2=E_0^2$, as in (10.4).

(ii) $\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}$ gives the $\boldsymbol{\varepsilon}$-component of $\mathbf{E}_{sc}$. Since $\boldsymbol{\varepsilon}$ is not necessarily in the direction of $\mathbf{E}_{sc}$, $\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2$ is not necessarily $\left|E_{sc}\right|^2$.

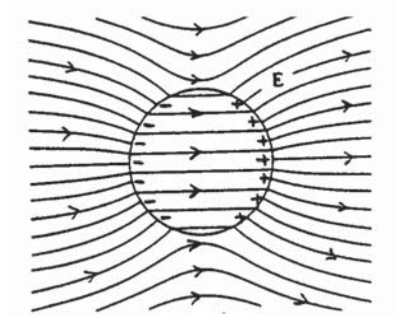
Example 1: Scattering by a uniform dielectric sphere with $\mu = \mu_0$, arbitrary ε, and radius $a \ll \lambda$.



source



relative ε



total $\mathbf{E}$

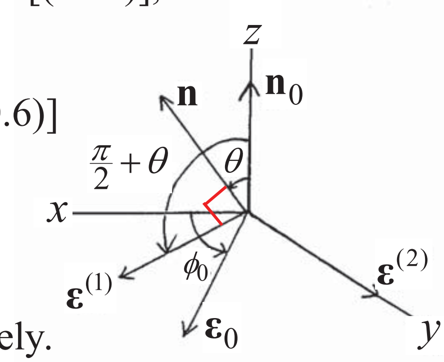
$$\begin{cases} \mathbf{p} = 4\pi\varepsilon_0 (\frac{\varepsilon_r - 1}{\varepsilon_r + 2}) a^3 \mathbf{E}_{inc} \quad [\mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0 \cdot \mathbf{x}}] \\ \mathbf{m} = 0 \ \ \text{[See discussion following Exercise 1]} \end{cases} \quad (4.56)\ \&\ (10.5)$$

Sub. (10.5) into $\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \boldsymbol{\varepsilon}^* \cdot \mathbf{p} + (\mathbf{n} \times \boldsymbol{\varepsilon}^*) \cdot \cancel{\mathbf{m}} / c \right|^2$ (10.4)

$$\Rightarrow \frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \right|^2 \qquad (10.6)$$

From $\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = \dfrac{r^2 \frac{1}{2Z_0} \left| \boldsymbol{\varepsilon}^* \cdot \mathbf{E}_{sc} \right|^2}{\frac{1}{2Z_0} \left| \boldsymbol{\varepsilon}_0^* \cdot \mathbf{E}_{inc} \right|^2}$ [(10.3)], we have obtained

$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \right|^2$ [(10.6)]



$\Rightarrow$ Let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(1)}$ & $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(2)}$ in (10.6), we get $\frac{d\sigma_{\parallel}}{d\Omega}$ & $\frac{d\sigma_{\perp}}{d\Omega}$, i.e. $\frac{d\sigma}{d\Omega}$ in polarization states $\boldsymbol{\varepsilon}^{(1)}$ & $\boldsymbol{\varepsilon}^{(2)}$ ($\parallel$ & $\perp$ to $\mathbf{n}$-$\mathbf{n}_0$ plane), respectively.

If 2 unit vectors have directions: $(\theta,\ \phi)$ and $(\theta',\ \phi')$, then, the dot product ($\cos\gamma$) of the 2 unit vectors is given by [see Eq. (1), Ch. 3]

$$\cos\gamma = \sin\theta \sin\theta' \cos(\phi - \phi') + \cos\theta\cos\theta'$$

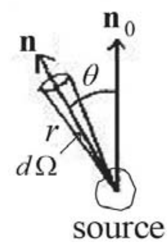
$\Rightarrow$ For $\boldsymbol{\varepsilon}_0 = (\frac{\pi}{2},\ \phi_0)$, $\boldsymbol{\varepsilon}^{(1)} = (\frac{\pi}{2} + \theta,\ 0)$, and $\boldsymbol{\varepsilon}^{(2)} = (\frac{\pi}{2},\ \frac{\pi}{2})$, we have

$$\begin{cases} \boldsymbol{\varepsilon}^{(1)} \cdot \boldsymbol{\varepsilon}_0 = \sin(\frac{\pi}{2} + \theta) \sin\frac{\pi}{2} \cos(0 - \phi_0) + \cos(\frac{\pi}{2} + \theta)\cos\frac{\pi}{2} = \cos\phi_0 \cos\theta \\ \boldsymbol{\varepsilon}^{(2)} \cdot \boldsymbol{\varepsilon}_0 = \sin\frac{\pi}{2} \sin\frac{\pi}{2} \cos(\frac{\pi}{2} - \phi_0) + \cos\frac{\pi}{2}\cos\frac{\pi}{2} = \sin\phi_0 \end{cases}$$

Thus, $$\begin{cases} \dfrac{d\sigma_{\parallel}}{d\Omega} = k^4 a^6 \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \left|\boldsymbol{\varepsilon}^{(1)} \cdot \boldsymbol{\varepsilon}_0\right|^2 = k^4 a^6 \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \cos^2\phi_0 \cos^2\theta \\ \dfrac{d\sigma_{\perp}}{d\Omega} = k^4 a^6 \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \left|\boldsymbol{\varepsilon}^{(2)} \cdot \boldsymbol{\varepsilon}_0\right|^2 = k^4 a^6 \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \sin^2\phi_0 \end{cases}$$

Assume that the incident radiation has a fixed direction $\mathbf{n}_0$, but is unpolarized (i.e. ϕ_0 is random). We take the average over ϕ_0.

$$\Rightarrow \begin{cases} \left\langle \dfrac{d\sigma_{\parallel}}{d\Omega} \right\rangle_{\phi_0} = \dfrac{1}{2\pi}\int_0^{2\pi} \dfrac{d\sigma_{\parallel}}{d\Omega} d\phi_0 = \dfrac{k^4 a^6}{2} \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \cos^2\theta \\ \left\langle \dfrac{d\sigma_{\perp}}{d\Omega} \right\rangle_{\phi_0} = \dfrac{1}{2\pi}\int_0^{2\pi} \dfrac{d\sigma_{\perp}}{d\Omega} d\phi_0 = \dfrac{k^4 a^6}{2} \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \end{cases} \qquad (10.7)$$



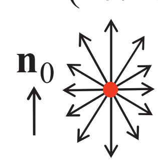
Define $$\Pi(\theta) \equiv \frac{\left\langle \frac{d\sigma_{\perp}}{d\Omega} \right\rangle_{\phi_0} - \left\langle \frac{d\sigma_{\parallel}}{d\Omega} \right\rangle_{\phi_0}}{\left\langle \frac{d\sigma_{\perp}}{d\Omega} \right\rangle_{\phi_0} + \left\langle \frac{d\sigma_{\parallel}}{d\Omega} \right\rangle_{\phi_0}} = \frac{\sin^2\theta}{1+\cos^2\theta}, \qquad (10.9)$$

which gives the degree of polarization of the scattered radiation, e.g. in the $\theta = \frac{\pi}{2}$ direction, $\Pi = 1$; hence, the radiation is 100% linearly polarized ($\perp$ to the $\mathbf{n}$-$\mathbf{n}_0$ plane).

Rewrite $$\begin{cases} \left\langle \dfrac{d\sigma_{\parallel}}{d\Omega} \right\rangle_{\phi_0} = \dfrac{1}{2\pi}\int_0^{2\pi} \dfrac{d\sigma_{\parallel}}{d\Omega} d\phi_0 = \dfrac{k^4 a^6}{2} \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \cos^2\theta \\ \left\langle \dfrac{d\sigma_{\perp}}{d\Omega} \right\rangle_{\phi_0} = \dfrac{1}{2\pi}\int_0^{2\pi} \dfrac{d\sigma_{\perp}}{d\Omega} d\phi_0 = \dfrac{k^4 a^6}{2} \left|\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \end{cases} \qquad [(10.7)]$$

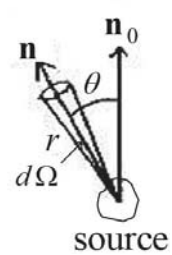
$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\phi_0} = \left\langle \frac{d\sigma_{\perp}}{d\Omega} \right\rangle_{\phi_0} + \left\langle \frac{d\sigma_{\parallel}}{d\Omega} \right\rangle_{\phi_0} = \frac{k^4 a^6}{2} \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 (1+\cos^2\theta) \qquad (10.10)$$

$\Rightarrow$ Same $\left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\phi_0}$ pattern in forward/backward directions.



The meaning of σ (scattering cross section) becomes clear if we integrate (10.10) over all directions to obtain

$$\langle \sigma \rangle_{\phi_0} = \int \left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\phi_0} d\Omega = \frac{8\pi}{3} k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \qquad (10.11)$$



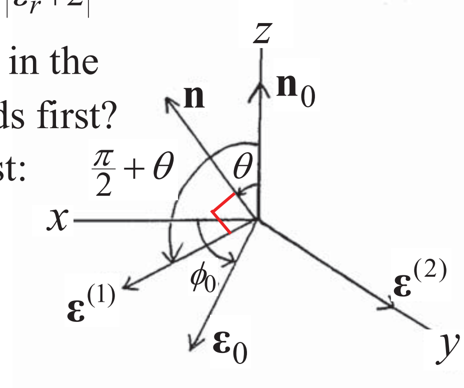
$$(10.11) \Rightarrow \frac{\langle \sigma \rangle_{\phi_0}}{\pi a^2} = \frac{8}{3} k^4 a^4 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \ll 1 \quad [ka \ll 1]$$

$\Rightarrow$ Scattering cross-section $\langle \sigma \rangle_{\phi_0} \ll$ geometrical cross-section πa^2

$\Rightarrow$ Only a small fraction of the wave incident on sphere is scattered.

Question:

In $\left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\phi_0} = \left\langle \frac{d\sigma_\perp}{d\Omega} \right\rangle_{\phi_0} + \left\langle \frac{d\sigma_\parallel}{d\Omega} \right\rangle_{\phi_0} = \frac{k^4 a^6}{2} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 (1 + \cos^2\theta)$ [(10.10)],



we add 2 scattered powers of the same ω in the same direction. Shouldn't we add the fields first?

To find out why, we add the fields first:

$$\mathbf{E}_{sc} = E_{sc}^{(1)} \boldsymbol{\varepsilon}^{(1)} + E_{sc}^{(2)} \boldsymbol{\varepsilon}^{(2)} \quad [\boldsymbol{\varepsilon}^{(1,2)} \perp \mathbf{n}]$$

$$\Rightarrow \mathbf{H}_{sc} = \mathbf{n} \times \mathbf{E}_{sc} / Z_0 \quad [(10.2)]$$

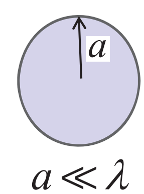
$$= [\mathbf{n} \times E_{sc}^{(1)} \boldsymbol{\varepsilon}^{(1)} + \mathbf{n} \times E_{sc}^{(2)} \boldsymbol{\varepsilon}^{(2)}] / Z_0$$

$$\Rightarrow \mathbf{E}_{sc} \times \mathbf{H}_{sc}^* = [E_{sc}^{(1)} \boldsymbol{\varepsilon}^{(1)} + E_{sc}^{2} \boldsymbol{\varepsilon}^{(2)}] \times [\mathbf{n} \times E_{sc}^{(1)*} \boldsymbol{\varepsilon}^{(1)} + \mathbf{n} \times E_{sc}^{(2)*} \boldsymbol{\varepsilon}^{(2)}] / Z_0$$

$$= \frac{1}{Z_0} \left(\left| E_{sc}^{(1)} \right|^2 + \left| E_{sc}^{2} \right|^2 \right) \mathbf{n} \text{ [sum of 2 powers], if } \boldsymbol{\varepsilon}^{(1)} \perp \boldsymbol{\varepsilon}^{(2)}$$

Thus, if the polarizations of 2 waves (same ω, same direction) are orthogonal, we may add their powers directly. Since there is no other component of $\mathbf{E}_{sc}$, $\frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega}$ is the *total* power radiated into $d\Omega$.

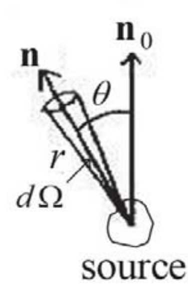
Example 2: Scattering by a perfectly conducting sphere with radius $a \ll \lambda$.



Rewrite

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2} \left| \boldsymbol{\varepsilon}^* \cdot \mathbf{p} + (\mathbf{n} \times \boldsymbol{\varepsilon}^*) \cdot \mathbf{m}/c \right|^2 \quad [(10.4)]$$

The incident wave will induce both $\mathbf{p}$ and $\mathbf{m}$ on the sphere given by



$$\begin{cases} \mathbf{p} = 4\pi\varepsilon_0 a^3 \mathbf{E}_{inc} & [(10.12)] \\ \mathbf{m} = -2\pi a^3 \mathbf{H}_{inc} & [(10.13)] \end{cases} \quad \text{[See Exercise 2]}$$

$$\text{Sub. } \begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0 \cdot \mathbf{x}} \\ \mathbf{H}_{inc} = \mathbf{n}_0 \times \mathbf{E}_{inc} / Z_0 \quad \left(Z_0 \equiv \sqrt{\mu_0 / \varepsilon_0} \right) \end{cases} \quad [(10.1)]$$

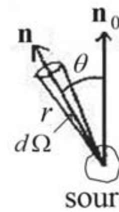
for $\mathbf{E}_{inc}$, $\mathbf{H}_{inc}$ in (10.12), (10.13), then sub. the results into (10.4)

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = k^4 a^6 \left| \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 - \tfrac{1}{2} (\mathbf{n} \times \boldsymbol{\varepsilon}^*) \cdot (\mathbf{n}_0 \times \boldsymbol{\varepsilon}_0) \right|^2 \qquad (10.14)$$

Rewrite $\frac{d\sigma}{d\Omega}=k^4a^6\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0-\frac{1}{2}(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right|^2$ [(10.14)]

$$\Rightarrow\begin{cases}\left\langle\dfrac{d\sigma_{\parallel}}{d\Omega}\right\rangle_{\phi_0}=\dfrac{k^4a^6}{2}(\cos\theta-\frac{1}{2})^2\\[2ex] \left\langle\dfrac{d\sigma_{\perp}}{d\Omega}\right\rangle_{\phi_0}=\dfrac{k^4a^6}{2}(1-\frac{1}{2}\cos\theta)^2\end{cases}\quad\begin{bmatrix}\text{obtained by following}\\ \text{the same steps as in}\\ \text{Example 1}\end{bmatrix}\qquad(10.15)$$

$$\Rightarrow\begin{cases}\left\langle\dfrac{d\sigma}{d\Omega}\right\rangle_{\phi_0}=\left\langle\dfrac{d\sigma_{\parallel}}{d\Omega}\right\rangle_{\phi_0}+\left\langle\dfrac{d\sigma_{\perp}}{d\Omega}\right\rangle_{\phi_0}=k^4a^6[\frac{5}{8}(1+\cos^2\theta)-\cos\theta] & (10.16)\\[2ex] \Pi(\theta)=\dfrac{3\sin^2\theta}{5(1+\cos^2\theta)-8\cos\theta}\quad[\text{maximal at }\theta=60°] & (10.17)\end{cases}$$



$$\langle\sigma\rangle_{\phi_0}=\int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0}d\Omega=\frac{10}{3}\pi k^4a^6\ll\pi a^2\quad[ka\ll 1]$$

Again, we find $\langle\sigma\rangle_{\phi_0}\ll\pi a^2$. By geometric optics, the scatterer (a conductor) would be opaque to waves, i.e. the portion of the wave incident on the sphere would have been totally blocked. This example shows that our intuition based on geometric optics completely breaks down when $\lambda\gg a$, where we need physical optics (as above).

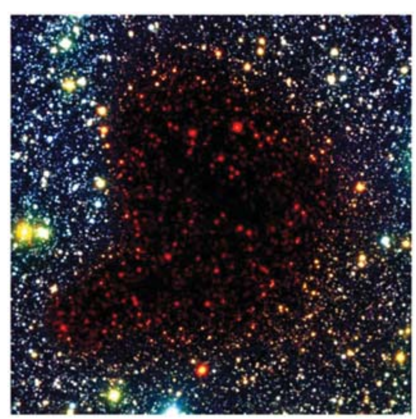
$$\text{Rewrite}\begin{cases}\text{Dielectric sphere: }\langle\sigma\rangle_{\phi_0}=\dfrac{8\pi}{3}k^4a^6\left|\dfrac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\quad[ka\ll 1]\\[2ex] \text{Conducting sphere: }\langle\sigma\rangle_{\phi_0}=\dfrac{10}{3}\pi k^4a^6\quad[ka\ll 1]\end{cases}$$

$\Rightarrow\langle\sigma\rangle_{\phi_0}$ is proportional to k^4 (or ω^4) [Rayleigh's law (p. 457)].

The dark cloud (gases and dust) at B68 in different wavelengths

visible light ← telescope

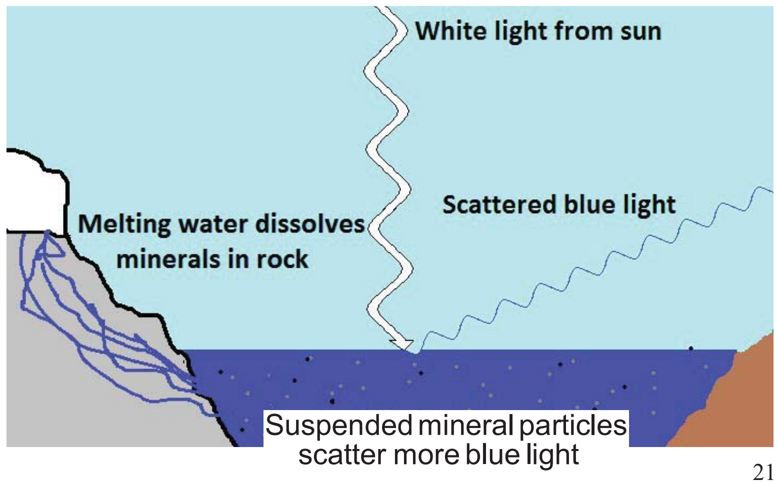
visible light & near-infrared telescope →



Alves, Lada, and Lada, 2001, Nature

Question 1: Why are the backlights of a car in red color?

Question 2: Why are mountain lakes so blue?

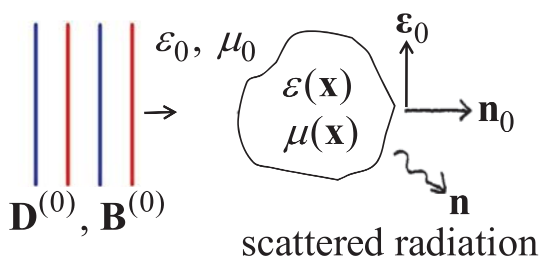


10.2 Perturbation Theory of Scattering

Model:

Consider a plane wave:

$$\begin{cases} \mathbf{D}^{(0)}(\mathbf{x}) = \boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0 \cdot \mathbf{x}} \\ \mathbf{B}^{(0)}(\mathbf{x}) = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \mathbf{n}_0 \times \mathbf{D}^{(0)}(\mathbf{x}) \end{cases} \tag{1}$$



incident to a slightly non-uniform region (the scatterer) with

$$\begin{cases} \varepsilon(\mathbf{x}) = \varepsilon_0 + \delta\varepsilon(\mathbf{x}) \\ \mu(\mathbf{x}) = \mu_0 + \delta\mu(\mathbf{x}) \end{cases}$$

[Note the difference btn. $\boldsymbol{\varepsilon}_0$ (polarization vector) and ε_0 (permittivity)]

The scatterer can be of any shape and size under the condition:

$$\delta\varepsilon(\mathbf{x}) \ll \varepsilon_0, \ \delta\mu(\mathbf{x}) \ll \mu_0, \tag{2}$$

where ε_0, μ_0 characterize the uniform region outside the scatterer.

Comparison between the model here and that in Sec. 10.1:

$$\begin{cases} \text{Here, the scatterer can have any size, but } \delta\varepsilon(\mathbf{x}) \ll \varepsilon_0, \ \delta\mu(\mathbf{x}) \ll \mu_0. \\ \text{In Sec. 10.1, the scatterer can have any } \varepsilon, \ \mu, \text{ or } \sigma, \text{ but size} \ll \lambda. \end{cases}$$

General Theory : Let $\mathbf{E}$ & $\mathbf{H}$ be the sum of incident and scattered $\mathbf{E}$ & $\mathbf{H}$, respectively. We manipulate the field equations as follows:

$$\nabla\times\mathbf{E}=i\omega\mathbf{B} \;\Rightarrow\; \nabla\times\nabla\times\varepsilon_0\mathbf{E}-i\omega\varepsilon_0\nabla\times\mathbf{B}=0 \quad \text{(3a)}$$

$$\nabla\times\mathbf{H}=-i\omega\mathbf{D} \;\Rightarrow\; i\omega\varepsilon_0\nabla\times\mu_0\mathbf{H}=\mu_0\varepsilon_0\omega^2\mathbf{D} \quad \text{(3b)}$$

$$\text{(3a)}+\text{(3b)}\Rightarrow\; \nabla\times\nabla\times\varepsilon_0\mathbf{E}-i\omega\varepsilon_0\nabla\times(\mathbf{B}-\mu_0\mathbf{H})=\mu_0\varepsilon_0\omega^2\mathbf{D} \quad \text{(4)}$$

$$\nabla\times\nabla\times\mathbf{D}=\nabla(\underbrace{\cancel{\nabla\cdot\mathbf{D}}}_{\rho_{free}})-\nabla^2\mathbf{D}=-\nabla^2\mathbf{D} \quad \text{(5)}$$

(4) − (5) gives

$$\nabla^2\mathbf{D}+\mu_0\varepsilon_0\omega^2\mathbf{D}=\overbrace{-\nabla\times\nabla\times(\mathbf{D}-\varepsilon_0\mathbf{E})-i\varepsilon_0\omega\nabla\times(\mathbf{B}-\mu_0\mathbf{H})}^{\text{a small quantity [see (10.29)]}} \quad (10.22)$$

Thus, the purpose of the above manipulation is to obtain a small quantity on the RHS, so we may later use the perturbation theory.

Question : Can $\mathbf{D}$ still be a large quantity? Yes, $\mathbf{D}$ contains $\mathbf{D}^{(0)}$ of the incident plane wave, which satisfies $\nabla^2\mathbf{D}^{(0)}+\mu_0\varepsilon_0\omega^2\mathbf{D}^{(0)}=0$.

Let $k^2=\mu_0\varepsilon_0\omega^2$

$$\Rightarrow(\nabla^2+k^2)\mathbf{D}=-\nabla\times\nabla\times(\mathbf{D}-\varepsilon_0\mathbf{E})-i\varepsilon_0\omega\nabla\times(\mathbf{B}-\mu_0\mathbf{H}) \quad (10.23)$$

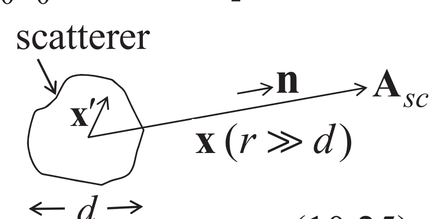
Rewrite $(\nabla^2+k^2)\mathbf{D}=-\nabla\times\nabla\times(\mathbf{D}-\varepsilon_0\mathbf{E})-i\varepsilon_0\omega\nabla\times(\mathbf{B}-\mu_0\mathbf{H})$ [(10.23)]

$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ gives $G(\mathbf{x},\mathbf{x}')=\dfrac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ [(6.40)]

$$\Rightarrow\mathbf{D}=\mathbf{D}^{(0)}+\frac{1}{4\pi}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\begin{Bmatrix}\nabla'\times\nabla'\times(\mathbf{D}-\varepsilon_0\mathbf{E})\\+i\varepsilon_0\omega\nabla'\times(\mathbf{B}-\mu_0\mathbf{H})\end{Bmatrix}, \quad (10.24)$$

where $\mathbf{D}^{(0)}$ is the field of the incident plane wave, which satisfies the homogeneous Helmholtz eq. [i.e. $\nabla^2\mathbf{D}^{(0)}+\mu_0\varepsilon_0\omega^2\mathbf{D}^{(0)}=0$]

At $r\gg d$ (d : scatterer size), we have



$$|\mathbf{x}-\mathbf{x}'|\approx r-\mathbf{n}\cdot\mathbf{x}'\Rightarrow\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\approx\frac{e^{ik(r-\mathbf{n}\cdot\mathbf{x}')}}{r-\cancel{\mathbf{n}\cdot\mathbf{x}'}}$$

$\Rightarrow\mathbf{D}$ can be written as $\mathbf{D}\approx\mathbf{D}^{(0)}+\mathbf{A}_{sc}\dfrac{e^{ikr}}{r}$, (10.25)

$$\text{where}\quad \mathbf{A}_{sc}\equiv\frac{1}{4\pi}\int d^3x'\,\underbrace{e^{-ik\mathbf{n}\cdot\mathbf{x}'}}_{\uparrow}\begin{Bmatrix}\nabla'\times\nabla'\times(\mathbf{D}-\varepsilon_0\mathbf{E})\\+i\varepsilon_0\omega\nabla'\times(\mathbf{B}-\mu_0\mathbf{H})\end{Bmatrix} \quad (10.26)$$

due to path differences of scattered waves ($\because kd$ no longer $\ll 1$)

Rewrite

$$\mathbf{A}_{sc} = \frac{1}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\nabla'\times\nabla'\times(\mathbf{D}-\varepsilon_0\mathbf{E}) + i\varepsilon_0\omega\nabla'\times(\mathbf{B}-\mu_0\mathbf{H})\right] \quad [(10.26)]$$

Using the following algebra:

$$\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\nabla'\times\mathbf{a}(\mathbf{x}) \quad [\mathbf{a}(\mathbf{x}) \text{ is any vector function of } \mathbf{x}.]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{e}_x\left(\frac{\partial a_z}{\partial y'} - \frac{\partial a_y}{\partial z'}\right) + \mathbf{e}_y\left(\frac{\partial a_x}{\partial z'} - \frac{\partial a_z}{\partial x'}\right) + \mathbf{e}_z\left(\frac{\partial a_y}{\partial x'} - \frac{\partial a_x}{\partial y'}\right)\right]$$

integration by parts

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[i\mathbf{e}_x(k_y a_z - k_z a_y) + \mathbf{e}_y(\cdots) + \mathbf{e}_z(\cdots)\right]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} i(\mathbf{k}\times\mathbf{a}) = \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} ik(\mathbf{n}\times\mathbf{a})$$

$\Rightarrow$ The end result is to replace "∇'" with "$ik\mathbf{n}$".

we obtain from (10.26)

$$\mathbf{A}_{sc} = \frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} - \frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\} \quad (10.27)$$

Note: $\mathbf{A}_{sc}$ is the scattering amplitude (NOT a vector potential). It gives the scattered field through $\mathbf{D}_{sc} = \mathbf{A}_{sc}\frac{e^{ikr}}{r}$.

10.2 Perturbation Theory of Scattering (*continued*)

Rewrite $\mathbf{D} \approx \mathbf{D}^{(0)} + \mathbf{A}_{sc}\frac{e^{ikr}}{r}$ [(10.25)] and

$$\mathbf{A}_{sc} = \frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} - \frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\} \quad [(10.27)]$$

In the definition: $\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = \dfrac{r^2\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}\right|^2}$ [(10.3)], we let

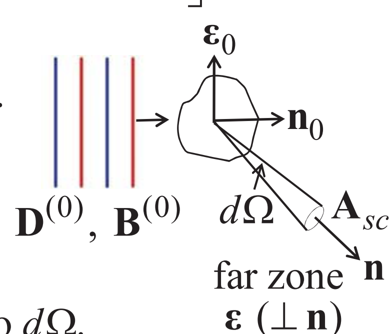
$$\mathbf{E}_{sc} = \frac{\mathbf{D}_{sc}}{\varepsilon_0} = \mathbf{A}_{sc}\frac{e^{ikr}}{\varepsilon_0 r};\quad \mathbf{E}_{inc} = \frac{\mathbf{D}_{inc}}{\varepsilon_0} = \frac{\mathbf{D}^{(0)}}{\varepsilon_0} = \boldsymbol{\varepsilon}_0\frac{D_0}{\varepsilon_0}e^{ik\mathbf{n}_0\cdot\mathbf{x}} \quad [(1)]$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = \frac{\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}\right|^2}{\left|\mathbf{D}^{(0)}\right|^2} \quad \left[\begin{array}{l}\boldsymbol{\varepsilon}\text{: polarization vector of}\\ \text{the scattered radiation}\end{array}\right] \quad (10.28)$$

Note:

1. $\mathbf{A}_{sc}$ is in the direction of $\mathbf{D}_{sc}$ [see above].

2. In (10.3), $\boldsymbol{\varepsilon}$ is not necessarily the direction of $\mathbf{E}_{sc}$. In (10.28), we have chosen $\boldsymbol{\varepsilon}$ to be the direction of $\mathbf{D}_{sc}$. Hence, $\frac{d\sigma}{d\Omega}$ in (10.28) accounts for all the scattered power into $d\Omega$.



Born Approximation: Rewrite

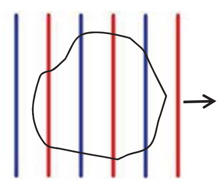
$$\mathbf{A}_{sc}=\frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n}-\frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\}[(10.27)]$$

Note: $\mathbf{D},\ \mathbf{H}$ depend on $\mathbf{A}_{sc}\Rightarrow$ (10.27) is an integral eq., not a solution.

$$\begin{matrix}\text{Linear}\\ \text{medium}\end{matrix}\Rightarrow\begin{cases}\mathbf{D}=[\varepsilon_0+\delta\varepsilon(\mathbf{x})]\mathbf{E}\\ \mathbf{B}=[\mu_0+\delta\mu(\mathbf{x})]\mathbf{H}\end{cases}\Rightarrow\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E}=\delta\varepsilon(\mathbf{x})\mathbf{E}\\ \mathbf{B}-\mu_0\mathbf{H}=\delta\mu(\mathbf{x})\mathbf{H}\end{cases}\qquad(10.29)$$

(10.29) $\Rightarrow$ The RHS of (10.27) is proportional to small quantities $\delta\varepsilon$ and $\delta\mu$, which allows us to use the perturbation theory, i.e. to 1st order in $\delta\varepsilon$ and $\delta\mu$, we may use the unperturbed fields as follows:

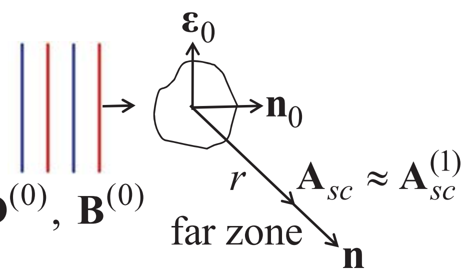
$$\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E}=\delta\varepsilon(\mathbf{x})\mathbf{E}\approx\delta\varepsilon(\mathbf{x})\mathbf{E}^{(0)}=\dfrac{\delta\varepsilon(\mathbf{x})}{\varepsilon_0}\mathbf{D}^{(0)}\\ \mathbf{B}-\mu_0\mathbf{H}=\delta\mu(\mathbf{x})\mathbf{H}\approx\delta\mu(\mathbf{x})\mathbf{H}^{(0)}=\dfrac{\delta\mu(\mathbf{x})}{\mu_0}\mathbf{B}^{(0)}\end{cases}\qquad(10.30)$$



This is called the Born approximation, with which the integral equation in (10.27) becomes a solution for $\mathbf{A}_{sc}$.

$$\text{Sub}\begin{cases}\mathbf{D}^{(0)}(\mathbf{x})=\boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}\\ \mathbf{B}^{(0)}(\mathbf{x})=\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathbf{n}_0\times\mathbf{D}^{(0)}(\mathbf{x})=\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathbf{n}_0\times\boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}\end{cases}\begin{bmatrix}\text{incident}\\ \text{fields, (1)}\end{bmatrix}$$

$$\text{into}\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E}\approx\dfrac{\delta\varepsilon(\mathbf{x})}{\varepsilon_0}\mathbf{D}^{(0)}\\ \mathbf{B}-\mu_0\mathbf{H}\approx\dfrac{\delta\mu(\mathbf{x})}{\mu_0}\mathbf{B}^{(0)}\end{cases}[(10.30)]$$



$$\Rightarrow\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E}\approx\dfrac{\delta\varepsilon(\mathbf{x})}{\varepsilon_0}\boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}\\ \mathbf{B}-\mu_0\mathbf{H}\approx\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{\delta\mu(\mathbf{x})}{\mu_0}\mathbf{n}_0\times\boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}\end{cases}$$

Sub. this eq. into

$\boxed{\omega=k/\sqrt{\mu_0\varepsilon_0}}$

$$\mathbf{A}_{sc}=\frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n}-\frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\}[(10.27)]$$

$$\Rightarrow\mathbf{A}_{sc}\approx\underbrace{\mathbf{A}_{sc}^{(1)}}_{\text{under Born approx.}}=\frac{k^2}{4\pi}D_0\int d^3x' e^{ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{x}'}\begin{Bmatrix}(\mathbf{n}\times\boldsymbol{\varepsilon}_0)\times\mathbf{n}\dfrac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0}\\ -\mathbf{n}\times(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\dfrac{\delta\mu(\mathbf{x}')}{\mu_0}\end{Bmatrix}\qquad(6)$$

Rewrite $\mathbf{A}_{sc}^{(1)} = \frac{k^2}{4\pi} D_0 \int d^3x' \underbrace{e^{ik(\mathbf{n}_0-\mathbf{n})\cdot\mathbf{x}'}}_{\text{phase factor}} \left\{ \begin{array}{l} (\mathbf{n}\times\boldsymbol{\varepsilon}_0)\times\mathbf{n}\,\frac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0} \\ -\mathbf{n}\times(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\,\frac{\delta\mu(\mathbf{x}')}{\mu_0} \end{array} \right\}$ [(6)]

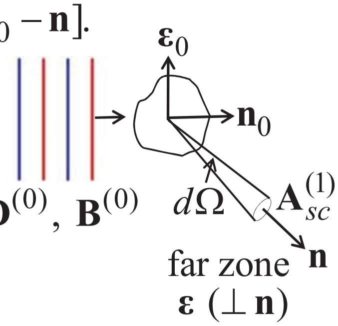
$e^{ik\mathbf{n}_0\cdot\mathbf{x}'}$: Phase of the incident wave in exciting the scatterer point at $\mathbf{x}'$
$e^{-ik\mathbf{n}\cdot\mathbf{x}'}$: Phase adjustment to scattered wave from $\mathbf{x}'$ to $\mathbf{x}$ [see (10.26)]

Dot by $\frac{\boldsymbol{\varepsilon}^*}{D_0}$; use $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ & $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

$$\Rightarrow \quad \frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'} \left\{ \begin{array}{l} \boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\,\frac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0} \\ +(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\,\frac{\delta\mu(\mathbf{x}')}{\mu_0} \end{array} \right\} \qquad (10.31)$$

where $\mathbf{q} \equiv k(\mathbf{n}_0 - \mathbf{n})$ [$q \le 2k$ & $\mathbf{q}$ depends on $\mathbf{n}_0 - \mathbf{n}$].

In $\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2}$ [(10.28)], let $\mathbf{A}_{sc} = \mathbf{A}_{sc}^{(1)}$

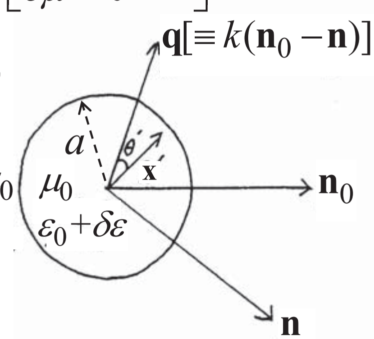


$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}|^2}{|\mathbf{D}^{(0)}|^2} \quad \left[\begin{array}{l}\text{under Born} \\ \text{approx.}\end{array}\right] \qquad (7)$$

Example: Scattering by a sphere with $\delta\varepsilon = const$ and $\delta\mu = 0$

$$(10.31) \Rightarrow \quad \frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\,\frac{\delta\varepsilon}{\varepsilon_0}\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'} \quad \left[\begin{array}{l}\delta\varepsilon = const \\ \delta\mu = 0\end{array}\right]$$

$$\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'} = \int_0^a r'^2 dr' \int_0^{2\pi} d\phi' \int_{-1}^{1} d\underbrace{\cos\theta'}_{y}\, e^{iqr'\overbrace{\cos\theta'}^{y}}$$



$$= 2\pi\int_0^a r'^2 dr' \left[\frac{1}{iqr'} e^{iqr'y}\right]\Bigg|_{y=-1}^{y=1}$$

$$= \frac{4\pi}{q}\int_0^a r'\sin(qr')dr' = 4\pi\left[-\frac{a\cos qa}{q^2} + \frac{\sin qa}{q^3}\right]$$

$$\Rightarrow \frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = k^2\,\frac{\delta\varepsilon}{\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\left[\frac{\sin qa - qa\cos qa}{q^3}\right] \text{ (p. 465)} \quad \boxed{\begin{array}{l} q \text{ depends} \\ \text{on } \mathbf{n}_0 - \mathbf{n}\end{array}} \qquad (8a)$$

If $a \ll \lambda$ (hence $qa \ll 1$), then, $\lim\limits_{qa\to 0}\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = k^2 a^3\,\frac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$ (8b)

In this limit, $\mathbf{q}\cdot\mathbf{x}' \ll 1 \Rightarrow e^{i\mathbf{q}\cdot\mathbf{x}'} \approx 1 \Rightarrow$ Radiated fields from all points of the sphere have ~0 phase difference (considered in phase) $\Rightarrow$ 100% constructive (or coherent) superposition

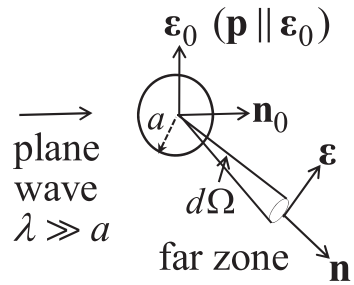
Sub. $\lim_{qa\to 0} \frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = k^2 a^3 \frac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$ into $(\frac{d\sigma}{d\Omega})_{\text{Born}} = \frac{\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}\right|^2}{\left|\mathbf{D}^{(0)}\right|^2}$

$$\Rightarrow \lim_{qa\to 0}(\frac{d\sigma}{d\Omega})_{\text{Born}} \approx k^4 a^6 \left|\frac{\delta\varepsilon}{3\varepsilon_0}\right|^2 \left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 \begin{bmatrix}\delta\varepsilon \ll \varepsilon_0 \\ a \ll \lambda\end{bmatrix} \qquad (10.32)$$

Comparison with $\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 \begin{bmatrix}\text{any } \varepsilon \\ a \ll \lambda\end{bmatrix}$ [(10.6)]

(10.6) reduces to (10.32) in the limit $\varepsilon_r = \varepsilon/\varepsilon_0 \to 1$ (i.e. $\delta\varepsilon \ll \varepsilon_0$)
$\Rightarrow$ (10.32) is dipole radiation. $\Rightarrow$ $\boldsymbol{\varepsilon}$ is on the $\mathbf{n}$-$\boldsymbol{\varepsilon}_0$ ($\mathbf{n}$-$\mathbf{p}$) plane [(9.22)].

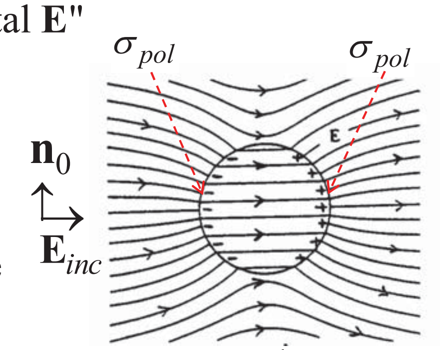
Question: (10.6) and (10.32) both apply to a dielectric sphere with $a \ll \lambda$. (10.6) is valid for any ε. (10.32) is valid for $\delta\varepsilon \ll \varepsilon_0$. A physical effect is included in (10.6) [but not in (10.32)] that keeps $d\sigma/d\Omega$ at a finite value in the limit $\varepsilon \to \infty$ while $d\sigma/d\Omega$ in (10.32) diverges as $\delta\varepsilon \to \infty$. What is it and why? (see next page).



Discussion: Rewrite
$$\begin{cases} \lim_{qa\to 0}(\frac{d\sigma}{d\Omega})_{\text{Born}} \approx k^4 a^6 \left|\frac{\delta\varepsilon}{3\varepsilon_0}\right|^2 \left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 & [(10.32)] \\ \frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 & [(10.6)] \end{cases}$$

The induced $\mathbf{p}$ is proportional to the "total $\mathbf{E}$" *inside* the sphere as seen by the molecule.

The total $\mathbf{E}$ in (10.6) is the sum of the "self $\mathbf{E}$" (due to σ_{pol} on the sphere) and the "incident $\mathbf{E}_{inc}$". The self $\mathbf{E}$ is opposite to $\mathbf{E}_{inc}$. Hence, the total $\mathbf{E}$ inside the sphere is smaller than $\mathbf{E}_{inc}$. As $\varepsilon \to \infty$, the self $\mathbf{E}$ almost completely cancels the incident $\mathbf{E}_{inc}$, so the induced $\mathbf{p}$ does not diverge and $d\sigma/d\Omega$ in (10.6) remains finite.



total $\mathbf{E}$

The total $\mathbf{E}$ in (10.32) is $\mathbf{E} \approx \mathbf{E}^{(0)}$, $\mathbf{B} \approx \mathbf{B}^{(0)}$ [(10.30)], i.e. the self $\mathbf{E}$ and field cancellation are neglected. Hence, $d\sigma/d\Omega \to \infty$ as $\delta\varepsilon \to \infty$. This is the physical reason why (10.32) is valid only for $\delta\varepsilon \ll \varepsilon_0$.

Blue Sky: Scattering by gases (first quantitatively treated by Rayleigh; p. 465: "a masterful physicist at work")

Review of Ch. 4: An external $\mathbf{E}$ will induce a *microscopic* dipole moment $\mathbf{p}$ on a molecule given by $\mathbf{p} = \gamma_{mol}\varepsilon_0\mathbf{E}$ [(4.72)+(4.73)].

Assume only one type of molecules, each with dipole moment $\mathbf{p}$. In Ch. 4, we let $\mathbf{P}(\mathbf{x}) = N(\mathbf{x})\mathbf{p}$ [(4.28)], where $N(\mathbf{x})$ is the *macroscopic* molecular density at $\mathbf{x}$ and hence $\mathbf{P}(\mathbf{x})$ is the total dipole momemt per unit volume at $\mathbf{x}$. Then, $\mathbf{D}(\mathbf{x}) = \varepsilon_0\mathbf{E}(\mathbf{x}) + \underbrace{\mathbf{P}(\mathbf{x})}_{N(\mathbf{x})\gamma_{mol}\varepsilon_0\mathbf{E}(\mathbf{x})}$ [(4.34)]

$\Rightarrow \mathbf{D}(\mathbf{x}) = \varepsilon_0\mathbf{E}(\mathbf{x}) + N(\mathbf{x})\gamma_{mol}\varepsilon_0\mathbf{E}(\mathbf{x})$

$\Rightarrow \mathbf{D}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x})$, where $\varepsilon(\mathbf{x}) = \varepsilon_0[1 + N(\mathbf{x})\gamma_{mol}]$ [*macroscopic* ε]

$$\Rightarrow \qquad \delta\varepsilon(\mathbf{x}) = \varepsilon_0 N(\mathbf{x})\gamma_{mol} \quad \text{[macroscopic } \delta\varepsilon\text{]} \qquad (9)$$

e.g. for air, $\begin{cases} N(\text{sea level}) \approx 2.7\times10^{25}/\text{m}^3 \\ \gamma_{mol} \approx 10^{-29}\ \text{m}^3 \end{cases}$ (p. 163) $\Rightarrow \begin{cases} N\gamma_{mol} \sim 10^{-4} \\ \delta\varepsilon \ll \varepsilon_0 \end{cases}$

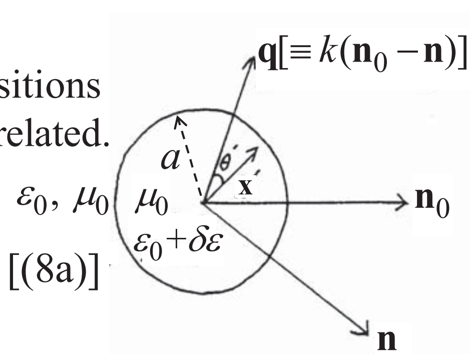
Correlated and uncorrelated (independent) radiations:

In $\varepsilon(\mathbf{x})$, if all scatterers have fixed positions, the scattered fields from all points are *correlated*. How they superpose (constructively or destructively) depends on the object shape and the radiation direction $\mathbf{n}$.

Example 1: Correlated scattering by *macroscopic* $\delta\varepsilon$

$$\delta\varepsilon(\mathbf{x}) = \varepsilon_0 N(\mathbf{x})\gamma_{mol} \quad [(9)]$$

In this case, all scatterers have fixed positions (as in a solid). So the enitted fields are correlated. The dielectric sphere is an example:



$$\begin{cases} \dfrac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = k^2\dfrac{\delta\varepsilon}{\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\left[\dfrac{\sin qa - qa\cos qa}{q^3}\right] \quad [(8a)] \\ \lim\limits_{qa\to0}\dfrac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = k^2a^3\dfrac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0) \quad [(8b)] \end{cases}$$

In (8b) (applicable to $d \ll \lambda$), $\mathbf{A}_{sc}^{(1)} \propto a^3 \propto$ total no of molecules in the sphere $\Rightarrow$ Fields from all points superpose most coherently.

Example 2: Uncorrelated scattering by air molecules:

The Poynting vector at an observation point due to the scattered fields from a large no. of air molecules located at $\mathbf{x}_i$ $(i = 1, 2, \cdots)$ is

$$\mathbf{S} = \tfrac{1}{2}\mathrm{Re}(\sum_i \mathbf{E}_i \times \sum_{i'} \mathbf{H}_{i'}^*) = \tfrac{1}{2}\mathrm{Re}(\sum_i \sum_{i' \neq i} \mathbf{E}_i \times \mathbf{H}_{i'}^*) + \tfrac{1}{2}\mathrm{Re}(\sum_i \mathbf{E}_i \times \mathbf{H}_i^*)$$

Air molecules are *randomly* distributed. Random path differences (between different $\mathbf{x}_i$'s and observation pt.) result in a random phase relation between $\mathbf{E}_i$ & any $\mathbf{H}_{i'}$ with $i' \neq i$. "Large no" + "randomness"

$$\Rightarrow \sum_i \sum_{i' \neq i} \mathbf{E}_i \times \mathbf{H}_{i'}^* = 0 \text{ (by statistics)} \Rightarrow \mathbf{S} = \tfrac{1}{2}\mathrm{Re}(\sum_i \mathbf{E}_i \times \mathbf{H}_i^*)$$

i.e. all particles radiate independently (i.e. *uncorrelated* radiation). The observed power is the sum of powers radiated by individual molecules.

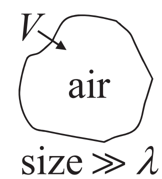
To express the randomness, we write $N(\mathbf{x})$ *microscopically* as

$$N(\mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{x}_i) \; [\Rightarrow \int_{\text{unit volume}} \sum_i \delta(\mathbf{x} - \mathbf{x}_i) d^3x = N]$$

Then, the macroscopic $\delta\varepsilon(\mathbf{x}) = \varepsilon_0 N(\mathbf{x}) \gamma_{mol}$ [(9)] can be written

$$\delta\varepsilon(\mathbf{x}) = \varepsilon_0 \sum_i \gamma_{mol} \delta(\mathbf{x} - \mathbf{x}_i) \quad [\text{microscopic } \delta\varepsilon] \qquad (10.33)$$

Scattering cross-section: Rewrite $\delta\varepsilon(\mathbf{x}) = \varepsilon_0 \sum_i \gamma_{mol} \delta(\mathbf{x} - \mathbf{x}_i)$ [(10.33)]



The microscopic $\delta\varepsilon(\mathbf{x})$ in (10.33) fluctuates wildly in $\mathbf{x}$, but the the perturbation theory is still valid because the corresponding macroscopic $\delta\varepsilon(\mathbf{x})$ [(9)] $\ll \varepsilon_0$. Consider a volume V of air with size $\gg \lambda$ and random $\mathbf{x}_i$ of molecules.

Let $\delta\mu = 0$ in (10.31) $\Rightarrow \dfrac{\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}^{(1)}}{D_0} = \dfrac{k^2}{4\pi} \int_V d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'} \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \dfrac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0}$

$$\delta\varepsilon(\mathbf{x}') = \varepsilon_0 \sum_i \gamma_{mol} \delta(\mathbf{x}' - \mathbf{x}_i) \Rightarrow \frac{\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi} \gamma_{mol} \sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i} \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 \qquad (10)$$

sum over all $\mathbf{x}_i$ in volume V

Sub. (10) into $(\frac{d\sigma}{d\Omega})_{\text{Born}} = \left|\boldsymbol{\varepsilon}^* \cdot \mathbf{A}_{sc}^{(1)}\right|^2 / \left|\mathbf{D}^{(0)}\right|^2$ [(7)]

$$\Rightarrow (\frac{d\sigma}{d\Omega})_{\text{Born}} = \frac{k^4}{16\pi^2} \left|\gamma_{mol}\right|^2 \left|\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0\right|^2 F(\mathbf{q}) \begin{bmatrix} \text{by all molecules} \\ \text{in volume } V \end{bmatrix}, \text{ where}$$

$$F(\mathbf{q}) \equiv \left|\sum_i e^{i\mathbf{q}\cdot\mathbf{x}_i}\right|^2 = \underbrace{\sum_i \sum_{i'} e^{i\mathbf{q}\cdot(\mathbf{x}_i - \mathbf{x}_{i'})}} = \underbrace{\text{total no of molecules in } V}_{F(\mathbf{q}) \text{ indep. of } \mathbf{q}} \quad (10.19)$$

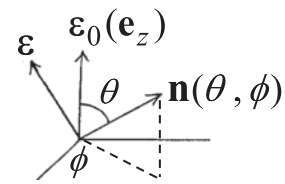
Sum of $i \neq i'$ terms vanishes due to random distribution of $\mathbf{x}_i$

Rewrite $(\frac{d\sigma}{d\Omega})_{\text{Born}} = \frac{k^4}{16\pi^2}|\gamma_{mol}|^2 |\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2 F(\mathbf{q}) \quad \left[\begin{array}{l} F(\mathbf{q}) : \text{total no of} \\ \text{molecules in } V \end{array}\right]$

We now relate the microscopic γ_{mol} to macroscopic ε (electrical permittivity), n (index of refraction), and N (molecular density).

$$\varepsilon = \varepsilon_0 (1 + N\gamma_{mol}) \Rightarrow \gamma_{mol} = \frac{\frac{\varepsilon}{\varepsilon_0} - 1}{N} = \frac{n^2 - 1}{N} \underset{\boxed{n \approx 1}}{\approx} \frac{2(n-1)}{N} \qquad (11)$$

$$\Rightarrow (\frac{d\sigma}{d\Omega})_{\text{Born}} = \frac{k^4}{4\pi^2 N^2}|n-1|^2 |\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0|^2 F(\mathbf{q})$$



Let $\boldsymbol{\varepsilon}_0 = \mathbf{e}_z$ (direction of $\mathbf{D}^{(0)}$) and $\mathbf{n} = (\theta, \phi)$. For dipole radiation, $\boldsymbol{\varepsilon}$ (direction of $\mathbf{E}_{sc}$) is in the $\mathbf{n}$-$\mathbf{p}$ plane [see (9.22)], which is the $\mathbf{n}$-$\boldsymbol{\varepsilon}_0$ plane ($\because \mathbf{p} \parallel \boldsymbol{\varepsilon}_0$). Hence, $\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}_0 = \cos(\pi/2 - \theta) = \sin\theta$ [indep. of ϕ]

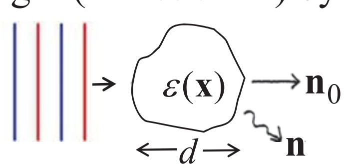
$\Rightarrow$ Total scattering cross section *per molecule* is

$$\sigma = \frac{1}{F(\mathbf{q})}\int (\frac{d\sigma}{d\Omega})_{\text{Born}} d\Omega = \frac{k^4}{4\pi^2 N^2}|n-1|^2 \int_0^{2\pi} d\phi \underbrace{\int_{-1}^{1} d\cos\theta |\overbrace{\boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}_0}^{\sin\theta}|^2}_{\boxed{\int_{-1}^{1} \sin^2\theta d\cos\theta = \frac{4}{3}}}$$

$$= \frac{2k^4}{3\pi N^2}|n-1|^2 \quad \text{[for a single molecule]} \qquad (10.34)$$

Rewrite $\sigma = \frac{2k^4}{3\pi N^2}|n-1|^2 \quad \left[\begin{array}{l} \text{scattering cross-section} \\ \text{of a single molecule} \end{array}\right] \quad$ [(10.34)]

$$\sigma \propto k^4 \Rightarrow \begin{cases} \text{Violet light } (\lambda \approx 410 \text{ nm}) \text{ is scattered more than} \\ \text{red light } (\lambda \approx 650 \text{ nm}) \text{ by a factor of } (\frac{650}{410})^4 \approx 6.3. \end{cases}$$



Sky at daytime

Question: Why does the sky look blue, not violet?

Attenuation coefficient: Rewrite $\sigma = \frac{2k^4}{3\pi N^2}|n-1|^2 \begin{bmatrix}\text{for a single}\\ \text{molecule}\end{bmatrix}$

Let I be the intensity of the incident wave. Then,

$$\begin{cases} I\sigma \text{ is the power scattered by a single molecule} \\ IN\sigma \text{ is the power scattered by a unit volume of molecules} \end{cases}$$

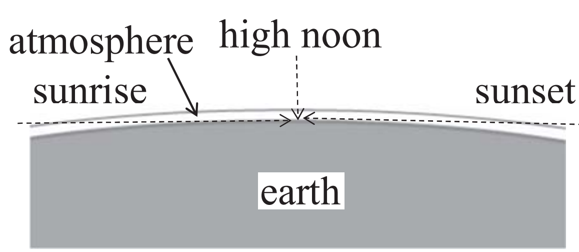
$$\Rightarrow \frac{dI}{dx} = -IN\sigma = -I\alpha, \text{ with } \alpha \equiv N\sigma = \frac{2k^4}{3\pi N}|n-1|^2 \begin{bmatrix}\text{attenuation}\\ \text{coefficient}\end{bmatrix} \quad (10.35)$$

Question: $n-1 \approx \frac{1}{2}N\gamma_{mol}$ [(11)], so (10.35) $\Rightarrow \alpha \propto N$ for the same γ_{mol}. But for a fixed $n-1$ (or $N\gamma_{mol}$), then (10.35) $\Rightarrow \alpha \propto \frac{1}{N}$. Why?

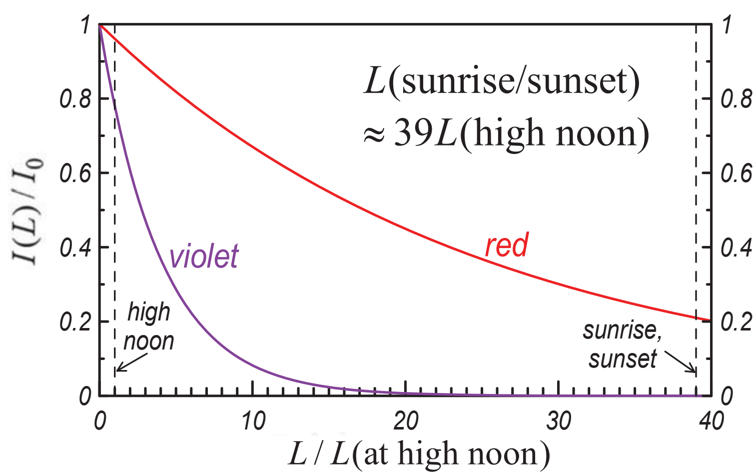
Molecules radiate independently, but *e*'s in each molecule radiate coherently [i.e. power $\propto$ (no. of molecular e's)^2]. If gas 1 and gas 2 have the same total no. of e's per unit volume ($\Rightarrow n_1, n_2$ largely the same), but $N_1(\text{gas 1}) = 10N_2(\text{gas 2})$, then a gas 2 molecule contains 10 times more e's and scatters 100 times more power than a gas 1 molecule. So gas 1, though 10 times higher in molecular density, scatters 10 times less total power. This is why $\alpha \propto \frac{1}{N}$ for a fixed *n*.

Atomspheric scattering:

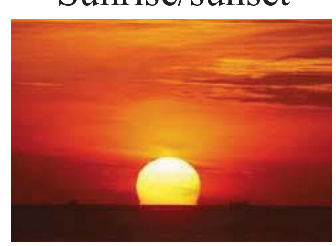
Sunlight enters air at $x = 0$ with intensity I_0 and passes a distance L to reach the ground at $x = L$ with intensity $I(L)$.



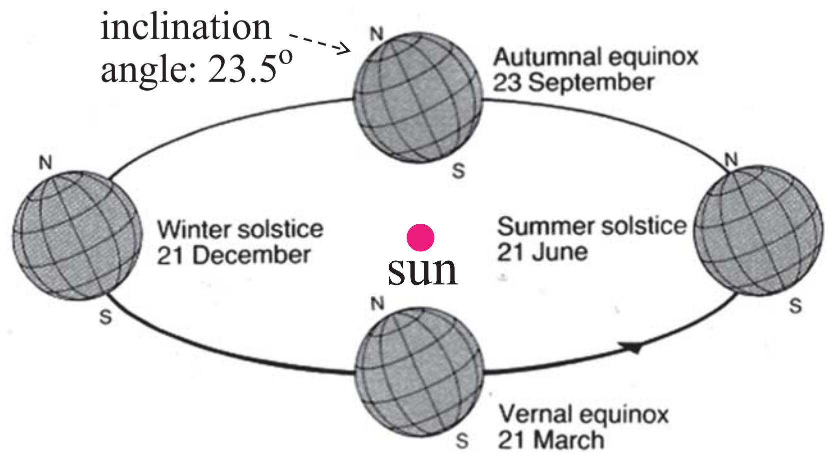
$$\frac{dI(x)}{dx} = -I(x)\alpha(x) \quad \Rightarrow \quad I(L) = I_0 e^{-\int_0^L \alpha(x)dx} \quad [\alpha(\text{violet}) \approx 6.3\alpha(\text{red})]$$



Sunrise/sunset



Astronauts orbiting the earth see a redder sunrise/sunset. Why?

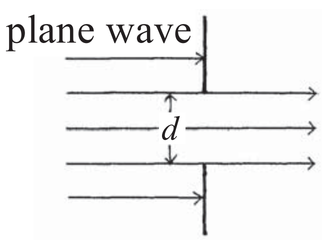


from "Atlas of the Solar System", Royal Astro. Soc.

Questions:

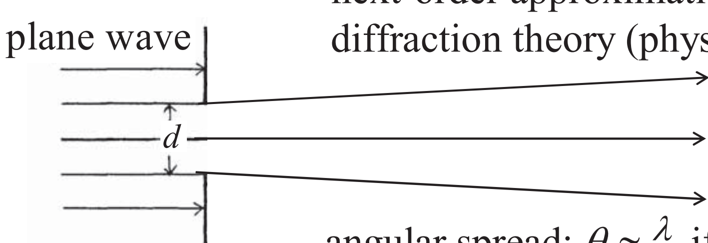
1. Why is it more likely to get a sunburn in the summer?
2. The sunlinght is scattered by the atmosphere, but its energy reaches the earth. What's the main reason for hot summer and cold winter?

10.5 Scalar Diffraction Theory



lowest-order approximation,
ray tracing (geometrical optics)

next-order approximation,
diffraction theory (physical optics)



angular spread: $\theta \approx \frac{\lambda}{d}$ if $d \gg \lambda$

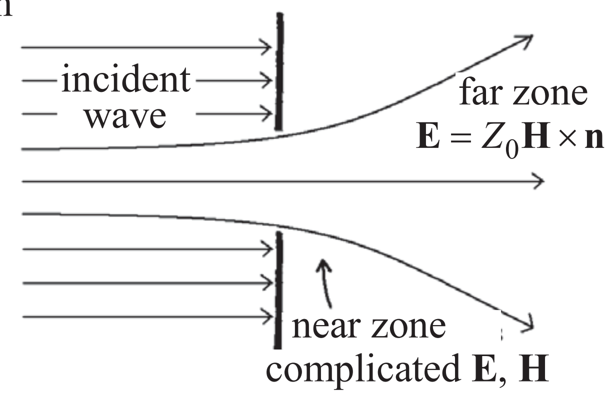
Nature of the Diffraction Problem: Scattering and diffraction theories both deal with fields due to induced $\mathbf{J}$ and ρ. In general, the treatments are different. In scattering, we obtain the fields from $\mathbf{J}$ and ρ directly. In diffraction, we obtain the fields indirectly through the b.c.'s to get the next-order correction to the geometrical optics, which works best when $\lambda \ll d$ (see p. 478).

Justification of the Scalar Diffraction Theory: Physically, **J** and ρ induced on the aperture material by the incident wave generate EM fields (also dissipating some of the incident wave). Far from the edges of the aperture, **J**, ρ principally result in reflection of the incident wave, whereas **J**, ρ near the edges of the aperture produce fields that enter into the diffraction region together with the incident wave. The superposed fields form the diffraction pattern.

In the far zone (distance from aperture >> d >> λ), the fields take the form of an EM wave:

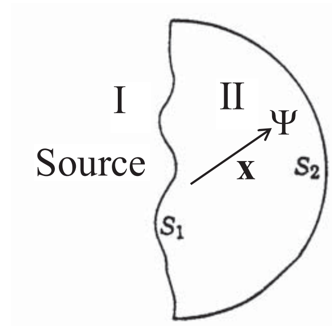
$$\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n} \quad \text{[see (9.19)]}$$

where Z_0 is the impedance of free space, and **n** is the direction of wave propagation.

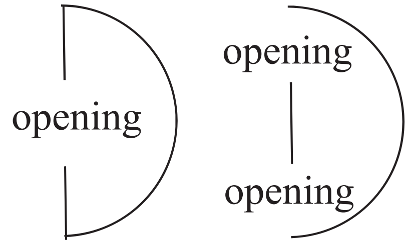


Thus, **E**, **H**, and **n** are mutually orthogonal, and the amplitudes of **E** and **H** have a known ratio Z_0. Therefore, one component of the fields gives most of the information (phase and intensity, but not the polarization) about the far fields. This justifies a scalar theory for the diffraction phenomenon and explains why it has been the basis of most of the work on diffraction.

The Kirchhoff Integral Formula: In Sec. 10.1, we calculate the *scattered* fields directly from the induced **J**, ρ, which radiate mostly through their dipole moments. In this section, the *diffracted* fields are generated by the induced **J**, ρ (mostly on the edges of the aperture), but **J**, ρ do not appear explicitly in field equations. They are implicit in boundary conditions. The Kirchhoff integral formula expresses the diffracted fields in terms of the boundary fields. Determination of the near-zone fields requires accurate handling of the b.c.'s (very few cases can be solved completely). However, the far-zone fields can be fairly accurately determined with crude b.c.'s.



Consider a volume (Region II in upper figure) enclosed by surfaces S_1 and S_2. There are opening(s) (lower figures) on S_1. S_2 is generally taken to be at ∞, in which case S_1 is an infinite surface.



The source (e.g. a plane or spherical wave) is incident on S_1 from the outside (Region I). It generates diffracted fields in Region II. Let $\Psi(\mathbf{x},t)=\Psi(\mathbf{x})e^{-i\omega t}$ be a component of $\mathbf{E}$ or $\mathbf{B}$ in Region II, which gives the phase and intensity of $\mathbf{E}$ or $\mathbf{B}$, but not the polarization. $\Psi(\mathbf{x})$ obeys

$$(\nabla^2+k^2)\Psi(\mathbf{x})=0,\ k=\frac{\omega}{c} \qquad (10.73)$$

We shall express $\Psi(\mathbf{x})$ in terms of Ψ, $\frac{\partial\Psi}{\partial n}$ on S_1, S_2 by using

Green's thm.: $\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n})da \quad (1.35)$

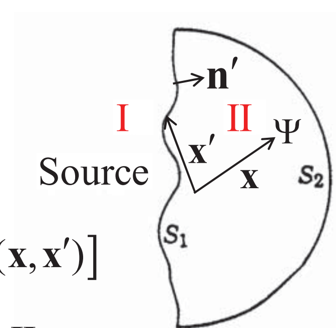
Apply $\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n})da \qquad (1.35)$

to Region II. Let $\psi=\Psi$ and $\phi=G(\mathbf{x},\mathbf{x}')=\frac{e^{ikR}}{4\pi R}$ $(\mathbf{R}=\mathbf{x}-\mathbf{x}')$ (10.76)

where $G(\mathbf{x},\mathbf{x}')$ satisfies $(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-\delta(\mathbf{x}-\mathbf{x}')$ (6.36), (10.74) with outgoing wave b.c.

$$\int_V d^3x'[G(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\Psi(\mathbf{x}')}^{-k^2\Psi(\mathbf{x}')}-\Psi(\mathbf{x}')\overbrace{\nabla'^2G(\mathbf{x},\mathbf{x}')}^{-k^2G(\mathbf{x},\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}')}]$$

$$=-\oint_{S_1+S_2}da'[G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')-\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')]$$



Hence, at an observation point $\mathbf{x}$ inside region II,

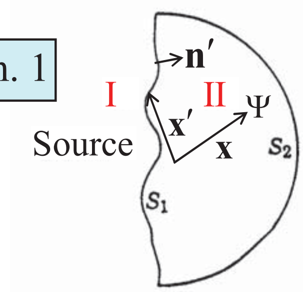
$$\Psi(\mathbf{x})=\oint_{S_1+S_2}da'[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')-G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')] \qquad (10.75)$$

Note: $\mathbf{n}'$ is inwardly directed into the volume instead of outwardly directed as in (1.35).

Rewrite

$$\Psi(\mathbf{x}) = \oint_{S_1+S_2} da'\left[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla' G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')\right] \quad (10.75)$$

$$G = \frac{e^{ikR}}{4\pi R} \Rightarrow \begin{cases} \frac{d}{dR}G = ik\frac{e^{ikR}}{4\pi R} - \frac{e^{ikR}}{4\pi R^2} \\ \nabla' R = \nabla'\left|\mathbf{x}-\mathbf{x}'\right| = -\nabla\left|\mathbf{x}-\mathbf{x}'\right| \overset{\boxed{\text{Eq. (1), Ch. 1}}}{=} -\frac{\mathbf{R}}{R} \end{cases}$$



$$\Rightarrow \nabla' G(\mathbf{x},\mathbf{x}') = \left(\frac{d}{dR}G\right)\nabla' R = -\frac{e^{ikR}}{4\pi R} ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}$$

$$\Rightarrow \Psi(\mathbf{x}) = \frac{-1}{4\pi}\oint_{S_1+S_2} da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \quad (10.77)$$

Let S_2 be at ∞. So Ψ on $S_2 \propto \frac{1}{R} \Rightarrow$ The S_2 portion of the integral in (10.77) is $\propto \frac{1}{R} \to 0$. (10.77) then gives the <u>Kirchhoff integral formula</u>:

$$\Psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \quad (10.79)$$

$\Psi(\mathbf{x})$ in Region II is now expressed in terms of Ψ and $\frac{\partial\Psi}{\partial n}$ on S_1.

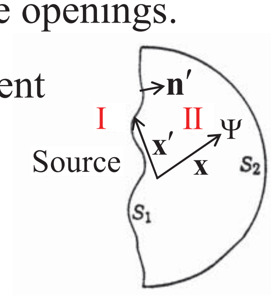
Note: Any opening on S_1 is part of S_1.

Kirchhoff Approximation: Rewrite (10.79),

$$\Psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \quad (10.79)$$

(10.79) is an integral equation for Ψ. It becomes a solution for Ψ under the Kirchhoff approximation, which consists of

1. Ψ and $\frac{\partial\Psi}{\partial n}$ vanish everywhere on S_1 except in the openings.
2. Ψ and $\frac{\partial\Psi}{\partial n}$ in the openings are those of the incident wave in the absence of any obstacles.

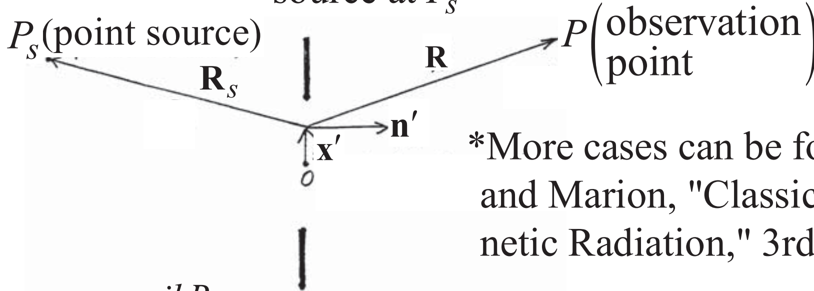


There are, however, mathematical inconsistencies with the Kirchhoff approximation:

1. If Ψ and $\frac{\partial\Psi}{\partial n}$ vanish on any finite surface, then $\Psi = 0$ everywhere (true for both Laplace and Helmholtz equations).
2. (10.79) does not yield on S_1 the assumed values of Ψ and $\frac{\partial\Psi}{\partial n}$.

Approximations made here work best for $\lambda \ll d$, and fail badly for $\lambda \sim d$ or $\lambda > d$ (d: size of the aperture or obstacle). See p.478.

*A Special Case**: Diffraction of spherical waves emitted by a point source at P_s



*More cases can be found in Heald and Marion, "Classical Electromagnetic Radiation," 3rd ed., Ch. 12.

$$\Psi(\mathbf{x}') = \frac{e^{ikR_s}}{R_s} \quad \text{[by Kirchhoff approximation]} \tag{12}$$

$$\Rightarrow \quad \nabla'\Psi(\mathbf{x}') = -\frac{e^{ikR_s}}{R_s} ik(1+\frac{i}{kR_s})\frac{\mathbf{R}_s}{R_s} \tag{13}$$

Sub. (12), (13) into

$$\Psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} da' \frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}') + ik(1+\frac{i}{kR})\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \tag{10.79}$$

and assume kR & $kR_s \gg 1$ and hence neglect $O(\frac{1}{kR})$ and $O(\frac{1}{kR_s})$ terms.

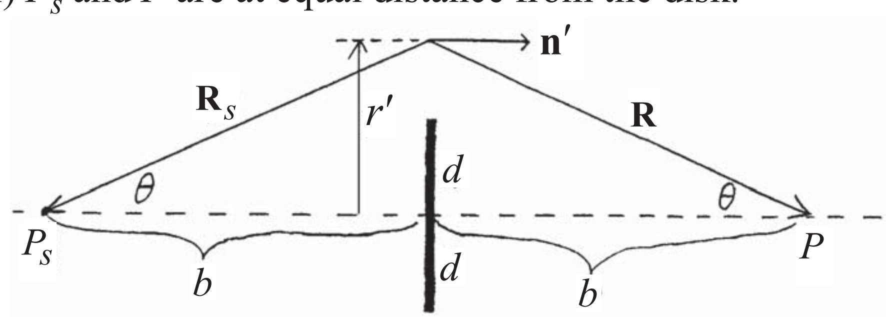
$$\Rightarrow \quad \Psi(P) = \frac{ik}{4\pi}\int_{S_1} da' \frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}'\cdot(\frac{\mathbf{R}_s}{R_s} - \frac{\mathbf{R}}{R}) \tag{14}$$

As we will find from the following example, the scalar diffraction theory agrees with observations, although it is highly artificial.

Example: Diffraction by a circular disk. For simplicity, we assume

(i) P_s and P are on the axis of the disk.

(ii) P_s and P are at equal distance from the disk.

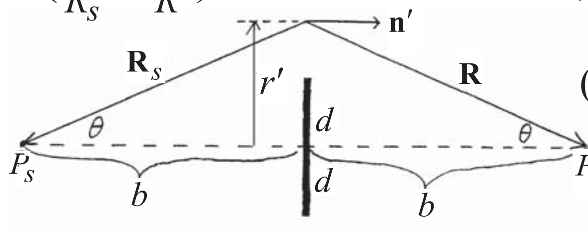


$$\Rightarrow \begin{cases} R_s = R \\ da' = 2\pi r'dr' \begin{pmatrix} R^2 = r'^2 + b^2 \Rightarrow \; r'dr' = RdR \\ \text{Hence, } da' = 2\pi RdR \end{pmatrix} \\ \mathbf{n}'\cdot\frac{\mathbf{R}_s}{R_s} = -\cos\theta = -\frac{b}{R}, \quad \mathbf{n}'\cdot\frac{\mathbf{R}}{R} = \cos\theta = \frac{b}{R} \end{cases} \tag{15}$$

Sub. $R_s = R$, $da' = 2\pi r' dr'$, $\mathbf{n}' \cdot \frac{\mathbf{R}_s}{R_s} = -\frac{b}{R}$, $\mathbf{n}' \cdot \frac{\mathbf{R}}{R} = \frac{b}{R}$ [(15)] into

$$\Psi(P) = \frac{ik}{4\pi}\int_{S_1} da' \frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}' \cdot \left(\frac{\mathbf{R}_s}{R_s} - \frac{\mathbf{R}}{R}\right) \quad (14)$$

$$\Rightarrow \Psi(P) = -ikb\int_{\sqrt{d^2+b^2}}^{\infty} \frac{e^{2ikR}}{R^2} dR \quad (16)$$



Integrating by parts

$$\left[\int_{a_1}^{a_2} u\,dv = uv\Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} v\,du, \text{ with } u = \frac{1}{R^2} \text{ and } dv = e^{2ikR} dR\right]$$

$$\Psi(P) = -ikb\left[\frac{e^{2ikR}}{2ikR^2}\Bigg|_{\sqrt{d^2+b^2}}^{\infty} + \frac{1}{2ik}\int_{\sqrt{d^2+b^2}}^{\infty} \frac{e^{2ikR}}{R^3} dR\right]$$

(integrating by parts again)

$$= -ikb\Bigg[\frac{e^{2ikR}}{2ikR^2}\Bigg|_{\sqrt{d^2+b^2}}^{\infty} - \underbrace{\frac{e^{2ikR}}{4k^2R^3}\Bigg|_{\sqrt{d^2+b^2}}^{\infty}}_{\text{negligible } (\because kR \gg 1)} + \cdots\Bigg] \approx \frac{be^{2ik\sqrt{d^2+b^2}}}{2(d^2+b^2)} \quad (17)$$

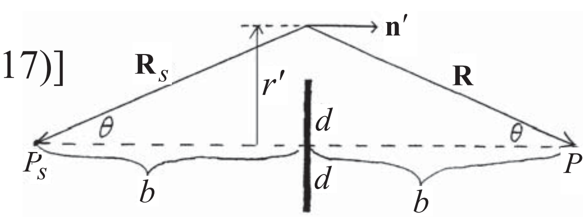
Questions:

(i) Intensity at P: $I(P) \propto |\Psi(P)|^2 = \frac{b^2}{4(d^2+b^2)^2}$ (18)

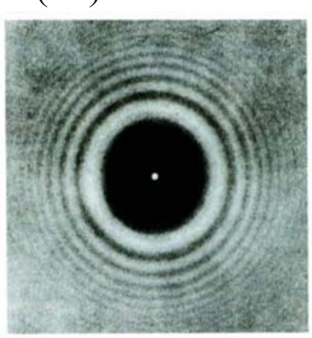
Since $I(P) > 0$ for all b, there is always a bright spot (Fresnel bright spot) at any point on the axis. What is the physical reason?

(ii) From $\Psi(P) \approx \frac{be^{2ik\sqrt{d^2+b^2}}}{2(d^2+b^2)}$ [(17)]

we get $\lim\limits_{d \to 0} \Psi(P) = \frac{e^{2ikR}}{2b}$



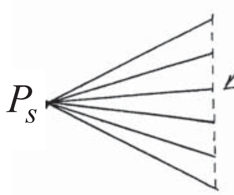
i.e. (17) becomes an exact solution. What is the mathematical reason?



← The diffraction pattern of a disk (from Halliday, Resnick, and Walker). Note the Fresnel bright spot at the center of the pattern. The concentric diffraction rings are not predictable by (18), which applies only to fields on the axis.

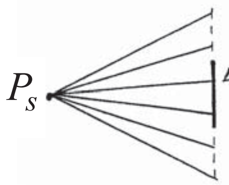
10.8 Babinet's Principle

Rewrite $\Psi(\mathbf{x}) = \frac{-1}{4\pi}\int_{S_1} da' \frac{e^{ikR}}{R}\mathbf{n}' \cdot \left[\nabla'\Psi(\mathbf{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right]$ (10.79)



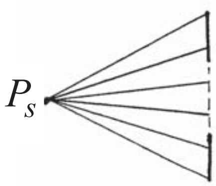
no diffraction screen, imagimary surface

$\cdot\ \Psi(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



diffraction screen

$\cdot\ \Psi_a(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



complementary diffraction screen

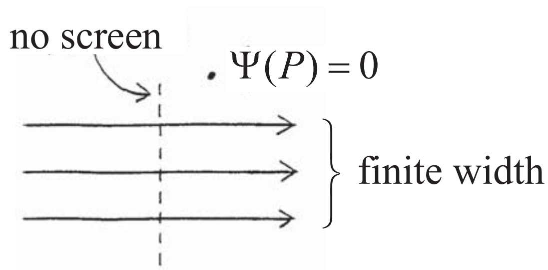
$\cdot\ \Psi_b(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$

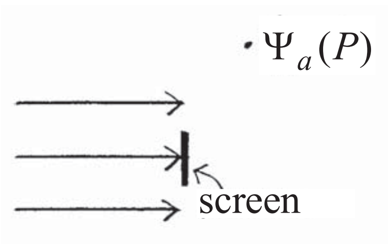
By Kirchoff's approx., for all 3 cases, Ψ, $\frac{\partial\Psi}{\partial n}$ in (10.79) are those of the unperturbed source P_s on dashed surfaces and 0 on obstacles.

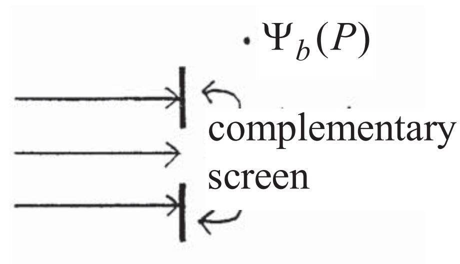
$\Rightarrow\ \Psi(P) = \Psi_a(P) + \Psi_b(P)$ [Babinet's principle]

10.8 Babinet's Principle (*continued*)

Example: a light beam of finite width





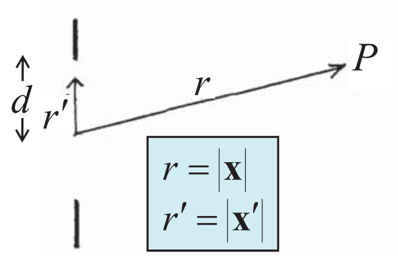


Babinet's principle: $\Psi(P) = \Psi_a(P) + \Psi_b(P) = 0$

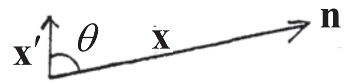
$$\Rightarrow\ \Psi_a(P) = -\Psi_b(P)$$

Fresnel and Fraunhofer Diffraction: (see p.491)

For $r \gg d$, in integrals such as (10.79), $R(=|\mathbf{x}-\mathbf{x}'|)$ can be approximated by $r(=|\mathbf{x}|)$ everywhere except in e^{ikR}, where the phase angle kR must be evaluated more accurately.



Consider 3 length scales: r, d, λ and apply the binomial expansion [Eq. (2), Ch. 1]



$$R = |\mathbf{x}-\mathbf{x}'| = (r^2 - 2rr'\cos\theta + r'^2)^{1/2}$$

$$= r[1-(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})]^{1/2} = r[1-\tfrac{1}{2}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2}) - \tfrac{1}{8}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})^2 + \cdots]$$

$$= r[1-\tfrac{\mathbf{n}\cdot\mathbf{x}'}{r} + \tfrac{1}{2}(\tfrac{r'^2}{r^2} - \tfrac{(\mathbf{n}\cdot\mathbf{x}')^2}{r^2}) + \cdots] = r - \mathbf{n}\cdot\mathbf{x}' + \tfrac{1}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2 + \cdots]$$

$$\Rightarrow \quad kR = O(kr) + O(kd) + O(\tfrac{kd^2}{r}) + \cdots$$

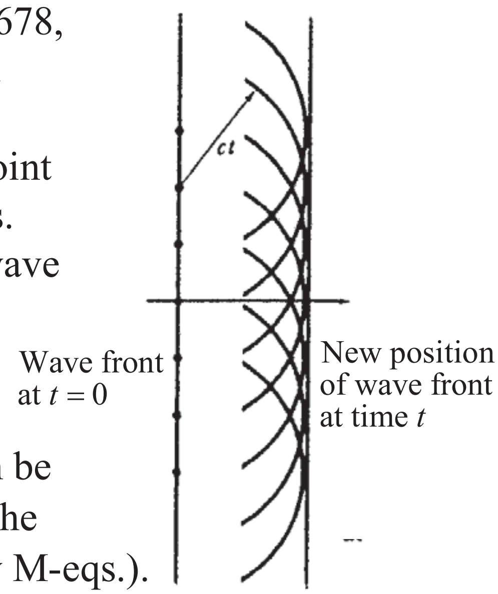
If the 3rd and higher terms are neglected, we have the Fraunhofer diffraction (far field). If the 3rd term is kept, but higher order terms are neglected, we have the Fresnel diffraction (near field).

Appendix A. Relevant College Physics Topics

(Ref.: Halliday, Resnick, and Walker, "Fundamentals of Physics")

Huygens' Principle: (proposed in 1678, 9 years before Newton's mechanics and 187 years before Maxwell's theory)

All points on a wave front serve as point sources of spherical secondary wavelets. After a time t, the new position of the wave front will be that of a surface tangent to these secondary wavelets.



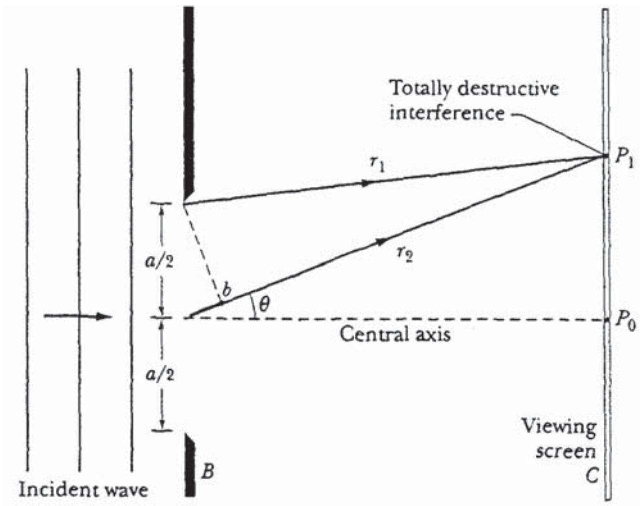
Interference and Diffraction:

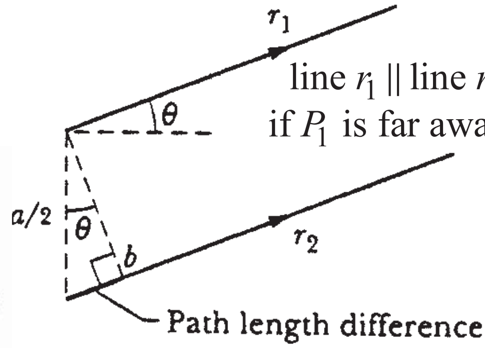
Both interference and diffraction can be interpreted by Huygens' principle plus the principle of superposition (not easily by M-eqs.).

Interference: superposition of two Huygens' wavelets.

Diffraction: superposition of a continuous distribution of Huygens' wavelets.

Superposition of Diffracted Waves : (Halliday, Resnick, and Walker)





line r_1 || line r_2 if P_1 is far away.

Total destructive interference occurs at point P_1, when

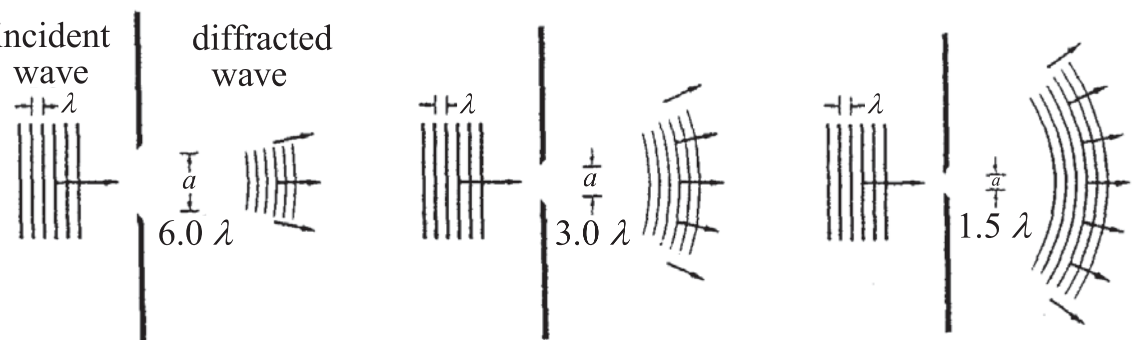
$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$ or $a\sin\theta = \lambda$

$\Rightarrow\ \theta \approx \frac{\lambda}{a}$, if $\theta \ll 1$

$\Rightarrow$ The smaller the $\frac{\lambda}{a}$ ratio, the narrower the angular width (θ) of the diffracted wave.

57

Geometrical Optics vs Physical Optics :



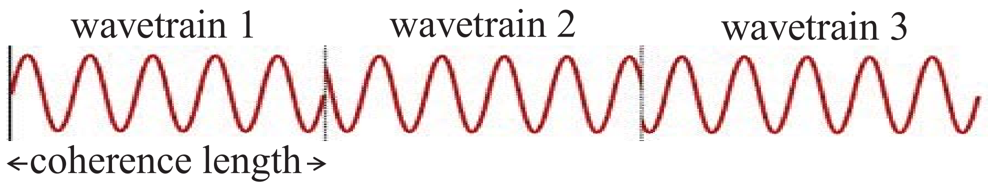
(from Halliday, Resnick, and Walker)

<u>Geometric optics:</u> EM waves travel in approximately straight lines if dimensions of the obstacle (such as dielectrics, mirrors and lenses) or aperture are much greater than the wavelength.

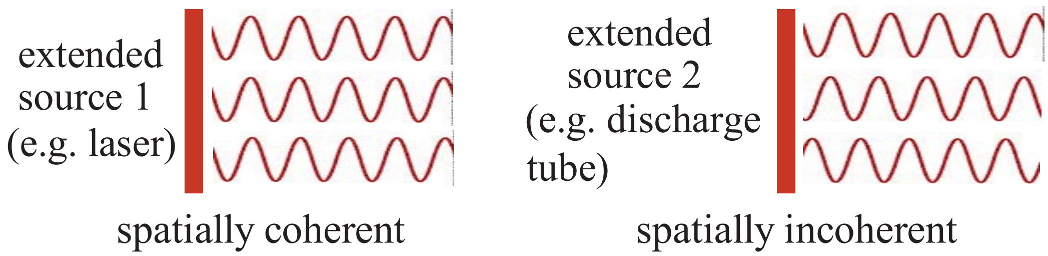
<u>Physical optics:</u> EM waves are diffracted if the dimensions of the obstacle or aperture are comparable or smaller than the wavelength.

Coherence:

1. A monochromatic (single ω) wave [$\sin(\omega t+\phi)$] is temporally coherent if its phase constant (ϕ) does not change in time.
   A light wave typically consists of a series of wavetrains, each being temporally coherent with a coherence length from 1-100 cm (discharge lamp) up to 30 km (the most coherent laser).

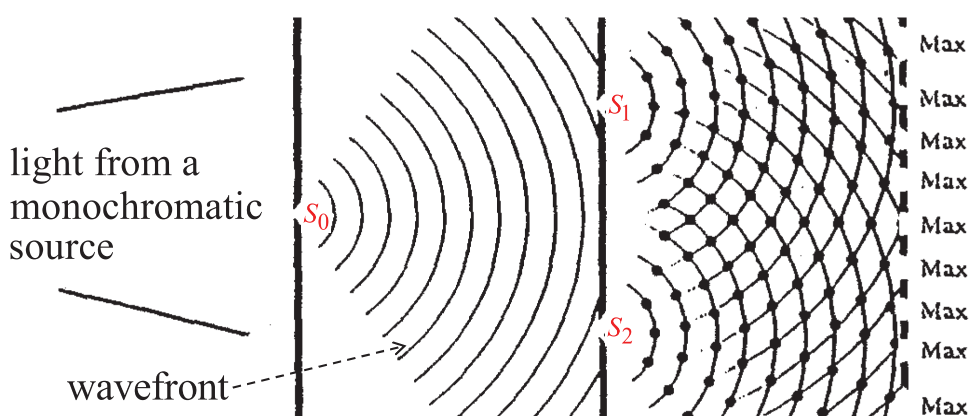


2. An extended source is spatially coherent if monochromatic waves emitted at all points are in phase (same ϕ).



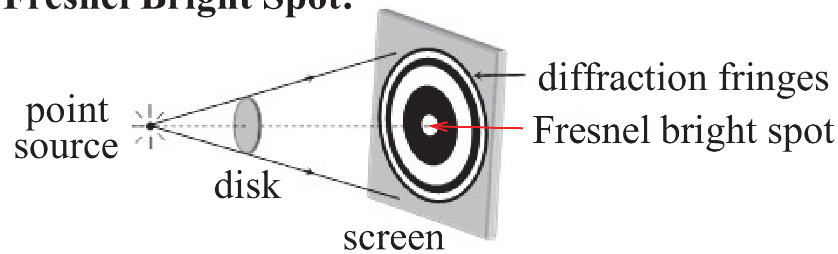
3. Two waves of the same ω [$\sin(\omega t+\phi_1)$, $\sin(\omega t+\phi_2)$] are temporally coherent with each other if their phase difference ($\phi_1-\phi_2$) remains constant in time (ϕ_1 and ϕ_2 are not necessarily constant in time, i.e. each wave may be temporally incoherent).

Young's experiment (1801):



The wave from S_0 may or may not be temporally coherent, but each wavefront is spatially coherent. Hence, the 2 waves from S_1 and S_2 are temporally coherent with each other.

The Fresnel Bright Spot:



Young's experiment did not put to rest the particle theory of light. In 1818 (47 years before Maxwell's theory), a 30-year old engineer Fresnel submitted a paper on the wave theory of light to a competition organized by the French Academy of Sciences. Poisson, one of the judges and a believer in the particle theory of light, pointed out the "absurd" consequence of Fresnel's wave theory by arguing that Huygens' wavelets from the edge of a disk will arrive in phase at the center of the disk's shadow on a screen, hence resulting in a bright spot there. Another judge, Arago, did an experiment which showed the bright spot. Fresnel won the competition. Interestingly, the bright spot is often referred to as the Poisson spot.