Chapter 3: Boundary-Value Problems in Electrostatics: II

We first cover some special functions commonly used in physics, with an emphasis on their properties.

See Secs. 3.2, 3.5, & 3.6 or M&W (Ch. 7) for their derivations.

Highly recommended handbooks:

1. Gradshteyn & Ryzhik, "Table of Integrals, Series, and Products".
2. Abramowitz & Stegun, "Handbook of Mathematical Functions".

3.2 Legendre Equation and Legendre Polynomials

Legendre Equation:

$$\frac{d}{dx}[(1-x^2)\frac{du}{dx}]+\nu(\nu+1)u=0, \qquad \overbrace{-1\le x\le 1}^{\text{range for most phys. problems}} \tag{3.10}$$

The Legendre eq. often appears in physics problems in spherical coordinates. It has the solution: $u(x)=AP_\nu(x)+BQ_\nu(x)$,

where $\begin{cases} P_\nu(x) \text{ is the } \underline{\text{Legendre function of the 1st kind.}} \\ Q_\nu(x) \text{ is the } \underline{\text{Legendre function of the 2nd kind.}} \end{cases}$ (3.11)-(3.14)

Rewrite $\begin{cases} \frac{d}{dx}[(1-x^2)\frac{du}{dx}]+\nu(\nu+1)u=0 \qquad -1\le x\le 1 \qquad (3.10) \\ u(x)=AP_\nu(x)+BQ_\nu(x) \;\; [P_\nu,\, Q_\nu\text{: linearly indep.}] \end{cases}$

$Q_\nu(x)$ diverges as $x\to\pm 1$ (p. 97, bottom). Hence, Q_ν appears in physics problems only when $x=\pm 1$ are outside the region of interest.

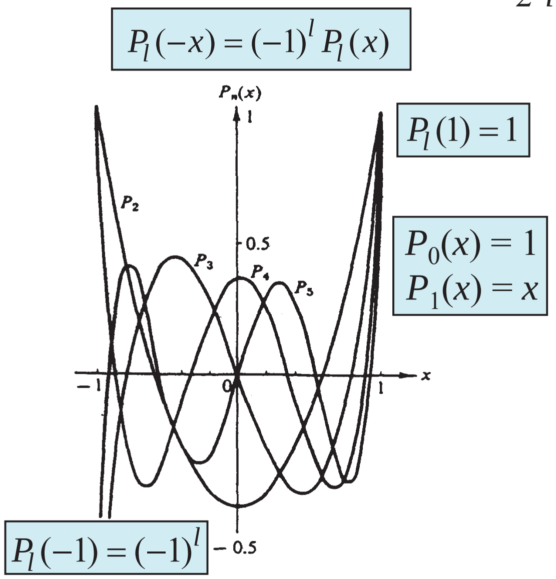
$P_\nu(x)$ $\begin{cases} \text{is finite for } |x|<1 \text{ and } x=1 \\ \text{diverges at } x=-1 \text{ unless } \nu \text{ is an integer} \end{cases}$ (see p. 105)

$\Rightarrow$ If the region of interest includes $x=-1$, the condition that $P_\nu(x)$ be finite at $x=-1$ requires ν to be an integer (denoted by l).

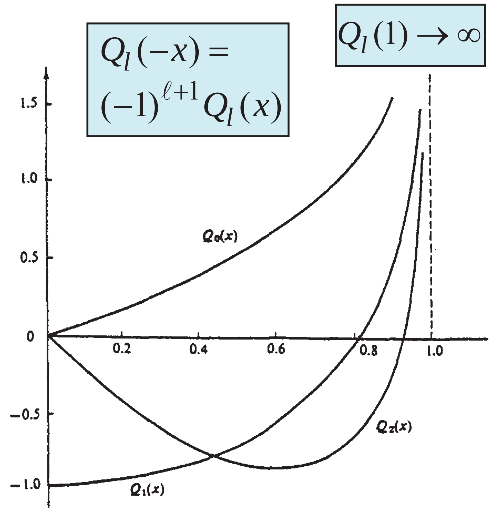
The form of the Legendre eq. is unchanged if $\nu\to-\nu-1$. Hence, $P_{-\nu-1}(x)=P_\nu(x)\Rightarrow$ When $\nu=l$ (an integer), negative l is redundant. Thus, $l=0,\ 1,\ 2\cdots$ for which $P_l(x)$ becomes a polynomial (next page).

Note: In physics, the argument x of $P_\nu(x)$ and $Q_\nu(x)$ is usually real and in the range $-1\le x\le 1$. In mathematics, the argument z of $P_\nu(z)$ and $Q_\nu(z)$ is in general complex ($z=x+iy$). ν is also in general a complex number (See Gradshteyn & Ryzhik, Secs. 8.7-8.9).

Legendre Polynomial : $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l, \ l = 0, 1, 2...$ (3.16)



Lengendre polynomials $P_2(x)$ - $P_5(x)$
Abramowitz & Stegun, p. 780



Second Lengendre functions $Q_0(x)$, $Q_1(x)$, and $Q_2(x)$
Abramowitz & Stegun, p. 339

The set $P_l(x)$ is orthogonal: $\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{l'l}$ (3.21)

It is complete in index $l \Rightarrow$ Any function $f(x)$ can be expanded as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \ [-1 \le x \le 1] \quad (3.23)$$

See (A.13-17) for general rules on "orthogonality" & "completeness".

3.5 Associated Legendre Functions and the Spherical Harmonics

Associated Legendre Equation :

$$\frac{d}{dx}[(1-x^2)\frac{du}{dx}] + [\nu(\nu+1) - \frac{m^2}{1-x^2}]u = 0 \ , \ \text{for } -1 \le x \le 1 \quad (3.9)$$

It has the solution: $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$, where

$$\begin{cases} P_\nu^m \text{ is the } \underline{\text{associated Legendre function of the 1st kind}}. \\ Q_\nu^m \text{ is the } \underline{\text{associated Legendre function of the 2nd kind}}. \end{cases} \quad (3.50)$$

Properties of P_ν, Q_ν, P_ν^m, and Q_ν^m can be found in Gradshteyn & Ryzhik (Secs. 8.7-8.9) and Abramowitz & Stegun (Ch. 8).

Rewrite $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$

$Q_\nu^m(x = \pm 1)$ diverges. Q_ν^m appears in physics problems only when $x = \pm 1$ is outside the region of interest (Q_ν, Q_ν^m not used in Jackson).

$P_\nu^m(x)$ is finite on the interval $-1 \le x \le 1$ only when

$$\begin{cases} \nu \text{ is zero or a positive integer } (\nu = l = 0,\ 1,\ 2...) \text{ and} \\ m = -l,\ -(l-1), \ldots,\ -1,\ 0,\ 1, \ldots,\ (l-1),\ l \end{cases} \quad \text{[p. 107]}$$

i.e. $u(x) = P_l^m(x)$ with $l = |m|,\ |m|+1,\ \ldots$[Assume m is a fixed integer]

Under these conditions, we have

$$P_l^m(x) = \tfrac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}} \left(\tfrac{d}{dx}\right)^{l+m} (x^2-1)^l \tag{3.50}$$

with the properties:
$$\begin{cases} P_l^0(x) = P_l(x) \\ P_l^m(-x) = (-1)^{l+m} P_l^m(x) \\ P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) & (3.51) \\ \int_{-1}^1 P_{l'}^m(x) P_l^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'} & (3.52) \end{cases}$$

The set $P_l^m(x)$ is complete in index l, i.e. any function $f(x)$ can be expanded as $\quad f(x) = \sum_{l=|m|}^{\infty} C_l P_l^m(x) \begin{bmatrix} -1 \le x \le 1 \\ m: \text{ a fixed integer} \end{bmatrix}$

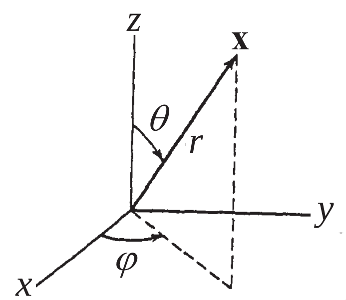
Question: Why is $P_l^m(x)$ complete in index l (not m)? See (A.20).

Spherical Harmonics $Y_{lm}(\theta, \varphi)$:

$$Y_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}, \tag{3.53}$$

where $l = 0$ or a positive integer; $m = -l,\ -(l-1), \ldots,\ 0, \ldots,\ (l-1),\ l$

Examples:
$$\begin{cases} Y_{0,0}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \\ Y_{1,-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi} \\ Y_{1,0}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_{1,1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \end{cases}$$



Properties of spherical harmonics:

Rewrite the spherical harmonics:

$$Y_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi} \quad [(3.53)]$$

(i) Using the orthogonality relation,

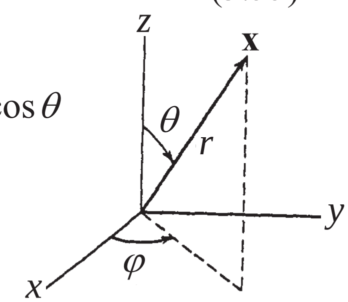
$$\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} \qquad (3.52)$$

we can show that the spherical harmonics are orthonormal, i.e

$$\int d\Omega Y_{l'm'}^*(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}, \qquad (3.55)$$

where

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta$$



(ii) The set $Y_{lm}(\theta,\varphi)$ is complete, i.e. any function $g(\theta,\varphi)$ can be expanded as

$$g(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_{lm}(\theta,\varphi) \qquad (3.58)$$

Multiply both sides by $Y_{lm}^*(\theta,\varphi)$, integrate over θ,φ, and make use of $\int d\Omega Y_{l'm'}^*(\theta,\varphi)Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$ [(3.55)].

$$\Rightarrow \quad A_{lm} = \int d\Omega Y_{lm}^*(\theta,\varphi)g(\theta,\varphi)$$

Sub. A_{lm} into (3.58) gives the following expression for $g(\theta,\varphi)$:

$$g(\theta,\varphi) = \int d\Omega' \left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]g(\theta',\varphi')$$

$$\Rightarrow \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta') \qquad (3.56)$$

Note: 1. This is a 2-D example of the general relation in (2.35).

2. (3.56) [as (2.35)] shows that an infinite sum of smooth functions $Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$ can add up to a singularity as $\theta,\ \varphi \to \theta',\ \varphi'$.

3.5 Associated Legendre Functions and the Spherical Harmonics (*continued*)

(iii) Other properties of $Y_{lm}(\theta,\varphi)$:

$$\begin{cases} Y_{l,-m}(\theta,\varphi)=(-1)^m Y_{lm}^*(\theta,\varphi) \\ Y_{l,0}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta) \end{cases}$$

This can be seen from the definition of $Y_{lm}(\theta,\varphi)$:

$$Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi} \tag{3.53}$$

and the relations:

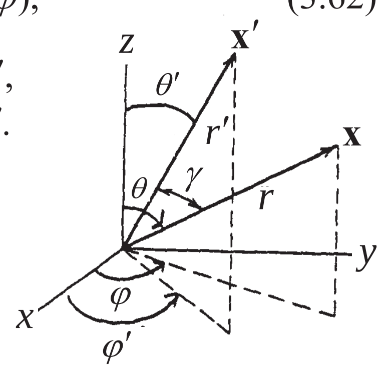
$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \tag{3.51}$$

$$P_l^0(x)=P_l(x)$$

3.6 Addition Theorem for Spherical Harmonics

The addition theorem for spherical harmonics is derived on pp. 110-111. Here we write the theorem without derivation:

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi), \tag{3.62}$$



where (θ,φ), (θ',φ') are directions of $\mathbf{x}$, $\mathbf{x}'$, respectively. γ is the angle between $\mathbf{x}$ & $\mathbf{x}'$.

Setting $l=1$ in (3.62) gives

$$P_1(\cos\gamma)=\frac{4\pi}{3}[Y_{1,-1}^*(\theta',\varphi')Y_{1,-1}(\theta,\varphi)$$
$$+Y_{1,0}^*(\theta',\varphi')Y_{1,0}(\theta,\varphi)$$
$$+Y_{1,1}^*(\theta',\varphi')Y_{1,1}(\theta,\varphi)]$$

Using $P_1(\cos\gamma)=\cos\gamma$, $Y_{1,-1}=\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$, $Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\theta$,

and $Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, we obtain a useful expression:

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{1}$$

Bessel Functions (see Sec. 3.7)

The Bessel eq. often appears in physics problems in cyclidrical coordinates. It has the form $\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + (1 - \frac{\nu^2}{x^2})u = 0$ (3.77)

with the solutions $\begin{cases} J_\nu(x): \text{ \underline{Bessel function of the 1st kind}} & (3.82) \\ N_\nu(x): \text{ \underline{Bessel function of the 2nd kind}} & (3.85) \end{cases}$

From $J_\nu(x)$ and $N_\nu(x)$, we may define the Hankel functions:

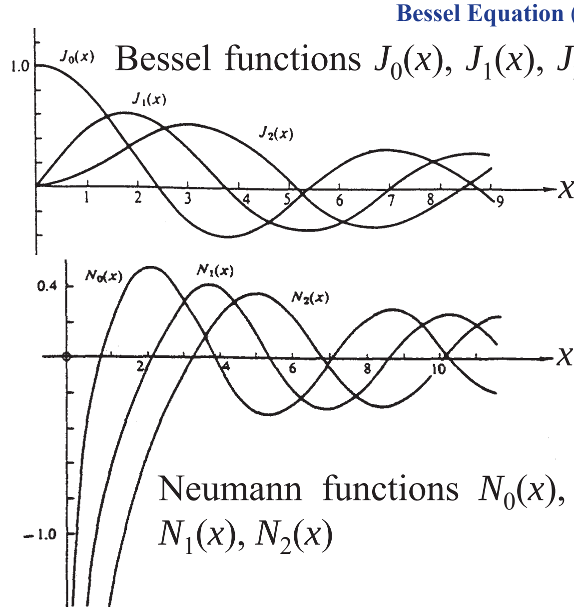
$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \end{cases} \qquad (3.86)$$

and the modified Bessel functions of the 1st kind (I_ν) and 2nd kind (K_ν). I_ν and K_ν are Bessel functions of imaginary argument.

$$\begin{cases} I_\nu(x) = i^{-\nu} J_\nu(ix) & (3.100) \\ K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

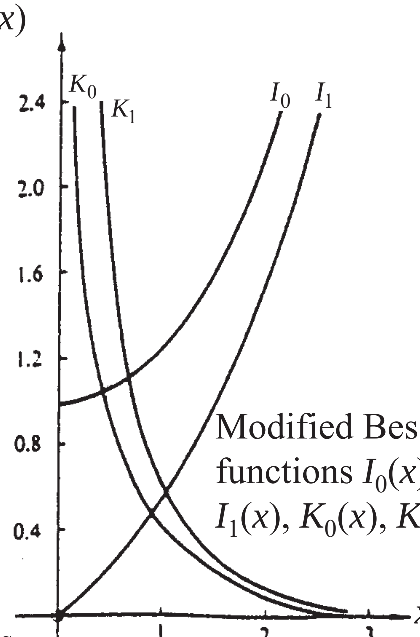
$I_\nu(x)$ & $K_\nu(x)$ are solus. of (3.77) with $x \to ix$ (used in Sec. 3.11)

Properties of these funcs. can be found on pp. 112-116, Gradshteyn & Ryzhik (Secs. 8.4-8.5), and Abramowitz & Stegun (Ch. 9).



Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$

Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$



Modified Bessel functions $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$

$N_\nu(x)$ and $K_\nu(x)$ diverge as $x \to 0$. Hence, they appear in physics problems only when $x = 0$ is outside the region of interest.

(Figures from Abramowitz & Stegun)

In $\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+(1-\frac{\nu^2}{x^2})u=0$ [(3.77)], replacing x with $k\rho$ gives a 2nd form of the Bessel eq.: $\frac{d^2u}{d\rho^2}+\frac{1}{\rho}\frac{du}{d\rho}+(k^2-\frac{\nu^2}{\rho^2})u=0$ (3.75)

with the solution: $u(\rho)=AJ_\nu(k\rho)+BN_\nu(k\rho)$. Assume the following

b.c.'s: $\begin{cases} 1:\ u(0)=\text{finite} \Rightarrow B=0; \\ 2:\ u(a)=0 \Rightarrow J_\nu(ka)=0 \Rightarrow ka=x_{\nu n},\ n=1,2,3\cdots, \text{ where} \\ \quad x_{\nu n} \text{ is the } n\text{-th root of } J_\nu(x)=0 \text{ (see p. 114).} \\ \quad \Rightarrow J_\nu(k\rho)=J_\nu(k_{\nu n}\rho), \text{ where } k_{\nu n}\equiv x_{\nu n}/a,\ n=1,2,3\cdots \end{cases}$

The $J_\nu(k_{\nu n}\rho)$ set are

Why the factor ρ here? See (A.22)

$\begin{cases} \text{orthogonal: } \int_0^a J_\nu(k_{\nu n'}\rho)J_\nu(k_{\nu n}\rho)\rho d\rho=\frac{a^2}{2}[J_{\nu+1}(\underbrace{k_{\nu n}a}_{x_{\nu n}})]^2\delta_{n'n} & (3.95) \\ \text{complete: } f(\rho)=\sum\limits_{n=1}^{\infty}C_nJ_\nu(k_{\nu n}\rho) \text{ for any } f(\rho) & (3.96) \end{cases}$

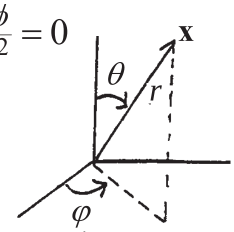
Questions: 1. (3.96) regards $J_\nu(k_{\nu n}\rho)$ as a complete set, but p.114 says "$\sqrt{\rho}\ J_\nu(k_{\nu n}\rho)$ form an orthogonal set". Any inconsistency? See (A.22).
2. Why are $J_\nu(k_{\nu n}\rho)$ orthogonal/complete in index n (not ν)? See (A.23).

3.1 Laplace Equation in Spherical Coordinates

(We will first cover Appendix A before going into this section)

Laplace eq. in spherical coordinates (see Jackson, back cover):

$$\nabla^2\phi(\mathbf{x})=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}=0$$



Let $\phi(\mathrm{x})=\frac{U(r)}{r}P(\theta)Q(\varphi)$

$$\Rightarrow PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

Multiply by $\frac{r^2\sin^2\theta}{UPQ}$

The φ-dependence is isolated within this term, so this term must be a constant. Let it be $-m^2$.

$$\Rightarrow \sin^2\theta\,[\underbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}_{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0 \qquad (3.3)$$

Dividing all terms by $\sin^2\theta$, we see that the r-dependence is isolated within this term. So this term must be a constant. Let it be $\nu(\nu+1)$.

$$\text{Rewrite } \sin^2\theta\,[\overbrace{\tfrac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0\ [(3.3)]$$

The equation for $Q(\varphi)$ is: $\dfrac{d^2Q}{d\varphi^2}+m^2Q=0$ $\begin{bmatrix}\text{an eigenvalue}\\ \text{problem}\end{bmatrix}$ (3.4)

$$\Rightarrow Q=e^{im\varphi},e^{-im\varphi} \quad (3.5)$$

m is to be determined from the b.c.

The equation for $P(\theta)$ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\nu(\nu+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \quad (3.6)$$

Let $x=\cos\theta$, then the equation takes the form of the associated Legendre equation:

$$\frac{d}{dx}(1-x^2)\frac{dP}{dx}+\left[\nu(\nu+1)-\frac{m^2}{1-x^2}\right]P=0 \quad \begin{bmatrix}\text{an eigenvalue}\\ \text{problem}\end{bmatrix} \quad (3.9)$$

$$\Rightarrow P=\begin{Bmatrix}P_\nu^m(x)\\ Q_\nu^m(x)\end{Bmatrix}=\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}$$

ν is to be determined from the b.c.

$$\text{Rewrite } \sin^2\theta\,[\overbrace{\tfrac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0\ [(3.3)]$$

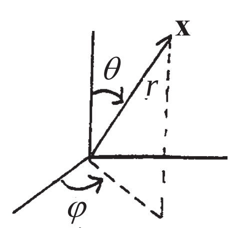
The equation for $U(r)$ is: $\dfrac{d^2U}{dr^2}-\dfrac{\nu(\nu+1)}{r^2}U=0$ (3.7)

$$\Rightarrow U=r^{\nu+1},r^{-\nu}\Rightarrow\frac{U}{r}=r^{\nu},r^{-\nu-1}$$

Since ν is determined from the b.c. for (3.6), this is not an eigenvalue problem.

Thus, $\nabla^2\phi(\mathbf{x})=0$

$$\Rightarrow\ \phi=\begin{Bmatrix}r^{\nu}\\ r^{-\nu-1}\end{Bmatrix}\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}e^{im\varphi}\\ e^{-im\varphi}\end{Bmatrix}, \quad (2)$$



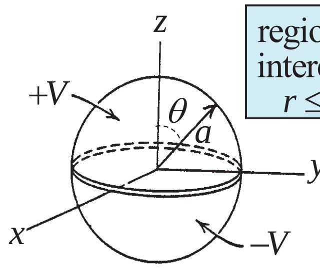
where each bracket represents a linear combination of the two functions inside [because $\nabla^2\phi(\mathbf{x})=0$ is linear and homogeneous].

Note: (2) is the solution of $\nabla^2\phi(\mathbf{x})=0$ without consideration of b.c.'s. ν and m in (2) are arbitrary constants until we apply the b.c.'s.

3.3 Boundary-Value Problems with Azimuthal Symmetry

Problem 1: Find ϕ *inside* 2 hemispheres held at opposite potentials (This will result in $E=\infty$ at $r=a$ and $\theta=\pi/2$, hence unrealistic.)

$$\nabla^2\phi=0,\quad \phi(a,\theta)=\begin{cases}+V, & 0\le\theta<\frac{\pi}{2}\\ -V, & \frac{\pi}{2}<\theta\le\pi\end{cases}$$



region of interest: $r\le a$

$$\phi=\begin{Bmatrix}r^{\nu}\\ r^{-\nu-1}\end{Bmatrix}\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}e^{im\varphi}\\ e^{-im\varphi}\end{Bmatrix}$$

(i) ϕ is indep. of φ. $\Rightarrow m=0$

(ii) ϕ is finite at $\theta=0$ and π.

$\Rightarrow$ Eigenvalue prob. in θ [see (A.18)]

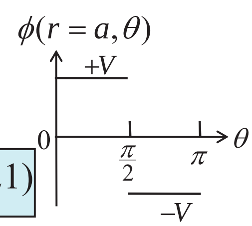
$\Rightarrow\ \nu=l=0,\ 1,\ 2,\ldots$ and drop Q_ν^m

(iii) ϕ is finite at $r=0$. $\Rightarrow$ drop $r^{-\nu-1}$

$$\Rightarrow\quad \phi(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

Note:
1. $P_\nu(-1)\to\infty$ unless ν is an integer (p.105.)
2. There is no negative l because $P_{-l-1}(x)=P_l(x)$.
3. $Q_\nu(x)\to\infty$ as $x\to\pm1$.

$$\text{b.c.}\ \phi(r=a,\theta)=\sum_l A_l a^l P_l(\cos\theta)=\begin{cases}+V, & 0\le\theta<\frac{\pi}{2}\\ -V, & \frac{\pi}{2}<\theta\le\pi\end{cases}$$



$$\boxed{\int_{-1}^{1}P_l(x)P_{l'}(x)dx=\tfrac{2}{2l+1}\delta_{ll'}\quad(3.21)}$$

$$\Rightarrow\int_{-1}^{1}P_l(\cos\theta)\phi(r=a,\theta)d\cos\theta=A_l a^l\underbrace{\int_{-1}^{1}P_l^2(\cos\theta)d\cos\theta}=A_l a^l\,\tfrac{2}{2l+1}$$

$$\Rightarrow A_l=\tfrac{V}{a^l}\tfrac{2l+1}{2}\left[\int_0^1P_l(\cos\theta)d\cos\theta-\int_{-1}^{0}P_l(\cos\theta)d\cos\theta\right]$$

$$=\begin{cases}\dfrac{V}{a^l}\dfrac{\left(-\frac{1}{2}\right)^{\frac{l-1}{2}}(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}, & \text{for odd } l\\ 0, & \text{for even } l\end{cases}$$

$\boxed{(2n+1)!!=(2n+1)(2n-1)(2n-3)\cdots\times3\times1}$

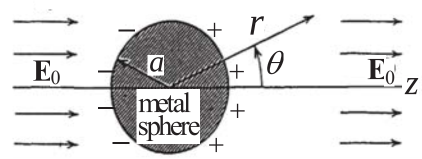
$\boxed{\text{pp. 99-100}}$

$$\Rightarrow\phi(r,\theta)=V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3(\cos\theta)+\cdots\right],\quad r\le a\qquad(3.36)$$

To find ϕ for $r>a$, replace $\left(\frac{r}{a}\right)^l$ in (3.36) by $\left(\frac{a}{r}\right)^{l+1}$ [see (2.27)]

Problem 2: A conducting sphere of radius a with net charge Q is in a uniform $\mathbf{E}_0$ ($=E_0\mathbf{e}_z$). Find ϕ $(r \geq a)$ and σ on the surface.

$$\rho = 0 \Rightarrow \phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_{\nu}^{m}(\cos\theta) \\ Q_{\nu}^{m}(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$



(i) ϕ is indep. of φ. $\Rightarrow m = 0$

(ii) ϕ is finite at $\theta = 0$ and π. $\Rightarrow$ Eigenvalue prob. in θ [see (A.18)]

$\Rightarrow \ \nu = l = 0,\ 1,\ 2, \ldots$ and drop Q_{ν}^{m}

$$\Rightarrow \ \phi(r,\theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta)$$

This term gives $\mathbf{E}_0$ (external field)

b.c.: As $r \to \infty$, $\phi = \underbrace{-E_0 r\cos\theta}_{\text{external field}} + \underbrace{\frac{Q}{4\pi\varepsilon_0 r}}_{\text{far field of net chrage } Q} = \overbrace{-E_0 z} + \frac{Q}{4\pi\varepsilon_0 r}$

$$\text{Thus, } \begin{cases} P_1(\cos\theta) = \cos\theta \Rightarrow A_1 = -E_0,\ A_{l\neq 1} = 0 \\ P_0(\cos\theta) = 1 \Rightarrow B_0 = \dfrac{Q}{4\pi\varepsilon_0} \quad (B_{l\neq 0} \text{ yet to be determined}) \end{cases}$$

Question: The b.c. shows $\phi \to \infty$ as $z \to \infty$. What's the reason?

$$\Rightarrow \phi(r,\theta) = -E_0 r\cos\theta + \frac{Q}{4\pi\varepsilon_0 r} + \underbrace{\sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)}$$

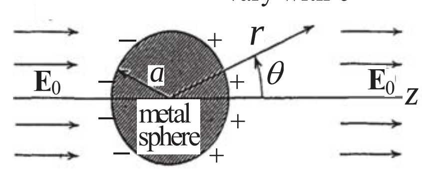
Combine the $P_1(\cos\theta) = \cos\theta$ term here with $-E_0 r\cos\theta$.

b.c.: $\phi(r = a) = const.$

$$\Rightarrow \phi(r = a) = \underbrace{(-E_0 a + \frac{B_1}{a^2})}_{0} \underbrace{\cos\theta}_{\substack{\text{vary} \\ \text{with } \theta}} + \frac{Q}{4\pi\varepsilon_0 a} + \sum_{l=2}^{\infty} \underbrace{B_l}_{0} a^{-(l+1)} \underbrace{P_l(\cos\theta)}_{\text{vary with } \theta}$$

For $\phi(r = a) = const.$ (i.e. indep. of θ), we must have

$B_1 = E_0 a^3$ and $B_{l\geq 2} = 0$



$$\Rightarrow \phi(r,\theta) = -E_0 r\cos\theta + \frac{Q}{4\pi\varepsilon_0 r} + \underbrace{E_0 \frac{a^3}{r^2}\cos\theta}_{\text{due to induced } \sigma \text{ on the sphere}} \tag{3}$$

$$\begin{cases} \mathbf{E}(\text{inside}) = 0 \\ \text{\& Gauss's law} \end{cases} \Rightarrow \sigma = -\varepsilon_0 \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2} \quad \text{[see (1.22)]}$$

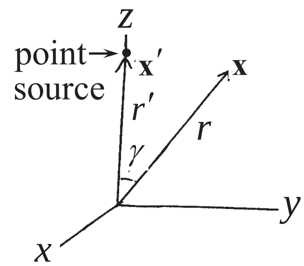
Questions: 1. The field inside the sphere due to σ is $-\mathbf{E}_0$. Why?

2. Why is Q uniformly distributed? (See the prob. in Sec. 2.3).

Problem 3: Find ϕ of $\nabla^2\phi = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ in infinite space.

First, assume the point source lies on the z-axis at a distance r' from the origin and divide the space into 2 regions: $r < r'$ and $r > r'$.

Since the source is on the boundary ($r = r'$), we have $\nabla^2\phi = 0$ in each region, both having the solution: $\phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\gamma) \\ Q_\nu^m(\cos\gamma) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$



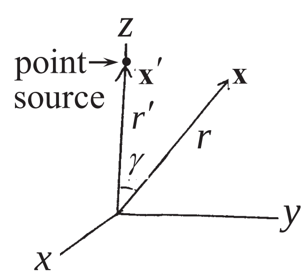
(i) ϕ is indep. of φ. $\Rightarrow$ $m = 0$

(ii) ϕ is finite at $\gamma = 0$ and π. $\Rightarrow \nu = l = 0, 1, 2, \ldots$ and drop Q_ν^m

(iii) ϕ is finite $\begin{cases} \text{at } r = 0. & \Rightarrow \text{drop } r^{-l-1} \text{ in region } r < r' \\ \text{as } r \to \infty. & \Rightarrow \text{drop } r^{l} \text{ in region } r > r' \end{cases}$

$$\Rightarrow \phi = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma), & r < r' \\ \sum\limits_{l=0}^{\infty} B_l \dfrac{1}{r^{l+1}} P_l(\cos\gamma), & r > r' \end{cases}$$

The formal method to solve for A_l and B_l (hence obtain the solution for all $\mathbf{x}$) is to match the b.c. at $r = r'$ (as will be done in Sec. 3.9). Here we obtain A_l and B_l by exploiting the fact that we already know the solution is ϕ = 1/ |**x**–**x**'| [(1.31)]. So, by the uniqueness theorem, we have



$$\phi = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma) & , \ r < r' \\ \sum\limits_{l=0}^{\infty} B_l \dfrac{1}{r^{l+1}} P_l(\cos\gamma) , & r > r' \end{cases}$$

For $\gamma = 0$, we have $P_l(1) = 1$ and $|\mathbf{x}-\mathbf{x}'| = |r - r'|$. Hence,

$$\frac{1}{|r-r'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l & , \ r < r' \\ \sum\limits_{l=0}^{\infty} B_l \dfrac{1}{r^{l+1}} , & r > r' \end{cases}$$

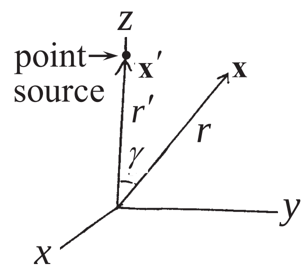
$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$

Let $n=-1$, $x=1$, and $y = r/r'$ or r'/r.

$$\Rightarrow \frac{1}{|r-r'|} = \begin{cases} \dfrac{1}{r'-r} = \dfrac{1}{r'}\dfrac{1}{1-\frac{r}{r'}} = \dfrac{1}{r'}\sum\limits_{l=0}^{\infty}\left(\dfrac{r}{r'}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}} = \sum\limits_{l=0}^{\infty} A_l r^l, & r<r' \\[2ex] \dfrac{1}{r-r'} = \dfrac{1}{r}\dfrac{1}{1-\frac{r'}{r}} = \dfrac{1}{r}\sum\limits_{l=0}^{\infty}\left(\dfrac{r'}{r}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}} = \sum\limits_{l=0}^{\infty} B_l \dfrac{1}{r^{l+1}}, & r>r' \end{cases}$$

from last page

$$\Rightarrow A_l = \frac{1}{r'^{l+1}};\ B_l = r'^l$$

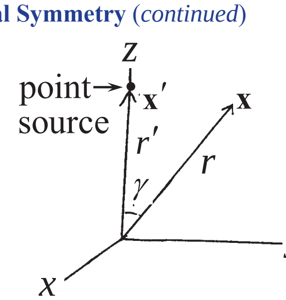


$$\Rightarrow \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}P_l(\cos\gamma), & r<r' \\[2ex] \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}P_l(\cos\gamma), & r>r' \end{cases}$$

or $$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma), \quad \text{[two equations in one]} \qquad (3.38)$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r'.

Rewrite $$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma), \qquad (3.38)$$

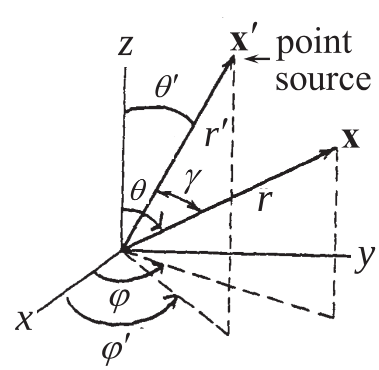


which is derived with the point source located on the z-axis (upper figure). Note that each term on the RHS of (3.38) is a smooth function of $\mathbf{x}$ satisfying $\nabla^2\phi(\mathbf{x})=0$ in regions $r>r'$ and $r<r'$, but they add up to a sigularity as $\mathbf{x}$ approaches $\mathbf{x}'$ from any direction.

The RHS of (3.38) depends only on

$$\begin{cases}\text{(1) the magnitudes } (r,\ r') \text{ of } \mathbf{x} \text{ and } \mathbf{x}' \\ \text{(2) the angle } (\gamma) \text{ between } \mathbf{x} \text{ and } \mathbf{x}'\end{cases},$$

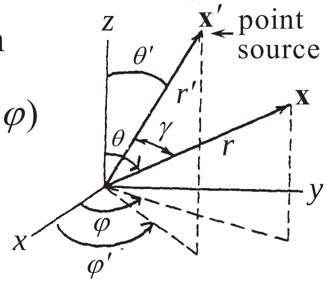
which suggests that we may convert (3.38) into a general form which holds for the point source at an arbitrary point (lower figure).



The general form may be readily obtained by way of the addition theorem (next page).

Sub. the RHS of the addition theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.62)$$



for $P_l(\cos\gamma)$ in $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \dfrac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$,

we get $$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.70)$$

So, we started with a physics problem (the potential of a point source in infinite space), but ended up with a mathematical relation in (3.70).

Question: Why write a simple function $\phi = \dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$ in such a complicated form? (See next problem.)

Problem 4: Find the potential due to total charge q, which is uniformly distributed on a circular ring of radius a.

Let $\rho(\mathbf{x}) = K\delta(\theta-\alpha)\delta(r-c)$ in spherical coordinates

$$q = \int \rho(\mathbf{x})d^3x$$

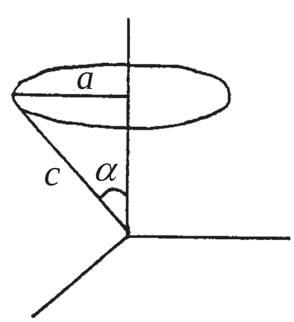
$$= K\int \delta(\theta-\alpha)\delta(r-c)\overbrace{r^2\sin\theta drd\theta d\varphi}^{d^3x}$$

$$= 2\pi Kc^2\sin\alpha$$

$$\Rightarrow \quad K = \frac{q}{2\pi c^2\sin\alpha}$$

$$\Rightarrow \rho(\mathbf{x}) = \frac{q}{2\pi c^2\sin\alpha}\delta(\theta-\alpha)\delta(r-c)$$

$$= \frac{q}{2\pi c^2}\delta(\cos\theta-\cos\alpha)\delta(r-c)$$



$$\boxed{\delta[f(x)] = \frac{\delta(x-a)}{|f'(a)|}}$$

$$\rho(\mathbf{x}') = \frac{q}{2\pi c^2}\delta(\cos\theta' - \cos\alpha)\delta(r'-c)$$
$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$
$$= \frac{q}{2\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\int_V r'^2 dr' d\cos\theta' d\varphi' \begin{bmatrix} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \\ \cdot\delta(\cos\theta'-\cos\alpha)\delta(r'-c) \end{bmatrix},$$

where $Y_{lm}(\theta',\varphi') = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta')e^{im\varphi'}$ [(3.53)]

Use $\int_0^{2\pi} e^{im\varphi'}d\varphi' = \begin{cases} 0, & \text{if } m \neq 0 \\ 2\pi, & \text{if } m = 0 \end{cases}$ and $P_l^{m=0}(\cos\theta') = P_\ell(\cos\theta')$

$$\Rightarrow \phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\int_0^\infty r'^2 dr' \int_{-1}^{1} d\cos\theta' \begin{bmatrix} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta')P_l(\cos\theta) \\ \cdot\delta(\cos\theta'-\cos\alpha)\delta(r'-c) \end{bmatrix}$$
$$= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta)$$

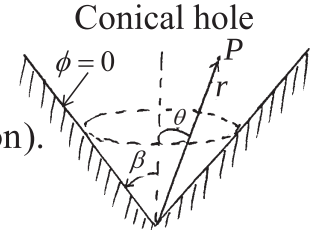
Jackson uses a slightly different method to derive this. See p.103.

3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point

Consider a "conical hole" or "sharp point" with a conducting boundary ($\phi = 0$). Assume the region of interest is source-free.

$$\nabla^2\phi = 0 \Rightarrow \phi = \begin{Bmatrix} r^\nu \\ r^{-\nu-1} \end{Bmatrix}\begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix}\begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

Conical hole

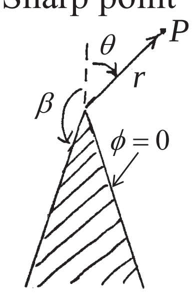


(i) Geometry and b.c. indep. of φ (by assumption).
$\Rightarrow$ $m = 0$

(ii) $Q_\nu^m(\cos\theta)$ diverges at $\theta = 0$ ($\cos\theta = 1$).

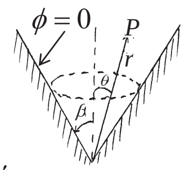
$\Rightarrow$ drop $Q_\nu^m(\cos\theta) \Rightarrow$ $\phi = \begin{Bmatrix} r^\nu \\ r^{-\nu-1} \end{Bmatrix} P_\nu(\cos\theta)$

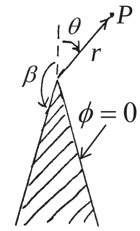
Sharp point



Note: $P_\nu(x)$ diverges at $x = -1$ unless $\nu =$ integer.
However, our region of interest is $0 \leq \theta < \pi$.
$\Rightarrow \cos\theta = -1$ is outside the region of interest.
$\Rightarrow \nu$ is not required to be an integer.

Rewrite: $\phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} P_\nu(\cos\theta)$





(iii) ϕ is finite at $r=0$.

$$\Rightarrow \begin{cases} \text{(a) demand } \nu > 0 \text{ and drop } r^{-\nu-1} \Rightarrow \phi = r^{\nu} P_\nu(\cos\theta) \\ \text{or (b) demand } -\nu-1>0 \text{ and drop } r^{\nu} \Rightarrow \phi = r^{-\nu-1} \underbrace{P_\nu(\cos\theta)}_{=P_{-\nu-1}(\cos\theta)} \\ \qquad \Rightarrow \phi = r^{-\nu-1} P_{-\nu-1}(\cos\theta) \end{cases}$$

$\Rightarrow$ Either option (a) or option (b) gives $\phi = r^{\nu} P_\nu(\cos\theta),\ \nu>0$

(iv) $\phi = 0$ at $\theta=\beta \Rightarrow P_\nu(\cos\beta)=0 \Rightarrow \underbrace{\nu}_{\text{eigenvalue}} = \nu_1,\ \nu_2,\ \nu_3, \dots\ (\nu>0)$

$$\Rightarrow \phi(r,\theta) = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \overset{\boxed{r\to 0}}{\approx} A_1 r^{\nu_1} P_{\nu_1}(\cos\theta), \qquad (3.44)$$

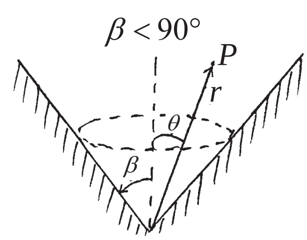
$\boxed{\nu_1\text{: smallest eigenvalue}}$

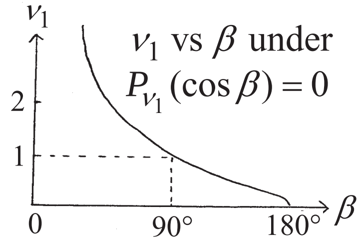
Question: Is $P_{\nu_k}(\cos\theta)$ a complete set in the region $0 \le \theta \le \beta$? Yes.

$\because$ b.c. $P_\nu(\cos\beta)=0$ makes the operator in (3.9) Hermitian [see (A.12)].

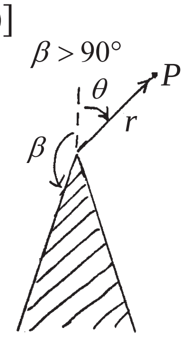
Note: $P_{\nu_k}(\cos\theta)$ is a set specific to and most useful for this problem.

$$r \to 0 \Rightarrow \begin{cases} E_r = -\frac{\partial\phi}{\partial r} \approx -\frac{\partial}{\partial r} A_1 r^{\nu_1} P_{\nu_1}(\cos\theta) \\ \quad = -\nu_1 A_1 r^{\nu_1-1} P_{\nu_1}(\cos\theta) \propto r^{\nu_1-1} \\ E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \approx -\frac{1}{r}\frac{\partial}{\partial\theta} A_1 r^{\nu_1} P_{\nu_1}(\cos\theta) \\ \quad = A_1 r^{\nu_1-1} \sin\theta P'_{\nu_1}(\cos\theta) \propto r^{\nu_1-1} \\ \sigma = -\varepsilon_0 E_\theta(\theta=\beta) \quad [\mathbf{E}(\theta>\beta)=0, \text{ see } (1.22)] \\ \quad \approx -A_1\varepsilon_0 r^{\nu_1-1} \sin\beta P'_{\nu_1}(\cos\beta) \propto r^{\nu_1-1} \end{cases}$$





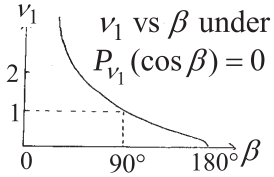
$$\begin{cases} \nu_1 > 1, \text{ if } \beta < 90° \\ \nu_1 = 1, \text{ if } \beta = 90° \\ \nu_1 < 1, \text{ if } \beta > 90° \end{cases}$$

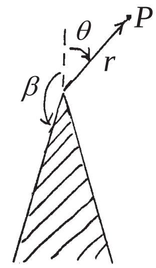


Discussion:

1. If $\beta < 90^{\circ}$ (conical hole, $\nu_1 > 1$), E & $\sigma \to 0$ as $r \to 0$.

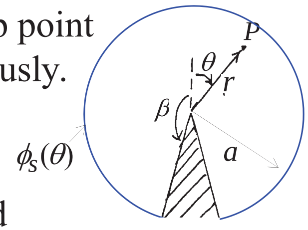
Rewrite $\begin{cases} E_r \propto A_1 r^{\nu_1 - 1} \\ E_\theta \propto A_1 r^{\nu_1 - 1} \\ \sigma \propto A_1 r^{\nu_1 - 1} \end{cases}$ as $r \to 0$





2. If $\beta > 90^{\circ}$ (sharp point, $\nu_1 < 1$), E & $\sigma \to \infty$ as $r \to 0$.

Large E-field (> 3×10^4 V/cm) can cause the air to break down to form a conducting path for the sharp point (e.g. lightning rod) to discharge slowly & continuously.



3. Rewrie $\phi(r,\theta) = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta)$ [(3.44)]

To draw conclusions 1 & 2 above, we only need $A_1 \neq 0$, which requires the b.c. $\phi(r = a, \theta) \neq 0$. So,

$$\phi(r = a, \theta) = \sum_{k=1}^{\infty} A_k a^{\nu_k} P_{\nu_k}(\cos\theta),$$

which can be used to determine all A_k in (3.44).

$\because P_{\nu_k}(\cos\theta)$ are linearly indep. See Ch. 2, Eqs. (3a,b).

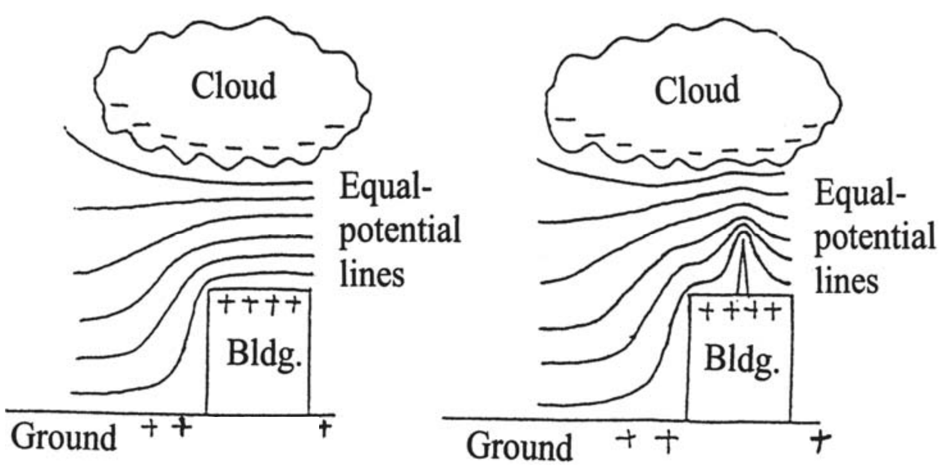
Question: If $\phi(r = a, \theta) = 0$ for all θ, then $A_k = 0$ for all k. Why?

3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point (*continued*)

Question: At the sharp point ($r \to 0$), $E \to \infty$. Is this physical?

Since atoms are finite in size, the lightning rod can't be perfectly sharp. Hence, ϕ is finite at the tip. On a clear day with a small $\Delta\phi$ between the ground the clouds, the lightning rod will not discharge.

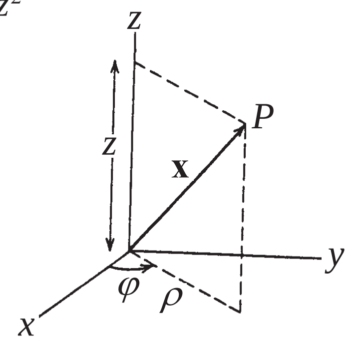
A physical picture of the lightning rod



3.7 Laplace Equation in Cylindrical Coordinates

Laplace eq. in cylindrical coordinates (see Jackson, back cover):

$$\nabla^2\phi(\mathbf{x})=0\Rightarrow\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2}+\frac{\partial^2\phi}{\partial z^2}=0$$



Let $\phi(\mathrm{x})=R(\rho)Q(\varphi)Z(z)$

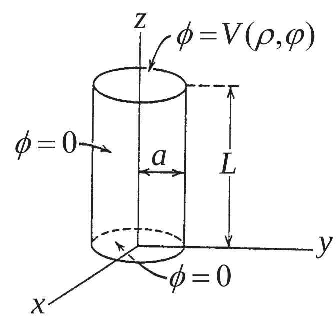
$$\Rightarrow\begin{cases}\dfrac{\partial^2 Z}{\partial z^2}-k^2Z=0\Rightarrow Z=e^{\pm kz}\\[2ex]\dfrac{\partial^2 Q}{\partial\varphi^2}+\nu^2Q=0\Rightarrow Q=e^{\pm i\nu\varphi}\\[2ex]\dfrac{\partial^2 R}{\partial\rho^2}+\frac{1}{\rho}\dfrac{\partial R}{\partial\rho}+\left(k^2-\dfrac{\nu^2}{\rho^2}\right)R=0\\[2ex]\Rightarrow R=J_\nu(k\rho)+N_\nu(k\rho)\ \text{(see pp. 112-116 or lecture notes p. 13).}\end{cases}$$

$$\Rightarrow\phi=\begin{Bmatrix}J_\nu(k\rho)\\N_\nu(k\rho)\end{Bmatrix}\begin{Bmatrix}e^{i\nu\varphi}\\e^{-i\nu\varphi}\end{Bmatrix}\begin{Bmatrix}e^{kz}\\e^{-kz}\end{Bmatrix}$$

3.8 Bounday-Value Problems in Cylindrical Coordinates

Example 1: Potential inside a charge-free cylinder (see figure) with b.c.'s: $\phi(z=L)=V(\rho,\varphi)$ and $\phi=0$ on other surfaces.

$$\nabla^2\phi(\mathbf{x})=0\Rightarrow\phi=\begin{Bmatrix}J_\nu(k\rho)\\N_\nu(k\rho)\end{Bmatrix}\begin{Bmatrix}e^{i\nu\varphi}\\e^{-i\nu\varphi}\end{Bmatrix}\begin{Bmatrix}e^{kz}\\e^{-kz}\end{Bmatrix}$$



(i) $Z(z)=Ae^{kz}+Be^{-kz}$

$\phi=0$ at $z=0\ \Rightarrow Z(0)=0\Rightarrow B=-A$

$\Rightarrow Z(z)=A\left(e^{kz}-e^{-kz}\right)=A'\sinh kz$

(ii) $\phi(\varphi)=\phi(\varphi+2\pi)$

$\Rightarrow\nu=m=$ integer

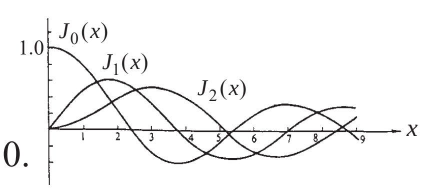
This is an eigenvalue problem in φ and ρ, but not in z. Why?

(iii) ϕ is finite at $\rho=0$. $\Rightarrow$ drop $N_m(k\rho)\Rightarrow R=J_m(k\rho)$

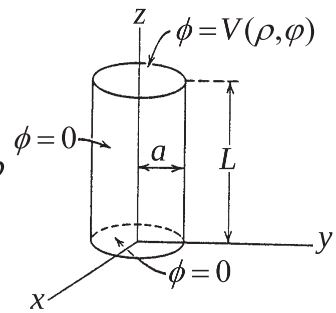
(iv) $\phi=0$ at $\rho=a\Rightarrow J_m(ka)=0$

$\Rightarrow k=k_{mn}=\dfrac{x_{mn}}{a},\ n=1,\ 2,\ 3\ldots$

where x_{mn} is the n-th root of $J_m(x)=0$.



Thus, we expand the solution as follows



$$\phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z) \cdot (A_{mn}\sin m\varphi + B_{mn}\cos m\varphi$$

(v) $\phi(\rho,\varphi,z=L) = V(\rho,\varphi)$

$$\Rightarrow V(\rho,\varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L) \cdot (A_{mn}\sin m\varphi + B_{mn}\cos m\varphi)$$

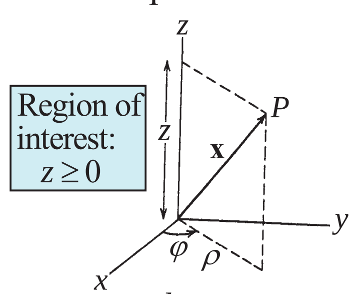
Operate both sides with $\int_0^{2\pi} d\varphi \int_0^a \rho d\rho J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi \\ \cos m\varphi\end{Bmatrix}$ and make use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and the relation: $\int_0^a J_m(k_{mn'}\rho)J_m(k_{mn}\rho)\rho d\rho = \frac{a^2}{2}[J_{m+1}(k_{mn}a)]^2\delta_{n'n}$ (3.95)

$$\Rightarrow \begin{Bmatrix} A_{mn} \\ B_{mn}\end{Bmatrix} = \frac{2\mathrm{cosech}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^{2\pi} d\varphi \int_0^a \rho d\rho V(\rho,\varphi) J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi \\ \cos m\varphi\end{Bmatrix}$$

Note: Use $\frac{1}{2}B_{0n}$ for the $m=0$ term in the $\phi(\rho,\varphi,z)$ series .

Example 2: Potential in the charge-free semi-infinite space $z \geq 0$

subject to the b.c. $\begin{cases}\phi(\rho,\varphi,z=0) = V(\rho,\varphi) \\ \phi(\rho\to\infty,\varphi,z\to\infty) = 0\end{cases}$



$$\nabla^2\phi(\mathbf{x}) = 0 \Rightarrow \phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho)\end{Bmatrix}\begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi}\end{Bmatrix}\begin{Bmatrix} e^{kz} \\ e^{-kz}\end{Bmatrix}$$

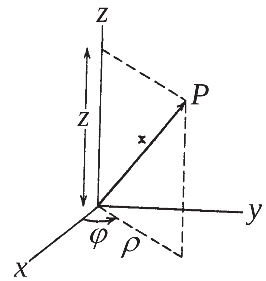
(i) ϕ remains finite as $z \to \infty$. $\Rightarrow$ drop $e^{kz} \Rightarrow Z(z) = Ae^{-kz}$

(ii) $\phi(\varphi) = \phi(\varphi + 2\pi)$ $\Rightarrow \nu = m =$ integer

(iii) ϕ is finite at $\rho = 0$. $\Rightarrow$ drop $N_m(k\rho) \Rightarrow R = J_m(k\rho)$

(iv) $\phi = 0$ at $\rho \to \infty$ $\Rightarrow J_m(k\cdot\infty) = 0$ $\Rightarrow \begin{cases}\text{Continuous eigenvalue } k; \\ k \text{ series} \to k \text{ integral}\end{cases}$

$$\Rightarrow \phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\int_0^\infty dk e^{-kz} J_m(k\rho)[A_m(k)\sin m\varphi + B_m(k)\cos m\varphi] \tag{3.106}$$



Rewrite (3.106) with variable k changed to k':

$$\phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\int_0^{\infty}dk'e^{-k'z}J_m(k'\rho)$$
$$\cdot[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi]$$

(v) $\phi(\rho,\varphi,z=0)=V(\rho,\varphi)$

$$\Rightarrow\ V(\rho,\varphi)=\sum_{m=0}^{\infty}\int_0^{\infty}dk'J_m(k'\rho)[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi]$$

Operating both sides with $\int_0^{2\pi}d\varphi\int_0^{\infty}\rho d\rho J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$ and making use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and the relation: $\int_0^{\infty}xJ_m(kx)J_m(k'x)dx=\frac{1}{k}\delta(k-k')$ (3.108)

$$\Rightarrow\ \begin{Bmatrix}A_m(k)\\ B_m(k)\end{Bmatrix}=\frac{k}{\pi}\int_0^{2\pi}d\varphi\int_0^{\infty}\rho d\rho V(\rho,\varphi)J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix} \qquad (3.109)$$

For $m=0$, use $\frac{1}{2}B_0(k)$ in series (3.106).

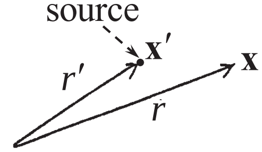
3.9 Expansion of Green Functions in Spherical Coordinates

The Green function for an electrostatic potential problem with Dirichlet b.c.'s satisfies

$$\nabla^2G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with $G(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}$ on the boundary surface.

Question: Jackson p.120 states the b.c. as "$G(\mathbf{x},\mathbf{x}')=0$ for either $\mathbf{x}$ or $\mathbf{x}'$ on the boundary surface." Why?



Case 1: Green function in infinite space

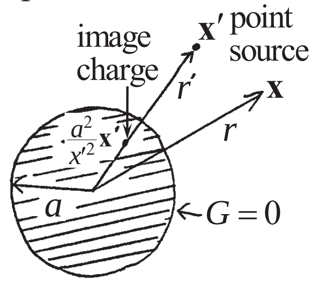
In Sec. 1.10, we have the solution:

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|},$$

which can be expanded in spherical coordinates as (Sec. 3.3)

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)\quad[(3.70)]$$

Case 2: Green function outside a conducting sphere



By the method of images, we have obtained the Green function in Sec. 2.6,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \quad [(2.16)]$$

The first term in (2.16) is expanded in (3.70).

The second term can be expanded using (3.70). Since $|\mathbf{x}| > \left|\frac{a^2}{x'^2}\mathbf{x}'\right|$, we substitute $r_> = |\mathbf{x}| = r$ and $r_< = \left|\frac{a^2}{x'^2}\mathbf{x}'\right| = \frac{a^2}{r'}$ into (3.70) to obtain

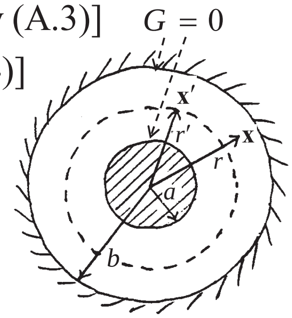
$$\frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{a\left(\frac{a^2}{r'}\right)^l}{r' r^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$\boxed{r'}$

$$\Rightarrow\; G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \quad (3.114)$$

Case 3: Green function inside a spherical shell

$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ [inhomogeneous D.E. by (A.3)]
with homogeneous b.c.'s $G(r = a \;\&\; \mathrm{b}) = 0$ [by (A.4)]



We will now solve the problem by a systematic method: method of expansion.

Write $\delta(\mathbf{x}-\mathbf{x}')$ in spherical coordinates,

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta'),$$

where r',θ',φ' (arbitrary constants) are coordinates of the source ($\mathbf{x}'$).

Apply $\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$ [(3.56)]

$$\Rightarrow \delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \quad (3.117)$$

Note: In (3.117), we have decomposed a "unit point charge" into an infinite number of spherical "charge layers", all of which have smooth charge distributions [with $Y_{lm}(\theta,\varphi)$ dependence] on the $r = r'$ surface.

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

$$= -4\pi \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (4)$$

variable constant constants variables

The RHS of (4) suggests that we try the following form for $G(\mathbf{x},\mathbf{x}')$:

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r,r')\underbrace{Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}_{\text{same as RHS of (4)}} \qquad (5)$$

Why not $g_{lm}(r,r')$? See (3.120).

Sub. (5) into LHS of (4). Note $Y_{lm}(\theta,\varphi) \propto P_l^m(\cos\theta)e^{im\varphi}$ and use

$$\begin{cases} \dfrac{d^2}{d\varphi^2}e^{im\varphi} = -m^2 e^{im\varphi} \\ \left[\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}(\sin\theta\dfrac{d}{d\theta}) - \dfrac{m^2}{\sin^2\theta}\right]P_l^m(\cos\theta) = -\ell(\ell+1)P_l^m(\cos\theta) \ [(3.6)] \end{cases}$$

Y_{lm} is complex, but $\sum_{m=-l}^{l} Y_{lm}$ is real.

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 G}{\partial\varphi^2}$$

$-m^2 G$

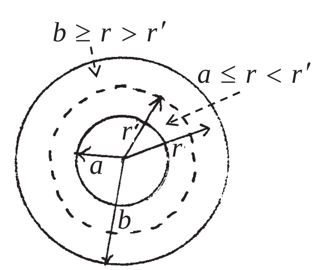
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')] - \frac{l(l+1)}{r^2}g_l(r,r')\right]Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (6)$$

Equate the RHS of (4) and (6). $Y_{lm}(\theta,\varphi)$'s are orthogonal, hence linearly indep. [lectures notes, Ch. 2, Eqs. (3a,b)]. Thus, the coefficients of each $Y_{lm}(\theta,\varphi)$ term on the RHS of (4) and (6) are equal.

$$\Rightarrow \quad \frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')] - \frac{l(l+1)}{r^2}g_l(r,r') = -\frac{4\pi}{r^2}\delta(r-r'), \qquad (3.120)$$

which turns out to be an equation indep. of the index m.

To solve (3.120), divide the space into $a \le r < r'$ and $b \ge r > r'$. In each region, (3.120) reduces to



$$\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')] - \frac{l(l+1)}{r^2}g_l(r,r') = 0$$

$$\Rightarrow \quad g_l(r,r') = \begin{cases} Ar^l + Br^{-l-1}, & a \le r < r' \\ A'r^l + B'r^{-l-1}, & b \ge r > r' \end{cases}$$

We will use 4 b.c.'s to determine A, B, A', and B'. But first note:

If $a = 0$, B must be set to 0 for $g_l(r,r')$ to be finite at $r = 0$. (7)

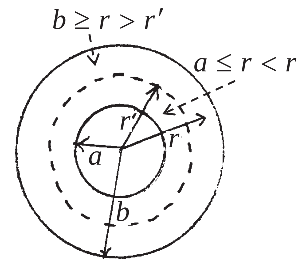
b.c. (i): $g_l(r=a,r')=0 \;\Rightarrow g_l(r,r')=A\big(r^l-\underbrace{\tfrac{a^{2l+1}}{r^{l+1}}}\big), \quad a \le r < r'$

$\boxed{\text{If } a=0, \text{ this term does not exist [see (7)].}}$

b.c. (ii): $g_l(r=b,r')=0 \;\Rightarrow g_l(r,r')=B'\big(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\big), \quad b \ge r > r'$

b.c. (iii): $g_l(r,r')$ is continuous at $r=r'$. Physical reason: E is finite at $r=r'$. $\Rightarrow \phi$ [or $g_l(r,r')$] is continuous.

Thus, $\quad A\big(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\big)=B'\big(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\big)$



$$\Rightarrow \frac{A}{B'}=\frac{\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}}{r'^l-\frac{a^{2l+1}}{r'^{l+1}}} \Rightarrow \begin{cases} A=C\big(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\big) \\ B'=C\big(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\big) \end{cases}$$

$$\Rightarrow g_l(r,r')=\begin{cases} C\big(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\big)\big(r^l-\frac{a^{2l+1}}{r^{l+1}}\big), & a \le r < r' \\ C\big(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\big)\big(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\big), & b \ge r > r' \end{cases}$$

$$=C\big(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\big)\big(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\big) \quad \boxed{r_<(r_>)\text{: smaller (larger) of } r \,\&\, r'} \quad (3.122)$$

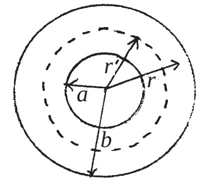
We need one more condition to get the remaining constant C in

$$g_l(r,r')=C\big(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\big)\big(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\big) \quad [(3.122)]$$

Rewrite $\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')]-\frac{l(l+1)}{r^2}g_l(r,r')=-\frac{4\pi}{r^2}\delta(r-r') \quad [(3.120)]$

b.c. (iv): Physically, $E_r\,(\propto \frac{d}{dr}g_l)$ is discontinuous across the charge layer at $r=r'$. Mathematically, we integrate (3.120) from $r'-\varepsilon$ to $r'+\varepsilon$ $(\varepsilon \to 0)$ to bring out the meaning of $\delta(r-r')$, hence the E_r discontinuity.

Multiply (3.120) by r and integrate across r'

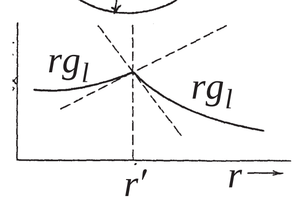


$$\Rightarrow \frac{d}{dr}[rg_l(r,r')]_{r'+\varepsilon}-\frac{d}{dr}[rg_l(r,r')]_{r'-\varepsilon}=-\frac{4\pi}{r'}$$

$$\Rightarrow -\frac{C}{r'}[1-(\tfrac{a}{r'})^{2l+1}][l+(l+1)(\tfrac{r'}{b})^{2l+1}]$$

$\boxed{\text{use (3.122)}}$

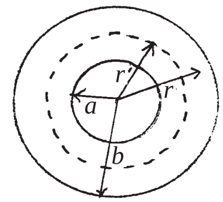
$$-\frac{C}{r'}[(l+1)+l(\tfrac{a}{r'})^{2l+1}][1-(\tfrac{r'}{b})^{2l+1}]=-\frac{4\pi}{r'}$$



$$\Rightarrow C=\frac{4\pi}{(2l+1)[1-(\frac{a}{b})^{2l+1}]}$$

Sub. C into $g_l(r,r') = C(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}})(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})$ [(3.122)]

$$\Rightarrow g_l(r,r') = \frac{4\pi}{(2l+1)[1-(\frac{a}{b})^{2l+1}]}(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}})(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})$$



$$\Rightarrow G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r,r') Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \quad [(5)]$$

$$= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)[1-(\frac{a}{b})^{2l+1}]}(r_<^l - \underbrace{\frac{a^{2l+1}}{r_<^{l+1}}})(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}) \qquad (3.125)$$

If $a = 0$, this term does not exist [see (7)].

Limiting case 1: $a = 0$ & $b \to \infty$, (3.125) $\Rightarrow$ (3.70)

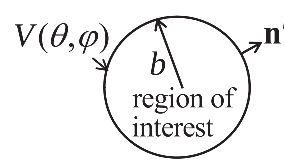
$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \quad [(3.70)]$$

Limiting case 2: $a \neq 0$ & $b \to \infty$, (3.125) $\Rightarrow$ (3.114)

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1}[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}(\frac{a^2}{rr'})^{l+1}] Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta,\varphi) \quad [(3.114)]$$

3.10 Solution of Potential Problems with the Spherical Green Function Expansion

Example 1: Potential inside a charge-free sphere of radius b subject to the b.c. $\phi(r=b) = V(\theta,\varphi)$



Since we already have the Green function [(3.125)] for this problem (inhomogeneous due to the b.c.), it is convenient to use the formal solution derived in Sec. 1.10:

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\mathbf{x}')}_{=0} G(\mathbf{x},\mathbf{x}') d^3x' - \varepsilon_0 \oint_S \phi(\mathbf{x}') \frac{\partial}{\partial n'} G(\mathbf{x},\mathbf{x}') da' \quad [(1.44)]$$

There is no charge inside.

$$\Rightarrow \qquad \phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \phi(\mathbf{x}') \frac{\partial}{\partial n'} G(\mathbf{x},\mathbf{x}') da' \qquad (8)$$

Note: $\frac{\partial}{\partial n'}$ is a derivative along $\mathbf{n}'$. In deriving (1.44), $\mathbf{n}'$ is required to be $\perp$ to the boundary surface and pointing outward from the region of interest. So we have $\frac{\partial}{\partial n'} = \frac{\partial}{\partial r'}$ for this example.

Rewrite $\phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$ [(8)], where

$$G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)[1-(\frac{a}{b})^{2l+1}]}(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}})(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})$$

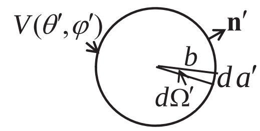
In (8), r' is on S

For this example, $a = 0$, $r_> = r'(= b)$, and $r_< = r(\le b)$, hence

$$G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l,m}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)r^l(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})$$

$$\Rightarrow \frac{\partial G}{\partial r'} = 4\pi\sum_{l,m}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)r^l(-\frac{l+1}{r'^{l+2}} - \frac{lr'^{l-1}}{b^{2l+1}})$$

$$\Rightarrow \left.\frac{\partial G}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2}\sum_{l,m}(\frac{r}{b})^l Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (9a)$$



$$\phi(\mathbf{x}')\big|_S = \phi(r' = b) = V(\theta',\varphi') \qquad (9b)$$

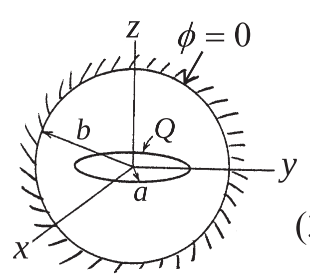
$$da' = b^2 d\Omega' \qquad (9c)$$

Sub. (9a-c) into $\phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$ [(8)], we get

$$\phi(\mathbf{x}) = \sum_{l,m}[\int V(\theta',\varphi')Y_{lm}^*(\theta',\varphi')d\Omega'](\frac{r}{b})^l Y_{lm}(\theta,\varphi) \qquad (3.128)$$

Example 2: Potential due to a uniformly charged ring of radius a and total charge Q located on the x-y plane inside a grounded conducting sphere of radius b

In spherical coordinates, the x-y plane is the $\theta = \pi/2$ (or $\cos\theta = 0$) plane. The charge exists only at $r = a$ on the $\cos\theta = 0$ plane. Hence, $\rho(\mathbf{x})$ can be written as



$$\rho(\mathbf{x}) = \frac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta) \qquad (3.129)$$

The potental is given by

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \quad [(1.44)]$$

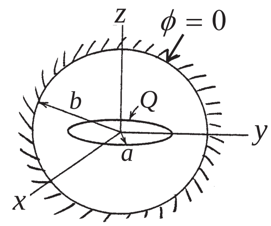
No inner conductor in this problem $\Rightarrow$ (3.125) reduces to

$$G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \qquad (10)$$

Symmetry in $\varphi \Rightarrow m=0$. Hence,

$$Y_{lm}(\theta,\varphi) \to Y_{l0}(\theta,\varphi) = \sqrt{\tfrac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$\Rightarrow\ G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\tfrac{1}{r_>^{l+1}} - \tfrac{r_>^l}{b^{2l+1}}\right) \tag{11}$$



Sub. (11) and $\rho(\mathbf{x}) = \dfrac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta)$

into $\phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\int d^3x' \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')$, we obtain

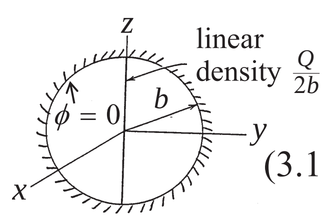
$$\phi(\mathbf{x}) = \frac{Q}{8\pi^2\varepsilon_0 a^2}\int r'^2 dr' d\cos\theta' d\varphi' \begin{bmatrix} \delta(r'-a)\delta(\cos\theta') \\ \cdot \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\tfrac{1}{r_>^{l+1}} - \tfrac{r_>^l}{b^{2l+1}}\right) \end{bmatrix}$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)P_l(\cos\theta) r_<^l \left(\tfrac{1}{r_>^{l+1}} - \tfrac{r_>^l}{b^{2l+1}}\right) \tag{3.130}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and a.

Example 3: Potential due to a uniformly charged line of length 2b and total charge Q located on the z-axis inside a grounded conducting sphere of radius b.

In spherical coordinates, the charge exists only at $\theta = 0$ and π (or $\cos\theta = \pm 1$). Hence,



$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right] \tag{3.132}$$

Question: Is this a unformly charged line? (Yes. See Prob. 1 below)

The potential is given by

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \quad [(1.44)]$$

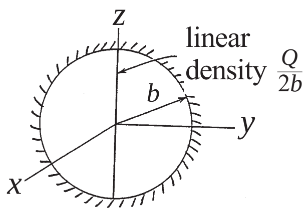
No inner conductor + symmetry in $\varphi \Rightarrow G(\mathbf{x},\mathbf{x}')$ is the same as (11):

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\tfrac{1}{r_>^{l+1}} - \tfrac{r_>^l}{b^{2l+1}}\right) \tag{11}$$

Sub. (3.132) and (11) into $\phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x'$, we obtain

$$\phi(\mathbf{x}) = \frac{Q}{8\pi\varepsilon_0 b}\int r'^2 dr' d\cos\theta' d\varphi' \left[\begin{array}{l}\dfrac{\delta(\cos\theta'-1)+\delta(\cos\theta'+1)}{2\pi r'^2} \\ \cdot \sum\limits_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\end{array}\right]$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\sum_{l=0}^{\infty}\left[P_l(1)+P_l(-1)\right]P_l(\cos\theta)\underbrace{\int_0^b r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)dr'}_{} \quad (3.133)$$



$$= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^r r'^l dr' + r^l \int_r^b \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)dr'$$

$$= \frac{2l+1}{l(l+1)}\left[1-\left(\tfrac{r}{b}\right)^l\right]$$ *Note*: This gives $\frac{0}{0}$ for $l = 0$.

By L'Hospital's rule, the $l = 0$ term is given by $\ln(\frac{b}{r})$ (see p. 124).

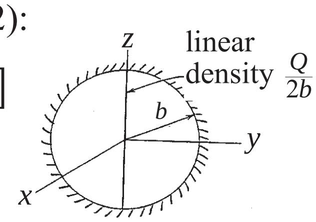
$P_l(-1) = (-1)^l$ and $P_l(1) = 1 \Rightarrow$ Odd l terms cancel. $\Rightarrow$ Let $l = 2j$.

$$\Rightarrow \phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r}\right) + \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left[1-\left(\frac{r}{b}\right)^{2j}\right]P_{2j}(\cos\theta)\right\} \quad (3.136)$$

Problem 1: Show the charge density in (3.132):

$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

represents a unifomly charged line along z.



Solution: The total charge is

$$\int \rho(\mathbf{x})d^3x = \frac{Q}{2b}\int_0^b r^2 dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\varphi \frac{\delta(\cos\theta-1)+\delta(\cos\theta+1)}{2\pi r^2}$$

$$= \frac{Q}{2b}\left[\int_0^b dr\int_{-1}^1 d\cos\theta\, \underbrace{\delta(\cos\theta-1)}_{\theta=0,\ +z\text{-axis}} + \int_0^b dr\int_{-1}^1 d\cos\theta\, \underbrace{\delta(\cos\theta+1)}_{\theta=\pi,\ -z\text{-axis}}\right]$$

$$= \frac{Q}{2b}\int_{-b}^{b} dz \Rightarrow$$ Each dz contributes equally $\Rightarrow$ uniform distribution

Question: We have let $\int_{-1}^1 d\cos\theta\ \delta(\cos\theta \pm 1) = 1$. But $\cos\theta$ does not cross -1 or 1. Why is the integral equal to 1?

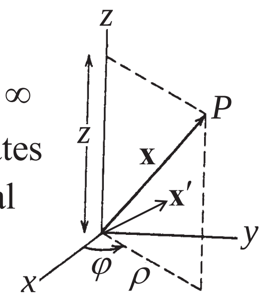
Answer: This issue can be resolved by a limiting procedure, i.e. letting

$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}\cdot\lim_{\varepsilon\to 0}\left\{\delta[\cos\theta-(1-\varepsilon)] + \delta[\cos\theta+(1-\varepsilon)]\right\}$$

3.11 Expansion of Green Functions in Cylindrical Coordinates

Consider the Green equation in infinite space:

$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$, with $G(\mathbf{x},\mathbf{x}') = 0$ as $|\mathbf{x}| \to \infty$



We just obtained the solution in spherical coordinates [(3.125), $a \to 0$, $b \to \infty$]. We now solve it in cylindrical coordinates in the same way, but in *infinite* space.

Write $\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$

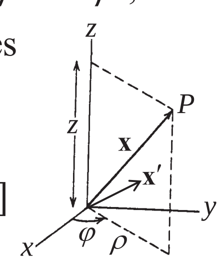
with
$$\begin{cases} \delta(\varphi-\varphi') = \frac{1}{2\pi}\sum\limits_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} & \left[\text{A special case of (2.35): } \sum\limits_{n=1}^{\infty} U_n^*(\xi')U_n(\xi) = \delta(\xi-\xi')\right] \\ \delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} & \\ \qquad = \frac{1}{\pi}\int_0^{\infty} dk \cos[k(z-z')] & \left[\text{An extension of (2.35) to continuous index } k. \text{ Also, see (6) of Ch. 2.}\right] \end{cases}$$

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (12)$$

Since $e^{im\varphi}$ and e^{ikz} are complete sets, we may expand $G(\mathbf{x},\mathbf{x}')$ in variables φ and z (*Note*: Eigenvalue k is continuous)

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk g_m(k,\rho,\rho') e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (3.140)$$

where the coefficient $g_m(k,\rho,\rho')$ depends on m, k, ρ and ρ', but only ρ is treated as a variable. Sub. (3.140) into (12) gives



$$\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right) \cdot g_m(k,\rho,\rho') e^{im(\varphi-\varphi')}\cos[k(z-z')]$$

$$= -\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (13)$$

On the LHS, $\frac{\partial^2}{\partial\varphi^2} \to -m^2$, $\frac{\partial^2}{\partial z^2} \to -k^2$, $\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$

$$\Rightarrow \left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \left(k^2+\frac{m^2}{\rho^2}\right)\right] g_m(k,\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho-\rho') \qquad (3.141)$$

In regions $\rho < \rho'$ & $\rho > \rho'$: $[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - (k^2 + \frac{m^2}{\rho^2})]g_m(k,\rho,\rho') = 0$

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} AI_m(k\rho) + BK_m(k\rho), & \rho < \rho' \\ A'I_m(k\rho) + B'K_m(k\rho), & \rho > \rho' \end{cases} \quad \begin{bmatrix}\text{See Jackson,} \\ \text{(3.98)-(3.101)}\end{bmatrix}$$

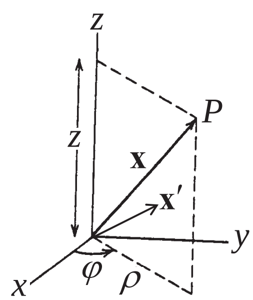
(i) g_m is finite at $\rho = 0. \Rightarrow B = 0$

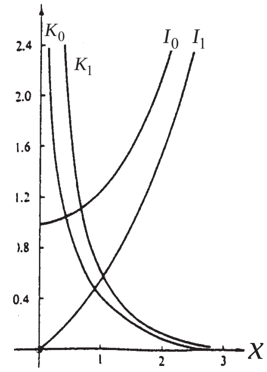
(ii) g_m is finite as $\rho \to \infty. \Rightarrow A' = 0$

(iii) g_m is continuous at $\rho = \rho'$.

$\Rightarrow AI_m(k\rho') = B'K_m(k\rho')$

$$\Rightarrow \frac{A}{B'} = \frac{K_m(k\rho')}{I_m(k\rho')} \Rightarrow \begin{cases} A = CK_m(k\rho') \\ B' = CI_m(k\rho') \end{cases}$$





Modified Bessel functions I_m, K_m

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} CK_m(k\rho')I_m(k\rho), & \rho < \rho' \\ CI_m(k\rho')K_m(k\rho), & \rho > \rho' \end{cases}$$

$$= CI_m(k\rho_<)K_m(k\rho_>)$$

(iv) Rewrite $[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - (k^2 + \frac{m^2}{\rho^2})]g_m(k,\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho - \rho')$ (3.141)

Multiply (3.141) by ρ and integrate form $\rho' - \varepsilon$ to $\rho' + \varepsilon$ ($\varepsilon \to 0$).

$$\Rightarrow \left.\frac{dg_m}{d\rho}\right|_{\rho'+\varepsilon} - \left.\frac{dg_m}{d\rho}\right|_{\rho'-\varepsilon} = -\frac{4\pi}{\rho'} \quad [g_m = CI_m(k\rho_<)K_m(k\rho_>)] \qquad (3.143)$$

$$\Rightarrow Ck[I_m(k\rho')K_m'(k\rho') - K_m(k\rho')I_m'(k\rho')] = -\frac{4\pi}{\rho'}$$

Wronskian

Gradshteyn & Ryzhik, Sec. 8.474

Use $\mathrm{W}[I_m(x), K_m(x)] \equiv I_m(x)K_m'(x) - I_m'(x)K_m(x) = -\frac{1}{x}$ (3.147)

$$\Rightarrow Ck(\frac{-1}{k\rho'}) = -\frac{4\pi}{\rho'} \Rightarrow C = 4\pi \Rightarrow g_m(k,\rho,\rho') = 4\pi I_m(k\rho_<)K_m(k\rho_>)$$

Sub. $g_m(k,\rho,\rho')$ into (3.140), we obtain the solution for $G(\mathbf{x},\mathbf{x}')$:

$$\Rightarrow G(\mathbf{x},\mathbf{x}') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

Since $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$, by the uniqueness theorem, we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>) \quad (3.148)$$

3.12 Eigenfunction Expansion for Green Functions

Eigenfunction Expansion of Green Function in 3 Dimensions :

We have obtained the Green function for the Poisson eq. by the method of eigenfunction expansion in 2 dim. [e.g. (3.118), in θ, φ]. Here, we develop a general technique to obtain the Green function by eigenfunction expansion in 3 dim. Consider the Green function for a general inhomogeneous D.E. with homogeneous b.c.'s:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (3.156)$$

a given real function   a given constant

We shall solve (3.156) by expanding $G(\mathbf{x},\mathbf{x}')$ and $\delta(\mathbf{x}-\mathbf{x}')$ in eigenfunctions of a related problem formulated as follows.

same $f(\mathbf{x})$ as in (3.156)   an eigenvalue to be determined by the b.c., not the same λ as in (3.156)

$$\nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi(\mathbf{x}) = 0 \qquad (3.153)$$

with the same boundary surface and homogeneous b.c. as for (3.156).

Rewrite $\quad \nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi(\mathbf{x}) = 0 \quad$ [(3.153)]

Assume b.c.'c make $[\nabla^2 + f(\mathbf{x})]$ a *Hermitian* operator [see (A.11), (A.12)], and $\psi_n(\mathbf{x})$ are the 3-D (normalized) eigenfunctions, we have

$$\int_V \psi_m^*(\mathbf{x})\psi_n(\mathbf{x}) d^3x = \delta_{mn} \quad \text{[see (A.13)]} \qquad (3.155)$$

and ψ_n form a *complete* set [see (A.17)] with *real* eigenvalues λ_n.

Write $G(\mathbf{x},\mathbf{x}') = \sum_n a_n(\mathbf{x}')\psi_n(\mathbf{x}) \qquad (3.157)$

Sub. (3.157) and $\delta(\mathbf{x}-\mathbf{x}') = \sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$ [see (2.35)] into $\nabla^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ [(3.156)], we obtain

$$\sum_n a_n(\mathbf{x}')\{\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda]\psi_n(\mathbf{x})\} = -4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

ψ_n satisfies $\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0,$

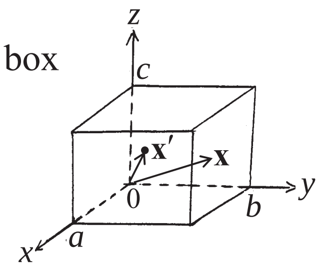
$$\Rightarrow \sum_n a_n(\mathbf{x}')(\lambda - \lambda_n)\psi_n(\mathbf{x}) = -4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

Question : Have we made use of $\int \delta(\mathbf{x}-\mathbf{x}')d^3x = 1$?

$$\Rightarrow a_n(\mathbf{x}') = 4\pi\frac{\psi_n^*(\mathbf{x}')}{\lambda_n - \lambda} \Rightarrow G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n - \lambda} \qquad (3.160)$$

We now specialize (3.156) to the Green function for the Poisson eq., i.e. $\nabla^2 G(\mathbf{x},\mathbf{x}') + [\underbrace{f(\mathbf{x})}_{0} + \underbrace{\lambda}_{0}]G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$

Example 1: Green function for a rectangular box



$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with $G(\mathbf{x},\mathbf{x}') = 0$ at $\begin{cases} x = 0 \text{ and } a \\ y = 0 \text{ and } b \\ z = 0 \text{ and } c \end{cases}$

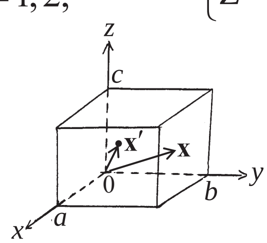
Consider the corresponding eigenvalue problem [(3.153) with $f(\mathbf{x}) = 0$ and $\lambda \to k^2$]: $\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0$ with the same b.c.'s

Let $\psi(\mathbf{x}) = X(x)Y(y)Z(z) \Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_l^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_m^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_n^2} + k^2 = 0$

$$\Rightarrow \begin{cases} X(x) = Ae^{ik_l x} + Be^{-ik_l x} \\ Y(y) = Ce^{ik_m y} + De^{-ik_m y} \\ Z(z) = Ee^{ik_n z} + Fe^{-ik_n z} \end{cases} \text{ with } k^2 = k_l^2 + k_m^2 + k_n^2$$

$$\text{b.c.} \begin{cases} X(x) = 0 \text{ at } x = 0 \ \& \ a \\ Y(y) = 0 \text{ at } y = 0 \ \& \ b \\ Z(z) = 0 \text{ at } z = 0 \ \& \ c \end{cases} \Rightarrow \begin{cases} k_l = \frac{l\pi}{a}, \ l = 1, 2, \cdots \\ k_m = \frac{m\pi}{b}, \ m = 1, 2, \cdots \\ k_n = \frac{n\pi}{c}, \ n = 1, 2, \cdots \end{cases} \ \& \ \begin{cases} X = \sin\frac{l\pi x}{a} \\ Y = \sin\frac{m\pi y}{b} \\ Z = \sin\frac{n\pi z}{c} \end{cases}$$

$$\Rightarrow k^2 = k_{lmn}^2 = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$



$$\Rightarrow \psi(\mathbf{x}) = \sqrt{\frac{8}{abc}} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \tag{3.166}$$

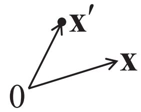
Sub. $\psi(\mathbf{x})$ into $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_j \frac{\psi_j^*(\mathbf{x}')\psi_j(\mathbf{x})}{\lambda_j - \lambda}$ [(3.160)], we obtain

$$\boxed{\sum_j \to \sum_{l,m,n} \ ; \ \lambda_j \to k_{lmn}^2; \ \lambda = 0}$$

$G(\mathbf{x},\mathbf{x}')$

$$= \frac{32}{\pi abc} \sum_{l,m,n=1}^{\infty} \frac{\sin\frac{l\pi x'}{a} \sin\frac{l\pi x}{a} \sin\frac{m\pi y'}{b} \sin\frac{m\pi y}{b} \sin\frac{n\pi z'}{c} \sin\frac{n\pi z}{c}}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \tag{3.167}$$

Example 2: Green function for infinite space



$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G(\mathbf{x},\mathbf{x}') = 0 \text{ as } |\mathbf{x}| \to \infty$$

Instead of treating it as an eigenvalue problem (as in Jackson), we use the Fourier transform. Let the Fourier transform of $G(\mathbf{x},\mathbf{x}')$ be

$$G(\mathbf{k},\mathbf{x}') = \frac{1}{(2\pi)^{3/2}}\int G(\mathbf{x},\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x \quad \begin{bmatrix}\text{3-D extension of (2.45)} \\ \mathbf{x}' \text{ is treated as a } const.\end{bmatrix}$$

Then, the Fourier transform of $\nabla G(\mathbf{x},\mathbf{x}')$ is

$$\frac{1}{(2\pi)^{3/2}}\int \nabla G(\mathbf{x},\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$$

$$= \frac{1}{(2\pi)^{3/2}}\int \left(\tfrac{\partial}{\partial x}\mathbf{e}_x + \tfrac{\partial}{\partial y}\mathbf{e}_y + \tfrac{\partial}{\partial z}\mathbf{e}_z\right)G(\mathbf{x},\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$$

[integrate by parts and use $G(\pm\infty,\mathbf{x}') = 0$]

$$= \frac{1}{(2\pi)^{3/2}}\int i\mathbf{k}G(\mathbf{x},\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x = i\mathbf{k}G(\mathbf{k},\mathbf{x}') \tag{14a}$$

Similarly, the Fourier transform of $\nabla^2 G(\mathbf{x},\mathbf{x}')$ is

$$\frac{1}{(2\pi)^{3/2}}\int \nabla^2 G(\mathbf{x},\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x = -k^2 G(\mathbf{k},\mathbf{x}') \tag{14b}$$

The Fourier transform of $\delta(\mathbf{x}-\mathbf{x}')$ is

$$\frac{1}{(2\pi)^{3/2}}\int \delta(\mathbf{x}-\mathbf{x}')e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x = \frac{1}{(2\pi)^{3/2}}e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

Thus, Fourier transforming both sides of $\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ gives $\quad -k^2 G(\mathbf{k},\mathbf{x}') = -4\pi\frac{1}{(2\pi)^{3/2}}e^{-i\mathbf{k}\cdot\mathbf{x}'}$

$$\Rightarrow G(\mathbf{k},\mathbf{x}') = \frac{2}{(2\pi)^{1/2}}\frac{e^{-i\mathbf{k}\cdot\mathbf{x}'}}{k^2} \quad \text{[solution in } \mathbf{k}\text{-space]}$$

A Fourier inverse transform [(2.44)] gives the solution in $\mathbf{x}$-space:

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{(2\pi)^{3/2}}\int G(\mathbf{k},\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}}d^3k = \frac{1}{2\pi^2}\int \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2}d^3k$$

Question: Does $G(\mathbf{x},\mathbf{x}')$ contain any more or less information than $G(\mathbf{k},\mathbf{x}')$?

Since $G(\mathbf{x},\mathbf{x}') = 1/|\mathbf{x}-\mathbf{x}'|$, by the uniquess theorem, we get another mathematical identity for $1/|\mathbf{x}-\mathbf{x}'|$ in infinite space [in addition to (3.70) & (3.148)]:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi^2}\int d^3k\,\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \tag{3.164}$$

Solution of Inhomogeneous D. E. by the Green Function Method :

To show the usefulness of the 3-D Green function just obtained, we consider an inhomogeneous linear D.E. [see (A.2) & (A.6)]:

$$\nabla^2 u(\mathbf{x}) + [f(\mathbf{x}) + \lambda] u(\mathbf{x}) = -4\pi S(\mathbf{x}) \leftarrow \boxed{\text{distributed source}} \quad (15)$$

wth homogeneous b.c.'s We have shown that the solution for

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad (3.156)$$

is $$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n - \lambda}, \quad (3.160)$$

where $\psi_n(\mathbf{x})$ is the eigenfunction of $\nabla^2 \psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n] \psi_n(\mathbf{x}) = 0.$

By the principle of linear superposition [cf. (1.3) & (1.5), Ch. 1],

we get the solution: $$u(\mathbf{x}) = \int_V G(\mathbf{x}, \mathbf{x}') S(\mathbf{x}') d^3 x' \quad (16)$$

We may verify (16) to be the solution if we operate both sides of (16) with $\nabla^2 + f(\mathbf{x}) + \lambda$ and apply (3.156) to the RHS.

Note: If $\lambda = \lambda_n$, there is no solution unless $\int_V \psi_n^*(\mathbf{x}) S(\mathbf{x}) d^3 x = 0.$

Appendix A. Eigenvalue Problem

(Ref.: Mathews and Walker, "Math. Meth. of Phys.," 2nd Ed., Ch. 9)

Terminology and Definitions :

(i) Linear differential operator: L is a linear differential operator if

$$L(au_1 + bu_2) = aLu_1 + bLu_2 \quad (A.1)$$

(u_1 and u_2: arbitrary functions; a and b: arbitrary constants.)

Examples of linear L: $\frac{d^n}{dx^n}$, $\frac{d}{dx} p(x) \frac{d}{dx} - q(x)$

(ii) Linear D.E.: The D.E. is linear if it can be put in the form:

$$\sum_{n=0}^{N} f_n(x) \frac{d^n u}{dx^n} = g(x) \quad [f_n(x),\ g(x):\ \text{given functions}], \quad (A.2)$$

in which the dependent variable u in all terms is of the 1st or 0 degree (only u^0 & u; no u^2, u^3, etc). 3-D example : $\nabla^2 \phi = -\rho/\varepsilon_0$

(iii) Homogeneous linear D.E.: The above equation with $g(x) = 0$, i.e.

$$\sum_{n=0}^{N} f_n(x) \frac{d^n u}{dx^n} = 0 \quad [\text{All terms 1st degree in } u] \quad \text{3-D example} : \nabla^2 \phi = 0$$

$\Rightarrow$ If each u_n $(n = 1, 2, \cdots)$ satisfies the D. E., so does $\sum_n a_n u_n$. (A.3)

(iv) Homogeneous b.c.:

If u satisfies the b.c., so does au (examples on next page). (A.4)

(v) Homogeneous linear boundary-value problem:

Here, the word "problem" refers to a D.E. with a "region of interest" and "b.c.'s".

A homogeneous linear boundary-value problem is a problem governed by a *homog*. and *linear* D.E. with *homog*. b.c.'s.

$\Rightarrow$
1. If u is a solution (i.e. it satisfies the "homog. linear D.E." and "homog. b.c's"), so is au.
2. If there are multiple solutions u_n $(n = 1, 2, \cdots)$, any linear combination of u_n (i.e. $\sum_n a_n u_n$) is also a solution. (A.5)

Note: A problem can be inhomogeneous because either the b.c. or the D.E. is inhomogeneous (M&W, p. 218 and p. 268). (A.6)

Example: The prob. in Sec. 2.9 is inhomogeneous due to the b.c.'s.

Formulation of an Eigenvalue Problem :

An eigenvalue problem involving the differential operator* consists of [see M&W, Eq. (9.9)]

$\underbrace{\text{a linear homog. D.E. of the form}}$ + $\underbrace{\text{homog. b.c.'s of the form}}$ (A.7)

$Lu(x) = \lambda h(x) u(x),\ a \le x \le b$	$u(a) = 0$ & $u(b) = 0$
L: linear differential operator	or $u'(a) = 0$ & $u'(b) = 0$
λ: eigenvlue	or $u(a) = u(b)$ & $u'(a) = u'(b)$
u: eigenfunction	or $u(a)$ & $u(b)$ are finite (i.e. *any* finite number, not a single *fixed* number)
$h(x)$: density function, a given real function with $h(x) \ge 0$	

*There are also eigenvalue problems which involve the matrix or integral operator (see M&W, pp. 261-262).

Definition of Hermitian Operator :

L is a Hermitian operator if

$$\int_a^b u_1^*(x)Lu_2(x)dx = \left[\int_a^b u_2^*(x)Lu_1(x)dx\right]^*, \qquad (A.8)$$

where u_1 and u_2 are arbitrary functions obeying the homog. b.c.'s.

Example 1: $L = \dfrac{d^2}{dx^2}$ (A.9)

is Hermitian if $\left. u_1^* \dfrac{du_2}{dx}\right|_a^b = 0 \;\&\; \left. u_2 \dfrac{du_1^*}{dx}\right|_a^b = 0$ (A.10)

e.g. $u_{1,2}(a) = 0$ or $\left.\dfrac{du_{1,2}}{dx}\right|_a = 0$ plus $u_{1,2}(b) = 0$ or $\left.\dfrac{du_{1,2}}{dx}\right|_b = 0.$

Proof: $\int_a^b u_1^*(x)\dfrac{d^2}{dx^2}u_2(x)dx = \cancel{\left. u_1^*\dfrac{du_2}{dx}\right|_a^b} - \int_a^b \dfrac{du_1^*}{dx}\dfrac{du_2}{dx}dx$

$\boxed{\text{integration by parts}}$ $\boxed{\text{integration by parts}}$

$$= \cancel{\left. -u_2\frac{du_1^*}{dx}\right|_a^b} + \int_a^b u_2\frac{d^2}{dx^2}u_1^* dx = \left[\int_a^b u_2^*\frac{d^2}{dx^2}u_1 dx\right]^* \Rightarrow \text{Satisfy (A.8)}$$

Example 2: $L = \dfrac{d}{dx}p(x)\dfrac{d}{dx} - q(x)$ $\left[\begin{array}{l}\text{Sturm-Liouville}\\ \text{differential operator}\end{array}\right]$ (A.11)

$\boxed{\text{real functions}}$

is Hermitian if the b.c.'s on $u(x)$ & $u'(x)$ or boundary values of $p(x)$

result in $\left. u_1^* p\dfrac{du_2}{dx}\right|_a^b = 0 \;\;\&\;\; \left. u_2 p \dfrac{du_1^*}{dx}\right|_a^b = 0$ (A.12)

Proof: $\int_a^b u_1^* L u_2 dx = \int_a^b u_1^* \frac{d}{dx}(p\frac{d}{dx}u_2)dx - \int_a^b q u_1^* u_2 dx$

$\boxed{\text{integration by parts}}$

$$\overset{(A.12)}{=} \cancel{\left. u_1^* p \frac{d}{dx}u_2\right|_a^b} - \int_a^b \frac{du_2}{dx}\, p\, \frac{du_1^*}{dx}dx - \int_a^b q u_1^* u_2 dx$$

$\boxed{\text{integration by parts}}$

$$\overset{(A.12)}{=} \cancel{\left. -u_2 p\frac{d}{dx}u_1^*\right|_a^b} + \int_a^b u_2\frac{d}{dx}(p\frac{d}{dx}u_1^*)dx - \int_a^b q u_1^* u_2 dx$$

$$= \left[\int_a^b u_2^* L u_1 dx\right]^* \Rightarrow \text{Satisfy (A.9)}$$

Note: (A.11) is a differential operator commonly found in physics.

Properties of Eigenvalue Problem with Hermitian Operator :

1. L is Hermitian $\Rightarrow \lambda_n$'s are real and u_n's are orthogonal (A.13)

Proof : Let u_i, u_j be eigenfunctions belonging to eigenvalues λ_i, λ_j, respectively, i.e. $\begin{cases} Lu_i = \lambda_i h u_i \\ Lu_j = \lambda_j h u_j \end{cases}$

Then, $Lu_i = \lambda_i h u_i \Rightarrow \int_a^b u_j^* L u_i dx = \lambda_i \int_a^b u_j^* u_i h dx$

Use the Hermitian property of L — real

$$\text{LHS} = \int_a^b u_j^* L u_i dx = [\int_a^b u_i^* L u_j dx]^* = [\lambda_j \int_a^b u_i^* u_j h dx]^* = \lambda_j^* \int_a^b u_i u_j^* h dx$$

$$\Rightarrow \ (\lambda_i - \lambda_j^*) \int_a^b u_i u_j^* h dx = 0 \ \ [h \geq 0] \quad \text{(A.14)}$$

$$\Rightarrow \begin{cases} i = j \Rightarrow \lambda_i - \lambda_i^* = 0 \ \& \ \lambda_j - \lambda_j^* = 0 \Rightarrow \lambda_i \ \& \ \lambda_j \text{ are real.} \\ i \neq j \Rightarrow u_i \ \& \ u_j \text{ are orthogonal in the sense } \int_a^b u_i u_j^* h dx = 0 \quad \text{(A.15)} \end{cases}$$

Note the presence of the density function $h(x)$

$\Rightarrow$ With (A.13-15), no need to prove (3.19) on p.98 & (3.94) on p.115.

2. The eigenvalue problem is a *linear* and *homog*. boundary-value problem. $\Rightarrow$ If u_n's are solutions, $\sum_n a_n u_n$ is also a solution. (A.16)

3. If L is Hermitian, u_n form a complete set. (A.17)

The following quotes, though not proofs, make this very clear.

Jackson, p.68: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete."

M&W, p.265: "It is possible to expand any function, obeying the appropriate conditions, in a series of eigenfunctions. That is, the eigenfunctions of a Hermitian operator form a complete set under very general conditions. We shall not prove this property here but it is in fact true for all the commonly encountered differential eqs. in physics."

See M&W p.173 for the meaning of "appropriate conditions", which principally apply to functions in mathematics. In physics, we may simply say that a complete set of eigenfunctions can represent any function. They are thus powerful building blocks of physical quantities.

Examples : Here we examine some previous problems in the context of an eigenvalue problem.

Example 1: $\dfrac{d^2X}{dx^2}=-\alpha^2 X$, b.c.'s: $X(0)=X(a)=0$

$\Rightarrow X(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$

> These 2 b.c.'s make d^2/dx^2 Hermitian [see (A.9), (A.10)].

$$\text{b.c.'s}\begin{cases} X(0)=0\Rightarrow B=-A\Rightarrow X(x)=A(e^{i\alpha x}-e^{-i\alpha x})=A'\sin\alpha x \\ X(a)=0\Rightarrow \alpha=\alpha_n=\frac{\pi n}{a},\ n=1,2,\dots\quad [\alpha_n\text{: eigenvalues}]\end{cases}$$

$\Rightarrow \sin\alpha_n x$ $(n=1,2,\dots)$ form a set of eigenfunctions.

Note the following general properties of an eigenvalue problem:

a. The D.E. & b.c.'s are both homogeneous. $\Rightarrow$ Each eigenfunction $(\sin\alpha_n x)$ multiplied by any constant A_n is still a solution.

b. Eigenvalues $(\alpha_n=n\pi/a)$ are determined by the b.c.'s.

c. d^2/dx^2 is Hermitian $\Rightarrow$ All eigenvalues α_n are real.

d. d^2/dx^2 is Hermitian $\Rightarrow$ The set of $\sin\alpha_n x$ are orthogonal.

e. d^2/dx^2 is Hermitian $\Rightarrow$ The set of $\sin\alpha_n x$ are complete, i.e. any function $f(x)$ can be expanded as as $f(x)=\sum\limits_{n=1}^{\infty}A_n\sin\alpha_n x$

Example 2: Eigenvalue problem involving the Legendre equation (Jackson Sec. 3.2 and 3.4, M&W Sec. 7.1)

$$\frac{d}{dx}[(1-x^2)\frac{du}{dx}]+\nu(\nu+1)u=0\ [(3.10)],\ -1\le x\le 1,\ \begin{cases}u(-1)=\text{finite}\\ u(1)=\text{finite}\end{cases}\quad (\text{A.18})$$

This is an eigenvalue problem of the form:

$$\overbrace{[\frac{d}{dx}\underset{\substack{\uparrow\\ \boxed{1-x^2}}}{p(x)}\frac{d}{dx}-\underset{\substack{\uparrow\\ \boxed{0}}}{q(x)}]}^{L}u(x)=\lambda\overbrace{\underset{\substack{\uparrow\\ \boxed{-\nu(\nu+1)}}}{h(x)}}^{1}u(x)$$

> Whether L is Hermitian depends on the form of L and the b.c.'s.

Question : 1. Is L Hermitian?

For L to be Hermitian, we need $u_i^*(x)p(x)\dfrac{du_j(x)}{dx}\Big|_{-1}^{1}=0$ [(A.12)].

(A.12) is satisfied here because $p=1-x^2=0$ at $x=\pm 1$ although $u_{i,j}(\pm 1)=$ finite $(\neq 0)$.

2. What is the eigenvalue? Strictly, $-\nu(\nu+1)$ is the eigenvalue. But we shall loosely call ν an eigenvalue (see M&W, p.262).

Rewrite

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]+\nu(\nu+1)u=0,\ -1\le x\le 1,\ \begin{cases}u(-1)=\text{finite}\\ u(1)=\text{finite}\end{cases}\qquad \text{(A.18)}$$

(A.18) has the solution (lecture notes, p. 1):

$$u(x)=AP_\nu(x)+BQ_\nu(x)$$

b.c.'s "$u(x=\pm 1)=\text{finite}$" require $B=0$ and $\nu=l=0,\ 1,\ 2\cdots$.

Thus, the solution is $\quad u(x)=P_l(x)$ with $l=0,\ 1,\ 2,\ldots$

Since L is Hermitian, the set $u(x)$ are orthogonal in the sense:

$$\int_a^b u_i(x)u_j^*(x)h(x)dx=0,\quad \text{if } i\neq j\ \ [\text{(A.15)}]$$

So, with $h(x)=1$, we have $\quad \int_{-1}^{1}P_{l'}(x)P_l(x)dx=\frac{2}{2l+1}\delta_{l'l}\qquad$ (3.21)

Eigenfunctions of a Hermitian operator form a complete set.

$\Rightarrow P_l(x)$ is complete in index ℓ, i.e. any function $f(x)$ can be expanded as

$$f(x)=\sum_{l=0}^{\infty}A_lP_l(x)\ \ [-1\le x\le 1]\qquad \text{(A.19)}\ \text{(3.23)}$$

Example 3: Eigenvalue problem involving the associated Legendre equation (Jackson Sec. 3.5, M&W Sec. 7.1)

$$\frac{d}{dx}[(1-x^2)\frac{du}{dx}]+[\nu(\nu+1)-\frac{m^2}{1-x^2}]u=0\,[(3.9)],\ -1\le x\le 1,\ \begin{cases}u(-1)=\text{finite}\\ u(1)=\text{finite}\end{cases}$$

A question arises as to whether ν or m is the eigenvalue. This can be resolved by putting the equation in the eigenvalue problem format:

$$\overbrace{[\frac{d}{dx}\underbrace{p(x)}_{\boxed{1-x^2}}\frac{d}{dx}-\underbrace{q(x)}_{\boxed{m^2/(1-x^2)}}]}^{L}u(x)=\underbrace{\lambda}_{\boxed{-\nu(\nu+1)}}\overbrace{h(x)}^{1}u(x)$$

For the same reason as in Example 2, L here is a Hermitian operator.

Thus, ν is the eigenvalue, which is to be determined from b.c.'s.

The associated Legendre eq. has the solution (lecture notes, p. 4): $u(x)=AP_\nu^m(x)+BQ_\nu^m(x)$. For $u(x=\pm 1)=\text{finite}$, we require $B=0$, $\nu=l=0,\ 1,\ 2\cdots$, and $m=-l,\ -(l-1)\cdots-1,\ 0,\ 1,\cdots,\ (l-1),\ l$. Thus,

$$u(x)=P_l^m(x)\ \text{with}\ l=|m|,\ |m|+1,\ |m|+2,\ldots$$

Appendix A. Eigenvalue Problem (*continued*)

Rewrite $\quad u(x) = P_l^m(x)$ with $l = |m|,\ |m|+1,\ |m|+2, \ldots$

Since the operator L is Hermitian, l is the eigenvalue, and $P_l^m(x)$ is the eigenfunction, $P_l^m(x)$ is orthogonal in index ℓ (not m). Thus, with the density function $h(x) = 1,$ we have

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'} \qquad (3.52)$$

Also, because of the Hermitian property of the operator L, $P_l^m(x)$ is complete in eigenvalue index ℓ, i.e. any function $f(x)$ can be expanded as

$$f(x) = \sum_{l=|m|}^{\infty} C_l P_l^m(x) \quad [\text{see M\&W, p.175.}] \qquad (A.20)$$

Appendix A. Eigenvalue Problem (*continued*)

Example 4: Eigenvalue problem involving the Bessel equation
(Jackson Secs. 3.7 and 3.8; M&W Sec. 7.2)

$$\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)u = 0 \ [(3.75)],\ 0 \le \rho \le a,\ \text{b.c.} \begin{cases} u(0) = \text{finite} \\ u(a) = 0 \end{cases}$$

This equation can be written: $\frac{d}{d\rho}\rho\frac{du}{d\rho} + \left(k^2\rho - \frac{\nu^2}{\rho}\right)u = 0 \qquad (A.21)$

Again, we have the question as to whether k or ν is the eigenvalue. Putting (A.21) in the format:

$$[\overbrace{\frac{d}{d\rho} \underset{\substack{\uparrow \\ \boxed{\rho}}}{p(\rho)} \frac{d}{d\rho} - \underset{\substack{\uparrow \\ \boxed{\nu^2/\rho}}}{q(\rho)}}^{L}]u(\rho) = \underset{\substack{\uparrow \\ \boxed{-k^2,\ \text{eigenvalue}}}}{\lambda} \overbrace{h(\rho)}^{\rho\,\text{(density function)}} u(\rho) = 0,$$

we see that k is the eigenvalue.

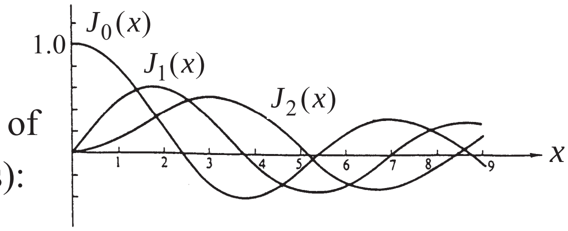
As shown on. p. 13 of lecture notes, $\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)u = 0$

has the solution $u(\rho) = AJ_\nu(k\rho) + BN_\nu(k\rho).$

Rewrite $\begin{cases} \dfrac{d^2u}{d\rho^2}+\dfrac{1}{\rho}\dfrac{du}{d\rho}+(k^2-\dfrac{\nu^2}{\rho^2})u=0,\ 0\le\rho\le a, & \begin{cases} u(0)=\text{finite} \\ u(a)=0 \end{cases} \\ u(\rho)=AJ_\nu(k\rho)+BN_\nu(k\rho) \end{cases}$

$u(0)=\text{finite}\Rightarrow B=0.$

$u(a)=0\Rightarrow J_\nu(ka)=0,$



which yields an infinite number of eigenvalues (and eigenfunctions):

$k=k_{\nu n},\ n=1,2,3\cdots,$

where $k_{\nu n}a=x_{\nu n}$ and $x_{\nu n}$ is the n-th root of $J_\nu(x)=0$ (see p. 114).

L is Hermitian. $\Rightarrow J_\nu(k_{\nu n}\rho)$ are orthogonal in index n:

$$\int_0^a J_\nu(k_{\nu n'}\rho)J_\nu(k_{\nu n}\rho)\rho d\rho=\frac{a^2}{2}[J_{\nu+1}(\underbrace{k_{\nu n}a}_{x_{\nu n}})]^2\delta_{n'n} \qquad \begin{matrix}\text{(A.22)}\\ \text{(3.95)}\end{matrix}$$

density function, see (A.15)

and complete in eigenvalue index n: $\quad f(\rho)=\sum_{n=1}^{\infty}C_nJ_\nu(k_{\nu n}\rho) \qquad \begin{matrix}\text{(A.23)}\\ \text{(3.96)}\end{matrix}$