Chapter 14: Radiation by Moving Charges

14.1 Liénard-Wiechert Potentials and Fields for a Point Charge

$\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ for a Point Charge:

Rewrite (6.16) and (6.16) in *Gaussian unit system* (see p. 782 for conversion formulae):

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -4\pi\rho & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} & (6.16) \end{cases}$$

As shown in lecture notes [Ch. 6, Eq. (7) & Sec. 9.1], the solutions are

$$\begin{Bmatrix} \Phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \rho(\mathbf{x}',t') \\ \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \end{Bmatrix} \quad (9.2)$$

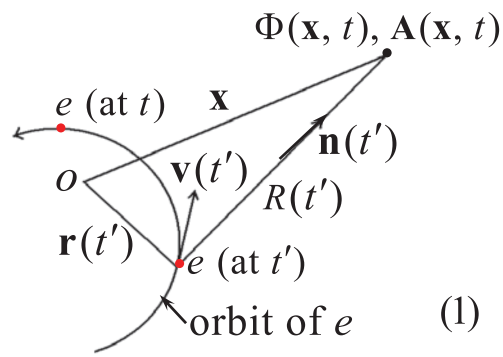
Note: For sources with $e^{-i\omega t}$ dependence (as in Chaps. 9 & 10), only $\mathbf{A}$ is needed to determine $\mathbf{E}$ & $\mathbf{B}$. Here, we need both Φ & $\mathbf{A}$.

Rewrite
$$\begin{Bmatrix} \Phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \rho(\mathbf{x}',t') \\ \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \end{Bmatrix} \quad (9.2)$$

Consider a point charge e (e carries a sign) moving along the orbit $\mathbf{r}(t')$ with velocity $\mathbf{v}(t') = \frac{d\mathbf{r}(t')}{dt'}$. Then,

$$\begin{cases} \rho(\mathbf{x}',t') = e\delta[\mathbf{x}'-\mathbf{r}(t')] \\ \mathbf{J}(\mathbf{x}',t') = e\mathbf{v}(t')\delta[\mathbf{x}'-\mathbf{r}(t')] \end{cases}$$

$$\Rightarrow \begin{cases} \Phi(\mathbf{x},t) = e\int dt' \frac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \frac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases} \quad (1)$$



e's orbit is arbitrary, including $\mathbf{v} = 0$ or *const*

where $R(t') = |\mathbf{x}-\mathbf{r}(t')|$ and $\boldsymbol{\beta}(t') = \mathbf{v}(t')/c$.

$\delta[t'+\frac{R(t')}{c}-t] \Rightarrow t'$ is the solution of $t'+\frac{R(t')}{c}-t = t'+\frac{|\mathbf{x}-\mathbf{r}(t')|}{c}-t = 0.$

Question: There is only one solution for t' for fixed $\mathbf{x}$ and t. Why?

Lienard-Wiechert Potentials for a Point Charge :

Rewrite
$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases} \quad (1)$$

$$\delta[f(x)] = \frac{\delta(x-x_i)}{|f'(x)|} \ [f(x_i)=0,\ \text{p.26}] \Rightarrow \delta[t'+\frac{R(t')}{c}-t] = \frac{\delta(t'-t_{ret})}{\left|\frac{d}{dt'}[t'+\frac{R(t')}{c}-t]\right|},$$

where t_{ret} (retarded time) is the solution of $t'+\frac{R(t')}{c}-t=0$.

"*ret*" $\Rightarrow t' = t_{ret}$

$$\Rightarrow \begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta(t'-t_{ret})}{R(t')\left|\frac{d}{dt'}[t'+\frac{R(t')}{c}-t]\right|} = \left[\dfrac{e}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta(t'-t_{ret})}{R(t')\left|\frac{d}{dt'}[t'+\frac{R(t')}{c}-t]\right|} = \left[\dfrac{e\boldsymbol{\beta}(t')}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \end{cases}, \quad (2)$$

where $f(t') \equiv t' + \frac{R(t')}{c}$ $\quad$ t (observation time) is at a fixed value. $\quad (3)$

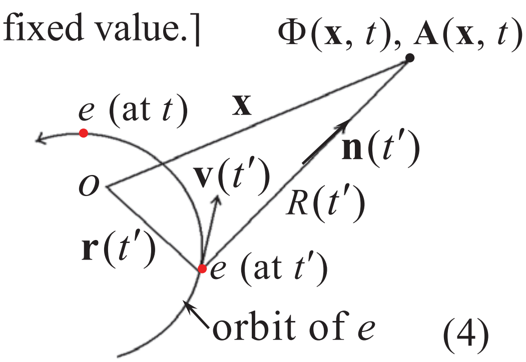
$$\frac{dR(t')}{dt'} = \frac{d|\mathbf{x}-\mathbf{r}(t')|}{dt'} = \frac{d}{dt'}[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}(t')\cdot\mathbf{r}(t')]^{\frac{1}{2}}$$

[$\mathbf{x}$ (observation point) is at a fixed value.]

$$= \frac{-2\mathbf{x}\cdot\frac{d}{dt'}\mathbf{r}(t') + 2\mathbf{r}(t')\cdot\frac{d}{dt'}\mathbf{r}(t')}{2[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + r^2(t')]^{\frac{1}{2}}}$$

$$= -\frac{\mathbf{v}(t')\cdot[\mathbf{x}-\mathbf{r}(t')]}{R(t')}$$

$$= -\mathbf{v}(t')\cdot\mathbf{n}(t') \quad \boxed{(4)} \quad (4)$$



$$\Rightarrow \frac{d}{dt'}f(t') = \frac{d}{dt'}[t' + \frac{R(t')}{c}] = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t') \equiv \kappa\ (>0) \quad (5)$$

Sub. (5) into (2) gives the Lienard-Wiechert potentials

$$\begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} \quad (14.8)$$

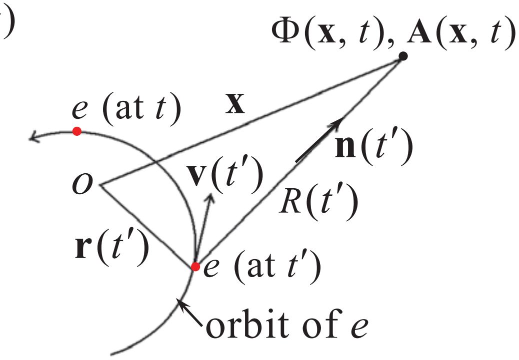
Quantities in the brackets are evaluated at the retarded time, i.e. solu. of $t' + \frac{R(t')}{c} - t = 0$.

E and B for a Point Charge: We have 2 expressions for Φ, $\mathbf{A}$:

$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases} (1); \quad \begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} \quad (14.8)$$

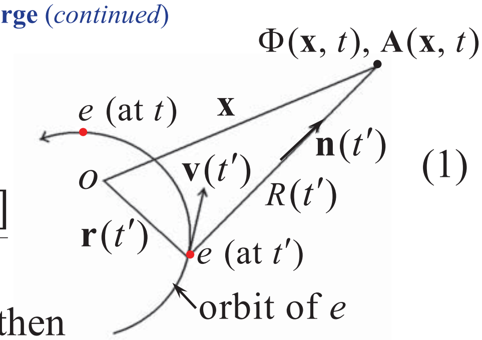
To calculate $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$

from $\begin{cases} \mathbf{E}(\mathbf{x},t) = -\nabla\Phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A}(\mathbf{x},t) \end{cases}$



we need to differentiate $\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ with respect to $\mathbf{x}$. The RHS of (14.8) depends on $\mathbf{x}$ through $\mathbf{n}$ and R, but the RHS of (1) depends on $\mathbf{x}$ through R only. Hence, it is more convenient to use (1).

Rewrite $\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases}$ (1)



Let $F(R)$ be any func. of $R(=|\mathbf{x}-\mathbf{r}(t')|)$, then

$$\nabla_{\mathbf{x}}F(R) = \frac{dF}{dR}\nabla_{\mathbf{x}}R = \frac{dF}{dR}\underbrace{\nabla_{\mathbf{x}}|\mathbf{x}-\mathbf{r}(t')|}_{=\mathbf{n}(t')} = \mathbf{n}(t')\frac{dF}{dR} \quad (6)$$

$$\boxed{\nabla_{\mathbf{x}}|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \text{ [lecture notes, Ch. 1, Eq. (1)]}}$$

$$(1)\ \&\ (6) \Rightarrow \begin{cases} \nabla\Phi(\mathbf{x},t) = e\int \mathbf{n}(t')\left\{\dfrac{-\delta[t'+\frac{R(t')}{c}-t]}{R^2(t')} + \dfrac{\delta'[t'+\frac{R(t')}{c}-t]}{cR(t')}\right\}dt' \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\dfrac{e}{c}\int \dfrac{\boldsymbol{\beta}(t')\delta'[t'+\frac{R(t')}{c}-t]}{R(t')}dt' \end{cases}$$

$$\Rightarrow \mathbf{E}(\mathbf{x},t) = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} = e\int\left\{\frac{\mathbf{n}}{R^2}\delta[t'+\frac{R(t')}{c}-t] + \frac{\boldsymbol{\beta}-\mathbf{n}}{Rc}\delta'[t'+\frac{R(t')}{c}-t]\right\}dt'$$

Rewrite $\mathbf{E}(\mathbf{x},t)=e\int\left\{\frac{\mathbf{n}}{R^2}\delta[t'+\frac{R(t')}{c}-t]+\frac{\boldsymbol{\beta}-\mathbf{n}}{Rc}\delta'[t'+\frac{R(t')}{c}-t]\right\}dt'$

Let $dt'=\frac{dt'}{df(t')}df(t')=\frac{1}{\kappa}df(t')$, where $\begin{cases} f(t')\equiv t'+\frac{R(t')}{c} & [(3)] \\ \kappa\equiv 1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t') & [(5)]\end{cases}$

$$\Rightarrow \mathbf{E}(\mathbf{x},t)=e\int\left\{\frac{\mathbf{n}}{\kappa R^2}\delta[f(t')-t]+\frac{\boldsymbol{\beta}-\mathbf{n}}{\kappa Rc}\delta'[f(t')-t]\right\}df(t') \quad \left[\begin{array}{l}\text{see note}\\ \text{below}\end{array}\right]$$

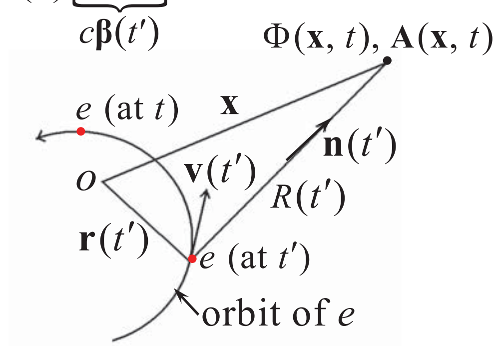
$$=e\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{c}\frac{d}{df(t')}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret} \qquad \boxed{\int g(x)\delta'(x-a)dx=-g'(a)}$$

$$=e\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret} \qquad \boxed{\frac{d}{df(t')}=\frac{dt'}{df(t')}\frac{d}{dt'}=\frac{1}{\kappa}\frac{d}{dt'}} \tag{7}$$

Note: Because of the $\delta[f(t')-t]$ and $\delta'[f(t')-t]$ factors in the integrand, the $\int\cdots df(t')$ integral above demands $f(t')[\equiv t'+\frac{R(t')}{c}]=t$, or $t'=t-\frac{R(t')}{c}$. Thus, as $[\cdots]_{ret}$ implies, $\mathbf{n}$, $\boldsymbol{\beta}$, R, κ in the integrand need to be evaluated at the retarded time t' (not at t), .

To simplify $\mathbf{E}$, we first evaluate $\frac{d\mathbf{n}(t')}{dt'}$ and $\frac{d}{dt'}(\kappa R)$.

$$\frac{d\mathbf{n}(t')}{dt'}=\frac{d}{dt'}\frac{\mathbf{x}-\mathbf{r}(t')}{R(t')}=-\underbrace{\frac{\mathbf{x}-\mathbf{r}(t')}{R^2(t')}}_{\frac{\mathbf{n}(t')}{R(t')}}\underbrace{\frac{dR(t')}{dt'}}_{-c\boldsymbol{\beta}\cdot\mathbf{n}\text{ by (4)}}-\frac{1}{R(t')}\underbrace{\frac{d\mathbf{r}(t')}{dt'}}_{c\boldsymbol{\beta}(t')}=\frac{c}{R}\left[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}\right] \tag{8a}$$



$\boxed{\kappa\equiv 1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')}$

$$\frac{d}{dt'}(\kappa R)=\frac{d}{dt'}\{[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]R\}$$

$$=(1-\boldsymbol{\beta}\cdot\mathbf{n})\underbrace{\frac{d}{dt'}R}_{-c\boldsymbol{\beta}\cdot\mathbf{n}}-R\frac{d}{dt'}(\boldsymbol{\beta}\cdot\mathbf{n})$$

$$=-c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n})-R\dot{\boldsymbol{\beta}}\cdot\mathbf{n}-R\boldsymbol{\beta}\cdot\frac{d\mathbf{n}}{dt'} \qquad \boxed{\text{Sub. (8) for } \frac{d\mathbf{n}}{dt'}}$$

$$=-c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n})-R\dot{\boldsymbol{\beta}}\cdot\mathbf{n}-c\left[(\mathbf{n}\cdot\boldsymbol{\beta})^2-\beta^2\right]$$

$$=-c(\boldsymbol{\beta}\cdot\mathbf{n})(1-\boldsymbol{\beta}\cdot\mathbf{n}+\boldsymbol{\beta}\cdot\mathbf{n})+c\beta^2-R\dot{\boldsymbol{\beta}}\cdot\mathbf{n}$$

$$=c\beta^2-c(\boldsymbol{\beta}\cdot\mathbf{n})-R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) \tag{8b}$$

$$\mathbf{E}(\mathbf{x},t)=e\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{c\kappa^2 R}\frac{d}{dt'}(\mathbf{n}-\boldsymbol{\beta})+\frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)\right]_{ret}$$

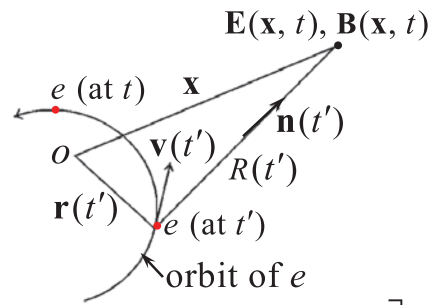
from (7)

Use (8a,b)

$$=e\left\{\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{c\kappa^2 R}\left[\frac{c}{R}[\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}]-\dot{\boldsymbol{\beta}}\right]-\frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa^3 R^2}\left[c\beta^2-c(\boldsymbol{\beta}\cdot\mathbf{n})-R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\right\}_{ret}$$

$$=e\left\{\frac{1}{R^2}\left[\underbrace{\frac{\mathbf{n}}{\kappa}+\frac{\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}}{\kappa^2}}-\frac{(\mathbf{n}-\boldsymbol{\beta})(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})}{\kappa^3}\right]+\frac{1}{R}\left[\frac{-\dot{\boldsymbol{\beta}}}{c\kappa^2}+\frac{(\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})}{c\kappa^3}\right]\right\}_{ret}$$

$$\boxed{=\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2}=\frac{(\mathbf{n}-\boldsymbol{\beta})[1-(\boldsymbol{\beta}\cdot\mathbf{n})]}{\kappa^3}\quad[\kappa\equiv 1-\boldsymbol{\beta}\cdot\mathbf{n}]}$$



$$=e\left\{\begin{aligned}&\frac{1}{\kappa^3 R^2}(\mathbf{n}-\boldsymbol{\beta})[(1-\boldsymbol{\beta}\cdot\mathbf{n})-(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})]\\&+\frac{1}{c\kappa^3 R}\left[-\dot{\boldsymbol{\beta}}(1-\boldsymbol{\beta}\cdot\mathbf{n})+(\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\end{aligned}\right\}_{ret}$$

$$=e\left[\frac{1}{\kappa^3 R^2}[(\mathbf{n}-\boldsymbol{\beta})(\underbrace{1-\beta^2}_{1/\gamma^2})]+\frac{1}{c\kappa^3 R}\left\{\underbrace{\mathbf{n}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})-\dot{\boldsymbol{\beta}}}_{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}-[\underbrace{\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})-\dot{\boldsymbol{\beta}}\cdot(\boldsymbol{\beta}\cdot\mathbf{n})}_{\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})}]\right\}\right]_{ret}$$

$$\Rightarrow \mathbf{E}(\mathbf{x},t)=e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}+\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret} \qquad (14.14)$$

To derive $\mathbf{B}(\mathbf{x},t)$, we rewrite (7):

$$\mathbf{E}(\mathbf{x},t)=e\left[\frac{\mathbf{n}}{\kappa R^2}+\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$\Rightarrow \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)=e\left[\frac{1}{\kappa c}\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$=-e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right)+\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret}$$

$$\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)$$

Use (8a)

$$=\frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})}{\kappa R}\right]-\frac{d\mathbf{n}}{dt'}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}$$

$$=-\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right)-\frac{c[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}]}{R}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}$$

$$=-\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right)-\frac{c\,\mathbf{n}\times\boldsymbol{\beta}}{R^2}$$

∇ operates on $R(t')$ only [only $R(t')$ depends on $\mathbf{x}$]

$$\mathbf{B}(\mathbf{x},t)=\nabla\times\mathbf{A}=e\int dt'\nabla\times\left[\frac{\boldsymbol{\beta}(t')\delta[t'+R(t')/c-t]}{R(t')}\right]$$

(1)

$$=e\int dt'\left[\nabla\frac{\delta[t'+R(t')/c-t]}{R(t')}\right]\times\boldsymbol{\beta}(t')$$

$\nabla\times\psi\mathbf{a}=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$

$$=e\int dt'\left[-\frac{\delta[t'+R(t')/c-t]}{R^2}+\frac{\delta'[t'+R(t')/c-t]}{cR}\right]\overbrace{\nabla R(t')}^{\mathbf{n}(t')}\times\boldsymbol{\beta}(t')$$

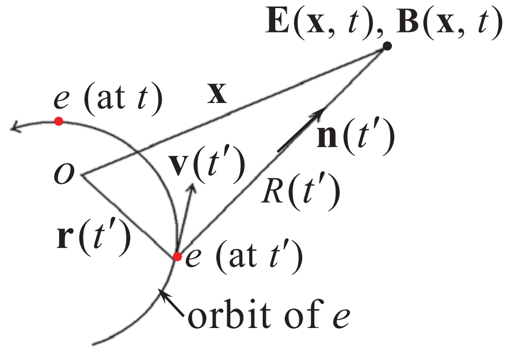
$$=-e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right)+\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret}$$

same steps as in deriving (7)

$$\Rightarrow \mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \qquad (14.13)$$

Rewrite
$$\begin{cases} \mathbf{E}(\mathbf{x},t) = e\left[\dfrac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} + \dfrac{e}{c}\left[\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret} & (14.14) \\ \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \quad [e \text{ carries a sign.}] & (14.13) \end{cases}$$

Discussion:



(i) We have derived (14.13) and (14.14) from (9.2), which applies to free space, in which waves of all frequencies travel at the same speed c.

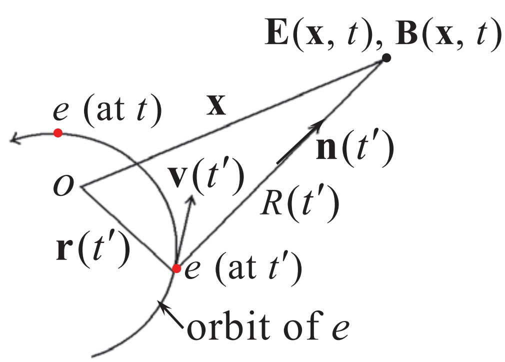
Hence, a multi-frequency signal originating from the retarded instant t' and retarded position $\mathbf{r}(t')$ arrives at $\mathbf{x}$ at the same instant t.

Note: 1. All signals travel at speed c independent of e's velocity.

2. t, t', $\mathbf{x}$, and $\mathbf{x}'$ are quantities in the same inertial frame.

3. Unit vector $\mathbf{n}(t')$ points from the *retarded* position to $\mathbf{x}$.

Rewrite
$$\begin{cases} \mathbf{E}(\mathbf{x},t) = \underbrace{e\left[\dfrac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}}_{\text{velocity field } (\propto \frac{1}{R^2})} + \underbrace{\dfrac{e}{c}\left[\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\substack{\text{acceleration field} \\ (\propto \frac{\dot{\beta}}{R} \text{ and } \perp \mathbf{n})}} & (14.14) \\ \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) & (14.13) \end{cases}$$



(ii) $\mathbf{E}$, $\mathbf{B}$ can be divided into

a. <u>Velocity fields</u> (1st term):

These are essentially static fields, which fall off as $1/R^2$.

$\mathbf{E}(\mathbf{x},t)$ reduces to Coulomb's field if $\boldsymbol{\beta}(t') = 0$. It is not along $\mathbf{n}(t')$ unless $\boldsymbol{\beta}(t') = 0$.

b. <u>Acceleration fields</u> (2nd term):

These are radiation fields for which $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, $\mathbf{n}(t')$ are mutually orthogonal and $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$ fall off as $1/R$.

$\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$ (*radiated by e* at t') $= 0$ if $\dot{\boldsymbol{\beta}}(t') = d\boldsymbol{\beta}(t')/dt' = 0$.

Rewrite

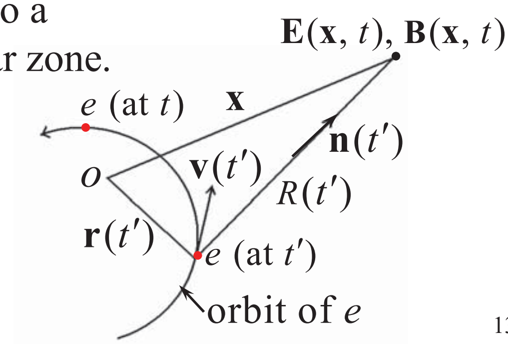
$$
\begin{cases}
\mathbf{E}(\mathbf{x},t) = e\underbrace{\left[\dfrac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})} + \dfrac{e}{c}\underbrace{\left[\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot{\beta}}{R}\text{ and }\perp\mathbf{n})}} & (14.14)\\
\mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) & (14.13)
\end{cases}
$$

(iii) In Ch. 9, for localized $\mathbf{J}$ and ρ with $e^{-i\omega t}$ dependence, we get

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik\sum_{l,m} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')j_l(kr')Y_{lm}^*(\theta',\phi')\ [(9.11)],$$

in which the field profile transitions from a (predominately) static field in spatial dependence in the near zone to a (predominately) radiation field in the far zone.

For the fields of a point charge [(14.13) and (14.14)] the "velocity" and "acceleration" fields appear in separate terms. Hence, they can be separated at all positions (0 to ∞).



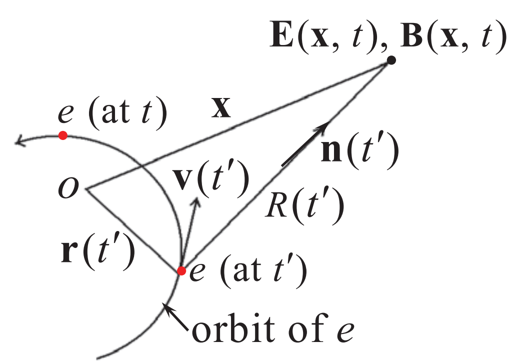
Rewrite

$$
\begin{cases}
\mathbf{E}(\mathbf{x},t) = e\underbrace{\left[\dfrac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})} + \dfrac{e}{c}\underbrace{\left[\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot{\beta}}{R}\text{ and }\perp\mathbf{n})}} & (14.14)\\
\mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) & (14.13)
\end{cases}
$$

(iv) Quantities in the brackets are to be evaluated at the retarded time t'.

The retarded quantities [t', $\boldsymbol{\beta}(t')$, $\mathbf{n}(t')$, $R(t')$] can be obtained in terms of $\mathbf{x}$ and t from the equation:

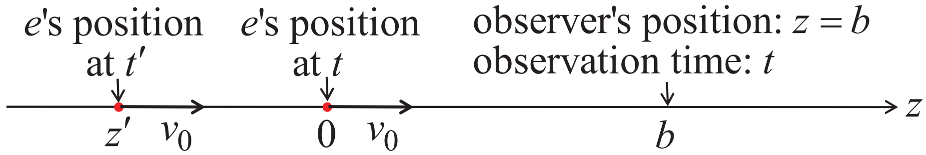
$$t' + \frac{R(t')}{c} - t = t' + \frac{|\mathbf{x}-\mathbf{r}(t')|}{c} - t = 0,$$



where $\mathbf{r}(t')$ is specified with respect to $\mathbf{x}$ and t (see *Exercise* below).

Thus, as indicated on the LHS of (14.13) and (14.14), the final expressions for $\mathbf{E}$ and $\mathbf{B}$ are functions of $\mathbf{x}$ and t only [see (14.17a) below as a specific example].

Exercise: A point charge e is moving at a constant velocity $v_0\mathbf{e}_z$ on the z-axis ($v_0 > 0$). At time t, it is at $z = 0$. An obsever is located at $z = b$ on the z-axis ($b > 0$). The "velocity field" of e oberved at time t and $z = b$ has its origin at the retarded time t' ($< t$) and the retarded position z' ($< b$). Find t' and z' in terms of t and b.



Charge e's orbit is $\quad z' = v_0(t'-t)$ (A)

The retarded time t' is the solution of $\; t' = t - \dfrac{|b-z'|}{c}$ (B)

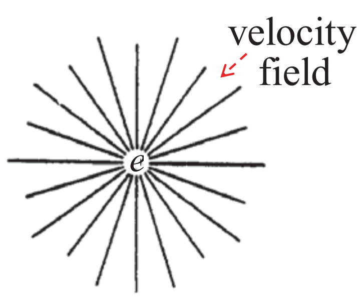
Sub. (A) into (B) $\Rightarrow \; t' = t - \dfrac{|b-v_0(t'-t)|}{c} \underset{\uparrow}{=} t - \dfrac{b-v_0(t'-t)}{c}$ (C)

(C) $\Rightarrow \; t' = t - \dfrac{b}{c-v_0}$ (retarded time) $\boxed{b>0,\ v_0>0,\ t'<t}$ (D)

(A), (D) $\Rightarrow \; z' = -v_0 \dfrac{b}{c-v_0}$ (retarded position)

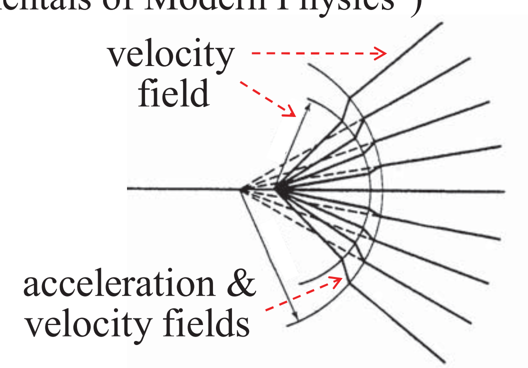
Note: The determination of t', z' requires knowledge of e's orbit.

A conceptual picture of velocity and acceleration fields:
(from R. M. Eisberg, "Fundamentals of Modern Physics")



E-field lines around a point charge e at rest:

$\mathbf{E} = \dfrac{e}{r^2}\mathbf{e}_r \begin{bmatrix}\text{in Gaussian} \\ \text{unit system}\end{bmatrix}$



Soild lines: **E**-field lines of the same e at rest at a new position, after a brief period of acceleration and deceleration

Fields between circles (not an exact plot) originate from e during a brief accelaration/decelaration period. It leaves e at speed c.

Other lines show pure velocity field (plotted exactly).

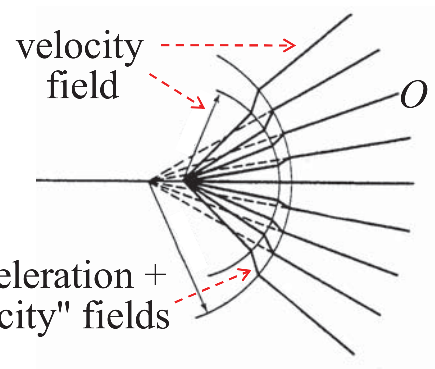
In order for Coulomb's law (left figure) to apply to all space ($r = 0 \to \infty$), e has to be at rest from $t = -\infty$ to the time of observation.

Rewrite

$$\mathbf{E}(\mathbf{x},t)=\underbrace{e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\ (\propto\frac{\dot{\beta}}{R}\text{ and }\perp\mathbf{n})}} \quad (14.14)$$

$$\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \quad (14.13)$$

(v) At a different t, an oberver at the same $\mathbf{x}$ will see different $\mathbf{E}(\mathbf{x},t)$ & $\mathbf{B}(\mathbf{x},t)$, which originate from a different retarded time t' and a different retarded position.



Consider the example on the right figure again. All fields move away from e at speed c. Thus, an observer at point O will first see the Coulomb field (a velocity field) from e at its old position, then "acceleration + velocity" fields (briefly), and finally the Coulomb field from e at its new position.

Fields of a Point Charge e Moving at $\mathbf{v} = const.$

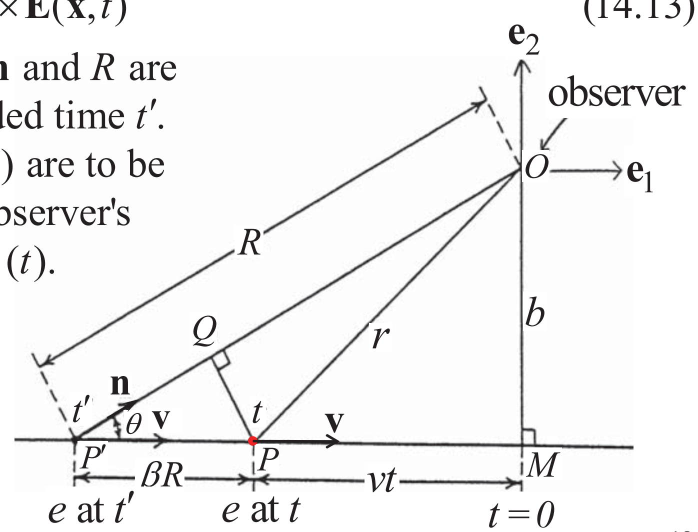
Rewrite

$$\mathbf{E}(\mathbf{x},t)=\underbrace{e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\underbrace{\cancel{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}}_{\substack{\text{acceleration field}\\ [\propto\frac{\dot{\beta}}{R}\text{ and }\perp\mathbf{n}(t')]}} \quad (14.14)$$

$$\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \quad (14.13)$$

In (14.13) and (14.14), $\mathbf{n}$ and R are to be evaluated at the retarded time t'. However, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are to be expressed in terms of the observer's position ($\mathbf{x}=b\mathbf{e}_2$) and time (t).

So our first job is to express the retarded quantities in terms of b and t from the knowledge that e moves at $\mathbf{v}=const.$



For a general solution, we obtain retarded quantities by geometry:

$P'P$ = distance between points P' and $P = v\frac{R}{c} = \beta R$

$P'Q = \beta R\cos\theta = \boldsymbol{\beta}\cdot\mathbf{n}R$

$OQ = R - P'Q = R(1-\boldsymbol{\beta}\cdot\mathbf{n})$

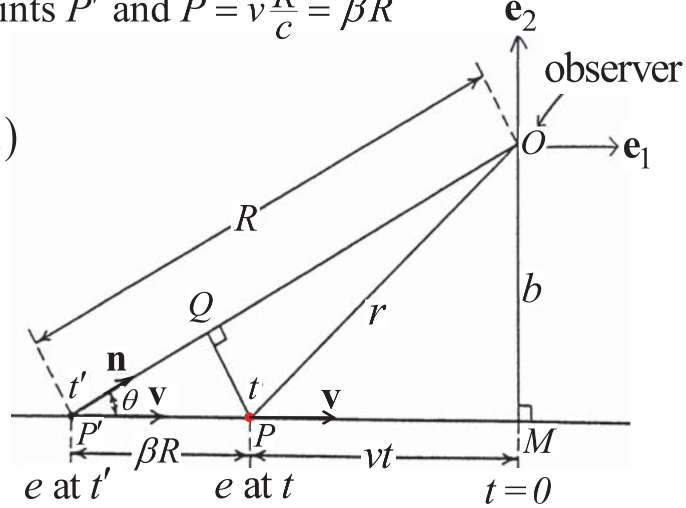
$$(OQ)^2 = \left[R(1-\boldsymbol{\beta}\cdot\mathbf{n})\right]^2_{ret}$$

$$= r^2 - (\underbrace{PQ}_{(P'P)^2\sin^2\theta})^2$$

$$= \overbrace{r^2}^{b^2+v^2t^2} - \beta^2\overbrace{R^2\sin^2\theta}^{b^2}$$

$$= b^2 + v^2t^2 - \beta^2 b^2 = \frac{1}{\gamma^2}(b^2+\gamma^2v^2t^2)$$

$$\Rightarrow \left[R(1-\boldsymbol{\beta}\cdot\mathbf{n})\right]_{ret} = \frac{1}{\gamma}(b^2+\gamma^2v^2t^2)^{1/2}$$



In the above, R and $\mathbf{n}$ are retarded quantities and $\boldsymbol{\beta}$ is a constant.

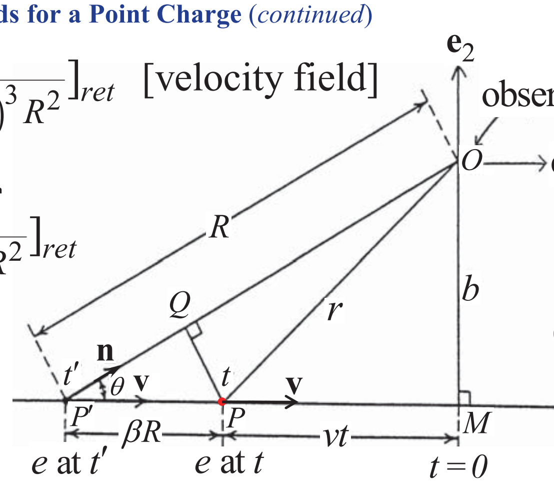
$$\mathbf{v} = const. \Rightarrow \mathbf{E} = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret} \quad \text{[velocity field]}$$

$$\Rightarrow E_2 = \mathbf{E}\cdot\mathbf{e}_2 = e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_2}^{b/R} - \overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_2}^{0}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}$$

$$= e\left[\frac{b}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^3}\right]_{ret} \quad (14.17b)$$

$$= \frac{e\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}} \quad (14.17a)$$

[same as (11.152)]



$\left[R(1-\boldsymbol{\beta}\cdot\mathbf{n})\right]_{ret} = \frac{1}{\gamma}(b^2+\gamma^2v^2t^2)^{1/2}$, last page

$$E_1 = \mathbf{E}\cdot\mathbf{e}_1 = e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_1}^{\cos\theta} - \overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_1}^{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret} = e\left[\frac{\cos\theta-\beta}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret} = \frac{-e\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

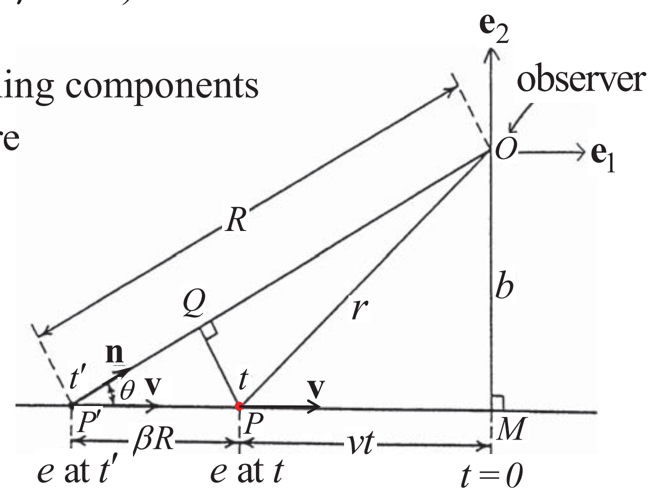
$E_3 = 0$ by symmetry.

$\cos\theta - \beta = \frac{\beta R - vt}{R} - \beta = -\frac{vt}{R}$.
$t < 0$ on the left side of the origin ($t = 0$)

$$\mathbf{B}(\mathbf{x},t) = \mathbf{n}(t') \times \mathbf{E}(\mathbf{x},t) = \left(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2\right) \times \left(E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2\right)$$
$$= \left(E_2 \cos\theta - E_1 \sin\theta\right)\mathbf{e}_3$$

So, the only nonvanishing component of $\mathbf{B}$ is B_3 given by

$$B_3 = E_2 \underbrace{\cos\theta}_{\frac{\beta R - vt}{R}} - E_1 \underbrace{\sin\theta}_{\frac{b}{R}} = \frac{e\gamma}{(b^2+\gamma^2 v^2 t^2)^{3/2}}\left[\frac{b}{R}(\beta R - vt) + vt\frac{b}{R}\right] = \beta E_2$$



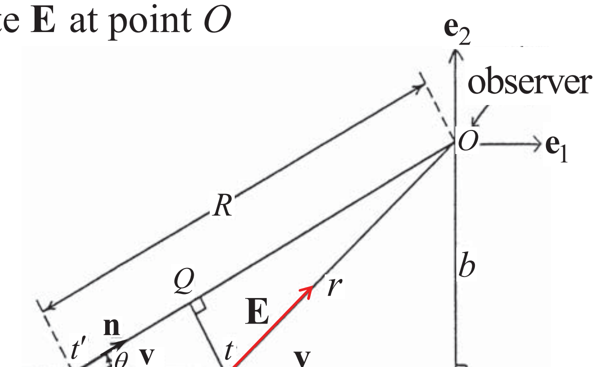
In summary, the non-vanishing components of $\mathbf{E}$, $\mathbf{B}$ at point O and tine t are

$$\begin{cases} E_1 = \dfrac{-e\gamma vt}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ B_3 = \beta E_2 \end{cases}$$

As desired, $\mathbf{E}$ and $\mathbf{B}$ are now expressed in terms of the observer's position ($\mathbf{x} = b\mathbf{e}_2$) and time ($t$).

Spatial field pattern: Rewrite $\mathbf{E}$ at point O

$$\begin{cases} E_1 = \dfrac{-e\gamma vt}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \end{cases}$$



$\Rightarrow \dfrac{E_1}{E_2} = \dfrac{-vt}{b}$ $\left(= \dfrac{|vt|}{b} \text{ if } t < 0\right)$

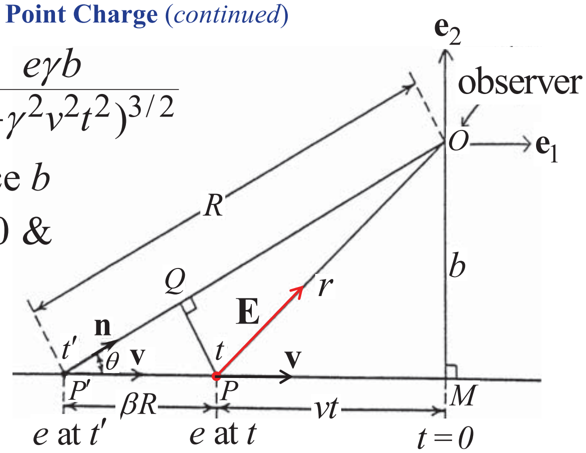
$\Rightarrow$ $\mathbf{E}$ (*velocity* field) is directed from point P (e's present position) to point O (the observation point).

$\Rightarrow$ Since b & t can have any $\pm$ values, this direction relation is true for any observation point around e, i.e. $\mathbf{E}$-field lines are straight lines originating from (or converging to) e at its instantaneous position P, as if e "*carries*" its velocity fields while in motion.

Note: As will be shown in Sec. 14.2, the "acceleration fields" of e will *leave* e as radiation fields, pointing from e at t' to the observer.

$$E_1 = \frac{-e\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}; \; E_2 = \frac{e\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

Let $t = 0$, we get E_2 at a distance b above e (point O_1 below). Let $b = 0$ & $vt = -b$, we get E_1 at a distance b in front of e (point O_2 below).

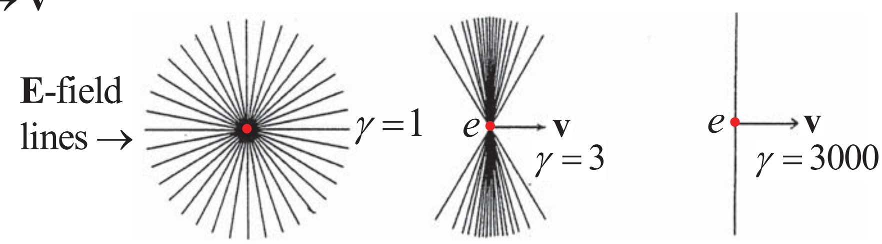


O_1: $E_2 (= E_\perp) = \dfrac{\gamma e}{b^2}$

b

P, e $\rightarrow \mathbf{v}$; b; O_2: $E_1 (= E_{||}) = \dfrac{e}{\gamma^2 b^2}$

$$\Rightarrow \begin{cases} \text{At same distance from } e, \\ E_\perp \text{ (at } O_1) = \gamma^3 E_{||} \text{ (at } O_2). \\ \perp, \; || \text{ are with respect to } \mathbf{v}. \end{cases} \quad (9)$$

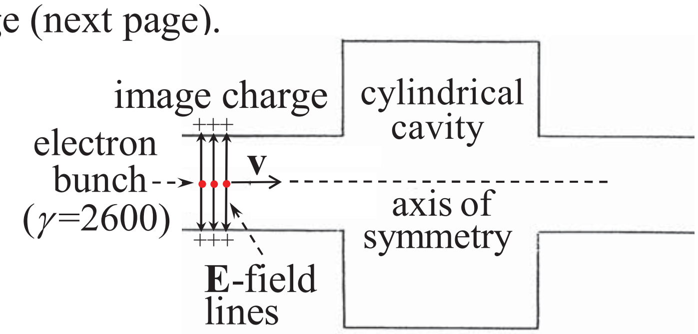


Electrodynamics in an Accelerator Cavity :

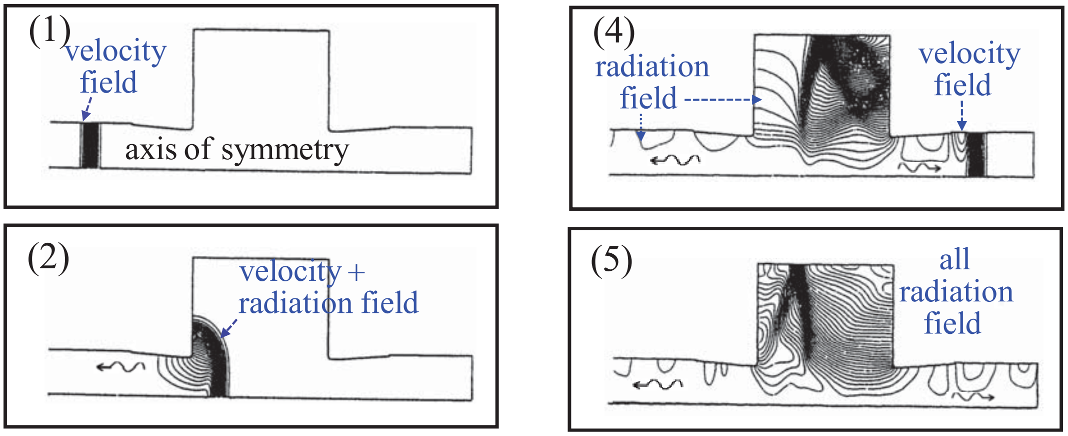
Particle accelerators normally accelerate charged particles by inputing microwaves into a cavity to generate a strong **E**-field.

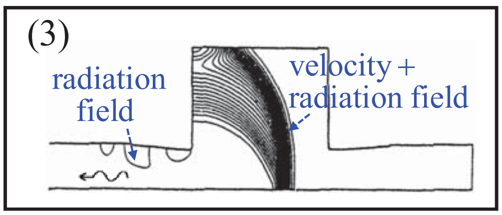
However, as shown in the figure below, charged particles can also generate their own fields in the cavity. An electron bunch moving at $\mathbf{v} = const$ with $\gamma = 2600$ is about to enter a cavity. Since $E_\perp = (2600)^3 E_{||}$, the e's **E**-field (velocity field) lines are essentially $\perp$ to **v**, originating from an electron and terminating at the + image charge on the metal wall. Hence, electrons hardly "see" each other.

Once the bunch enters the cavity, acceleration (radiation) fields will emerge (next page).



Fields in the cavity produced by a γ = 2600 electron bunch





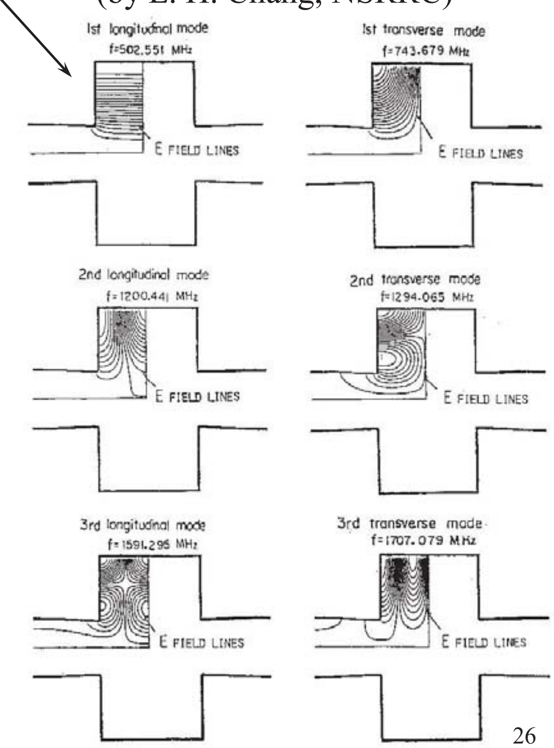
"Velocity" and "radiation" fields due to a single electron bunch (by Ch. Wang, NSRRC). Fields behind the bunch are called wake fields.

Question: How does the *e*-bunch get decelerated (hence radiate)?
Ans: Attractive forces due to + image charges on the wall.

The lowest order (TM_{010}) mode is excited by the injection of high power microwaves from a klystron. The axial electric field of this mode is used to accelerate the electrons.

Wake fields left in the cavity by the electron bunch can be viewed as the superposition of a complete set of cavity eigenmodes. One or more of the higher-order modes may thus be resonantly reinforced by a succession of electron bunches to grow to significant amplitude and interfere with the acceleration process.

E-field lines of several cavity modes (by L. H. Chang, NSRRC)

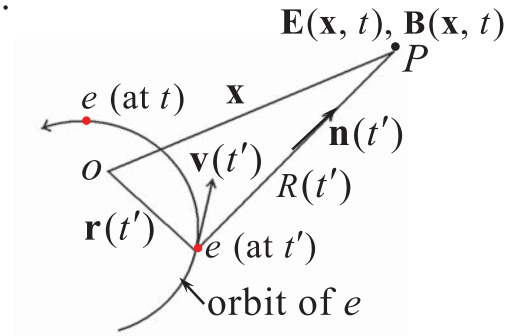


14.2 Total Power Radiated by an Accelerated Charge

Rewrite
$$\mathbf{E}(\mathbf{x},t) = \underbrace{e\left[\cancel{\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}}\right]_{ret}}_{\text{velocity field (neglect)}} + \underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\text{acceleration field}} \quad (14.14)$$

In Sec. 14.1, we have shown that e carries its "velocity fields" with it while moving at const. velocity.

From here on, we will study the "acceleration fields" of e, which will *leave* e as radiation fields, pointing from e at t' to the observer at P. The quantity of interest is now the power of radiation.



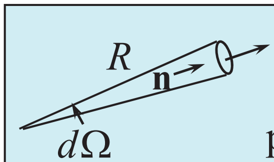
The Poynting vector received at P (point of observation) is

$$\mathbf{S}(\mathbf{x},t) = \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t) \overset{(14.13)}{=} \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)]$$
$$= \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t') \quad [\text{pointing along } \mathbf{n}(t')]$$

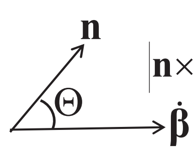
Larmor's Formula: Rewrite $\mathbf{E}(\mathbf{x},t) = \frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}$ $\left[\begin{array}{l}\text{accel.}\\ \text{field}\end{array}\right]$

Let $\beta \to 0$, then retarded $(\gamma,\ R,\ \boldsymbol{\beta},\ \mathbf{n}) \approx$ present $(\gamma,\ R,\ \boldsymbol{\beta},\ \mathbf{n})$

$$\Rightarrow \lim_{\beta\to 0}\mathbf{E}(\mathbf{x},t) \approx \frac{e}{cR}\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}) \Rightarrow \lim_{\beta\to 0}\mathbf{S}\cdot\mathbf{n} = \frac{c}{4\pi}|\mathbf{E}|^2\,\mathbf{n} = \frac{e^2}{4\pi c R^2}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2$$



$$dP = \underbrace{\mathbf{S}}_{\text{power/unit area}}\cdot\overbrace{\mathbf{n}R^2 d\Omega}^{\text{area over } d\Omega} \Rightarrow \frac{dP}{d\Omega} = \mathbf{S}\cdot\mathbf{n}R^2$$



$$|\mathbf{n}\times\dot{\boldsymbol{\beta}}| = \frac{|\dot{\mathbf{v}}|}{c}\sin\Theta$$

$$\Rightarrow \lim_{\beta\to 0}\frac{dP}{d\Omega} = (\lim_{\beta\to 0}\mathbf{S}\cdot\mathbf{n})R^2 = \frac{e^2}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2 = \frac{e^2}{4\pi c}|\mathbf{n}\times\dot{\boldsymbol{\beta}}|^2 \quad (14.20)$$

$$= \frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\sin^2\Theta \quad \left[\frac{\text{power radiated}}{\text{unit solid angle}},\ \text{peak at } \Theta = \frac{\pi}{2}\right] \quad (14.21)$$

$$\Rightarrow \lim_{\beta\to 0} P = \int\frac{dP}{d\Omega}d\Omega = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2 \quad [\text{Larmor's formula}] \quad (14.22)$$

$$= \frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{p}}{dt}\right|^2 \quad \left[\frac{d}{dt}\mathbf{p} = m\overbrace{\mathbf{v}}^{0}\frac{d}{dt}\gamma + m\overbrace{\gamma}^{1}\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}\right] \quad (14.23)$$

All quantities in Secs. 14.1-14.4 are real. Hence, $\left|\frac{d\mathbf{p}}{dt}\right|^2 = \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}$

A side note: *Radiative reaction (See Jackson, Sec. 16.1)*:

If a constant external force f is applied to an electron (initially at rest), the electron acceleration will be $a = f / m_e$ (Newton's law) and the radiated power by the electron will be $P = \frac{2e^2}{3c^3} a^2 = \frac{2e^2}{3m_e^2 c^3} f^2$

At time T, the electron will have a velocity $v = aT = \frac{fT}{m_e}$, and its energy (E) increases at the rate: $\frac{dE}{dt} = fv = \frac{f^2 T}{m_e}$ (accelerating power).

$$\Rightarrow \quad \frac{P}{\frac{dE}{dt}} = \frac{\text{radiated power}}{\text{accelerating power}} = \frac{2e^2}{3m_e c^3 T} = \frac{6.26\times10^{-24}}{T \text{ (in sec)}}$$

$\Rightarrow$ When $T < 6.26\times10^{-24}$ s, we have $P > \frac{dE}{dt}$, i.e. the radiated power exceeds the accelerating power (the electron has zero energy at $t = 0$).

Consequently, for a correct treatment of the electron's motion on the time scale of 6.26×10^{-24} s, a <u>radiative reaction</u> force [see (16.7)] should be added to the equation of motion.

Why is this correction ignored in so many calculations? This is discussed in Jackson Sec. 16-1. First, a completely satisfactory classical treatment of the radiative reaction effects does not exist (it touches the nature of elementary particles). Second, in almost all practical situations, we have a time scale $\gg 6.26\times10^{-24}$ s. Hence, the correction for such an extremely short time scale is unimportant (time scales for the need of a radiative reaction force are even shorter for ions). This explains why results calculated without the radiative reaction correction agree so well with the experiments.

Relativistic Generalization of Larmor's Formula :

Rewrite $\lim\limits_{\beta\to 0} P = \frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{p}}{dt}\right|^2$ $\begin{bmatrix}\text{Larmor's} \\ \text{formula}\end{bmatrix}$ (14.23)

The form of (14.23) *suggests* that we may generalize (14.23) to an exact eq. (for any β) as follows:

$$\begin{cases}\mathbf{p} \to \mathbf{P} = (\mathbf{p}, \frac{iE}{c}) \text{ (4-vector)} \\ dt \to d\tau \text{ (4-scalar)} \boxed{\text{See (10), Ch. 11}}\end{cases} \Rightarrow \underbrace{\frac{d\mathbf{p}}{dt}}_{\text{in (14.23)}} \to \frac{d\mathbf{P}}{d\tau} \Rightarrow P = \overbrace{\frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{P}}{d\tau}\right|^2}^{\text{Lorentz invariant}} \quad (14.24)$$

In terms of $\mathbf{p}$ & E, we have $P = \frac{2e^2}{3m^2c^3}\left[\left|\frac{d\mathbf{p}}{d\tau}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right]$ (14.25)

Converting to lab time by $d\tau = \frac{dt}{\gamma}$, we obtain

$$P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right] \quad \begin{bmatrix}\text{total instantaneous} \\ \text{radiation power}\end{bmatrix} \quad (10a)$$

As will be shown in (14.43), (14.47), (10a) agrees with the exact total P derived directly from (14.14). *Exercises* 1 & 2 below also show P is a Lorentz invariant. Hence, (14.24) [or (10a)] is valid.

Rewrite $P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ [(10a)]

(10a) can be written in 2 other forms [(10b) and (14.26) below]:

$$\frac{dE}{dt} = mc^2\frac{d}{dt}\gamma = mc^2\frac{d}{dt}\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}} = mc^2\frac{2\frac{p}{m^2c^2}\frac{d}{dt}p}{2(1+p^2/m^2c^2)^{\frac{1}{2}}}$$

$$\boxed{\begin{aligned}&\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \Rightarrow \gamma^2 = \left(1-\frac{v^2}{c^2}\right)^{-1} = \left(1-\frac{p^2}{\gamma^2m^2c^2}\right)^{-1} \\ &\Rightarrow \gamma^2 = 1+\frac{p^2}{m^2c^2} \Rightarrow \gamma = \left(1+p^2/m^2c^2\right)^{\frac{1}{2}}\end{aligned}}$$

$$= \frac{p}{\gamma m}\frac{dp}{dt} = v\frac{dp}{dt}$$

Sub. $v\frac{dp}{dt}$ for $\frac{dE}{dt}$ into (10a)

$$\Rightarrow P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right] \quad (10b)$$

Note: $\frac{d\mathbf{p}}{dt}$ is due to both direction and amplitude variations of $\mathbf{p}$, but $\frac{dp}{dt}$ is only due to amplitude variation of $\mathbf{p}$.

Rewrite $P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ [(10a)]

$$\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2$$

$$= m^2c^2\left|\boldsymbol{\beta}\frac{d\gamma}{dt} + \gamma\dot{\boldsymbol{\beta}}\right|^2 - m^2c^2\left(\frac{d\gamma}{dt}\right)^2$$

$$= m^2c^2\left[\beta^2\left(\frac{d\gamma}{dt}\right)^2 + 2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\gamma\frac{d\gamma}{dt} + \gamma^2\left|\dot{\boldsymbol{\beta}}\right|^2 - \left(\frac{d\gamma}{dt}\right)^2\right]$$

$$= m^2c^2\left[-\frac{1}{\gamma^2}\gamma^6\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 + 2\gamma^4\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 + \gamma^2\left|\dot{\boldsymbol{\beta}}\right|^2\right]$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-\frac{1}{2}} = -\frac{1}{2}\frac{-\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}-\dot{\boldsymbol{\beta}}\cdot\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{\frac{3}{2}}} = \gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$$

$$= \gamma^4m^2c^2\left[\left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 + (1-\beta^2)\left|\dot{\boldsymbol{\beta}}\right|^2\right] = \gamma^4m^2c^2\left[\left|\dot{\boldsymbol{\beta}}\right|^2 + \left(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\right)^2 - \beta^2\dot{\beta}^2\right]$$

$$= \gamma^4m^2c^2\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right]$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \Rightarrow \left|\mathbf{A}\times\mathbf{B}\right|^2 = A^2B^2 - \left|\mathbf{A}\cdot\mathbf{B}\right|^2$$

Sub. this into (10a)

$$\Rightarrow P = \frac{2e^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right] \qquad (14.26)$$

Linear Accelerators: [$\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$, i.e. $\mathbf{F}$ (accelerating force) $\parallel \mathbf{p}$]

Total radiated power:

Rewrite
$$\begin{cases} P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right] & (10b) \\ P = \frac{2e^2}{3c}\gamma^6\left[\left|\dot{\boldsymbol{\beta}}\right|^2 - \left|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\right|^2\right] & (14.26) \end{cases}$$

$$\mathbf{F}\parallel\mathbf{p} \Rightarrow \begin{cases} \left|\frac{d\mathbf{p}}{dt}\right|^2 = \left(\frac{dp}{dt}\right)^2 \Rightarrow \left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2 = (1-\beta^2)\left(\frac{dp}{dt}\right)^2 = \frac{1}{\gamma^2}\left(\frac{dp}{dt}\right)^2 \\ \boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = 0 \end{cases}$$

Thus,
$$\left.\begin{cases} (10b) \\ (14.26) \end{cases}\right\} \Rightarrow \begin{cases} P = \frac{2e^2}{3m^2c^3}\left(\frac{dp}{dt}\right)^2 & (14.27) \\ P = \frac{2e^2}{3c^3}\gamma^6\left(\frac{dv}{dt}\right)^2 & (11) \end{cases}$$

(14.27) and (11) are identical because, for $\mathbf{F}\parallel\mathbf{p}$, we have

$\frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \gamma^3 m\frac{dv}{dt}$ [see Ch. 11, Appendix A, Eq. (A.38)]

$$P=\frac{2e^2}{3m^2c^3}\left(\frac{dp}{dt}\right)^2\ [(14.27)] \text{ and } \frac{dp}{dt}=F=\frac{dE}{dx}\ [E=\gamma mc^2]$$

$$\Rightarrow P=\frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2 \qquad (14.28)$$

$$\Rightarrow \frac{P}{\frac{dE}{dt}}=\frac{\text{radiated power}}{\text{accelarating power}}=\frac{\frac{2e^2}{3m^2c^3}\frac{dE}{dx}\frac{dE}{vdt}}{\frac{dE}{dt}}\overset{\boxed{v\approx c}}{\approx}\underbrace{\frac{2e^2}{3m^2c^4}\frac{dE}{dx}}_{\text{in Gaussian units}} \qquad (14.29)$$

A convenient formula for linear electron accelerators ($\beta\approx 1$):

In (14.29), $e=4.803\times 10^{-10}$ statcoulomb, $m=m_e=9.109\times 10^{-28}$ g, $c=3\times 10^{10}$ cm/s. dE/dx is commonly expressed in MeV/m. Thus, use

$$\begin{cases}1\text{ erg}=6.242\times 10^5\text{ MeV}\\ 1\text{ cm}=10^{-2}\text{ m}\end{cases}\Rightarrow \frac{dE}{dx}\ (\text{in }\frac{\text{erg}}{\text{cm}})=\frac{10^{-2}}{6.242\times 10^5}\frac{dE}{dx}\ (\text{in }\frac{\text{MeV}}{\text{m}})$$

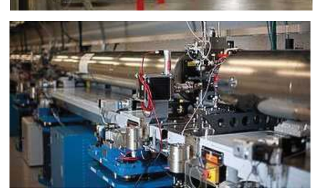
$$\Rightarrow \frac{P}{\frac{dE}{dt}}=3.68\times 10^{-15}\frac{dE}{dx}\ (\text{in }\frac{\text{MeV}}{\text{m}}) \quad [\text{for electrons with } \beta\approx 1]$$

Typically, $\frac{dE}{dx}<50\ \frac{\text{MeV}}{\text{m}}\Rightarrow\begin{cases}\text{Radiation losses are completely}\\ \text{negligible in linear accelerators.}\end{cases}$

The Stanford Linear Accelerator Center (SLAC):

Below is an aerial view of SLAC with the detector complex at the right side. Research at SLAC has produced 4 Nobel prizes.

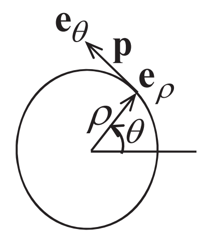
The main facility is a 3.2 km linear accelerator, in which 240 S-band klystrons (2.86 GHz, 65 MW each) are used to accelerate electrons/positrons up to 50 GeV in a series of cavities (accelerating force ≈ 20 MeV/m). Operational since 1966, it is still the world's longest linear accelerator.



Circular Accelerators: (e.g. synchrotron)

Rewrite $$P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right] \qquad \text{(10b)}$$

As just shown, radiation due to $\frac{dp}{dt}$ (i.e. for $\mathbf{F} \parallel \mathbf{p}$) is negligible. Thus, we neglect it in (10b) and consider only the force $\perp$ to $\mathbf{p}$ (e.g. a magnetic force). Then, $P \approx \frac{2e^2}{3m^2c^3}\gamma^2\left|\frac{d\mathbf{p}}{dt}\right|^2$



$$\frac{d\mathbf{p}}{dt} = \frac{d(p\mathbf{e}_\theta)}{dt} = p\frac{d}{dt}\mathbf{e}_\theta + \mathbf{e}_\theta \underbrace{\frac{dp}{dt}}_{\text{negligible}} \approx p\underbrace{\frac{d\mathbf{e}_\theta}{d\theta}}_{-\mathbf{e}_\rho}\underbrace{\frac{d\theta}{dt}}_{\omega} = -\omega p\mathbf{e}_\rho$$

$$\Rightarrow \left|\frac{d\mathbf{p}}{dt}\right| \approx \omega p \qquad \boxed{\omega = \frac{v}{\rho},\ p = \gamma m v}$$

$$\Rightarrow \quad P \approx \frac{2e^2}{3m^2c^3}\gamma^2\omega^2p^2 = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4 \quad \begin{bmatrix}\text{in Gaussian} \\ \text{units}\end{bmatrix} \qquad (14.31)$$

Note that (14.31) is an exact expression for P at any instant at which e experiences no force $\parallel$ to $\mathbf{p}$ (i.e. $\frac{dp}{dt} = 0$).

Convenient formulae for circular electron accelerators ($\beta \approx 1$):

$$\delta E = \frac{\text{radiated energy}}{\text{per particle per revolution}} = \frac{2\pi\rho}{v}P \underset{\substack{\uparrow \\ P = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4\ [(14.31)]}}{=} \frac{4\pi}{3v}\frac{e^2c}{\rho}\beta^4\gamma^4 \underset{\substack{\uparrow \\ \beta \approx 1}}{\approx} \frac{4\pi}{3}\frac{e^2}{\rho}\gamma^4$$

For electrons: $\gamma = \frac{\gamma m_e c^2}{m_e c^2} = \frac{E}{m_e c^2} = \frac{E\,(\text{in Mev})}{0.511}$ $[m_e c^2 = 0.511 \text{ MeV}]$

$$\Rightarrow \delta E\,(\text{in erg}) = \frac{4\pi}{3}\frac{e^2}{\rho}\gamma^4 = \frac{4\pi}{3}\frac{e^2\ (e \text{ in statcoulomb})}{\rho\ (\text{in cm})}\left[\frac{E\,(\text{in Mev})}{0.511}\right]^4$$

$(1 \text{ erg} = 6.242\times10^5 \text{ MeV},\ e = 4.803\times10^{-10} \text{ statcoulomb})$

$$\Rightarrow \frac{\delta E\,(\text{in MeV})}{6.242\times10^5} = \frac{4\pi}{3}\frac{(4.803\times10^{-10})^2}{100\rho\ (\text{in meter})}\left[\frac{E\,(\text{in Gev})\times10^3}{0.511}\right]^4$$

$$\Rightarrow \delta E\,(\text{in MeV}) \approx 8.85\times10^{-2}\frac{[E\ (\text{in GeV})]^4}{\rho\ (\text{in meter})} \quad \begin{bmatrix}\text{for electrons} \\ \text{with } \beta \approx 1\end{bmatrix} \qquad (14.33)$$

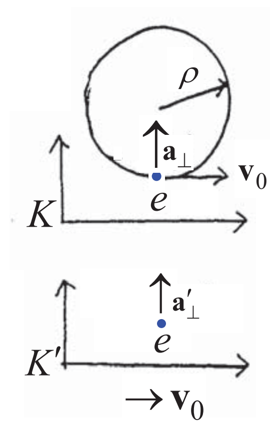
$$\approx \begin{cases} 1 \text{ keV}, & \text{for early synchrontrons (used as accelerators)} \\ 128 \text{ keV}, & \text{for the 1.5 GeV NSRRC synchrontron} \\ 8.85 \text{ MeV}, & \text{for the 10 GeV Cornell synchrontron} \end{cases} \begin{pmatrix}\text{used as} \\ \text{storage rings}\end{pmatrix}$$

Exercise 1 on Lorentz transformation: For $\mathbf{F} \perp \mathbf{p}$, we have $P = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4$ [(14.31)]. Show that, viewed in e's rest frame, the same total power is radiated (i.e. P is a Lorentz invariant).

Solution: Consider the instant when e is located at the bottom of its orbit. At this instant, e moves horizontally to the right at $\mathbf{v}_0$ (upper figure) and the acceleration $\mathbf{a}_\perp$ points vertically upward with $a_\perp = v_0^2/\rho$ (ρ: radius of the circle). Viewed in the instantaneous rest frame of e (lower figure), we have

$$\begin{cases} \mathbf{a}'_{\|} = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\underbrace{\mathbf{a}_{\|}}_{0} \\ \boxed{\mathbf{v}=\mathbf{v}_0} \\ \mathbf{a}'_\perp = \dfrac{1}{\gamma_0^2\left(1-\frac{v_0^2}{c^2}\right)^3}\underbrace{\left[\mathbf{a}_\perp - \frac{\mathbf{v}_0}{c^2}\times(\overbrace{\mathbf{a}}^{\mathbf{a}_\perp}\times\overbrace{\mathbf{v}}^{\mathbf{v}_0})\right]}_{\mathbf{a}_\perp(1-v_0^2/c^2)} \end{cases}$$

$\boxed{\text{Lecture Notes, Ch. 11, Eq. (A.22)}}$



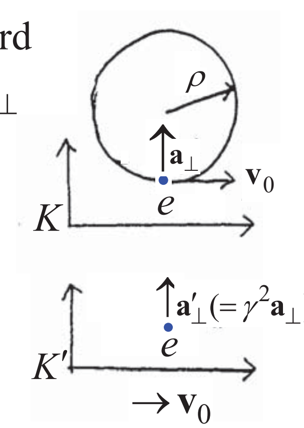
Thus, $\mathbf{a}'_{\|} = 0$ and $\mathbf{a}'_\perp = \gamma_0^2\mathbf{a}_\perp$.

Thus, the acceleration of e is vertically upward in both frames and they are related by $\mathbf{a}'_\perp = \gamma^2\mathbf{a}_\perp$

Since e is at rest in frame K', Larmor's formula in (14.23) becomes exact, which gives

$$P' = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}'|^2 = \frac{2e^2}{3c^3}|\mathbf{a}'_\perp|^2 = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2$$

$$= \frac{2e^2}{3c^3}\gamma^4\frac{v_0^4}{\rho^2} = \frac{2e^2c}{3\rho^2}\beta^4\gamma^4 \qquad \boxed{a_\perp = \frac{v_0^2}{\rho}}$$



This is identical to (14.31), which is the total power as viewed in the lab frame, thus providing an example that the *total* radiated power $P = \frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{P}}{d\tau}\right|^2$ [(14.24)] is a Lorentz invariant ($P = P'$).

Exercise 2 on Lorentz transformation: Show, by a Lorentz transformation, that the total radiated power by charge e in a linear accelerator ($\mathbf{F} \parallel \mathbf{p}$), $P = \frac{2e^2}{3c^3}\gamma^6(\frac{dv}{dt})^2$ [(11)], is a Lorentz invariant.

(11) is valid in all inertial frames. In the rest frame of e [which moves with e at its instantaneous velocity v in the frame for (11)], we have $\quad \frac{dv'}{dt'} = \gamma^3 \frac{dv}{dt}$ [(A.23) of lecture notes Ch. 11]

where $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$.

$$\Rightarrow P' = \frac{2e^2}{3c^3}(\frac{dv'}{dt'})^2$$

This is identical to Larmor's formula [(14.22)], which is exact in the rest frame of e. Thus, $P = \frac{2e^2}{3c^3}\gamma^6(\frac{dv}{dt})^2$ [(11)], is a Lorentz invariant.

14.3 Angular Distribution of Radiation Emitted by an Accelerated Charge

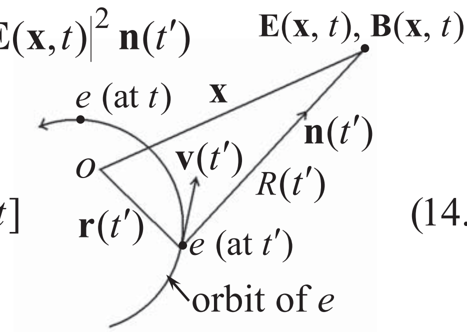
Rewrite (14.14): $\mathbf{E}(\mathbf{x},t) = \underbrace{e\left[\cancel{\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}}\right]_{ret}}_{\text{neglect}} + \underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\text{acceleration field}}$

$$\overbrace{\mathbf{S}(\mathbf{x},t)}^{\boxed{\text{power/unit area}}} = \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)$$

$$= \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)] = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t')$$

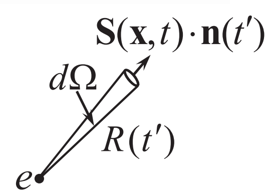
$$\Rightarrow \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2$$

$$= \frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret} \text{ [at } \mathbf{x},\, t] \qquad (14.35)$$



$$dP(t) = \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')R^2(t')d\Omega$$

$$\Rightarrow \frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2_{ret} \text{ [at } \mathbf{x},\, t] \qquad (12)$$



Note: $dP(t)/d\Omega$ is independent of $R(t')$.

Sec. 14.2 considers the *total* P radiated by e at $[\mathbf{r}(t'), t']$.

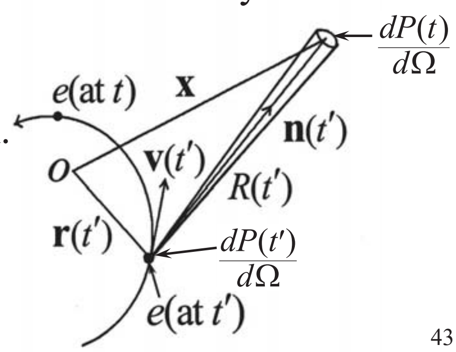
This section will find the *angular distribution of* P radiated by e of any β at $[\mathbf{r}(t'), t']$: $\dfrac{dP(t')}{d\Omega} = \dfrac{\text{power radiated by } e \text{ at } \mathbf{r}(t') \text{ and } t'}{\text{unit solid angle}}$

So far we only have the power *received* at $(\mathbf{x}, t)$:

$$\frac{dP(t)}{d\Omega} = \frac{\text{power received at } \mathbf{x} \text{ and } t}{\text{unit solid angle}} = \frac{e^2}{4\pi c}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2_{ret} \quad [(12)]$$

In general, $\dfrac{dP(t')}{d\Omega} \neq \dfrac{dP(t)}{d\Omega}$ for the following reason:

If e moves toward $\mathbf{x}$, a pulse of duration dt' emitted by e into $d\Omega$ will be seen by the observer at the other end of $d\Omega$ as a pulse of shorter duration ($dt < dt'$) due to pulse length compression.



Thus, to express $dP(t')/d\Omega$ in terms of $dP(t)/d\Omega$, we need to first determine the ratio of dt (pulse duration received) to dt' (pulse duration radiated).

$$t = t' + \frac{R(t')}{c} \begin{bmatrix} t: \text{observation time} \\ t': \text{emission time} \end{bmatrix} \text{ and } \frac{dR(t')}{dt'} = -\mathbf{v}(t')\cdot\mathbf{n}(t') \ [(4)]$$

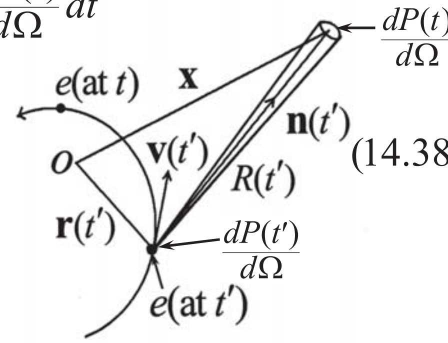
$$\Rightarrow \quad \frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t') \tag{13}$$

$\Rightarrow$ A pulse emitted over a duration dt' at $[\mathbf{r}(t'), t']$ is received at $(\mathbf{x}, t)$ as a pulse of duration $dt = dt'[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]$.

dt', dt are infinitesimal (i.e. $\to 0$). So the energy radiated into $d\Omega$ at $[\mathbf{r}(t'), t']$ in dt' reaches the other end of the same $d\Omega$ at $(\mathbf{x}, t)$ in dt.

By conservation of energy, $\dfrac{dP(t')}{d\Omega}dt' = \dfrac{dP(t)}{d\Omega}dt$

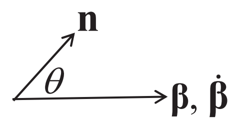
$$\Rightarrow \frac{dP(t')}{d\Omega} = \underbrace{\frac{dP(t)}{d\Omega}}_{\boxed{\frac{e^2}{4\pi c}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2_{ret} [(14.35b)]}} \overbrace{\frac{dt}{dt'}}^{\boxed{1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')}} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5} \tag{14.38}$$



Note: In (14.38), $dP(t')/d\Omega$, $\mathbf{n}$, $\boldsymbol{\beta}$, and $\dot{\boldsymbol{\beta}}$ are values at $[\mathbf{r}(t'), t']$.

Case 1: $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$

Rewrite (14.38): $\dfrac{dP(t')}{d\Omega} = \dfrac{e^2}{4\pi c}\dfrac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$



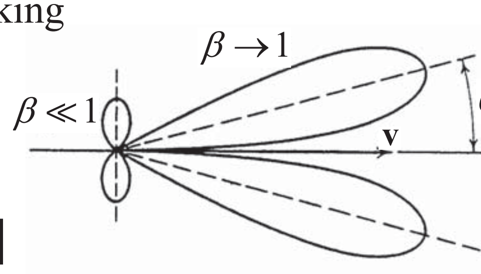
$$\left.\begin{cases}\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}} = 0 \\ \left|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right|^2 = \left|\dot{\boldsymbol{\beta}}\right|^2 \sin^2\theta\end{cases}\right\} \Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad (14.39)$$

Total radiated power: $P(t') = \int \dfrac{dP(t')}{d\Omega}d\Omega = \dfrac{2}{3}\dfrac{e^2}{c^3}\dot{v}^2\gamma^6, \qquad (14.43)$

which is identical to (14.27) and (11) [another verification of [(14.24)].

Angular distribution for low and high β:

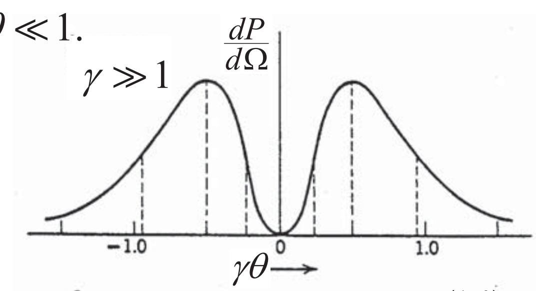
For $\beta \ll 1$, (14.39) reduces to Larmor's result: $\dfrac{dP(t')}{d\Omega} = \dfrac{e^2\dot{v}^2}{4\pi c^3}\sin^2\theta$ [(14.21)], with the radiation peaking at $\theta = 90^{\circ}$. But as $\beta \to 1$, the radiation is tipped forward, with the maximum intensity at



$$\theta_{\max} = \cos^{-1}\left[\tfrac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] \qquad (14.40)$$

Angular distribution for $\beta \to 1$:

As $\beta \to 1$, $\dfrac{dP(t')}{d\Omega}$ is confined to $\theta \ll 1$.



$$\Rightarrow 1-\beta\cos\theta \approx 1-\beta\left(1-\tfrac{1}{2}\theta^2\right)$$

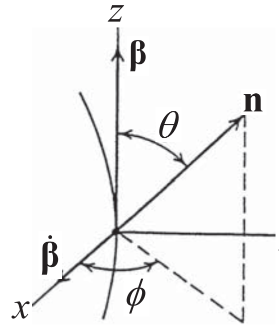
$$= 1-\beta+\frac{\overbrace{\beta}^{\approx 1}}{2}\theta^2 \approx \frac{(1-\beta)(1+\overbrace{\beta}^{\approx 1})}{2}+\frac{\theta^2}{2}$$

$$= \frac{1-\beta^2}{2}+\frac{\theta^2}{2} = \frac{1}{2\gamma^2}\left(1+\gamma^2\theta^2\right) \qquad (14)$$

$$\Rightarrow \lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\overbrace{\sin^2\theta}^{\approx\theta^2}}{(1-\beta\cos\theta)^5} \approx \frac{8}{\pi}\frac{e^2\dot{v}^2}{c^3}\gamma^8\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \qquad (14.41)$$

$$\Rightarrow \begin{cases}\theta_{\max} = \dfrac{1}{2\gamma} \text{ [angle of maximum intensity]} & (14.40) \\ <\theta^2>^{\frac{1}{2}} = \left[\dfrac{\int\theta^2\frac{dP(t')}{d\Omega}d\Omega}{\int\frac{dP(t')}{d\Omega}d\Omega}\right]^{\frac{1}{2}} = \dfrac{1}{\gamma} = \dfrac{mc^2}{E} \begin{bmatrix}\text{root mean} \\ \text{square angle}\end{bmatrix} & (14.42)\end{cases}$$

Case 2: $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$



Rewrite $\dfrac{dP(t')}{d\Omega} = \dfrac{e^2}{4\pi c} \dfrac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$ (14.38)

Let $\begin{cases} \boldsymbol{\beta} \parallel \mathbf{e}_z, \ \dot{\boldsymbol{\beta}} \parallel \mathbf{e}_x \\ \mathbf{n} = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z \end{cases}$

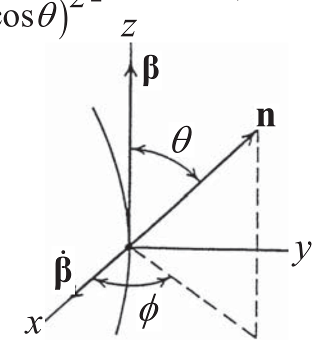
$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad (14.44)$$

Total radiated power: [integrating (14.44) over $d\Omega$]

$$P(t') = \int \frac{dP(t')}{d\Omega} d\Omega = \frac{2}{3}\frac{e^2}{c^3}|\dot{\mathbf{v}}|^2\gamma^4 \ \text{[total radiated power]}$$

$$= \begin{cases} \frac{2}{3}\frac{e^2 c}{\rho^2}\beta^4\gamma^4 & \begin{bmatrix}\text{identical to (14.31),} \\ \text{one more verification of (14.24)}\end{bmatrix} \\ \boxed{\uparrow \text{ use } \dot{v} = v^2/\rho} & \\ \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left|\frac{d\mathbf{p}}{dt}\right|^2 & \left[\text{use } \frac{d\mathbf{p}}{dt} = \gamma m\dot{\mathbf{v}}\right] \end{cases} \quad (14.47)$$

$$\text{Rewrite} \quad \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad (14.44)$$



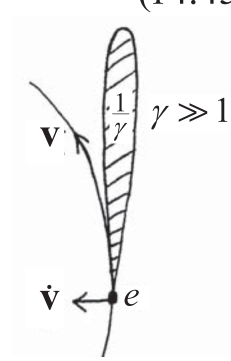
Angular distribution for $\beta \to 1$:

(14.44) suggests that $\dfrac{dP(t')}{d\Omega}$ peaks at $\theta \approx 0$ if $\beta \to 1$. Thus, for $\beta \to 1$, we use

$$1 - \beta\cos\theta \approx \frac{1}{2\gamma^2}(1+\gamma^2\theta^2) \quad [(14)]$$

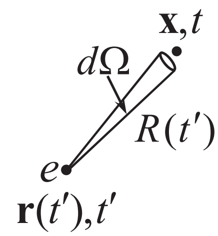
$$\Rightarrow \lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{2e^2}{\pi c^3}\gamma^6 \frac{|\dot{\mathbf{v}}|^2}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \quad (14.45)$$

$$\Rightarrow \begin{cases} \theta_{\max} = 0 \ \text{[angle of maximum intensity]} \\ <\theta^2>^{\frac{1}{2}} = \frac{1}{\gamma} \end{cases},$$



which indicates that the synchrotron radiation is in a narrow cone (like a searchlight) directed along **v**, the instantaneous velocity of *e*.

14.4 Radiation Emitted by a Charge in Arbitrary, Extremely Relativistic Motion



Secs. 14.2 and 14.3 have examined the radiated power from the viewpoint of the charge at $[\mathbf{r}(t'), t']$ and expresssed it in terms of the charge's instantaneous $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$.

From here on, we switch our viewpoint to the observer at $(\mathbf{x}, t)$. The emphasis will also be switched from the radiated power to the *frequency spectrum* of the signal at $(\mathbf{x}, t)$.

We cannot determine the spectrum from instantaneous quantities as those obtained in Secs. 14.2 and 14.3. We need to know the time history of the received signal.

In this section, we will estimate of the frequency width $(\Delta\omega)$ of the signal from its duration (Δt, a finite number), using the relation :

$$\Delta\omega \text{ (frequency width)} \cdot \Delta t \text{ (pulse duartion)} \sim 1$$

Example : The delta function in t-space (a zero width pulse) has an infinite spread in ω-space : $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega = \delta(t)$ [from (2.47)]

14.4 Radiation Emitted by a Charge with γ>>1 (*continued*)

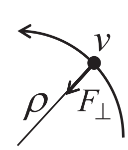
Rewrite
$$\begin{cases} P(t') = \dfrac{2e^2}{3m^2c^3}\left(\dfrac{dp}{dt}\right)^2, & \text{for } \dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta} \quad (14.27) \\ P(t') = \dfrac{2}{3}\dfrac{e^2}{m^2c^3}\gamma^2\left|\dfrac{d\mathbf{p}}{dt}\right|^2, & \text{for } \dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta} \quad (14.47) \end{cases}$$

which implies $P(\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}) = \gamma^2 P(\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta})$ for the same magnitude of the accelerating force [$\mathbf{F} \parallel \boldsymbol{\beta}$ in (14.27) and $\mathbf{F} \perp \boldsymbol{\beta}$ in (14.47)].

Hence, for a charge with $\gamma \gg 1$ in arbitrary motion, we may neglect $P(t')$ due to $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$ and consider only $P(t')$ due to $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$.

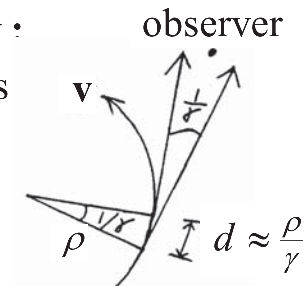
The instantaneous radius of curvature ρ can be expressed in terms of the perpendicular component of the acceleration $(\dot{v}_\perp)$ as follows.

$$F_\perp = \frac{\gamma m v^2}{\rho} = \gamma m \dot{v}_\perp \quad \left[\begin{array}{l}\text{For acceleration} \perp \text{to } \mathbf{v}\text{, the} \\ \text{effective mass is } \gamma m\text{. See} \\ \text{lecture notes, Ch. 11, Eq. (A.39).}\end{array}\right]$$



$$\Rightarrow \rho = \frac{v^2}{\dot{v}_\perp} \underset{\gamma \gg 1}{\approx} \frac{c^2}{\dot{v}_\perp} \quad (14.48)$$

Synchrotron Radiation-A Qualitative View :



For $\gamma \gg 1$, the radiation pulse at the observer's end is extremely short due to 2 effects:

Effect 1. Narrow angular width of radiation:

$$<\theta^2>^{\frac{1}{2}} \approx \frac{1}{\gamma}$$

$\Rightarrow$ The observer is illuminated by the light emitted in an arc of length $d \approx \frac{\rho}{\gamma}$, corresponding to an emission interval of $\Delta t' \approx \frac{\rho}{\gamma v} \approx \frac{\rho}{\gamma c}$.

Effect 2. Charge e is "chasing" its radiation. Hence, to the observer, the received pulse length is not $\Delta t'$, but a much shortened Δt given by

$$\Delta t = \Delta t' \frac{dt}{dt'} \overset{(13)}{=} \Delta t'[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')] = \Delta t'(1-\beta) \approx \Delta t' \frac{(1-\beta)(1+\beta)}{2} = \underbrace{\Delta t'}_{\frac{\rho}{\gamma c}} \frac{1}{2\gamma^2}$$

$$\Rightarrow \quad \Delta t \approx \frac{\rho}{2c\gamma^3} \quad \text{[pulse duration to the observer]}$$

$\Delta\omega\Delta t \sim 1 \Rightarrow$ The observer sees a broad spectrum of ω ranging from near 0 up to a critical frequency (ω_c) of

$$\omega_c \sim \frac{1}{\Delta t} \approx \frac{c}{\rho}\gamma^3 \qquad (14.50)$$

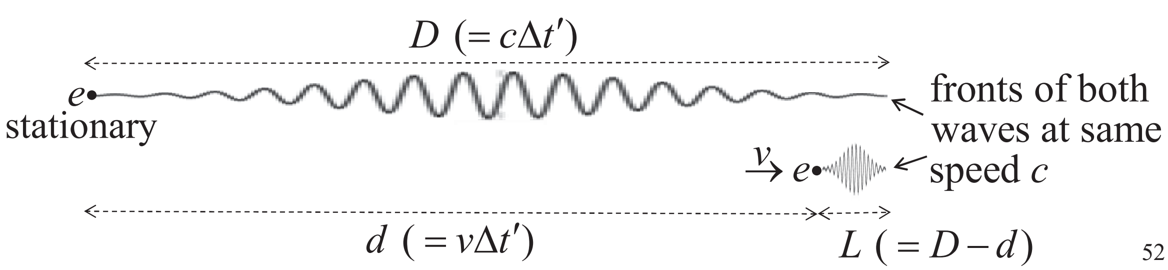
Physical picture of a relativistic e chasing its own radiation :

In effect 2 of last page, we find the pulse is shortened by a factor of $2\gamma^2$ because e is chasing its radiation. To see the physical picture, assume that an e with v (average velocity) $= 0$ has emitted a pulse of length $D = c\Delta t'$ over duration $\Delta t'$ (upper fig.). Consider another e with $v \approx c$. While radiating over the same duration $\Delta t'$, it moves a distance $d = v\Delta t'$(lower fig.). *Note* : The fronts of both waves move at speed c.

By chasing its radiation, the 2nd e compresses the radiated pulse to the length: $L = D - d = c\Delta t' - v\Delta t' = c(1-\frac{v}{c})\Delta t' \approx \frac{c\Delta t'}{2\gamma^2} = \frac{D}{2\gamma^2}$

$\Rightarrow$ The compressed L is indeed shorter than D by a factor of $2\gamma^2$.

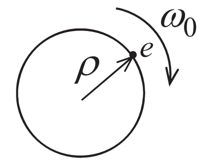
Note : The lower figure shows the effect of wavelength shortening.



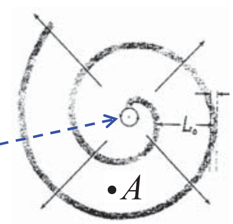
Rewrite $\omega_c \approx \frac{c}{\rho}\gamma^3$ [(14.50)]. If e is in circular motion with radius ρ and rotation frequency ω_0, then $\omega_0 \rho \approx c$ and (14.50) give $\omega_c \approx \omega_0 \gamma^3$.

Eexample: Cornell 10 GeV synchrotron

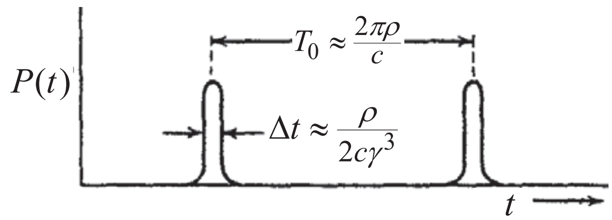
$$\begin{cases} \gamma \approx 2\times 10^4 \\ \omega_0 \approx 3\times 10^6 \text{ / sec} \end{cases} \Rightarrow \begin{cases} \omega_c \approx 2.4\times 10^{19} \text{ / sec} \\ \text{(16 keV x-rays)} \end{cases}$$



Since synchrotron radiation is highly directed (like the lighthouse emission), continuous radiation of a circulating e reaches an observer as sharp pulses at regular intervals of $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi\rho}{v} \approx \frac{2\pi\rho}{c}$



An e emits tangentially to its orbit. Far away, the radiation direction is essentially radially outward.

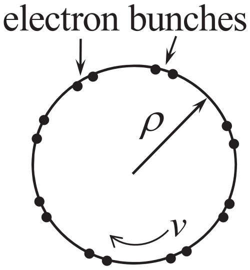


Observed at point A, the radiation comes as pulses, most appreciable on or near the orbital plane.

As a specific example of the pulse duration to the observer, consider again the Cornell 10 GeV synchrotron, for which we have $\omega_c \approx 2.4\times 10^{19}$ / sec

Since $\omega_c \sim \frac{1}{\Delta t}$, the pulse duration Δt from a single electron is incredibly short:

$$\Delta t \sim \frac{1}{\omega_c} \approx 4.2\times 10^{-20} \text{ sec} \quad \begin{bmatrix}\text{pulse duration} \\ \text{of a single } e\end{bmatrix}$$

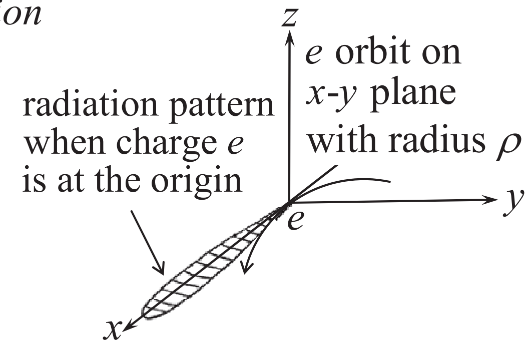


This explains the broad spectrum. However, the actual pluse in a synchrotron does not come from a single electron, but from an electron bunch of finite length (typically a few mm). Electrons in the bunch are *uncorrelated*, hence radiate *independently*. So the spectrum of the bunch is the same as that of a single electron, but the pulse duration (τ) equals the passage time ($\tau \approx \frac{\text{bunch length}}{c}$) of the electron bunch, e.g. for a bunch length of 6 mm, we have

$$\tau \approx 2\times 10^{-11} \text{ sec} \quad \text{[pulse duration of an } e \text{ bunch]}$$

Main properties of synchroyton radiation
(derived in Sec. 14.6 below)

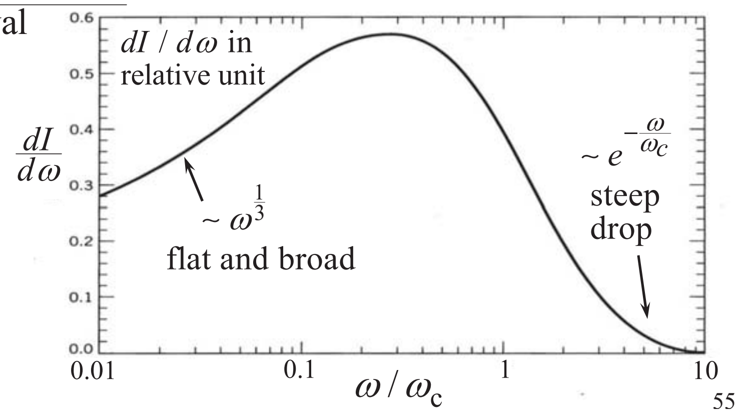
1. Let e be at the origin (see fig.) moving along $\mathbf{e}_x$ with velocity $\mathbf{v}$. Then, the radiation is confined to a narrow cone in the direction of $\mathbf{v}$ (or $\mathbf{e}_x$), with an angular spread of $|\theta| \leq \frac{1}{\gamma}$.



2. The radiation spectrum is shown in the lower figure, where

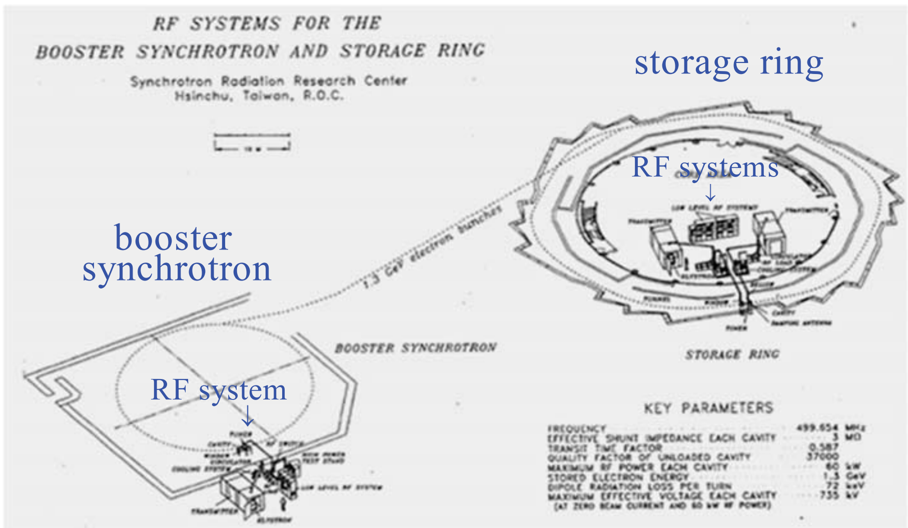
$$\begin{cases} \dfrac{dI}{d\omega} = \dfrac{\text{energy radiated in 1 revolution}}{\text{unit freq. interval}} \\ \omega_c \equiv 3\gamma^3 c/(2\rho) \end{cases}$$

3. The synchrotron radiation is ~7 times more polarized in the orbital plane, e.g. in the upper fig., $\mathbf{E}$ is largely oriented in the x-y plane.

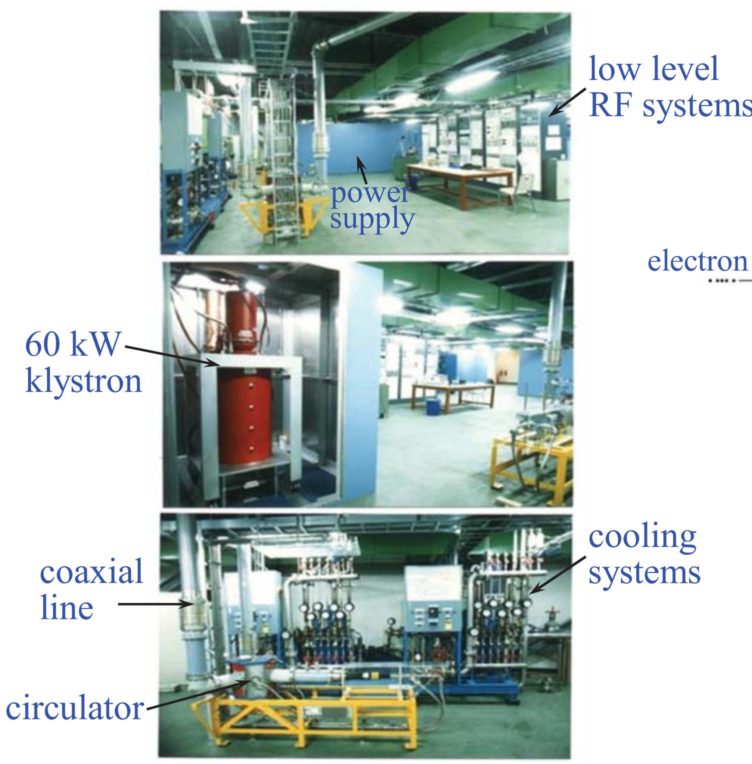


The Synchrotron as a Light Source : The synchrotron emits *intense radiation* with a *very broad frequency spectrum* in a beam of *extremely small angular spread* ($1/\gamma$). It is a unique research tool and can also be used for micro-fabrication and other applications. The photo below shows the light source facility at the National Synchrotron Radiation Research Center (NSRRC) in Taiwan.

Electron bunches are first accelerated to an energy of 1.5 GeV in the booster synchrotron, and then sent to the storage ring (also a synchrotron), where the energy is maintaind at 1.5 GeV while the electrons provide synchrotron radiation to users around the ring. The electrons are powered by 60-MHz microwaves from the RF systems.



The RF system



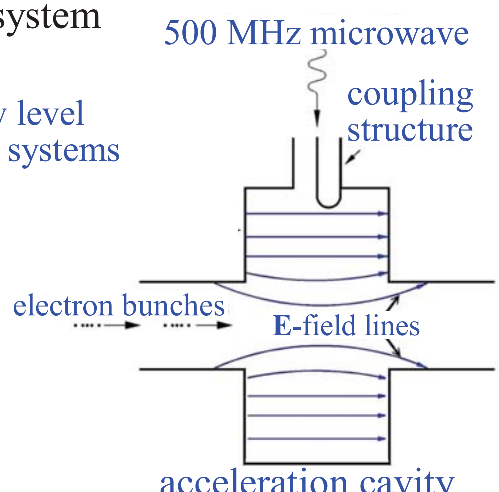
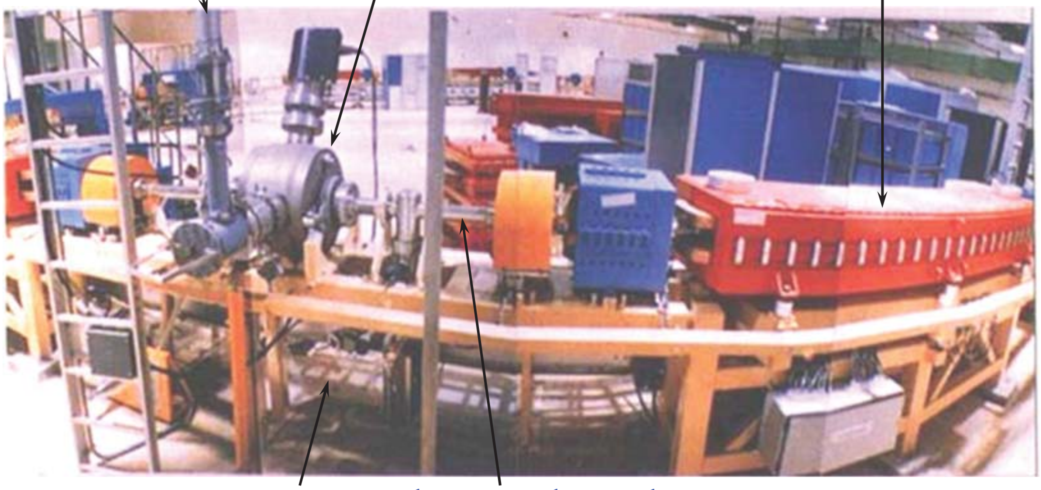


Photo of the NSRRC booster synchrotron showing some key components of the accelerator

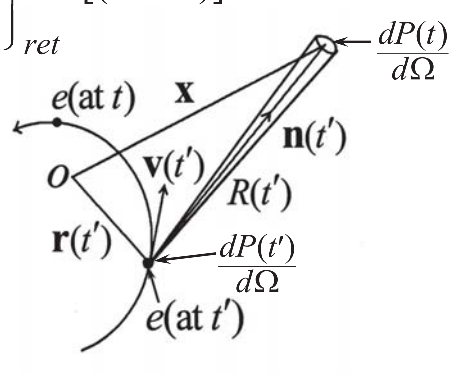


Research stations around the NSRRC storage ring

14.5 Distribution in Frequency and Angle of Energy Radiated by Accelerated Charges: Basic Results

Time-integration of the instantaneous $\frac{dP(t)}{d\Omega}$: Rewrite

$$\begin{cases} \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \dfrac{e^2}{4\pi c}\left\{\dfrac{1}{R^2}\left|\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret} \quad [(14.35)] \\ \dfrac{dP(t)}{d\Omega}\ \left[\dfrac{\text{power received at } \mathbf{x},\, t}{\text{unit solid angle}}\right] \\ \quad = R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') \\ \quad = \dfrac{e^2}{4\pi c}\left|\dfrac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2_{ret} \quad [(12)] \end{cases}$$

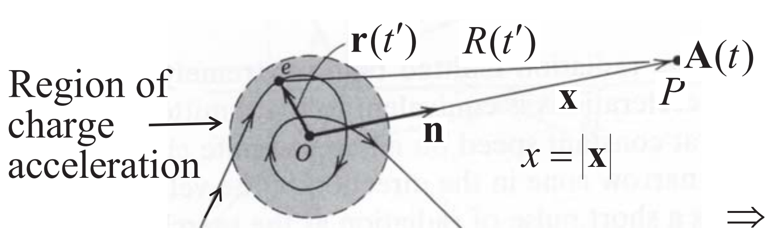


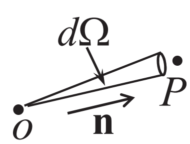
$$\text{Define } \mathbf{A}(t) \equiv \left(\frac{e^2}{4\pi c}\right)^{\frac{1}{2}}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{ret} \quad \left[= \left(\frac{c}{4\pi}\right)^{\frac{1}{2}} R(t')\mathbf{E}(\mathbf{x},t)\right] \quad (14.52)$$

$$\Rightarrow \frac{dP(t)}{d\Omega} = |\mathbf{A}(t)|^2 \quad [\mathbf{A}(t) \text{ is a definition, not a vector potential}] \quad (14.51)$$

Rewrite $\quad \frac{dP(t)}{d\Omega} = |\mathbf{A}(t)|^2 \quad [(14.51)]$

We assume that the observation point (P) is far enough away from the charge acceleration region ($\mathbf{x}$ is much larger than in the figure) so that $d\Omega$ from the charge acceleration region can be regarded as from a single point O to P (in the $\mathbf{n}$-direction).





P far away from O
$\Rightarrow$ all $d\Omega$ from O to P

Let $\frac{dW}{d\Omega}$ be the total energy per unit solid angle received at point P.

$$\Rightarrow \frac{dW}{d\Omega} = \int_{-\infty}^{\infty}\frac{dP(t)}{d\Omega}dt = \int_{-\infty}^{\infty}|\mathbf{A}(t)|^2 dt \quad \left[\frac{\text{total energy received}}{\text{unit solid angle}}\right] \quad (14.53)$$

Note: In general $\frac{dP(t)}{d\Omega} \neq \frac{dP(t')}{d\Omega}$, but $\int_{-\infty}^{\infty}\frac{dP(t)}{d\Omega}dt = \int_{-\infty}^{\infty}\frac{dP(t')}{d\Omega}dt'$, i.e. energy *received* at the end of $d\Omega$ (point P) = energy *radiated* into $d\Omega$.

Frequency spectrum: Rewrite $\frac{dW}{d\Omega}=\int_{-\infty}^{\infty}|\mathbf{A}(t)|^2\,dt$ [(14.53)]

The spectral information can be obtained by a Fourier transform:

$$\begin{cases}\mathbf{A}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{A}(t)e^{i\omega t}dt & (14.54)\\ \mathbf{A}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{A}(\omega)e^{-i\omega t}d\omega & (14.55)\end{cases}$$

Use (14.55) for $\mathbf{A}(t)$

$$\frac{dW}{d\Omega}=\int_{-\infty}^{\infty}|\mathbf{A}(t)|^2\,dt \overset{\downarrow}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega'\mathbf{A}^*(\omega')\cdot\mathbf{A}(\omega)\overbrace{\int_{-\infty}^{\infty}dte^{i(\omega'-\omega)t}}^{=2\pi\delta(\omega'-\omega)\text{ by (2.46)}}$$

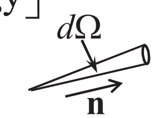
$$=\int_{-\infty}^{\infty}|\mathbf{A}(\omega)|^2\,d\omega \text{ [Parseval's thm., Lecture Notes, Sec. 2-8]} \quad (14.57a)$$

$$=\int_0^{\infty}2|\mathbf{A}(\omega)|^2\,d\omega \quad [\mathbf{A}(t)=real\Rightarrow\mathbf{A}(-\omega)=\mathbf{A}^*(\omega)] \quad (14.57b)$$

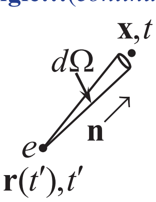
Physically, (14.57) shows $\frac{dW}{d\Omega}=\frac{\text{energy radiated at all }\omega\text{ in }\mathbf{n}\text{-direction}}{\text{unit solid angle}}$.

Thus, we write $\frac{dW}{d\Omega}=\int_0^{\infty}\frac{d^2I(\omega,\mathbf{n})}{d\omega d\Omega}d\omega$, $\left[\begin{array}{l}W,\ I\text{ are different}\\ \text{notations for energy}\end{array}\right]$ (14.58)

where $\frac{d^2I(\omega,\mathbf{n})}{d\omega d\Omega}=\frac{\text{energy radiated at freq. }\omega\text{ in }\mathbf{n}\text{-direction}}{\text{unit solid angle and unit freq. interval}}$



Rewrite $\begin{cases}\frac{dW}{d\Omega}=\int_0^{\infty}2|\mathbf{A}(\omega)|^2\,d\omega & [(14.57b)]\\ \frac{dW}{d\Omega}=\int_0^{\infty}\frac{d^2I(\omega,\mathbf{n})}{d\omega d\Omega}d\omega & [(14.58)]\end{cases}$



At 2 ends of $d\Omega$, energy received = energy radiated

$$\Rightarrow \frac{d^2I(\omega,\mathbf{n})}{d\omega d\Omega}=2|\mathbf{A}(\omega)|^2 \quad \left[\frac{\text{energy radiated at freq. }\omega\text{ in }\mathbf{n}\text{-direction}}{\text{unit solid angle and unit freq. interval}}\right] \quad (14.60)$$

Basic form of $\frac{d^2I(\omega,\mathbf{n})}{d\omega d\Omega}$: (main quantity of interest)

Sub. $\mathbf{A}(t)=\left(\frac{e^2}{4\pi c}\right)^{\frac{1}{2}}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{ret}$ [(14.52)]

into $\mathbf{A}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{A}(t)e^{i\omega t}dt$ [(14.54)]

$\mathbf{A}(\omega)$ is the Fourier transform of the *observed* signal $\mathbf{A}(t)$

$$\Rightarrow \mathbf{A}(\omega)=\left(\frac{e^2}{8\pi^2 c}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}e^{i\omega t}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{ret}dt$$

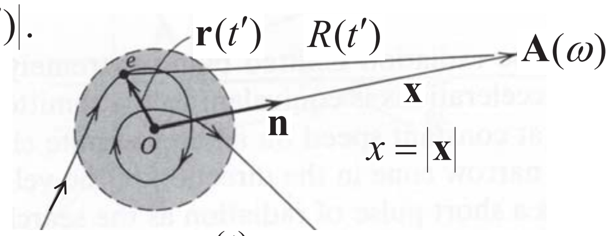
$t=t'+R(t')/c$
$dt=[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]dt'$

$$=\left(\frac{e^2}{8\pi^2 c}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}dt'e^{i\omega(t'+R(t')/c)}\frac{\mathbf{n}(t')\times\{[\mathbf{n}(t')-\boldsymbol{\beta}(t')]\times\dot{\boldsymbol{\beta}}(t')\}}{[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]^2} \quad (14.62)$$

We have assumed $|\mathbf{x}| \gg |\mathbf{r}(t')|$.

Hence, $\begin{cases} R(t') \approx x - \mathbf{n}\cdot\mathbf{r}(t') \\ \mathbf{n}(t') \approx \mathbf{n} = const. \end{cases}$

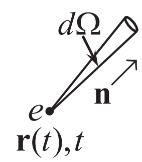
Change t' to t for brevity.



$$(14.62) \Rightarrow \mathbf{A}(\omega) \approx \left(\frac{e^2}{8\pi^2 c}\right)^{\frac{1}{2}} e^{i\frac{\omega x}{c}} \int_{-\infty}^{\infty} e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} dt$$

$$\Rightarrow \frac{d^2 I}{d\omega d\Omega} = 2|\mathbf{A}(\omega)|^2 = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt\right|^2 \quad (14.65)$$

$$= \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \left\{\frac{d}{dt}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]\right\} e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt\right|^2$$



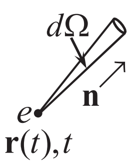
integration by parts

$$= \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \underbrace{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}_{\text{gives the direction of } \mathbf{E}} e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt\right|^2 \quad (14.67)$$

Thus, given the orbit $\mathbf{r}(t)$, we may evaluate $\frac{d^2 I}{d\omega d\Omega}$ from (14.67).

Discussion:

1. Rewrite $\displaystyle \frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt\right|^2 \quad (14.65)$



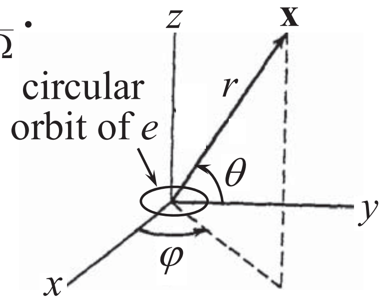
$$= \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt\right|^2 \quad (14.67)$$

(14.65) shows that contribution to $\frac{d^2 I}{d\omega d\Omega}$ comes only from the portion of the orbit with $\dot{\boldsymbol{\beta}} \neq 0$ (as expected), whereas (14.67) appears to indicate that the $\boldsymbol{\beta} = const$ portion also contributes. This question can be resolved by noting that, when $\boldsymbol{\beta} = const$, we have $\mathbf{r}(t) = \mathbf{r}_0 + \boldsymbol{\beta} ct$. Hence, $\omega[t - \frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}] = \omega t(1-\mathbf{n}\cdot\boldsymbol{\beta}) + \frac{\omega}{c}\mathbf{n}\cdot\mathbf{r}_0$, while $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = const$. Thus, the integrand oscillates in t with a constant amplitude, leading to vanishing contribution upon t-integration.

2. The integrands in (14.65) and (14.67) are different for the same t, but the integrated results are equal.

14.6 Frequency Spectrum of Radiation Emitted by a Relativistic Charged Particle in Instantaneous Circular Motion

Spatial / Temporal Dependence of $\frac{d^2I}{d\omega d\Omega}$ **:**



Consider a single e in circular motion on the x-y plane. Let the center of the orbit be the origin of coordinates and the observer's position be $\mathbf{x} = (r, \theta, \varphi)$.

By symmetry, as e moves on the circular orbit, an observer at the same r, θ but different φ will see an identical radiation pulse. The only difference is the pulse arrival time and its origin on the orbit. The arrival time is of no consequence since $\frac{d^2I}{d\omega d\Omega}$ is a time-integrated quantity [(14.67)]. Also, the radiation is emitted into $d\Omega$ (indep. of r). Hence, spatially, $\frac{d^2I}{d\omega d\Omega}$ depends only on θ.

The e emits identical radiation at any point on the orbit. We may thus choose a coordinate system to focus on a particular orbital point.

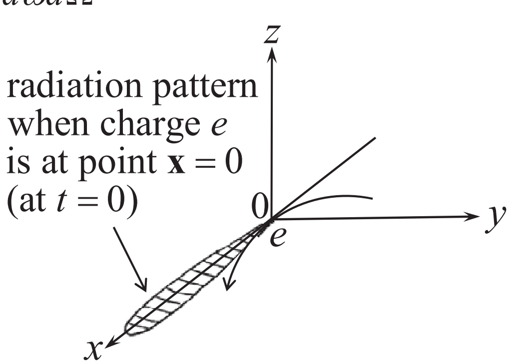
Detailed Analysis - an Exercise on Mathematical Techniques :

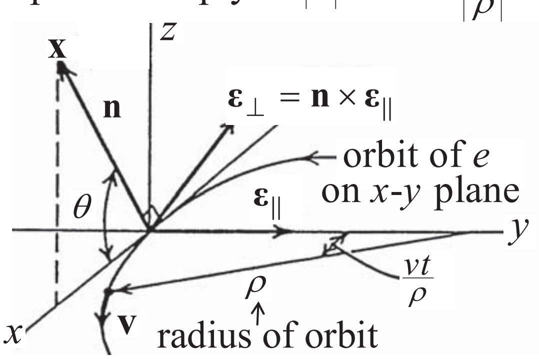
Coordinate system : $\frac{d^2I}{d\omega d\Omega}$ is the same for all e's and indep. of φ.

$\Rightarrow$ we may set up the following coordinates to focus on a single e:

1. Let the orbit be on the x-y plane. Put the origin of coordinates ($\mathbf{x} = 0$) on the orbit and let the x-axis be tangential to the orbit.
2. Set the time coordinate so that e is at $\mathbf{x} = 0$ at $t = 0$.
3. Put $\mathbf{x}$ on the x-z plane (since $\frac{d^2I}{d\omega d\Omega}$ depends only on θ).

$\frac{d^2I}{d\omega d\Omega}$ is highly directed at $\mathbf{e}_x \Rightarrow$ It peaks sharply at $|\theta| \ll 1$ & $\left|\frac{vt}{\rho}\right| \ll 1$.

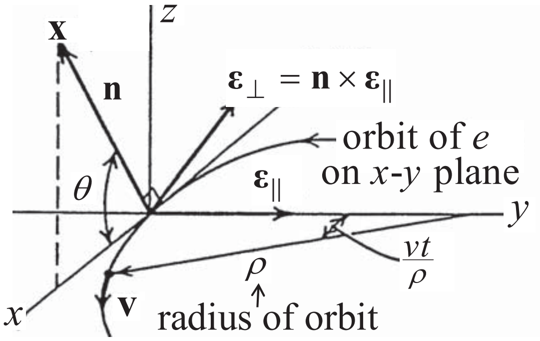




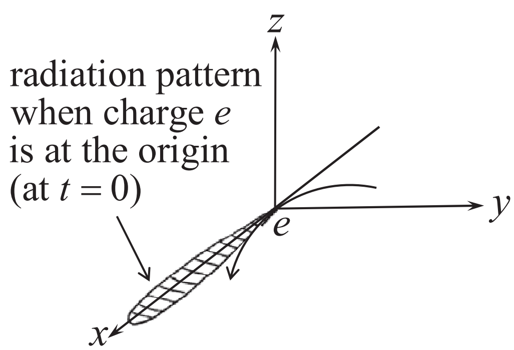
Definition of 3 unit vectors :

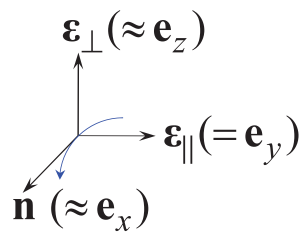
For later resolution of radiation polarization, we define 3 mutually orthogonal unit vectors as follows :

$$\begin{cases} \mathbf{n}\text{: direction of observation} \\ \boldsymbol{\varepsilon}_{\parallel} = \mathbf{e}_y \ \ (\parallel \text{ to the orbital plane}) \\ \boldsymbol{\varepsilon}_{\perp}\text{: a unit vector} \perp \text{to } \mathbf{n} \text{ and } \boldsymbol{\varepsilon}_{\parallel} \end{cases}$$



Since the maximum radiation is along $\mathbf{e}_x$, the relevant $\mathbf{n}$ is $\mathbf{n} \approx \mathbf{e}_x$. The corresponding $\boldsymbol{\varepsilon}_{\perp}$ is $\boldsymbol{\varepsilon}_{\perp} \approx \mathbf{e}_z$ (approx. $\perp$ to the orbital plane).

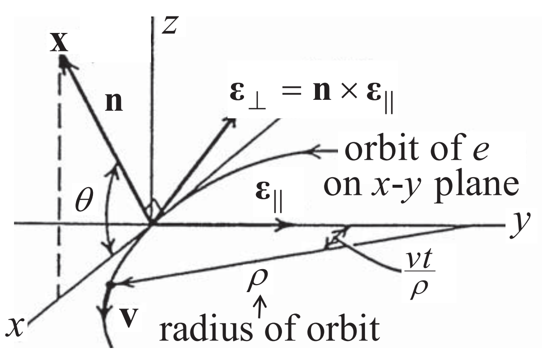




Evaluation of (14.67) : $\dfrac{d^2 I}{d\omega d\Omega} = \dfrac{e^2\omega^2}{4\pi^2 c}\left| \int_{-\infty}^{\infty} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]} dt \right|^2$

To evaluate (14.67), we need to express $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$ and $\mathbf{n}\cdot\mathbf{r}(t)$ as explicit functions of t.

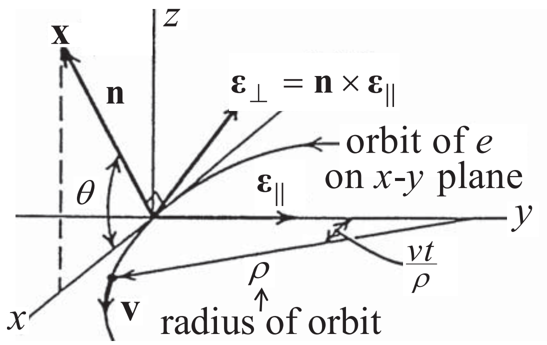
$$\begin{aligned} \boldsymbol{\beta} &= \beta_{\parallel}\boldsymbol{\varepsilon}_{\parallel} + \beta_{\perp}\boldsymbol{\varepsilon}_{\perp} + \beta_n \mathbf{n} \\ &= \beta[\sin\tfrac{vt}{\rho}\boldsymbol{\varepsilon}_{\parallel} - \cos\tfrac{vt}{\rho}\sin\theta\boldsymbol{\varepsilon}_{\perp} \\ &\quad + \cos\tfrac{vt}{\rho}\cos\theta\mathbf{n}] \end{aligned}$$



(15)

$$\begin{aligned} \mathbf{n}\times\boldsymbol{\beta} &= \beta[\sin\tfrac{vt}{\rho}(\mathbf{n}\times\boldsymbol{\varepsilon}_{\parallel}) \\ &\quad - \cos\tfrac{vt}{\rho}\sin\theta(\mathbf{n}\times\boldsymbol{\varepsilon}_{\perp})] \\ &= \beta[\sin\tfrac{vt}{\rho}\boldsymbol{\varepsilon}_{\perp} + \cos\tfrac{vt}{\rho}\sin\theta\boldsymbol{\varepsilon}_{\parallel}] \end{aligned}$$

$$\begin{aligned} \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) &= \beta[\sin\tfrac{vt}{\rho}(\mathbf{n}\times\boldsymbol{\varepsilon}_{\perp}) + \cos\tfrac{vt}{\rho}\sin\theta(\mathbf{n}\times\boldsymbol{\varepsilon}_{\parallel})] \\ &= \beta[-\sin\tfrac{vt}{\rho}\boldsymbol{\varepsilon}_{\parallel} + \cos\tfrac{vt}{\rho}\sin\theta\boldsymbol{\varepsilon}_{\perp}] \qquad \boxed{\mathbf{n}\times\boldsymbol{\varepsilon}_{\perp} = -\boldsymbol{\varepsilon}_{\parallel},\ \mathbf{n}\times\boldsymbol{\varepsilon}_{\parallel} = \boldsymbol{\varepsilon}_{\perp}} \qquad (14.71) \end{aligned}$$



Based on the knowledge that the radiation is confined to a narrow cone in the direction of $\mathbf{v}$, we have assumed:

1. The orbit of e is on the x-y plane. So the radiation is appreciable only for $|\theta|(\le \frac{1}{\gamma}) \approx 0$.

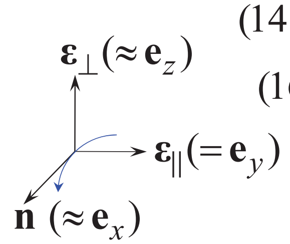
2. The observation point $\mathbf{x}$ is on the x-z plane. So the maximum radiation will be along $\mathbf{e}_x$, hence emitted at $t \approx 0$ when e moves on a very small arc ($|vt/\rho| \ll 1$) near the origin.

Thus, let $\beta \approx 1$, $|\theta| \ll 1$, & $\left|\frac{vt}{\rho}\right| \ll 1$ in

$$\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = \beta[-\sin\tfrac{vt}{\rho}\boldsymbol{\varepsilon}_\parallel + \cos\tfrac{vt}{\rho}\sin\theta\boldsymbol{\varepsilon}_\perp] \quad (14.71)$$

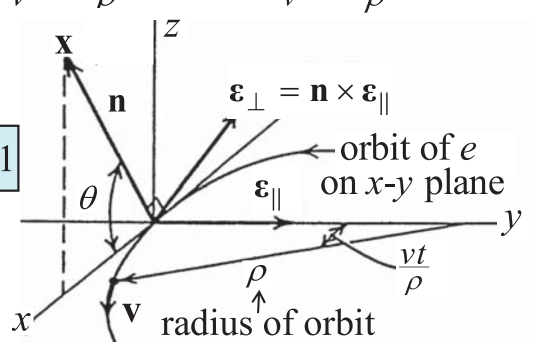
we obtain $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) \approx -\frac{ct}{\rho}\boldsymbol{\varepsilon}_\parallel + \theta\boldsymbol{\varepsilon}_\perp$, (16)

where $\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$ gives the polarization of $\mathbf{E}$. $\boldsymbol{\varepsilon}_\parallel$ is || to the orbital plane. Since $\theta \approx 0$, $\boldsymbol{\varepsilon}_\perp$ is approx. $\perp$ to the orbital plane.



$$\mathbf{r}(t) = \int_0^t c\boldsymbol{\beta}(\tau)d\tau \underset{\uparrow}{=} v[-\tfrac{\rho}{v}(\cos\tfrac{vt}{\rho}-1)\boldsymbol{\varepsilon}_\parallel - \tfrac{\rho}{v}\sin\tfrac{vt}{\rho}\sin\theta\boldsymbol{\varepsilon}_\perp + \tfrac{\rho}{v}\sin\tfrac{vt}{\rho}\cos\theta\mathbf{n}]$$

$\boxed{\mathbf{r}(t=0)=0}$ (15)



$$\Rightarrow \mathbf{n}\cdot\mathbf{r}(t) = \rho\sin\tfrac{vt}{\rho}\cos\theta$$

$$\Rightarrow \omega(t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}) = \omega[t-\tfrac{\rho}{c}\underbrace{\sin\tfrac{vt}{\rho}}_{\boxed{\approx \frac{vt}{\rho}-\frac{1}{6}(\frac{vt}{\rho})^3 \text{ if } \left|\frac{vt}{\rho}\right| \ll 1}}\overbrace{\cos\theta}^{\boxed{\approx 1-\frac{\theta^2}{2} \text{ if } |\theta|\ll 1}}]$$

$$\approx \omega\left\{t-\tfrac{\rho}{c}\left[\tfrac{vt}{\rho}-\tfrac{1}{6}(\tfrac{vt}{\rho})^3-\tfrac{\theta^2}{2}\tfrac{vt}{\rho}+\tfrac{1}{12}(\tfrac{vt}{\rho})^3\theta^2\right]\right\}$$

$$=\omega[\underbrace{t-\beta t}_{\frac{1-\beta^2}{1+\beta}t\approx\frac{t}{2\gamma^2}} + \underbrace{\frac{v^3t^3}{6c\rho^2}}_{\approx\frac{c^2t^3}{6\rho^2}} + \frac{\theta^2}{2}\beta t - \underbrace{\frac{1}{12}\frac{v^3t^3}{c\rho^2}\theta^2}_{\text{negligible}}]$$

(β can't be set to 1)

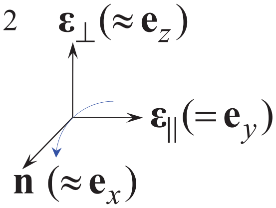
$\boxed{\beta \text{ can be set to 1 in this 3rd order term.}}$

$\boxed{\begin{array}{l}\sin\frac{vt}{\rho} \text{ and } \cos\theta \text{ must} \\ \text{be expanded to higher} \\ \text{order due to the near} \\ \text{concellation of lowest} \\ \text{order terms } t \text{ and } \beta t.\end{array}}$

$$\approx \frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}] \longleftarrow \boxed{\text{All are 3rd order terms.}} \quad (14.73)$$

$$\text{Sub}\begin{cases}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\approx-\frac{c}{\rho}t\boldsymbol{\varepsilon}_{\parallel}+\theta\boldsymbol{\varepsilon}_{\perp} & (16)\\ \omega(t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c})\approx\frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}] & (14.73)\end{cases}$$

$$\text{into }\frac{d^2I}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})e^{i\omega[t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}]}dt\right|^2 \quad (14.67)$$



$$\Rightarrow\frac{d^2I}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|-\boldsymbol{\varepsilon}_{\parallel}A_{\parallel}(\omega)+\boldsymbol{\varepsilon}_{\perp}A_{\perp}(\omega)\right|^2 \quad (14.74)$$

$$\text{where}\begin{cases}A_{\parallel}(\omega)=\frac{c}{\rho}\int_{-\infty}^{\infty}t\exp\{i\frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}]\}dt\\ A_{\perp}(\omega)=\theta\int_{-\infty}^{\infty}\exp\{i\frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}]\}dt\end{cases} \quad (14.75)$$

and "$\parallel$" & "$\perp$" refer to polarization "$\parallel$" & "$\perp$" to the orbital plane.

The integrands in (14.75) oscillate rapidly for large t and θ, giving negligible contributions. Hence, the radiation is appreciable only for $\theta\approx 0$, and it comes from the charge's motion at $t\approx 0$. This validates our assumptions of $t\approx 0$ and $\theta\approx 0$ in deriving (16) and (14.73).

$$\text{Rewrite (14.75):}\begin{cases}A_{\parallel}(\omega)=\frac{c}{\rho}\int_{-\infty}^{\infty}t\exp\{i\frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}]\}dt\\ A_{\perp}(\omega)=\theta\int_{-\infty}^{\infty}\exp\{i\frac{\omega}{2}[(\frac{1}{\gamma^2}+\theta^2)t+\frac{c^2t^3}{3\rho^2}]\}dt\end{cases}$$

$$\text{Define } x\equiv\frac{ct}{\rho}(\frac{1}{\gamma^2}+\theta^2)^{-\frac{1}{2}},\ \xi\equiv\frac{\omega\rho}{3c}(\frac{1}{\gamma^2}+\theta^2)^{\frac{3}{2}} \quad (14.76)$$

$$\Rightarrow A_{\parallel}(\omega)=\frac{\rho}{c}(\frac{1}{\gamma^2}+\theta^2)\int_{-\infty}^{\infty}x\exp[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)]dx$$

$$=\frac{\rho}{c}(\frac{1}{\gamma^2}+\theta^2)2i\underbrace{\int_0^{\infty}x\sin[\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)]dx}_{\text{Airy integral, }=\frac{1}{\sqrt{3}}K_{\frac{2}{3}}(\xi)}=\frac{2i\rho}{\sqrt{3}c}(\frac{1}{\gamma^2}+\theta^2)K_{\frac{2}{3}}(\xi) \quad (17a)$$

$K_{\frac{1}{3}}, K_{\frac{2}{3}}$: modified Bessel functions [See (3.101)]

Similarly,

$$A_{\perp}(\omega)=\frac{\rho}{c}\theta(\frac{1}{\gamma^2}+\theta^2)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp[i\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)]dx$$

$$=\frac{\rho}{c}\theta(\frac{1}{\gamma^2}+\theta^2)^{\frac{1}{2}}2\underbrace{\int_0^{\infty}\cos[\tfrac{3}{2}\xi(x+\tfrac{1}{3}x^3)]dx}_{\text{Airy integral, }=\frac{1}{\sqrt{3}}K_{\frac{1}{3}}(\xi)}=\frac{2\rho}{\sqrt{3}c}\theta(\frac{1}{\gamma^2}+\theta^2)^{\frac{1}{2}}K_{\frac{1}{3}}(\xi) \quad (17b)$$

Sub $A_{\|}(\omega)=\frac{2i\rho}{\sqrt{3}c}(\frac{1}{\gamma^2}+\theta^2)K_{\frac{2}{3}}(\xi)$ & $A_{\perp}(\omega)=\frac{2\rho}{\sqrt{3}c}\theta(\frac{1}{\gamma^2}+\theta^2)^{\frac{1}{2}}K_{\frac{1}{3}}(\xi)$

into $\frac{d^2I}{d\omega d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|-\boldsymbol{\varepsilon}_{\|}A_{\|}(\omega)+\boldsymbol{\varepsilon}_{\perp}A_{\perp}(\omega)\right|^2$ (14.74)

$$\Rightarrow \frac{d^2I}{d\omega d\Omega}=\frac{e^2}{3\pi^2c}(\frac{\omega\rho}{c})^2(\frac{1}{\gamma^2}+\theta^2)^2[K_{\frac{2}{3}}^2(\xi)+\frac{\theta^2}{1/\gamma^2+\theta^2}K_{\frac{1}{3}}^2(\xi)] \quad (14.79)$$

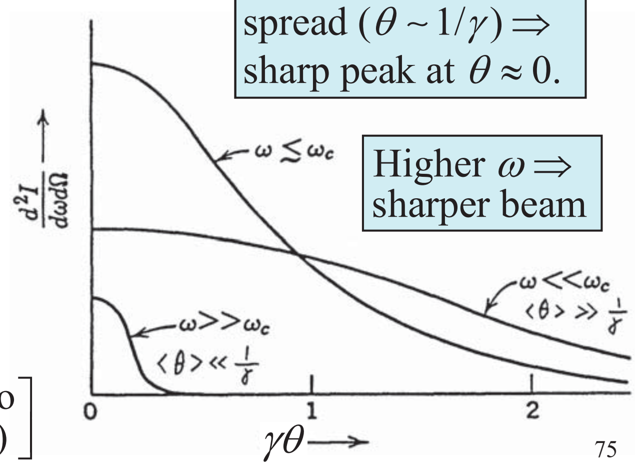
with $\xi=\frac{\omega\rho}{3c}(\frac{1}{\gamma^2}+\theta^2)^{\frac{3}{2}}$. (14.76)

Very narrow angular spread ($\theta \sim 1/\gamma$) $\Rightarrow$ sharp peak at $\theta \approx 0$.

$K(\xi)$ is negligible for $\xi \gg 1$ [(3.104)]. Thus, the critical freq. ω_{c} (beyond which $\frac{d^2I}{d\omega d\Omega}$ is small at any θ) may be defined by setting $\xi=\frac{1}{2}$ and $\theta=0$ in

$\xi \equiv \frac{\omega\rho}{3c}(\frac{1}{\gamma^2}+\theta^2)^{\frac{3}{2}}$ [(14.76)]

$\Rightarrow \omega_c \equiv \frac{3}{2}\gamma^3\frac{c}{\rho}$ (14.81) [see also (14.50)]



Higher $\omega \Rightarrow$ sharper beam

Energy radiated at all frequencies per unit solid angle: Rewrite

$$\frac{d^2I}{d\omega d\Omega}=\frac{e^2}{3\pi^2c}(\frac{\omega\rho}{c})^2(\frac{1}{\gamma^2}+\theta^2)^2[K_{\frac{2}{3}}^2(\xi)+\frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}K_{\frac{1}{3}}^2(\xi)] \quad [(14.79)]$$

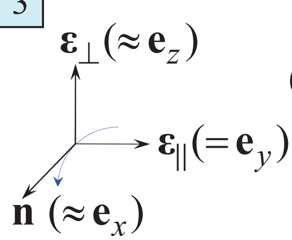
where $\xi=\frac{\omega\rho}{3c}(\frac{1}{\gamma^2}+\theta^2)^{\frac{3}{2}}$.

$$\Rightarrow \frac{dI}{d\Omega}=\int_0^\infty \frac{d^2I}{d\omega d\Omega}d\omega \left[=\frac{\text{energy radiated at all frequencies}}{\text{unit solid angle}}\right]$$

$$=\frac{9e^2}{\pi^2\rho}\frac{1}{(1/\gamma^2+\theta^2)^{5/2}}[\int_0^\infty \xi^2K_{\frac{2}{3}}^2(\xi)d\xi+\frac{\theta^2}{1/\gamma^2+\theta^2}\int_0^\infty \xi^2K_{\frac{1}{3}}^2(\xi)d\xi]$$

$$\boxed{\int_0^\infty \xi^2K_m^2(\xi)d\xi=\frac{\pi^2}{8}\left(\frac{\frac{1}{4}-m^2}{\cos m\pi}\right)=\begin{cases}\frac{5\pi^2}{144}, & m=\frac{1}{3}\\ \frac{7\pi^2}{144}, & m=\frac{2}{3}\end{cases}}$$

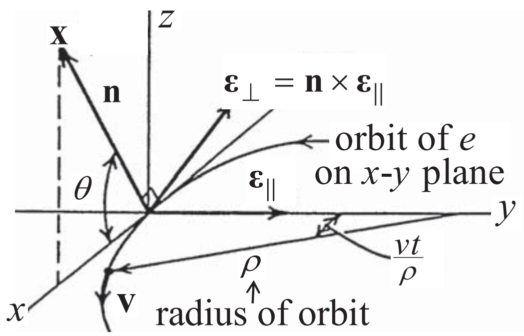
$$=\frac{7e^2}{16\rho}\frac{1}{(1/\gamma^2+\theta^2)^{5/2}}[\underbrace{1}_{\|\text{ polarization}}+\frac{5}{7}\underbrace{\frac{\theta^2}{1/\gamma^2+\theta^2}}_{\perp\text{ polarization}}] \quad (14.80)$$



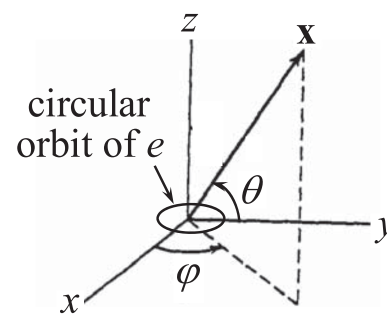
Total energy radiated by e in all directions in one revolution:

Rewrite $\frac{dI}{d\Omega}=\frac{7e^2}{16\rho}\frac{1}{(1/\gamma^2+\theta^2)^{5/2}}\left[1+\frac{5}{7}\frac{\theta^2}{1/\gamma^2+\theta^2}\right]$ [(14.80)]

In deriving (14.80), the origin of coordinates ($\mathbf{x}=0$) is at a point on the orbit. $\mathbf{x}$ is on the x-z plane. $\frac{dI}{d\Omega}$ originates from a short segment near $\mathbf{x}=0$. It is the energy radiated by e into $d\Omega$ along $\mathbf{x}$ when e is near the $\mathbf{x}=0$ point on its orbit.



By symmetry (see beginning of this section), e will radiate the same $\frac{dI}{d\Omega}$ (to a different direction) at any point on its orbit. Thus, to calculate the total energy radiated in all directions per revolution, we move the origin of coordinates to the center of e's orbit and integrate $\frac{dI}{d\Omega}$ over φ and θ.

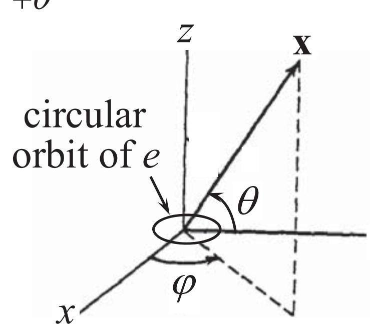


Rewrite $\frac{dI}{d\Omega}=\frac{7e^2}{16\rho}\frac{1}{(1/\gamma^2+\theta^2)^{5/2}}\left[1+\frac{5}{7}\frac{\theta^2}{1/\gamma^2+\theta^2}\right]$ [(14.80)]

$$\Rightarrow I\begin{pmatrix}\text{total energy radiated by } e \text{ in all}\\ \text{directions in one revolution}\end{pmatrix}$$

$$=\int\frac{dI}{d\Omega}d\Omega=\underbrace{\int_0^{2\pi}d\varphi}_{2\pi}\int_{-\pi/2}^{\pi/2}d\theta\frac{dI}{d\Omega}\cos\theta$$

$$=2\pi\int_{-\pi/2}^{\pi/2}\underbrace{\frac{dI}{d\Omega}}\cos\theta d\theta$$



$\boxed{\frac{dI}{d\Omega}\text{ sharply peaks at }\theta\approx 0\Rightarrow\text{Let }\cos\theta\approx 1;\ \int_{-\pi/2}^{\pi/2}\frac{dI}{d\Omega}d\theta\approx\int_{-\infty}^{\infty}\frac{dI}{d\Omega}d\theta}$

$$\approx 2\pi\int_{-\infty}^{\infty}\frac{dI}{d\Omega}d\theta=\frac{7\pi e^2}{8\rho}\int_{-\infty}^{\infty}\frac{1}{(\frac{1}{\gamma^2}+\theta^2)^{5/2}}\left[1+\frac{5}{7}\frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}\right]d\theta$$

$$=I_{\parallel}+I_{\perp}$$

(14.80) || polarization ⊥ polarization

This integral will yield $I_{\parallel}=7I_{\perp}$. This gives a very important property of the synchrotron radiation: It is strongly polarized in the orbital plane, i.e. $\mathbf{E}$ is largely oriented in the x-y plane.

Energy radiated per unit frequency interval in one revolution:

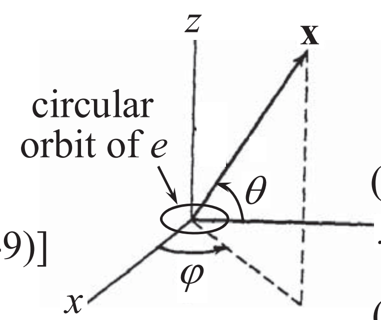
Rewrite $\frac{d^2I}{d\omega d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{\frac{2}{3}}^2(\xi) + \frac{\theta^2}{\frac{1}{\gamma^2}+\theta^2}K_{\frac{1}{3}}^2(\xi)\right]$ [(14.79)]

which depends only on θ.

$$\boxed{\xi = \frac{\omega\rho}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{3}{2}}}$$

$$\frac{dI}{d\omega} = \int \frac{d^2I}{d\omega d\Omega}d\Omega \quad \left[\frac{\text{energy radiated in 1 revolution}}{\text{unit frequency interval}}\right]$$

$$= \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \underbrace{\frac{d^2I}{d\omega d\Omega}}_{\text{Sharply peaks at } \theta \approx 0} \cos\theta d\theta$$



$$\approx 2\pi \int_{-\infty}^{\infty} \frac{d^2I}{d\omega d\Omega} d\theta \qquad (14.88)$$

[by Schwinger, Phy. Rev. **75**, 1912 (1949)]

$$= \sqrt{3}\frac{e^2}{c}\gamma\frac{\omega}{\omega_c}\int_{\frac{\omega}{\omega_c}}^{\infty} K_{\frac{5}{3}}(x)dx \qquad (14.91)$$

$$\boxed{\omega_c = \frac{3}{2}\gamma^3\frac{c}{\rho} \ [(14.81)]}$$

$$\approx \begin{cases} 3.25\frac{e^2}{c}\left(\frac{\omega\rho}{c}\right)^{\frac{1}{3}}, & \omega \ll \omega_c \qquad (14.89) \\ \sqrt{\frac{3\pi}{2}}\frac{e^2}{c}\gamma\left(\frac{\omega}{\omega_c}\right)^{\frac{1}{2}} e^{-\frac{\omega}{\omega_c}}, & \omega \gg \omega_c \qquad (14.90) \end{cases}$$

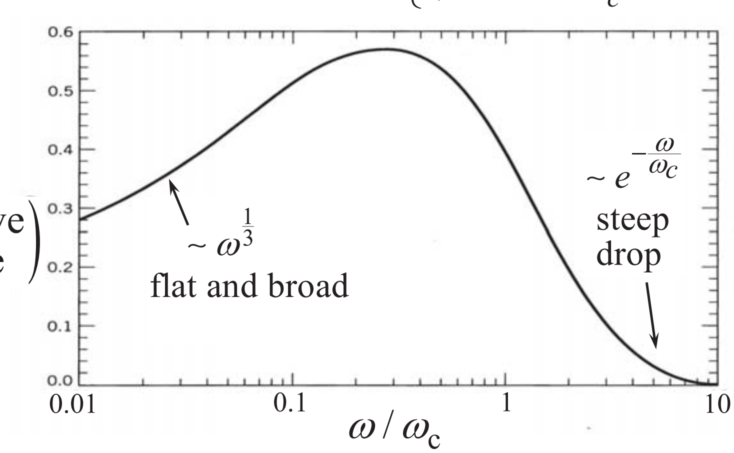
See p. 680 for a simpler derivation

Rewrite (14.89)-(14.91):

$$\frac{dI}{d\omega} = \frac{\text{energy radiated in 1 revolution}}{\text{unit frequency interval}}$$

$$= \sqrt{3}\frac{e^2}{c}\gamma\frac{\omega}{\omega_c}\int_{\frac{\omega}{\omega_c}}^{\infty} K_{\frac{5}{3}}(x)dx \approx \begin{cases} 3.25\frac{e^2}{c}\left(\frac{\omega\rho}{c}\right)^{\frac{1}{3}}, & \omega \ll \omega_c \\ \sqrt{\frac{3\pi}{2}}\frac{e^2}{c}\gamma\left(\frac{\omega}{\omega_c}\right)^{\frac{1}{2}} e^{-\frac{\omega}{\omega_c}}, & \omega \gg \omega_c \end{cases}$$

$\frac{dI}{d\omega}$ (relative scale)



← Synchrotron radiation spectrum

See p. 681 for quantum mechanical corrections as $\hbar\omega_c \to \gamma mc^2$.

Synchrotron Radiation in Nature :

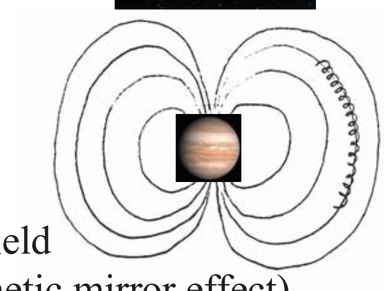
1. Crab nebula: 4000 light years away; expanding at 1300 km/sec; due to a supernova in 1054 visible in the daytime for 23 days, as recorded by Chinese astronomers in Sung (宋仁宗) dynasty.
   Measured radiation frequency: RF (radio frequency) to UV
   Strong polarization suggests synchrotron radiation.
   $\Rightarrow B \sim 10^{-4}$ G and electron energies up to 10^{13} eV

2. Jupiter (largest and most massive planet)
   Dipole magnetic field: ~ 1 Gauss (typical)
   Electron energies: $3-50$ MeV
   (both measured by space vehicles)

   $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$ [(5.69)]

   $\mathbf{m}$ (magnetic moment) antiparallel to $\mathbf{B}$

   $\Rightarrow$ **F** points in the direction of decreasing B-field
   $\Rightarrow$ Electrons trapped in Van Allen belts (magnetic mirror effect)
   $\Rightarrow$ Emitting synchrotron radiation up to ~ 1 GHz



14.8 Thomson Scattering of Radiation

Scattering of Incident Radiation by a Free Charge :

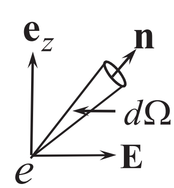
Assume a plane wave propagating along $\mathbf{e}_z$ (in free space) with

$$\mathbf{E}(\mathbf{x},t) = \boldsymbol{\varepsilon}_0 E_0 e^{ik_0 z - i\omega t},\ \mathbf{B}(\mathbf{x},t) = \mathbf{e}_z \times \mathbf{E}(\mathbf{x},t)\ \ \left[\boldsymbol{\varepsilon}_0 \perp \mathbf{e}_z\right] \qquad (18)$$

is incident on a free charge e with mass m initially at rest. Charge e will be accelerated and hence radiates. Assume $v \ll c$, the radiation frequency will be ω. This is a scattering problem as in Sec. 10.1.

The magnetic force is negligible if $v \ll c$. Thus, the partcile obeys the equation of motion: $\dot{\mathbf{v}}(t) = \frac{e}{m}\mathbf{E}(\mathbf{x},t) = \frac{e}{m}\boldsymbol{\varepsilon}_0 E_0 e^{ik_0 z - i\omega t}$ (19)

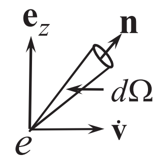
(19) is a nonlinear differential equation in t if e's position z varies with t. However, here we have $\boldsymbol{\varepsilon}_0$(or $\dot{\mathbf{v}}$) $\perp \mathbf{e}_z$; hence, e is at a constant z. This makes (19) the solution for $\dot{\mathbf{v}}(t)$.



Rewrite $\lim_{\beta\to 0}\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right|^2$ [(14.20)]. The *instantaneous* power emitted into a unit solid angle is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\mathbf{n}\times[\mathbf{n}\times\dot{\mathbf{v}}(t)]\right|^2\ \ [\dot{\mathbf{v}}(t) = \text{real part of the RHS of (19)}] \quad (20)$$

Rewrite $\begin{cases} \dot{\mathbf{v}}(t) = \frac{e}{m}\boldsymbol{\varepsilon}_0 E_0 e^{ik_0 z - i\omega t} & (19) \\ \frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\mathbf{n}\times[\mathbf{n}\times\dot{\mathbf{v}}(t)]|^2 & (20) \end{cases}$



Note: In (19), LHS=Re[RHS]. In (20), both $\mathbf{n}$ and $\dot{\mathbf{v}}(t)$ are real.

Let $\langle\cdots\rangle$ denote the time average over one wave period. To find $\left\langle\frac{dP}{d\Omega}\right\rangle$, we note if $\mathbf{X}(\mathbf{x},t) = \mathrm{Re}[\mathbf{X}(\mathbf{x})e^{-i\omega t}]$, then for real ω, we have

$$\langle \mathbf{X}(\mathbf{x},t)\cdot\mathbf{X}(\mathbf{x},t)\rangle = \tfrac{1}{2}|\mathbf{X}(\mathbf{x})|^2 \quad \text{[use (23), lecture notes, Ch. 6]}$$

$$\mathbf{n}\times[\mathbf{n}\times\dot{\mathbf{v}}(t)] = \mathrm{Re}[\tfrac{e}{m}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)E_0 e^{ik_0 z - i\omega t}] \text{ [by (19)]}$$

$$\Rightarrow \left\langle |\mathbf{n}\times[\mathbf{n}\times\dot{\mathbf{v}}(t)]|^2 \right\rangle = \frac{1}{2}\frac{e^2}{m^2}|E_0|^2[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)e^{ik_0 z}]\cdot[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0^*)e^{-ik_0 z}]$$

$$= \frac{e^2}{2m^2}|E_0|^2 |\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)|^2 \qquad (21)$$

$$\Rightarrow \left\langle\frac{dP}{d\Omega}\right\rangle \overset{(20)}{=} \frac{e^2}{4\pi c^3}\left\langle|\underbrace{\mathbf{n}\times[\mathbf{n}\times\dot{\mathbf{v}}(t)]}|^2\right\rangle \overset{(21)}{=} \frac{e^4}{8\pi m^2 c^3}|E_0|^2 |\underbrace{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)}|^2 \qquad (22)$$

gives direction of $\mathbf{E}_{sc}$ (see derivation of Larmor's Formula)

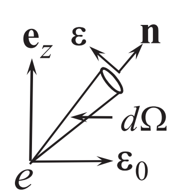
$\boldsymbol{\varepsilon}_0$: polarization vector of $\mathbf{E}$. $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$: polarization vectors of $\mathbf{E}_{sc}$.

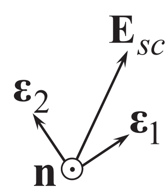
$\boldsymbol{\varepsilon}_0 \perp \mathbf{e}_z$. $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, and $\mathbf{n}$ are mutually orthogonal.

For linear polarization, $\boldsymbol{\varepsilon}_0$ is real and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1$ or $\boldsymbol{\varepsilon}_2$ (both real).

For circular polarization $\begin{cases} \boldsymbol{\varepsilon}_0 = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_x \pm i\boldsymbol{\varepsilon}_y);\ \boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0{}^* = 1 \\ \boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2);\ \boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^* = 1 \end{cases}$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^4}{8\pi m^2 c^3}|E_0|^2 |\underbrace{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)}_{\boxed{\text{gives direction of } \mathbf{E}_{sc}}}|^2 \quad [(22)],$$





$\Rightarrow$ The part of $\left\langle\frac{dP}{d\Omega}\right\rangle$ in polarization state $\boldsymbol{\varepsilon}$ is

$$\left\langle\frac{dP}{d\Omega}\bigg|_{\boldsymbol{\varepsilon}}\right\rangle = \frac{e^4}{8\pi m^2 c^3}|E_0|^2 |\boldsymbol{\varepsilon}^*\underbrace{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\varepsilon}_0)}_{\boxed{\mathbf{n}\cdot(\mathbf{n}\cdot\boldsymbol{\varepsilon}_0)-\boldsymbol{\varepsilon}_0}}]|^2 \underset{\boxed{\boldsymbol{\varepsilon}^*\perp\mathbf{n}}}{=} \frac{e^4}{8\pi m^2 c^3}|E_0|^2 |\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 \quad (14.122)$$

Incident fields: $\mathbf{E}(\mathbf{x},t) = \boldsymbol{\varepsilon}_0 E_0 e^{ik_0 z - i\omega t}$; $\mathbf{B}(\mathbf{x},t) = \mathbf{e}_z \times \mathbf{E}(\mathbf{x},t)$ [(18)]

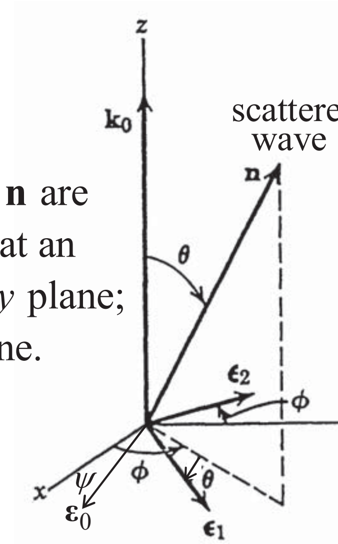
$$\Rightarrow \langle P\rangle = \frac{c}{4\pi}\frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{x})\times\mathbf{H}^*(\mathbf{x})]\cdot\mathbf{e}_z = \frac{c}{8\pi}|E_0|^2 \quad \begin{bmatrix}\text{average} \\ \text{incident } P\end{bmatrix} \qquad (23)$$

Differential Scattering Cross Section :

As in (10.3), we define the differential scattering cross section as

$$\left.\frac{d\sigma}{d\Omega}\right|_{\boldsymbol{\varepsilon}} = \frac{\dfrac{\text{radiated power in } \boldsymbol{\varepsilon}\text{-polarization}}{\text{unit solid angle}}}{\dfrac{\text{incident power}}{\text{unit area}}} = \frac{\left\langle \left.\dfrac{dP}{d\Omega}\right|_{\boldsymbol{\varepsilon}} \right\rangle}{\frac{c}{8\pi}\left|E_0\right|^2} = \left(\frac{e^2}{mc^2}\right)^2 \left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2$$ [(23)]

Let $\begin{cases} \boldsymbol{\varepsilon}_1 = \cos\theta(\mathbf{e}_x\cos\phi + \mathbf{e}_y\sin\phi) - \mathbf{e}_z\sin\theta \\ \boldsymbol{\varepsilon}_2 = -\mathbf{e}_x\sin\phi + \mathbf{e}_y\cos\phi \\ \boldsymbol{\varepsilon}_0 = \cos\psi\,\mathbf{e}_x + \sin\psi\,\mathbf{e}_y \quad [\psi: \text{any angle}] \end{cases}$

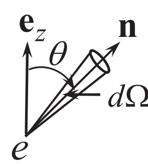


where (i) θ, ϕ give the direction of $\mathbf{n}$; (ii) $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \mathbf{n}$ are mutually orthogonal; (iii) $\boldsymbol{\varepsilon}_1$ is on the z-$\mathbf{n}$ plane at an angle θ below the x-y plane; (iv) $\boldsymbol{\varepsilon}_2$ is on the x-y plane; (v) ψ specifies the unit vector $\boldsymbol{\varepsilon}_0$ on the x-y plane.

$$\Rightarrow \begin{cases} \left.\dfrac{d\sigma}{d\Omega}\right|_{\boldsymbol{\varepsilon}_1} = \left(\dfrac{e^2}{mc^2}\right)^2 \cos^2\theta\cos^2(\phi-\psi) \\ \left.\dfrac{d\sigma}{d\Omega}\right|_{\boldsymbol{\varepsilon}_2} = \left(\dfrac{e^2}{mc^2}\right)^2 \sin^2(\phi-\psi) \end{cases}$$

For random ψ (unpolarized incident wave), we obtain the Thomson formula for scattering of radiation by a free charge:

$$\frac{d\sigma}{d\Omega} = \left\langle \left.\frac{d\sigma}{d\Omega}\right|_{\boldsymbol{\varepsilon}_1} + \left.\frac{d\sigma}{d\Omega}\right|_{\boldsymbol{\varepsilon}_2} \right\rangle_{\psi} = \left(\frac{e^2}{mc^2}\right)^2 \frac{1}{2}\left(1+\cos^2\theta\right) \tag{14.125}$$



Intergrating (14.125) over Ω yields the Thomson cross section

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \tag{14.126}$$

For electrons, $\dfrac{e^2}{m_e c^2} = 2.82\times10^{-13}$ cm [classical electron radius]

The Dual Nature of EM Radiation : waves or photons

$$\begin{cases} \hbar\omega: \text{ photon energy} \\ p = \dfrac{\hbar\omega}{c}: \text{ photon momentum} \end{cases}$$

The quantum mechanical (photon) treatments (e.g. the Compton formula, p. 696) show that the classical Thomson formula in (14.125) is valid when $p \ll mc$ (or $\hbar\omega \ll mc^2$).