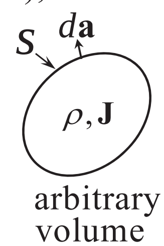
Chapter 5: Magnetostatics, Faraday's Law, Quasi-Static Fields

5.1 Introduction and Definitions

Magnetostatics was pioneered by Oersted (in 1819), Biot and Savart (in 1820), and Ampere (in 1820-1825).

We begin with the law of conservation of charge:



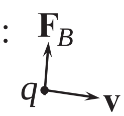
$$\int_V \nabla\cdot\mathbf{J}d^3x = \oint_S \mathbf{J}\cdot d\mathbf{a} = -\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\int_V \rho d^3x$$

$$\Rightarrow \nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \quad [\text{conservation of charge}] \qquad (5.2)$$

Magnetostatics is applicable under the static condition ($\frac{\partial\rho}{\partial t} = 0$).

Hence, (5.2) gives $\quad \nabla\cdot\mathbf{J} = 0 \;$ [for magnetoststics] $\qquad$ (5.3)

Assuming a magnetic force $\mathbf{F}_B$ is on a charge q moving at velocity $\mathbf{v}$, we define the magnetic induction $\mathbf{B}$ by the relation:



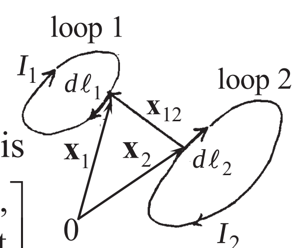
$$\mathbf{F}_B = q\mathbf{v}\times\mathbf{B} \quad [\text{definition of } \mathbf{B}],$$

which is consistent with the definition of $\mathbf{B}$ in (5.1) [see (5.71)].

5.2 Biot and Savart Law

The Biot-Savart law states that $d\mathbf{B}$ at $\mathbf{x}_1$ in loop 1 due to current I_2 on an element $d\ell_2$ at $\mathbf{x}_2$ in loop 2 is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I_2 \frac{d\ell_2\times\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3} \quad [\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2] \qquad (5.4)$$



Thus, the total $\mathbf{B}$ at $\mathbf{x}_1$ due to I_2 in loop 2 is

$$\mathbf{B} = \frac{\mu_0}{4\pi} I_2 \oint \frac{d\ell_2\times\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3} \quad \left[\begin{array}{l}\text{linear superposition,}\\ \text{an experimental fact}\end{array}\right] \qquad (1)$$

Assume I_1 in loop 1. The total force on loop 1 exerted by loop 2 is

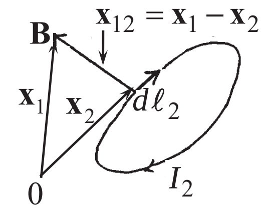
$$\mathbf{F}_{12} = I_1\oint d\ell_1\times\mathbf{B} \quad [\text{force on loop 1 exerted by loop 2}] \qquad (5.7)$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \underbrace{\frac{d\ell_1\times(d\ell_2\times\mathbf{x}_{12})}{|\mathbf{x}_{12}|^3}}$$

$$= \oint d\ell_2 \underbrace{\oint \frac{d\ell_1\cdot\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}}_{= -\oint d\ell_1\cdot\nabla\frac{1}{|\mathbf{x}_{12}|} = -\oint d\frac{1}{|\mathbf{x}_{12}|} = 0} - \oint\oint \frac{(d\ell_1\cdot d\ell_2)\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{(d\ell_1\cdot d\ell_2)\times\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3} \qquad (5.10)$$

5.3 Differential Equations of Magnetostatics and Ampere's Law



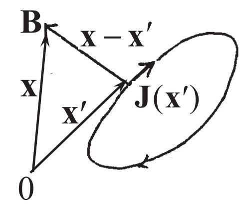
Gauss Law of Magnetism :

Rewrite (1): $\mathbf{B}=\frac{\mu_0}{4\pi}I_2\oint\frac{d\ell_2\times\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$ $\boxed{\text{cross section of wire}}$

Change $\mathbf{x}_1$ to $\mathbf{x}$, $\mathbf{x}_2$ to $\mathbf{x}'$, and let $I_2 d\ell_2=\mathbf{J}\,\overbrace{da}\,d\ell_2=\mathbf{J}d^3x$, we obtain

$$\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'=\frac{\mu_0}{4\pi}\int\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\times\mathbf{J}(\mathbf{x}')d^3x' \qquad (5.14)$$

$$\boxed{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}=-\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}}$$



$$\boxed{(\nabla\psi)\times\mathbf{a}=\nabla\times(\psi\mathbf{a})-\psi\nabla\times\mathbf{a}}$$

$$=\frac{\mu_0}{4\pi}\int\Big[\nabla\times\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{|\mathbf{x}-\mathbf{x}'|}\underbrace{\nabla\times\mathbf{J}(\mathbf{x}')}_{=0}\Big]d^3x'$$

$$=\frac{\mu_0}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad \boxed{\nabla\text{ operates on }\mathbf{x}} \qquad (5.16)$$

$$\Rightarrow \quad \nabla\cdot\mathbf{B}=0 \quad [\text{Gauss law of magnetism}] \qquad (5.17)$$

Ampere's Law : Rewrite (5.16):

$$\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

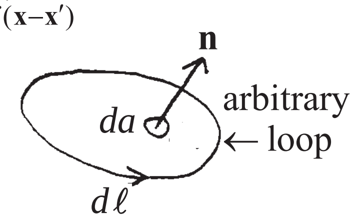
$$\boxed{\begin{aligned}&\int\nabla\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \qquad \overbrace{\nabla\text{ operates on }\mathbf{x}.}\\ &=\int\Big[\mathbf{J}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}+\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cancel{\nabla\cdot\mathbf{J}(\mathbf{x}')}\Big]d^3x'\\ &=-\int\mathbf{J}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'\\ &=\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\underbrace{\cancel{\nabla'\cdot\mathbf{J}(\mathbf{x}')}}_{(5.3)}d^3x'-\underbrace{\cancel{\int\nabla'\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}}_{\text{Use divergence thm. \& }\mathbf{J}(\infty)=0}\\ &=0\end{aligned}}$$

$$\Rightarrow\nabla\times\mathbf{B}(\mathbf{x})$$

$$=\frac{\mu_0}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$=\frac{\mu_0}{4\pi}\Big[\nabla\overbrace{\int\nabla\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}^{0}-\int\mathbf{J}(\mathbf{x}')\underbrace{\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}}_{-4\pi\delta(\mathbf{x}-\mathbf{x}')}d^3x'\Big]$$

$$\boxed{\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}}$$



$$\Rightarrow\nabla\times\mathbf{B}(\mathbf{x})=\mu_0\mathbf{J}(\mathbf{x}) \qquad (5.22)$$

$$\Rightarrow\underbrace{\int\nabla\times\mathbf{B}\cdot\mathbf{n}\,da}_{\oint\mathbf{B}\cdot d\ell}=\mu_0\underbrace{\int\mathbf{J}\cdot\mathbf{n}\,da}_{I\text{ (through the loop)}}$$

$$\Rightarrow\oint\mathbf{B}\cdot d\ell=\mu_0 I \quad \left[\begin{array}{l}\text{Ampere's law, a much more elaborate}\\ \text{representation of the Biot-Savart law}\end{array}\right] \qquad (5.25)$$

5.4 Vector Potential

Vector Potential :

Rewrite (5.16): $\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$

$$\Rightarrow \mathbf{B} = \nabla\times\mathbf{A}, \tag{5.27}$$

where $\mathbf{A}$ is the vector potential given by the general form:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \nabla\psi, \tag{5.28}$$

which shows that $\mathbf{A}$ may be freely transformed (without changing **B)** according to $\quad \mathbf{A}\to\mathbf{A}+\nabla\psi \quad$ (gauge transformation) $\quad$ (5.29)

We may exploit this freedom by choosing a ψ so that

$\nabla\cdot\mathbf{A} = 0$ [Coulomb gauge] $\qquad$ proved on previous page $\qquad$ (5.31)

$$\nabla\cdot(5.28) \Rightarrow \nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\overbrace{\int\nabla\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}^{0} + \nabla^2\psi = \nabla^2\psi,$$

Thus, $\nabla\cdot\mathbf{A} = 0 \Rightarrow \nabla^2\psi = 0$ everywhere $\Rightarrow \psi = const.$ (See p. 181.)

$$\Rightarrow \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \text{ [under Coulomb gauge]} \tag{5.32}$$

Another way to derive (5.32): Rewrite: $\begin{cases}\nabla\times\mathbf{B} = \mu_0\mathbf{J} & [(5.22)] \\ \mathbf{B} = \nabla\times\mathbf{A} & [(5.27)]\end{cases}$

$$\Rightarrow \nabla\times\nabla\times\mathbf{A} = \mu_0\mathbf{J}$$

$$\Rightarrow \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

Choose the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$)

$$\Rightarrow \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{5.31}$$

We again obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{5.32}$$

$$\boxed{\begin{aligned}&\text{cf. } \nabla^2\phi = -\frac{\rho}{\varepsilon_0} \\ &\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\end{aligned}}$$

Discussion :

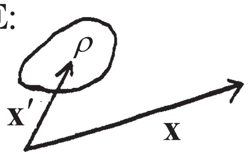
(5.32) applies to unbounded (infinite) space, i.e. the integration must include all currents, including those on boundaries. A boundary-value problem may be formulated from $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ and $\nabla\cdot\mathbf{B} = 0$ for region(s) with boundary surfaces and b.c.'s (See Sec. 5.9).

A Comparison of Electrostatics and Magnetostatics :

Electrostatics

Definition of **E**:

$$\mathbf{F}_E = q\mathbf{E}$$



Coulomb's law:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\overbrace{\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \qquad \nabla\times\mathbf{E} = 0}$$

$$\Downarrow \qquad\qquad \Downarrow$$

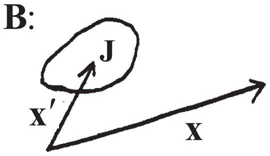
$$\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\varepsilon_0} \qquad \oint \mathbf{E}\cdot d\ell = 0$$

Gauss's law of electrostatics

Magnetostatics

Definition of **B**:

$$\mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$$



Biot-Savart law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\overbrace{\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0 \qquad \oint \mathbf{B}\cdot d\ell = \mu_0 I$$

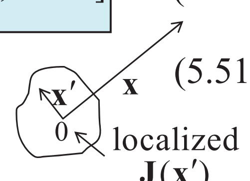
Gauss's Law of magnetism   Ampere's law

5.6 Magnetic Field of Localized Current Distribution, Magnetic Moment

Magnetic (Dipole) Moment :

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad \boxed{\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \frac{\cdots}{|\mathbf{x}|^5} + \cdots \text{ [Eq. (4), Ch. 4]}} \qquad (5.32)$$

$$= \frac{\mu_0}{4\pi}\Big[\frac{1}{|\mathbf{x}|}\underbrace{\int \mathbf{J}(\mathbf{x}')d^3x'}_{=0} + \frac{1}{|\mathbf{x}|^3}\mathbf{x}\cdot\underbrace{\int \mathbf{x}'\mathbf{J}(\mathbf{x}')d^3x'}_{=-\frac{1}{2}\int \mathbf{x}\times[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]d^3x'} + \frac{\cdots}{|\mathbf{x}|^5} + \cdots\Big] \qquad (5.51)$$



proved in Exercise 1 below

proved on p.185 under the conditions: 1. **J** is localized within volume of integration 2. $\nabla\cdot\mathbf{J} = 0$ [(5.3)]

$$= -\frac{\mu_0}{8\pi}\frac{\int \mathbf{x}\times[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]d^3x'}{|\mathbf{x}|^3} + \cdots$$

$$\approx \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \quad \left[\begin{array}{l}\text{If } \mathbf{x} \text{ is far}\\ \text{from } \mathbf{J}(\mathbf{x}').\end{array}\right] \qquad (5.55)$$

where $\mathbf{m} \equiv \frac{1}{2}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'$ [magnetic (dipole) moment] (5.54)

Note: 1. **m** is defined with respect to a *point of reference.* In (5.54), the point of reference is the origin of the coordinates ($\mathbf{x} = 0$).

2. "Localized" $\Rightarrow$ finite in size (10^{-10} cm^3 or as big as the galaxy).

Anti-symmetric unit tensor (ε_{ijk}): (used on p.185 and p.188)

$$\varepsilon_{ijk} \equiv \begin{cases} 0 & \text{, if two or more indices are equal} \\ 1 & \text{, if } i, j, k \text{ is an even permutation of } 1,2,3 \\ -1 & \text{, if } i, j, k \text{ is an odd permutation of } 1,2,3 \end{cases} \qquad (2)$$

Examples: $\varepsilon_{112} = 0$, $\varepsilon_{123} = 1$, $\varepsilon_{132} = -1$, $\varepsilon_{312} = 1$

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{jk} \varepsilon_{ijk} A_j B_k \ , \quad (\nabla\times\mathbf{A})_i = \sum_{jk} \varepsilon_{ijk} \tfrac{\partial}{\partial x_j} A_k$$

$$\begin{aligned} \nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \sum_{ijk} \varepsilon_{ijk} \tfrac{\partial}{\partial x_i}\left(A_j B_k\right) \\ &= \sum_{ijk}\left[\varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i}\right] \\ &= \sum_{ijk}\left[\varepsilon_{kij} B_k \frac{\partial A_j}{\partial x_i} - \varepsilon_{jik} A_j \frac{\partial B_k}{\partial x_i}\right] \\ &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \end{aligned}$$

Exercise 1: Prove that $\int \mathbf{J}(\mathbf{x}) d^3x = 0$ holds under 2 conditions:

(1) $\nabla\cdot\mathbf{J} = 0$ and (2) $\mathbf{J}$ is localized within volume of integration.

Proof: Since $\mathbf{J} = 0$ outside the volume of integration, we may extend the volume to ∞ without changing the integral value.

$$\int \mathbf{J}(\mathbf{x}) d^3x = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz (J_x \mathbf{e}_x + J_y \mathbf{e}_y + J_z \mathbf{e}_z)$$

Consider the x-component first: (P. 185 uses a different method)

$$\int J_x(\mathbf{x}) d^3x = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \underbrace{\int_{-\infty}^{\infty} J_x dx}$$

$$\boxed{= \cancel{x J_x \Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} x \frac{\partial J_x}{\partial x} dx}$$

$$= -\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} x \frac{\partial J_x}{\partial x} dx$$

$$= -\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} x\Big(\frac{\partial J_x}{\partial x} + \underbrace{\frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}\Big) dx = -\int x \overbrace{\nabla\cdot\mathbf{J}}^{0} d^3x = 0$$

The insertion of these 2 terms will not change the value of the integral because
$\int_{-\infty}^{\infty} (\frac{\partial J_y}{\partial y}) dy = J_y \Big|_{-\infty}^{\infty} = 0$ & $\int_{-\infty}^{\infty} (\frac{\partial J_z}{\partial z}) dz = J_z \Big|_{-\infty}^{\infty} = 0$

Similarly,

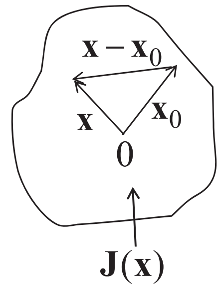
$\int J_y(\mathbf{x}) d^3x = \int J_z(\mathbf{x}) d^3x = 0$. Thus, $\int \mathbf{J}(\mathbf{x}) d^3x = 0$

Exercise 2: Prove that, under the condition $\int \mathbf{J}(\mathbf{x}) d^3x = 0$, the dipole mement $\mathbf{m}$ is independent of the point of reference.

Proof:

Let $\mathbf{m}(0)$ be the $\mathbf{m}$ defined with respect to the origin of the coordinates [as in (5.54)], i.e.

$$\mathbf{m}(0) = \tfrac{1}{2}\int \mathbf{x} \times \mathbf{J}(\mathbf{x}) d^3x$$



Let $\mathbf{m}(\mathbf{x}_0)$ be the $\mathbf{m}$ defined for the same current distribution $\mathbf{J}(\mathbf{x})$, but with respect to the point of reference at $\mathbf{x}_0$. Then,

$$\begin{aligned}\mathbf{m}(\mathbf{x}_0) &= \tfrac{1}{2}\int (\mathbf{x} - \mathbf{x}_0) \times \mathbf{J}(\mathbf{x}) d^3x \\ &= \tfrac{1}{2}\int \mathbf{x} \times \mathbf{J}(\mathbf{x}) d^3x - \tfrac{1}{2}\mathbf{x}_0 \times \overbrace{\int \mathbf{J}(\mathbf{x}) d^3x}^{0} \\ &= \mathbf{m}(0)\end{aligned}$$

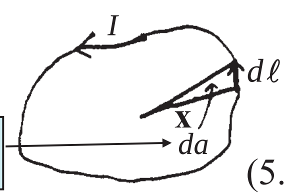
As will be shown in Sec. 5.7, it is desirable to choose a point of reference close to the center of $\mathbf{J}(\mathbf{x})$.

Example 1: Magnetic dipole moment of a *plane* loop

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x} \times \underbrace{\mathbf{J}(\mathbf{x}) d^3x} = \frac{I}{2}\oint \underbrace{\mathbf{x} \times d\boldsymbol{\ell}}_{2 \cdot da}$$

$\boxed{\mathbf{J} \cdot (\text{cross-section}) \cdot d\ell = I d\boldsymbol{\ell}}$

$\boxed{da: \text{ differential area}}$



$$\Rightarrow \begin{cases} |\mathbf{m}| = I \cdot (area) \\ \mathbf{m} \text{ is normal (by right hand rule) to the plane of the loop.} \end{cases} \quad (5.57)$$

Example 2: magnetic dipole moment of a number of charged particles in motion

$$\mathbf{J}(\mathbf{x}) = \sum_i q_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{x}_i)$$

$\boxed{\text{mechanical angular momentum } \mathbf{L}_i = M_i \mathbf{x}_i \times \mathbf{v}_i}$

$$\Rightarrow \mathbf{m} = \frac{1}{2}\int \mathbf{x} \times \mathbf{J}(\mathbf{x}) d^3x' = \frac{1}{2}\sum_i q_i \mathbf{x}_i \times \mathbf{v}_i = \sum_i \frac{q_i}{2M_i}\mathbf{L}_i \quad (5.58)$$

$\boxed{\text{if } q_i / M_i = e/M \text{ for all particles.}}$

$$= \frac{e}{2M}\mathbf{L} \quad (5.59)$$

$\boxed{\mathbf{L}\text{: total angular momentum}}$

For an orbiting electron in an atom, (5.59) relates its $\mathbf{L}$ to its $\mathbf{m}$.

Dipole Field: (due to the dipole moment $\mathbf{m}$ of a localized $\mathbf{J}$)

Rewrite $\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$ [(5.55)]

$$\boxed{\begin{aligned}\nabla\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3} &= \frac{1}{|\mathbf{x}|^3}\nabla\cdot\mathbf{x} + \mathbf{x}\cdot\nabla\frac{1}{|\mathbf{x}|^3}\\ &= \frac{3}{|\mathbf{x}|^3} - \mathbf{x}\cdot\frac{3\mathbf{x}}{|\mathbf{x}|^5} = 0\end{aligned}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\nabla\times(\mathbf{m}\times\frac{\mathbf{x}}{|\mathbf{x}|^3})$$

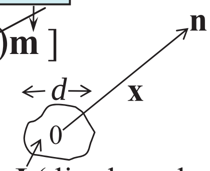
$$\boxed{\begin{aligned}\nabla\times(\mathbf{a}\times\mathbf{b}) &= \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a})\\ &\quad + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}\end{aligned}}$$

$\boxed{\because\ \mathbf{m} \text{ is a constant.}}$

$$= \frac{\mu_0}{4\pi}[-(\mathbf{m}\cdot\nabla)\frac{\mathbf{x}}{|\mathbf{x}|^3} + \cancel{\mathbf{m}\,\nabla\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3}} - \frac{\mathbf{x}}{|\mathbf{x}|^3}\cancel{\nabla\cdot\mathbf{m}} + \cancel{(\frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\nabla)\mathbf{m}}\,]$$

$$= \frac{\mu_0}{4\pi}[-m_x\frac{\partial}{\partial x}\frac{\mathbf{x}}{|\mathbf{x}|^3} - m_y\frac{\partial}{\partial y}\frac{\mathbf{x}}{|\mathbf{x}|^3} - m_z\frac{\partial}{\partial z}\frac{\mathbf{x}}{|\mathbf{x}|^3}]$$

$$= \frac{\mu_0}{4\pi}[-m_x(\frac{\mathbf{e}_x}{|\mathbf{x}|^3} - \mathbf{x}\frac{3x}{|\mathbf{x}|^5}) - (y) - (z)]$$

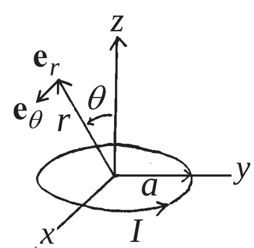


J (dipole and higher moments)

$$= \frac{\mu_0}{4\pi}\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3},\ \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad \boxed{\text{cf. } \mathbf{E}_{dipole}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}|^3}\ (4.13)} \qquad (5.56)$$

Note: (5.56) is a good approximation of the exact $\mathbf{B}$ if $|\mathbf{x}| \gg d$ [see (5.51)], implying that $\mathbf{B}$ of any localized $\mathbf{J}$ approaches a dipole field in the far zone ($|\mathbf{x}| \gg d$). However, (5.56) diverges (invalid) as $|\mathbf{x}| \to 0$.

As in the case of the electric dipole moment, here we have characterized a localized $\mathbf{J}$ by a constant quantity (magnetic moment $\mathbf{m}$), which turns an otherwise complicated calculation into a simple one, but with limited validity.



Example: A circular loop with current I

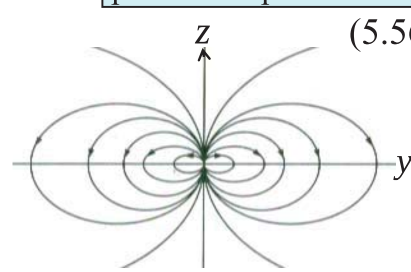
By (5.57), $\mathbf{m} = I\pi a^2\mathbf{e}_z$. The dipole field is

Unit vectors in spherical coordinate system are position-dependent.

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3} \qquad \boxed{\begin{aligned}\mathbf{n} &= \mathbf{x}/|\mathbf{x}| = \mathbf{e}_r\\ \mathbf{m} &= I\pi a^2\mathbf{e}_z\end{aligned}} \qquad (5.56)$$

$$= \frac{\mu_0}{4\pi}I\pi a^2\frac{3\mathbf{e}_r(\mathbf{e}_r\cdot\mathbf{e}_z) - \mathbf{e}_z}{r^3}$$

$$\boxed{\mathbf{e}_z = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta}$$



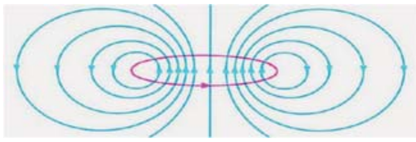
$$= \frac{\mu_0}{4\pi}\underbrace{I\pi a^2}_{M}\frac{2\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta}{r^3}\ [\text{for } r \gg a] \qquad (5.41)$$

Griffiths, p. 246
$\mathbf{m} = m\mathbf{e}_z$

Question: $\mathbf{B} \to \infty$ as $r \to 0 \Rightarrow$ The I-loop is not a pure dipole. Why?

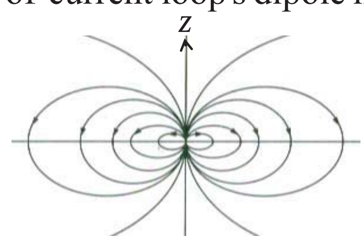
Comparison between **B** (*exact*) *and* **B** (*dipole*) *of a current loop*:

Exact **B** of a circular current loop



Benson, Fig. 30.9

B of current loop's dipole moment **m**

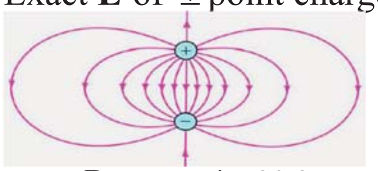


Griffiths p. 246
$\mathbf{m} = m\mathbf{e}_z$

A circular current loop is not a pure magnetic dipole. It consists of all multipole moments (including **m**). Its far field ≈ **B** of its **m**, but the near zone is dominated by fields of higher order moments.

Comparison between the exact **B** *of a current loop and the exact* **E** *of a pair of* ± *point charges*:

Exact **E** of ± point charges



Benson, Fig. 30.9

Far zone: **E** & **B** are dipole fields with same field lines in the *same* direction.
Near zone: **E** & **B** are multipole fields, much different and in *opposite* directions

5.7 Forces and Torque on and Energy of a Localized Current Distribution in an External Magnetic Induction

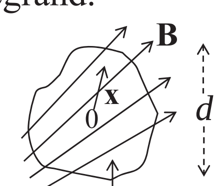
Magnetic Force on Localized J in External B:

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d^3x \tag{5.12}$$

where $\mathbf{B}(\mathbf{x})$ is due to sources other than $\mathbf{J}(\mathbf{x})$ in the integrand.

Taylor expansion about $\mathbf{x} = 0$ (Ch. 4 Appendix A):

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}(0) + (\mathbf{x}\cdot\nabla)\mathbf{B}(0) + \tfrac{1}{2}(\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla)\mathbf{B}(0) + \cdots$$



$$\textit{Note}: \begin{cases} (\mathbf{x}\cdot\nabla)\mathbf{B}(0) \Rightarrow (\mathbf{x}\cdot\nabla)\mathbf{B}(\mathbf{x})\big|_{\mathbf{x}=0} \\ (\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla)\mathbf{B}(0) \Rightarrow (\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla)\mathbf{B}(\mathbf{x})\big|_{\mathbf{x}=0} \\ \text{i.e. Differentiate } \mathbf{B}(\mathbf{x}) \text{ first, then take the value at } \mathbf{x}=0. \end{cases}$$

Question: Under what condition can the higher-order terms in the Taylor expansion be neglected?

Ans.: $(\mathbf{x}\cdot\nabla)\mathbf{B}(0) \sim d\frac{\partial B}{\partial \ell} \sim d\frac{B}{L}$, where d is the size of **J** and L is the scale length of B (i.e. the length for B to vary by ~B). Thus if $d \ll L$, the higher the order, the smaller the term is as compared with $\mathbf{B}(0)$.

Sub. $\mathbf{B}(\mathbf{x}) = \mathbf{B}(0) + (\mathbf{x}\cdot\nabla)\mathbf{B}(0) + \frac{1}{2}(\mathbf{x}\cdot\nabla)(\mathbf{x}\cdot\nabla)\mathbf{B}(0) + \cdots$

into $\mathbf{F} = \int \mathbf{J}(\mathbf{x})\times\mathbf{B}(\mathbf{x})d^3x$ [(5.12)]

$$\Rightarrow \mathbf{F} = [\underbrace{\int \mathbf{J}(\mathbf{x})d^3x}_{=0 \text{ (proved in Sec. 5.6)}}]\times\mathbf{B}(0) + \int \mathbf{J}(\mathbf{x})\times\left[(\mathbf{x}\cdot\nabla)\mathbf{B}(0)\right]d^3x + \underbrace{\cdots}_{\substack{\text{higher-order}\\ \text{terms (neglect)}}}$$

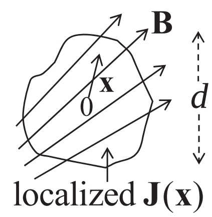
$$\approx \int \mathbf{J}(\mathbf{x})\times\left[(\mathbf{x}\cdot\nabla)\mathbf{B}(0)\right]d^3x = \nabla[\mathbf{m}\cdot\mathbf{B}(0)] \; (= \nabla[\mathbf{m}\cdot\mathbf{B}(\mathbf{x})]|_{\mathbf{x}=0})$$

$$= -\nabla U, \qquad \text{(5.69)}$$

See derivation on pp.188-189

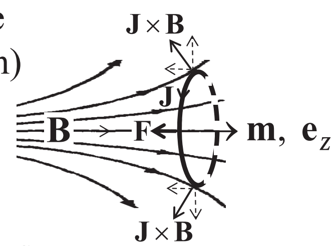
where $U = -\mathbf{m}\cdot\mathbf{B}(0) =$ potential energy [*origin*-dependent!]. (5.72)

Note: In (5.72), $\mathbf{m}$ is indep. of the choice of origin, but $\mathbf{B}(0)$ is the field at the origin. The neglect of higher-order terms in the B-field expansion requires us to choose the origin near the center of $\mathbf{J}(\mathbf{x})$ so that $\mathbf{B}(0)$ closely represents the field seen by $\mathbf{J}$ (see right fig.).



An example:

Consider a $\mathbf{B}$ symmetric about the z-axis (see figure). A circular wire (of uniform cross-section) has an axis coinciding with the z-axis. The wire carries a current $\mathbf{J}$ with $\mathbf{m}$ points along $+\mathbf{e}_z$.



In $U = -\mathbf{m}\cdot\mathbf{B}(0)$ [(5.72)], we choose $\mathbf{B}(0)$ to be the value of $\mathbf{B}$ at the center of the loop. Thus, $\mathbf{B}(0) = B(0)\mathbf{e}_z$ & $U = -mB(0) < 0$ [potential energy]. In $\mathbf{F} = -\nabla U$, $B(\mathbf{x})$ in U is operated by ∇ first, then $\mathbf{x}$ is put to 0 (see p. 189).

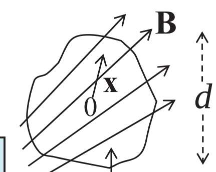
$\Rightarrow$ There is a force $\mathbf{F}$ on the wire along $-\mathbf{e}_z$, implying that the wire tends to move to the high-$\mathbf{B}$ region where U is smaller.

This can be interpreted as follows. Consider any two points on the wire which are 180^o apart. Divide the $\mathbf{J}\times\mathbf{B}$ force on each point into components $\perp$ and $\|$ to the axis: $(\mathbf{J}\times\mathbf{B})_\perp$ and $(\mathbf{J}\times\mathbf{B})_{\|}$. The $(\mathbf{J}\times\mathbf{B})_\perp$ forces at the two points cancel out, while the $(\mathbf{J}\times\mathbf{B})_{\|}$ forces add up. Thus, the net force $\mathbf{F}$ is along $-\mathbf{e}_z$.

Magnetic Torque on Localized J in External B:

$$\mathbf{N} = \int \mathbf{x} \times \mathbf{f}(\mathbf{x}) d^3x = \int \mathbf{x} \times [\mathbf{J}(\mathbf{x}) \times \underbrace{\mathbf{B}(\mathbf{x})}] d^3x \qquad (5.13)$$

$$\boxed{\mathbf{B}(0) + (\mathbf{x} \cdot \nabla)\mathbf{B}(0) + \cdots \approx \mathbf{B}(0)}$$



$$\approx \int \mathbf{x} \times [\mathbf{J}(\mathbf{x}) \times \mathbf{B}(0)] d^3x$$

$$\boxed{\nabla \cdot [|\mathbf{x}|^2 \mathbf{J}(\mathbf{x})] = \mathbf{J}(\mathbf{x}) \cdot \overbrace{\nabla |\mathbf{x}|^2}^{2\mathbf{x}} + |\mathbf{x}|^2 \overbrace{\nabla \cdot \mathbf{J}(\mathbf{x})}^{0} = 2\mathbf{x} \cdot \mathbf{J}(\mathbf{x})}$$

$$= \int [\mathbf{B}(0) \cdot \mathbf{x}] \mathbf{J}(\mathbf{x}) d^3x - \mathbf{B}(0) \int \overbrace{\mathbf{x} \cdot \mathbf{J}(\mathbf{x})} d^3x$$

$$= \underbrace{\int [\mathbf{B}(0) \cdot \mathbf{x}] \mathbf{J}(\mathbf{x}) d^3x} - \tfrac{1}{2} \mathbf{B}(0) \underbrace{\int \nabla \cdot [|\mathbf{x}|^2 \mathbf{J}(\mathbf{x})] d^3x} = \mathbf{m} \times \mathbf{B}(0) \qquad (5.71)$$

$= \underbrace{-\frac{1}{2}\int \mathbf{B}(0) \times [\mathbf{x} \times \mathbf{J}(\mathbf{x})] d^3x} = \mathbf{m} \times \mathbf{B}(0)$

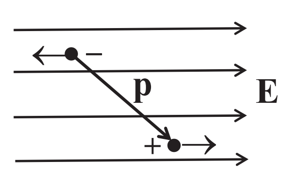
[Using the formula at the bottom of p.185, replacing $\mathbf{x}$ with $\mathbf{B}(0)$.]

$= \oint_S |\mathbf{x}|^2 \mathbf{J}(\mathbf{x}) \cdot d\mathbf{a} = 0$

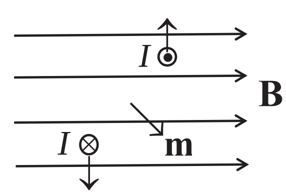
$\mathbf{J}$ is localized.
$\Rightarrow \mathbf{J} = 0$ on S

A Comparison of Electric and Magnetic Potential Energy, Force, and Torque in External Field :

Potential energy	*Force*	*Torque*
$U = -\mathbf{p} \cdot \mathbf{E}$ (4.24)	$\mathbf{F} = -\nabla U$	$\mathbf{N} = \mathbf{p} \times \mathbf{E}$
$U = -\mathbf{m} \cdot \mathbf{B}$ (5.72)	$\mathbf{F} = -\nabla U$	$\mathbf{N} = \mathbf{m} \times \mathbf{B}$



$\mathbf{N} = \mathbf{p} \times \mathbf{E}$



$\mathbf{N} = \mathbf{m} \times \mathbf{B}$

The torque due to $\mathbf{E}$ or $\mathbf{B}$ tends to orient $\mathbf{p}$ or $\mathbf{m}$ along the positive field direction (see figures). This will result in a state of minimum potential energy.

Questions:

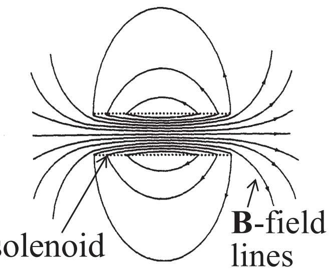
(1) Refer to the right figures. If an external torque rotates $\mathbf{p}$ or $\mathbf{m}$ clockwise, work will be done. Where does the energy go to?

(2) What is a "self-consistent" field? (See next page)

(3) How does a permanent magnet attract another permanent magnet?

Force in Self-Consistent B; Magnetic Pressure and Tension :

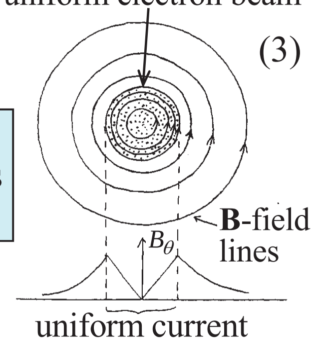
A self-consistent **B** is the field generated by the **J** under consideration plus $\mathbf{B}_{ext}$. Let $\mathbf{B}_{ext} = 0$. Using $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, we may express **f** (magnetic force per unit volume) entirely in terms of **B**.



$$\mathbf{f} \left(\frac{\text{magnetic force}}{\text{unit volume}}\right) = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\boxed{\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})}$$

$$= \underbrace{-\nabla \frac{B^2}{2\mu_0}} + \underbrace{\frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}} \quad \text{[see p. 320)]} \qquad (3)$$



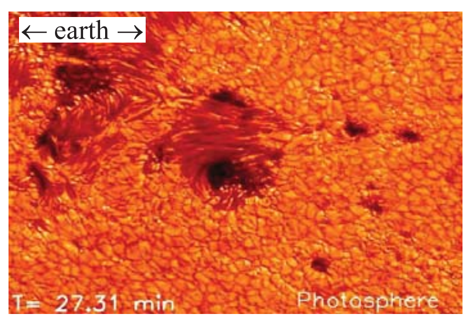
magnetic pressure force density, like $-\nabla(nkT)$ in air	magnetic tension force density, as if a curved **B**-field line tends to behaves like a rubber band.

So we have turned 2 familiar laws ($\mathbf{f} = \mathbf{J} \times \mathbf{B}$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) into 2 useful physical concepts.

Note : The 2 forces exactly cancel out in regions where $\mathbf{J} = 0$.

Question : Why will a wire collapse if it carries a sufficiently large I?

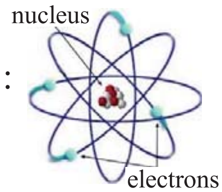
Pressure balance in the sunspot–application of magnetic pressure



Sunspots are dark, planet-sized regions on the surface of the Sun with T ≈ 3000-4500 K. They look dark because they are colder than the surrounding areas (about 5800 K).

The sun's average **B** is ~1 G. Sunspot's **B**-field is similar to the dipole field and can be as strong as 4000 G. A high **B**-field inhibits convection of the energy flux from the sun's interior, hence a lower T in the sunspot. Gas and magnetic pressures in the sunspot balance the outside pressure, resulting in a sunspot lifetime of days to weeks.

5.8 Macroscopic Equations, Boundary Conditions on B and H

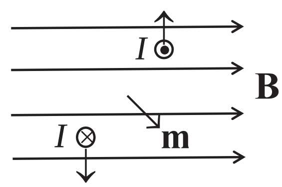


Molecular and Atomic Magnetic Dipole Moment :

Magnetic properties are complicated. We neglect the electron spin and consider only motion of electrons around the nucleus (atomic currents). Each orbiting electron forms a *microscopic* $\mathbf{m}$ in an atom (molecule). 3 cases are discussed below:

1. For most materials, these $\mathbf{m}$'s are randomly oriented with or without an external $\mathbf{B}$ $\Rightarrow$ no *macroscopic* magnetic moment.

2. For some other materials, the microscopic $\mathbf{m}$'s are randomly oriented in the absence of an external $\mathbf{B}$, but can be aligned to some degree by the torque ($\mathbf{N}$) of an external $\mathbf{B}$ toward the direction of $\mathbf{B}$ (see figure). A net molecular (atomic) magnetic moment is thus *induced*.



$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

3. There are also materials whose microscopic $\mathbf{m}$'s naturally form a *permanent* macroscopic magnetic moment without an external $\mathbf{B}$.

Magnetization:

The magnetization is the sum of all molecular (atomic) magnetic moments in a unit volume. It is defined as

$$\mathbf{M}(\mathbf{x}) = \sum_i N_i \langle \mathbf{m}_i \rangle \tag{5.76}$$

volume density of type i molecules

$\langle \mathbf{m}_i \rangle$: magnetic moment per type i molecule. We take the average over a small volume just in case each $\mathbf{m}_i$ is different (e.g. in a permanent magnet).

Note:

(1) Within a molecule, we have $\int \mathbf{J}(\mathbf{x}')d^3x' = 0$; hence, $\mathbf{m}_i$ & $\mathbf{M}$ are indep. of the reference point (proved earlier in Problem 2).

(2) As the definition of $\mathbf{P}$ in Ch. 4, (5.76) defines a *macroscopic* $\mathbf{M}$ in terms of *microscopic* $\mathbf{m}$'s ($\Rightarrow$ no divergent field by $\mathbf{M}$).

Since $\mathbf{M}$ is due to atomic electrons in orbital motion, a current density $\mathbf{J}_M$ may arise from $\mathbf{M}$ [as will be shown in (5.79) below].

The motion of free charges will also result in a current density, which we denote by $\mathbf{J}_{free}$ (Jackson denotes it by $\mathbf{J}$ in Sec. 5.8).

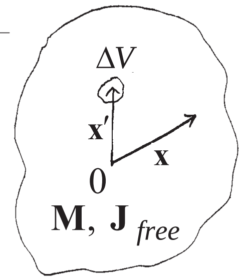
Macroscopic Equations :

Consider a medium with $\mathbf{M}$ (due to atomic currents) and $\mathbf{J}_{free}$ (due to free charges). We treat $\mathbf{M}$ (hence $\mathbf{J}_M$) and $\mathbf{J}_{free}$ separately. Write the vector potential $\Delta\mathbf{A}(\mathbf{x})$ due to an infinistesimal volume ΔV as

$$\Delta\mathbf{A}(\mathbf{x}) = \Delta\mathbf{A}_{free}(\mathbf{x}) + \Delta\mathbf{A}_M(\mathbf{x}),$$

where $\Delta\mathbf{A}_{free}(\mathbf{x})$ is due to $\mathbf{J}_{free}(\mathbf{x}')\Delta V$ and $\Delta\mathbf{A}_M(\mathbf{x})$ due to $\mathbf{M}(\mathbf{x}')\Delta V$.

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \Rightarrow \Delta\mathbf{A}_{free}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_{free}(\mathbf{x}')\Delta V}{|\mathbf{x}-\mathbf{x}'|}$$



For $\Delta\mathbf{A}_M(\mathbf{x})$, we do not yet have an expression for $\mathbf{J}_M$. So we use the dipole approx. [(5.55)]:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\overbrace{\mathbf{m}}^{\mathbf{m}\text{ at }\mathbf{x}=0}\times\mathbf{x}}{|\mathbf{x}|^3} \Rightarrow \Delta\mathbf{A}_M(\mathbf{x}) \approx \frac{\mu_0}{4\pi}\frac{\overbrace{\mathbf{M}(\mathbf{x}')\Delta V}^{\text{total }\mathbf{m}\text{ in }\Delta V\text{ at }\mathbf{x}=\mathbf{x}'}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\Rightarrow \Delta\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{\mathbf{J}_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]\Delta V$$

Question: $\Delta\mathbf{A}$ does not diverge as $\mathbf{x}\to\mathbf{x}'$. Why?

Rewrite $\Delta\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\frac{\mathbf{J}_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]\Delta V$. Let $\Delta V \to d^3x'$ and integrate over *all space* (p.192)

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{A}_{free}(\mathbf{x}) + \mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]d^3x',$$

where
$$\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$
$$= \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{x}')\times\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$
$$= \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \frac{\mu_0}{4\pi}\underbrace{\int \nabla'\times\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{=\oint_S \mathbf{n}\times\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da = 0}$$

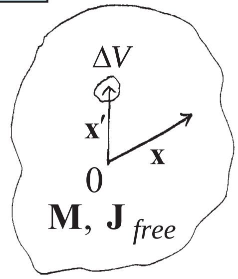
integration over all space

$\mathbf{a}\times\nabla\psi = \psi\nabla\times\mathbf{a} - \nabla\times(\psi\mathbf{a})$

($\mathbf{M} = 0$ on S)

$\int_V \nabla\times\mathbf{A}d^3x = \oint_S \mathbf{n}\times\mathbf{A}da$

$$= \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$



$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{A}_{free}(\mathbf{x}) + \mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_{free}(\mathbf{x}')+\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad (5.78)$$

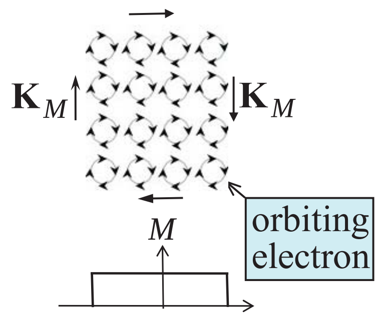
Rewrite $\mathbf{A}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\mathbf{J}_{free}(\mathbf{x}')+\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ [(5.78)]

In magnetostatics, only the electrical current $\mathbf{J}$ can produce $\mathbf{A}$ (or $\mathbf{B} = \nabla\times\mathbf{A}$). The equal footing of $\mathbf{J}_{free}$ and $\nabla\times\mathbf{M}$ in (5.78) suggests that $\nabla\times\mathbf{M}(\mathbf{x})$ must be a macroscopic $\mathbf{J}$ due to the orbital motion of atomic electrons.

Thus, we define a magnetization current density ($\mathbf{J}_M$) by

$$\mathbf{J}_M = \nabla\times\mathbf{M} \qquad (5.79)$$

Alignment (usually slightly) of the microscopic $\mathbf{m}$'s of orbiting electrons along a certain direction will result in a mascroscopic $\mathbf{M}$. The figure illustrates perfectly aligned $\mathbf{m}$'s (exaggerated!) with a uniform $\mathbf{M}$ (induced or permanent). A surface current ($\mathbf{K}_M$) is formed due to the abrupt dropoff of $\mathbf{M}$ on the boundary.



In the differentail form of Ampere's law: $\nabla\times\mathbf{B}(\mathbf{x}) = \mu_0\mathbf{J}(\mathbf{x})$ [(5.22)], separate free and atomic currents $\Rightarrow \nabla\times\mathbf{B} = \mu_0(\mathbf{J}_{free} + \nabla\times\mathbf{M})$ (5.80)

Defining a new quantity $\mathbf{H}$ (called the magnetic field)

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \quad \left[\begin{array}{l}\Rightarrow \text{Effects of atomic currents} \\ \text{are implicit in the } \mathbf{M} \text{ term in } \mathbf{H}.\end{array}\right] \qquad (5.81)$$

$\Rightarrow \quad \nabla\times\mathbf{H} = \mathbf{J}_{free}$ [macroscopic version of $\nabla\times\mathbf{B}(\mathbf{x}) = \mu_0\mathbf{J}(\mathbf{x})$] (5.82)

Question: Does $\mathbf{H}$ have a physical meaning?

Ans.: p. 193: "The fundamental fields are $\mathbf{E}$ and $\mathbf{B}$."; "The derived fields $\mathbf{D}$ and $\mathbf{H}$ are introduced as a matter of convenience."

Diamagnetic, Paramagnetic, and Ferromagnetic Substances :

The counterpart of (5.81) in electrostatics is $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ [(4.34)]. For small displacement of the bound electrons, we have the linear

relations: $\begin{cases}\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} & (4.36) \\ \mathbf{D} = \varepsilon\mathbf{E}, \text{ with } \varepsilon = \varepsilon_0(1+\chi_e) & (4.37), (4.38)\end{cases}$

For magnetic materials, $\mathbf{M}$ and $\mathbf{B}$ do not always have a linear relation. Possible relations between $\mathbf{B}$ and $\mathbf{H}$ are summarized below:

1. For diamagnetic and paramagnetic substances, $\mathbf{M}$ is proportional to $\mathbf{B}$ and we express the *linear* relation as

$$\mathbf{M} = \frac{\mu-\mu_0}{\mu\mu_0}\mathbf{B} \quad \begin{bmatrix} \mu > \mu_0 \Rightarrow \mathbf{M} \uparrow\uparrow \mathbf{B}, & \text{paramagnetic} \\ \mu < \mu_0 \Rightarrow \mathbf{M} \uparrow\downarrow \mathbf{B}, & \text{diamagnetic} \end{bmatrix} \tag{5}$$

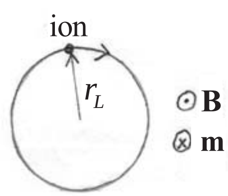
Substituting $\mathbf{M}$ into $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$ [(5.81)], we get the linear relation: $\quad \mathbf{B} = \mu\mathbf{H}$ [μ : magnetic permeability] (5.84)

Material	μ / μ_0
Gold	$1-3.4\times10^{-5}$
Copper	$1-9.7\times10^{-6}$
Iron (commercial 99Fe)	200 to 6000
Mu-metal (77Ni-16Fe-5Cu-2Cr)	20000 to 100000
Iron (pure 99.9Fe)	25000 to 350000

Question 1: How does a permanent magnet attract a piece of iron?

Question 2: The plasma (a gas of ions and electrons) is diamagnetic. Why?

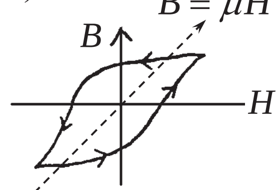
Cylclotron motion of an ion in a constant $\mathbf{B}$.



$\Rightarrow \mathbf{m} \uparrow\downarrow \mathbf{B}$

2. For the ferromagnetic substance, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$ [(5.81)] is still valid, but $\mathbf{B}$ and $\mathbf{H}$ have a *nonlinear* relation (solid line in the figure): $\quad \mathbf{B} = \mathbf{F}(\mathbf{H})$ (5.85)

$B = \mu H$ (linear relation)



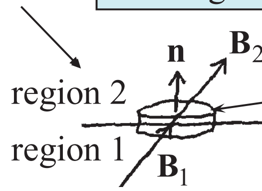
which also exhibits the hysteresis phenomenon (as shown by the solid lines above), i.e. $\mathbf{B}$ is not a single-valued function of $\mathbf{H}$.

Boundary Conditions :

(i) $\nabla\cdot\mathbf{B}=0 \Rightarrow \int_V \nabla\cdot\mathbf{B}d^3x=\oint_S \mathbf{B}\cdot d\mathbf{a}=0$

$\Rightarrow (\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n} \Rightarrow B_{\perp 1}=B_{\perp 2}$ (5.86)

$\mathbf{n}$: unit normal pointing from region 1 into region 2



(ii) $\nabla\times\mathbf{H}=\mathbf{J}_{free}$

$\Rightarrow \int\nabla\times\mathbf{H}\cdot\mathbf{n}'da=\int\mathbf{J}_{free}\cdot\mathbf{n}'da$

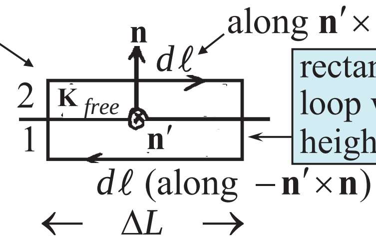
Stokes's thm. ($\mathbf{n}'$ & $d\ell$ follow right hand rule)

LHS $=\oint\mathbf{H}\cdot d\ell$ see right figure

$=(\mathbf{H}_2-\mathbf{H}_1)\cdot\overbrace{(\mathbf{n}'\times\mathbf{n})}\Delta L$

$=\mathbf{n}'\cdot[\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)]\Delta L$

$\boxed{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}$



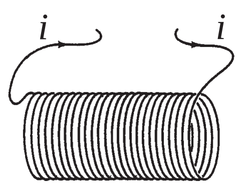
RHS $=\int\mathbf{J}_{free}\cdot\mathbf{n}'da=\mathbf{K}_{free}\cdot\mathbf{n}'\Delta L$

$\Rightarrow \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}_{free}$ (5.87)

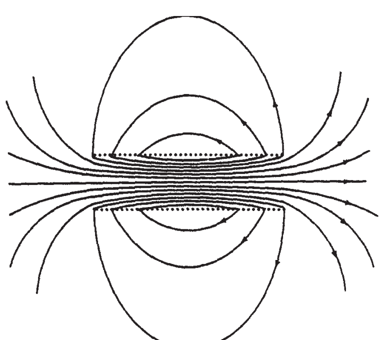
$\mathbf{K}_{free}$: surface current of *free* charges (unit: A/m)

Special case: $\mathbf{K}_{free}=0 \Rightarrow \mathbf{H}_{t2}=\mathbf{H}_{t1}$ t : tangential to surface (6)

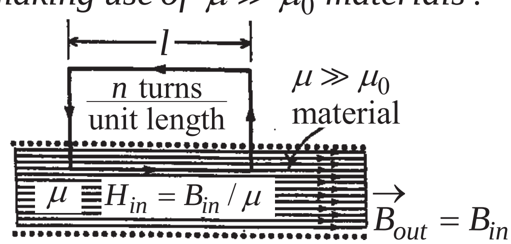
Producing a large B by making use of $\mu\gg\mu_0$ *materials* :



A solenoid.



B-field lines



Approximate the field by that of an infinite solenoid. So, H_{in}= constant.

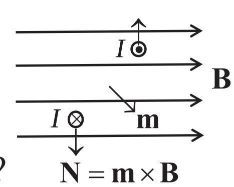
$\oint\mathbf{H}\cdot d\mathbf{l}=I_{free} \Rightarrow H_{in}l=nil$

$\Rightarrow H_{in}=ni \Rightarrow B_{in}=\mu H_{in}=\mu ni$

$B_\perp$ continuous $\Rightarrow B_{out}=B_{in}=\mu ni$

Question : $B_{out}=\mu ni$

$\Rightarrow$ Filling the solenoid core with $\mu\gg\mu_0$ material can greatly enhance B_{out} at the same i. Why?



5.9 Methods of Solving Boundary-Value Problems in Magnetostatics

Basic eqs. for $\mu \neq \mu_0 : \nabla \cdot \mathbf{B} = 0;\ \nabla \times \mathbf{H} = \mathbf{J}_{free}$ [(5.90)]. Unlike $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{free}$ [(5.22)], (5.90) can produce $\mathbf{B}$ even if $\mathbf{J}_{free} = 0$.

$\Rightarrow$ We may put (5.90) in forms for 2 types of boundary-value probs.

Type 1 : Linear medium with $\mu = const$ (in each region).

(a) Equation for vector potential $\mathbf{A}$ (with or without $\mathbf{J}_{free}$)

$$\mathbf{B} = \mu \mathbf{H} = \nabla \times \mathbf{A} \ \Rightarrow\ \mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}$$

use Coulomb gauge

$$\Rightarrow \nabla \times \mathbf{H} = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = \frac{1}{\mu}[\nabla(\cancel{\nabla \cdot \mathbf{A}}) - \nabla^2 \mathbf{A}] = \mathbf{J}_{free}$$

$$\Rightarrow \nabla^2 \mathbf{A} = -\mu \mathbf{J}_{free} \quad \text{[for any region of interest with b.c.'s]} \qquad (7)$$

(b) Equation for scalar potential (only for $\mathbf{J}_{free} = 0$)

$$\nabla \cdot \mathbf{B} = 0 \ \Rightarrow\ \mu \nabla \cdot \mathbf{H} = 0 \text{ and } \nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla \phi_M \qquad (5.93)$$

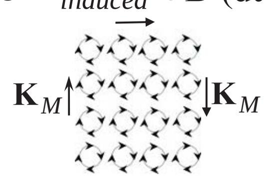
$$\Rightarrow \nabla^2 \phi_M = 0 \quad \text{[for any region of interest with b.c.'s]} \qquad (8)$$

Typically, we use (7) or (8) to solve for $\mathbf{A}$ or ϕ_M in each uniform region and find the coefficients by applying b.c.'s (5.86) and (5.87).

A comparison :

$\otimes \mathbf{B}_{ext}$ $\otimes \mathbf{M}_{induced}$ $\otimes \mathbf{B}$ (due to $\mathbf{M}$)

$$\begin{cases} \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}_{free} \ \text{[vacuum medium]} & (5.31) \\ \nabla^2 \mathbf{A} = -\mu \mathbf{J}_{free} \ \begin{bmatrix} \text{medium with} \\ \text{uniform } \mu \end{bmatrix} & (7) \end{cases}$$



$\mu > \mu_0$ medium

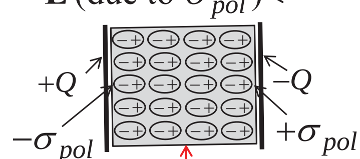
$\Rightarrow$ In a $\mu > \mu_0$ medium, the molecular magnetic dipoles tend to *increase* the ability of $\mathbf{J}_{free}$ to produce $\mathbf{B}$ by a factor of μ / μ_0 (upper figure).

In electrostatics, we have

$\mathbf{E}$ (due to Q) $\longrightarrow$

$\mathbf{E}$ (due to σ_{pol}) $\leftarrow$

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho_{free}}{\varepsilon_0} \ \text{[vacuum medium]} & (1.13) \\ \nabla \cdot \mathbf{E} = \dfrac{\rho_{free}}{\varepsilon} \ \begin{bmatrix} \text{dielectric} \\ \text{medium with} \\ \text{uniform } \varepsilon \end{bmatrix} & (4.39) \end{cases}$$

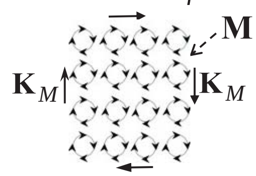


$\varepsilon > \varepsilon_0$ medium

$\Rightarrow$ In an $\varepsilon > \varepsilon_0$ medium, the molecular electric dipoles tend to *reduce* the ability of ρ_{free} to produce $\mathbf{E}$ by a factor of $\varepsilon / \varepsilon_0$ (lower figure).

Type 2 : Hard ferromagnets (permanent magnet) : $\mathbf{B}_{ext}=0, \mathbf{J}_{free}=0$

(a) Scalar potential



$$\nabla\times\mathbf{H}=\mathbf{J}_{free}=0 \Rightarrow \mathbf{H}=-\nabla\phi_M$$

$$\nabla\cdot\mathbf{B}=\mu_0\nabla\cdot(\mathbf{H}+\mathbf{M})=0 \quad [\text{Use } \mathbf{H}\equiv\tfrac{1}{\mu_0}\mathbf{B}-\mathbf{M}]$$

$$\Rightarrow \nabla^2\phi_M=\nabla\cdot\mathbf{M}=-\rho_M \quad [\text{for any region of interest with b.c.'s}] \qquad (5.95)$$

where $\rho_M\equiv-\nabla\cdot\mathbf{M}$ [ρ_M : a math. tool, not magnetic charge] (5.96)

$$\Rightarrow \phi_M=\frac{1}{4\pi}\int\frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-\frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad [\text{for all space}] \qquad (5.97)$$

$\psi\nabla'\cdot\mathbf{a}=\nabla'\cdot(\psi\mathbf{a})-\mathbf{a}\cdot\nabla'\psi$ with $\psi=1/|\mathbf{x}-\mathbf{x}'|$ & $\mathbf{a}=\mathbf{M}$
Apply divergence thm. to the $\nabla\cdot(\psi\mathbf{a})$ term (will get 0)

$$=\frac{1}{4\pi}\int\mathbf{M}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-\frac{1}{4\pi}\int\mathbf{M}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

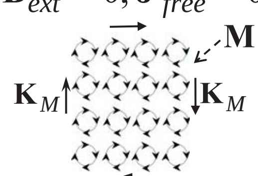
$\mathbf{a}\cdot\nabla\psi=\nabla\cdot(\psi\mathbf{a})-\psi\nabla\cdot\mathbf{a}$ with $\mathbf{a}=\mathbf{M}$ & $\psi=1/|\mathbf{x}-\mathbf{x}'|$
$\nabla\cdot\mathbf{M}(\mathbf{x}')=0$ ($\because\nabla$ operates on $\mathbf{x}'$)

$$=-\frac{1}{4\pi}\int\nabla\cdot\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-\frac{1}{4\pi}\nabla\cdot\int\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \qquad (5.98)$$

(b) Vector potential (for permanent magnet)

$$\nabla\times\mathbf{H}=\nabla\times(\frac{\mathbf{B}}{\mu_0}-\mathbf{M})=\mathbf{J}_{free}=0$$

$\mathbf{B}_{ext}=0, \mathbf{J}_{free}=0$



$\mathbf{H}\equiv\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}$ [(5.81)]

real current in permanent magnet

$$\Rightarrow \nabla\times\mathbf{B}=\mu_0\nabla\times\mathbf{M}=\mu_0\mathbf{J}_M,$$

where $\mathbf{J}_M\equiv\nabla\times\mathbf{M}$ [(5.79)]

$$\mathbf{B}=\nabla\times\mathbf{A}\Rightarrow\nabla\times\mathbf{B}=\nabla\times\nabla\times\mathbf{A}=\nabla(\overbrace{\cancel{\nabla\cdot\mathbf{A}}}^{\text{Use Coulomb gauge}})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}_M$$

$$\Rightarrow \nabla^2\mathbf{A}=-\mu_0\mathbf{J}_M \quad [\text{for any region of interest with b.c.'s}]$$

$$\Rightarrow \mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad [\text{for all space}] \qquad (5.102)$$

Note : (5.102) can be directly obtained from (5.78) by letting $\mathbf{J}_{free}=0$ in (5.78).

b.c. involving effective magnetic surface charge density σ_M:

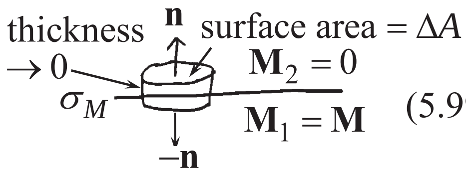
Rewrite $\nabla\cdot\mathbf{M}=-\rho_M$ [(5.96)]

$$\Rightarrow \int_v \nabla\cdot\mathbf{M}d^3x=\oint_S \mathbf{M}\cdot d\mathbf{a}=-\int_v \rho_M d^3x \text{ (see pillbox below)}$$

$$\Rightarrow (\overbrace{\mathbf{M}_2}^{=0}-\overbrace{\mathbf{M}_1}^{=\mathbf{M}})\cdot\mathbf{n}\Delta A=-\sigma_M\Delta A$$

$$\Rightarrow \sigma_M=\mathbf{n}\cdot\mathbf{M} \quad (5.99)$$

$\boxed{\text{a mathematical tool}}$



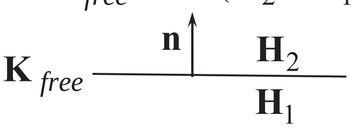
b.c. involving surface current density $\mathbf{K}_M$ *due to magnetization* $\mathbf{M}$:

In Sec. 5.8,

$\nabla\times\mathbf{H}=\mathbf{J}_{free}$

$\boxed{\text{real current}}$

$\Rightarrow \mathbf{K}_{free}=\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)$

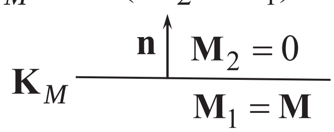


Here (by the same algebra),

$\nabla\times\mathbf{M}=\mathbf{J}_M$

$\boxed{\text{real current}}$

$\Rightarrow \mathbf{K}_M=\mathbf{n}\times(\overbrace{\mathbf{M}_2}^{=0}-\overbrace{\mathbf{M}_1}^{=\mathbf{M}})=\mathbf{M}\times\mathbf{n}$



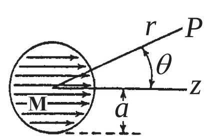
Note: There can be $\mathbf{B}$ even if $\mathbf{J}_{free}=0\Rightarrow$ many forms of eqs. & b.c.'s.

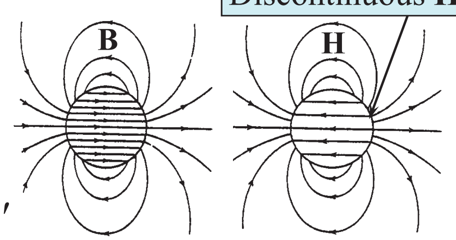
5.10 Uniformly Magnetized Sphere

Consider a permanent magnet with magnetization:

$$\mathbf{M}=\begin{cases} M_0\mathbf{e}_z\ , & r\le a\\ 0\ , & r>a\end{cases}$$

$\boxed{\text{Discontinuous } \mathbf{H}}$





$$\boxed{\begin{array}{l}\rho_M=-\nabla\cdot\mathbf{M}=0,\ r<a;\\ \sigma_M=\mathbf{n}\cdot\mathbf{M},\ r=a\end{array}}$$

$$\phi_M \underset{\uparrow\ \boxed{(5.97)}}{=} \frac{1}{4\pi}\int\frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{1}{4\pi}\oint_S\frac{\mathbf{n}\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da'$$

$$=\frac{M_0a^2}{4\pi}\int d\Omega'\frac{\overbrace{\cos\theta'}^{=\sqrt{\frac{4\pi}{3}}Y_{10}(\theta',\varphi')}}{|\mathbf{x}-\mathbf{x}'|}\underset{\text{use (3.70)}}{=}\frac{1}{3}M_0a^2\frac{r_<}{r_>^2}\cos\theta$$

$$=\begin{cases}\frac{1}{3}M_0r\cos\theta=\frac{1}{3}M_0z, & r\le a\\ \frac{1}{3}M_0a^3\frac{\cos\theta}{r^2}, & r>a\end{cases}$$

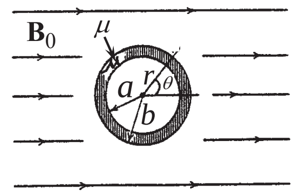
$$\boxed{\begin{array}{l}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\overset{(3.70)}{=}\frac{1}{r_>}+\frac{4\pi}{3}\frac{r_<}{r_>^2}\\ \cdot[Y_{1,-1}^*(\theta',\varphi')Y_{1,-1}(\theta,\varphi)\\ +Y_{10}^*(\theta',\varphi')\underbrace{Y_{10}(\theta,\varphi)}_{=\sqrt{\frac{3}{4\pi}}\cos\theta}\\ +Y_{11}^*(\theta',\varphi')Y_{11}(\theta,\varphi)]+\cdots\end{array}} \quad (5.104)$$

$$\Rightarrow\begin{cases}\mathbf{H}_{in}=-\nabla\phi_M(r\le a)=-\frac{1}{3}\mathbf{M}\\ \mathbf{B}_{in}=\mu_0\mathbf{H}_{in}+\mu_0\mathbf{M}=\frac{2}{3}\mu_0\mathbf{M}\ (\Rightarrow \mathbf{H}_{in}\uparrow\downarrow\mathbf{B}_{in})\end{cases} \quad (5.105)$$

$\mathbf{H}_{out}=-\nabla\phi_M(r>a)\Rightarrow$ *pure* dipole field with $\mathbf{m}=\frac{4\pi a^3}{3}\mathbf{M}$ [see (5.41)]

5.12 Magnetic Shielding, Spherical Shell of Permeable Material in a Uniform Field

Find $\mathbf{B}$ of a spherical μ-shell in an external $\mathbf{B}_0$.



$$\underbrace{\nabla^2 \phi_M = 0}_{(8)} \Rightarrow \phi_M = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

$$\Rightarrow \phi_M = \begin{cases} -H_0 r\cos\theta + \sum\limits_{l=0}^{\infty} \dfrac{\alpha_l}{r^{l+1}} P_l(\cos\theta)\ , \ r > b & (5.117) \\ \sum\limits_{l=0}^{\infty} (\beta_l r^l + \gamma_l \dfrac{1}{r^{l+1}}) P_l(\cos\theta), \quad a < r < b & (5.118) \\ \sum\limits_{l=0}^{\infty} \delta_l r^l P_l(\cos\theta), \qquad r < a & (5.119) \end{cases}$$

$-H_0 r\cos\theta$ gives the external $\mathbf{B}_0$.

boundary conditions

$$\begin{Bmatrix} \mathbf{H} = -\nabla\phi_M \ (5.93) \\ \mathbf{B} = \mu_0 \mathbf{H} \text{ (outside)} \\ \mathbf{B} = \mu \mathbf{H} \text{ (inside)} \end{Bmatrix} + \begin{Bmatrix} \mathbf{H}_{t2} = \mathbf{H}_{t1} \quad (6) \\ B_{\perp 1} = B_{\perp 2} \ (5.86) \end{Bmatrix} \Rightarrow \begin{cases} \left.\dfrac{\partial\phi_M}{\partial\theta}\right|_{b^+} = \left.\dfrac{\partial\phi_M}{\partial\theta}\right|_{b^-} \\ \left.\dfrac{\partial\phi_M}{\partial\theta}\right|_{a^+} = \left.\dfrac{\partial\phi_M}{\partial\theta}\right|_{a^-} \\ \mu_0 \left.\dfrac{\partial\phi_M}{\partial r}\right|_{b^+} = \mu \left.\dfrac{\partial\phi_M}{\partial r}\right|_{b^-} \\ \mu \left.\dfrac{\partial\phi_M}{\partial r}\right|_{a^+} = \mu_0 \left.\dfrac{\partial\phi_M}{\partial r}\right|_{a^-} \end{cases}$$

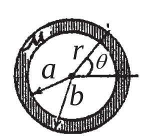
The shell is assumed to be a linear medium.

Pay atention to physics (next page) rather than coefficients evaluation.

$$\text{b.c.'s} \Rightarrow \begin{cases} \alpha_l = \beta_l = \gamma_l = \delta_l = 0 \text{ if } l \neq 1 \\ \alpha_1 = \dfrac{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}-1)(b^3-a^3)}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 \\ \delta_1 = \dfrac{-9\mu/\mu_0}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 \end{cases} \quad (5.121)$$

Neglect the field in region $a < r < b$

$$\Rightarrow \phi_M = \begin{cases} -H_0 r\cos\theta + \dfrac{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}-1)(b^3-a^3)}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 \dfrac{\cos\theta}{r^2}, & r > b \\ -9\dfrac{\mu}{\mu_0} \dfrac{1}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 r\cos\theta, & r < a \end{cases}$$



$$\Rightarrow \mathbf{H} = -\nabla\phi_M = \begin{cases} H_0\mathbf{e}_z + \text{a pure dipole field}, & r > b \\ 9\dfrac{\mu}{\mu_0} \dfrac{1}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0\mathbf{e}_z \text{ [uniform]}, & r < a \end{cases}$$

$$= H_0\mathbf{e}_z \text{ everywhere if } \mu = \mu_0 \text{ (as expected)}$$

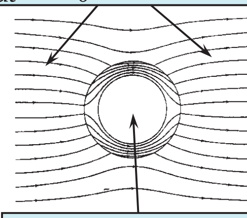
Limiting case 1:

$$\mu \gg \mu_0 \Rightarrow \ \phi_M \approx \begin{cases} -H_0 r\cos\theta + b^3 H_0 \dfrac{\cos\theta}{r^2}, & r > b \\ -\dfrac{9\mu_0}{2\mu(1-\frac{a^3}{b^3})} H_0 r\cos\theta, & r < a \end{cases} \qquad (5.122)$$

See (5.41)

$$\Rightarrow \mathbf{H} = -\nabla\phi_M \overset{\boxed{\mu \gg \mu_0}}{=} \begin{cases} H_0\mathbf{e}_z + \text{dipole field due to } \mathbf{m} = 4\pi b^3 H_0 \mathbf{e}_z, & r > b \\ \dfrac{9\mu_0}{2\mu(1-\frac{a^3}{b^3})} H_0 \mathbf{e}_z \ \text{[uniform]}, & r < a \end{cases}$$

$\mathbf{B}_{out} = \mathbf{B}_0 +$ dipole field



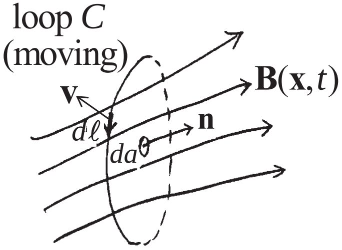
$\mathbf{B}_{in}$: uniform field ($\propto 1/\mu$ if $\mu \gg \mu_0$)

$\mathbf{B}_{in} \searrow$ as $\frac{\mu}{\mu_0} \nearrow$, implying $\mu > \mu_0$ materials tend to "absorb" B-field lines and thereby provide a *shielding* effect. High-μ materials can have μ/μ_0 as high as $10^3 - 10^6$.

Limiting case 2: $\mu = \mu_0$. $\mathbf{B} = \mathbf{B}_0$ everywhere (previous page), i.e. a static $\mathbf{B}$ penetrates into the shell (even a good conductor) as if there were no shell. *Note*: If $\partial\mathbf{B}/\partial t \neq 0$, the conductor's response will be far different.

5.15 Faraday's Law of Induction

Ampere's law links $\mathbf{B}$ & I. Faraday in 1831 discovered that a time-varying magnetic flux through an electrical circuit could induce an E-field around the circuit. This not only links $\mathbf{B}$ & $\mathbf{E}$, but also gives a new way to generate $\mathbf{E}$: a time-varying $\mathbf{B}$.



$d\ell$: infinitesimal length moving at $\mathbf{v}$ in lab frame

Let C be a closed *moving* loop, which can be an electrical circuit (as in the original experiments) or an imaginary loop in space (a far-reaching generalization). Different parts of C can move at different velocities. Let S be an arbitrary surface bounded by C.

Faraday's law (for a moving loop) states

$$\underbrace{\oint_C \mathbf{E}' \cdot d\ell}_{\text{electromotive force}} = -\frac{d}{dt}\underbrace{\int_S \mathbf{B}\cdot\mathbf{n}da}_{\text{magnetic flux through the loop}} \quad \text{[Faraday's law for a moving loop]}, \quad (5.141)$$

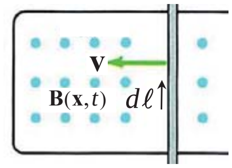
where $\mathbf{E}'$ at $d\ell$ is the electric field as viewed in the frame moving with the velocity ($\mathbf{v}$) of $d\ell$, and $\mathbf{B}$ is always viewed in the lab frame.

Note: In Jackson (5.135), (5.136), (5.138)-(5.140), $k = 1$ in SI units.

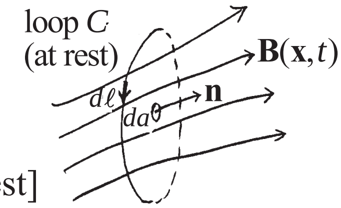
Rewrite $\oint_C \mathbf{E}'\cdot d\ell = -\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da$ [$\mathbf{E}'$: viewed in $d\ell$ frame], (5.141)

where $\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da = \frac{d}{dt}$(magnetic flux) is due to

$$\begin{cases}\text{time variation of } \mathbf{B}\text{, and / or} \\ \text{time variation of loop area}\end{cases} \begin{bmatrix}\text{upper} \\ \text{figure}\end{bmatrix}$$



If loop C (real circuit or imaginary loop) is *at rest* in the lab frame (lower figure), then in



$$(5.141), \begin{cases}\mathbf{E}' = \mathbf{E} \text{ (}\mathbf{E}\text{: viewed in lab frame)} \\ \frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}da = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da \text{ [for loop at rest]}\end{cases}$$

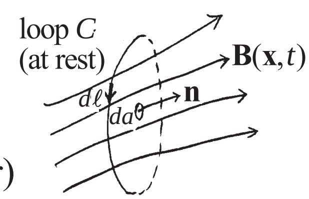
Thus, (5.141) $\Rightarrow \oint_C \mathbf{E}\cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da$ $\begin{bmatrix}\text{Faraday's law for} \\ \text{a loop at rest}\end{bmatrix}$ (5.139)

Stokes's thm.: $\oint_C \mathbf{E}\cdot d\ell = \int_S (\nabla\times\mathbf{E})\cdot\mathbf{n}da$ $\begin{bmatrix}\text{Directions of } d\ell \text{ and } \mathbf{n} \\ \text{follow right-hand rule.}\end{bmatrix}$

So (5.139) $\Rightarrow \int_S (\nabla\times\mathbf{E})\cdot\mathbf{n}da = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da$

$\Rightarrow \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ [Faraday's law in differential form] (5.143)

Questions: Rewrite $\oint_C \mathbf{E}\cdot d\ell = -\underbrace{\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}da}_{\boxed{\frac{d}{dt}\text{(mag. flux)}}}$ [loop at rest] [(5.139)]



(1) Can (5.139) determine the distribution of $\mathbf{E}$ in the loop?

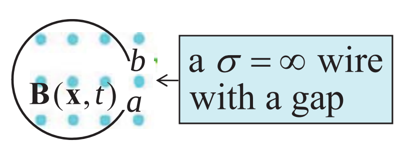
(2) If loop C is a closed wire with a resistor in it and $\sigma = \infty$ elsewhere, where is $\mathbf{E}$? (the resistor)

(3) Is $\oint_C \mathbf{E}\cdot d\ell$ entirely due to variation of an *external* magnetic flux?

Ans: No. Mag. flux variation due to loop corrent also affects it.

(4) If loop C is a closed wire with $\sigma = \infty$, then $\mathbf{E} = 0$ everywhere in the wire (for any $\mathbf{J}$), i.e. $\oint_C \mathbf{E}\cdot d\ell = 0$. By (5.139), the magnetic flux through the loop is a constant even if a time-varying $\mathbf{B}$ is externally applied. How is this possible? [See (3)]

(5) Consider a $\sigma = \infty$ wire with a gap with end points a and b (right figure). There is a time-varying $\mathbf{B}$ as shown in the fig. Unlike in a static $\mathbf{E}$, Here, $\int_a^b \mathbf{E}\cdot d\ell$ depends on the path from a to b. Why?

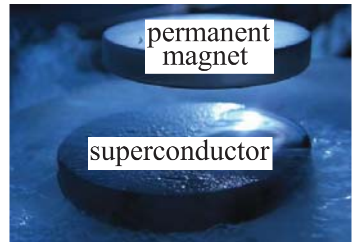


5.15 Faraday's Law of Induction (*continued*)

(6) What is Lenz's law (a consequence of Faraday's law)?

Ans: The direction of the current induced in a conductor by a changing B-field is such that the induced current produce a B-field to oppose the change.

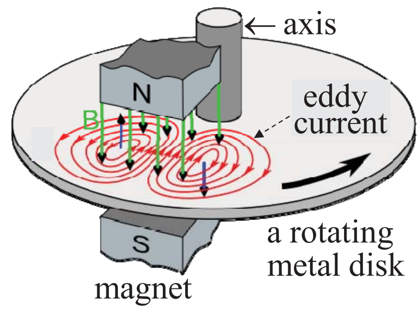
(7) The picture to the right shows a superconductor levitating a permanent magnet. Explain the mechanism in terms of Lenz's law (or Faraday's law) and the magnetic pressure.



Jubobroff sur Wikipedia

(8) What are eddy currents?

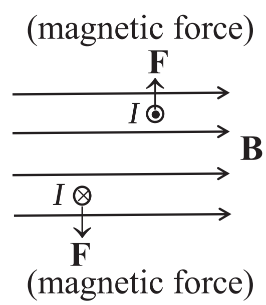
Ans: When a conductor sees a changing B-field in its frame, eddy currents are induced on the conductor to oppose the change (by Lenz's law). They flow in closed loops (hence the name eddy current) and consume Ohmic power.



(9) How does an eddy current brake work? ($\mathbf{J}\times\mathbf{B}$ force on the wheel).

5.15 Faraday's Law of Induction (*continued*)

(10) Consider a closed rigid wire at rest in a static B-field. The wire carries a current I. The torque (Γ) due to $\mathbf{J}\times\mathbf{B}$ magnetic forces $\mathbf{F}$ will rotate the wire to give it a K.E. (kinetic energy). Does this mean magnetic forces can do work?



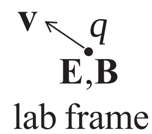
Ans : Yes. A rotation by $\Delta\theta$ ($\ll 1$) changes the K.E. by $\Delta(\text{K.E.}) = (\text{total } \Gamma)\cdot\Delta\theta$. However, as the wire rotates, an E-field is induced inside and $\mathbf{J}\cdot\mathbf{E}$ supplies Δ(K.E.).

Assume the i-th charged particle (e^- or ion) in the wire has a velocity $\mathbf{v}_i$ & charge q_i. So it experiences an electric force $q_i\mathbf{E}$ and a magnetic force $q_i\mathbf{v}_i\times\mathbf{B}$. Hence, the power delivered to this particle is $q_i\mathbf{v}_i\cdot\mathbf{E}+q_i(\mathbf{v}_i\times\mathbf{B})\cdot\mathbf{v}_i$. Since $(\mathbf{v}_i\times\mathbf{B})\cdot\mathbf{v}_i=0$, only the E-field produces the power. Assume n_i is the density of particles with q_i & $\mathbf{v}_i$. The power per unit volume delivered to a loop point is thus

$$P=\sum_i n_i q_i \mathbf{v}_i\cdot\mathbf{E}=\mathbf{J}\cdot\mathbf{E} \quad \text{[almost entirely due to } e^- s\text{]}$$

Part of P becomes P_{ohm} (or heat in the wire). The rest becomes the loop rotational K.E. due to the action of Γ.

Electric field viewed in lab and moving frames



Consider a charge q moving at velocity $\mathbf{v}$ relative to the lab frame. At the point of q, the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$, respectively. Thus, the force on q is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{[Both } \mathbf{E} \text{ and } \mathbf{B} \text{ are viewed in lab frame]} \qquad (10)$$

However, in q's rest frame (in which q has zero velocity), q experiences no magnetic force. By the Galilean transformation (which is valid for $v \ll c$), the force on q is the same in both the lab and rest frames. Hence, in its rest frame, q must see a different electric field $\mathbf{E}'$, which exerts the same $\mathbf{F}$ on q as in (10):

$$\mathbf{F} = q\mathbf{E}' \quad [\mathbf{E}' \text{ is viewed in } q\text{'s rest frame}] \qquad (11)$$

$$(10), (11) \Rightarrow \ \mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad [\text{relation between } \mathbf{E} \text{ and } \mathbf{E}'], \qquad (5.142)$$

which gives the relation between the electric fields at the same point as viewed in the lab frame and a frame moving at $\mathbf{v}$.

Note: (5.142) is the $v \ll c$ limit of the relativistic relation between $\mathbf{E}$ and $\mathbf{E}'$ [(11.149)].

5.16 Energy in the Magnetic Field

Let $\mathbf{B}$ be generated by a thin loop of uniform cross-section (σ) with a uniform (free) current density ($\mathbf{J}$). As $\mathbf{J}$ (hence $\mathbf{B}$) builds up from 0, an $\mathbf{E}$-field will be induced by $\frac{\partial \mathbf{B}}{\partial t}$, which does work on $\mathbf{J}$ at the rate:

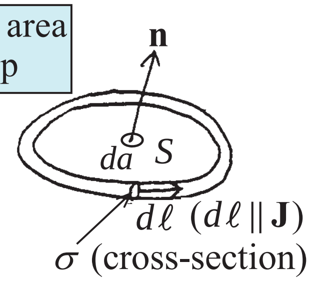
$J\sigma = I$ (current in the loop)

integration over the area encircled by the loop

$$\frac{dW}{dt} = -\overbrace{J\sigma}\oint \mathbf{E}\cdot d\boldsymbol{\ell} = -J\sigma\int_S [\underbrace{\nabla \times \mathbf{E}}_{-\partial \mathbf{B}/\partial t}]\cdot \mathbf{n}da$$

$$= J\sigma \int_S \mathbf{n}\cdot\frac{\partial \mathbf{B}}{\partial t} da$$

Stokes's thm.



$\Rightarrow \delta W$ [work done to generate $\delta\mathbf{B}$]

$$= J\sigma \int_S \mathbf{n}\cdot\delta\mathbf{B}da$$

$\delta\mathbf{B} = \nabla \times \delta\mathbf{A}$

$$= J\sigma \int_S [\nabla \times \delta\mathbf{A}]\cdot\mathbf{n}da$$

Stokes's thm.

$$= J\sigma \oint \delta\mathbf{A}\cdot d\boldsymbol{\ell}$$

$J\sigma d\boldsymbol{\ell} = \mathbf{J}\sigma d\ell = \mathbf{J}d^3x$

$$= \int_{loop} \delta\mathbf{A}\cdot\mathbf{J}d^3x$$

no $\mathbf{J}$ outside loop

$$= \int \delta\mathbf{A}\cdot\mathbf{J}d^3x \qquad (5.144)$$

Note: We assume a single loop here. Jackson assumes multiple loops. Question: Does the difference matter?

Rewrite $\delta W = \int \delta \mathbf{A} \cdot \mathbf{J} d^3x$ [work done to generate $\delta \mathbf{A}$] (5.144)

Let $\partial/\partial t \to 0$ (infinitesimal rate of buildup) $\Rightarrow$ Neglect $\partial \mathbf{D}/\partial t$ [see (6.2), Ch. 6)] $\Rightarrow$ $\mathbf{H}$ obeys the static law: $\nabla \times \mathbf{H} = \mathbf{J}$ [(5.90)]

$$\Rightarrow \delta W = \int \delta \mathbf{A} \cdot \mathbf{J} d^3x = \int \delta \mathbf{A} \cdot (\nabla \times \mathbf{H}) d^3x$$

$\boxed{\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})}$

$$= \int \mathbf{H} \cdot \underbrace{(\nabla \times \delta \mathbf{A})}_{\delta \mathbf{B}} d^3x + \underbrace{\int \nabla \cdot (\mathbf{H} \times \delta \mathbf{A}) d^3x}_{= \oint_S (\mathbf{H} \times \delta \mathbf{A}) \cdot d\mathbf{a} = 0}$$

For this integral to vanish, the integration must be over all space.

$$= \int \mathbf{H} \cdot \delta \mathbf{B} d^3x = \frac{1}{2} \int \delta(\mathbf{H} \cdot \mathbf{B}) d^3x \text{ [integrated over all space]}$$

for *linear* medium: $\mathbf{B} = \mu \mathbf{H}$ (not for ferromagnetic material)

$\Rightarrow$ Total work done to bring the field up from 0 to the final value $\mathbf{B}$:

$$W = \frac{1}{2} \int (\mathbf{H} \cdot \mathbf{B}) d^3x \quad \left[\begin{array}{l} \partial/\partial t \to 0 \text{ implies radiation loss} \to 0 \\ \Rightarrow W = \text{total magnetic field energy} \end{array} \right] \quad (5.148)$$

$\Rightarrow$ Postulate $w = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2\mu} |\mathbf{B}|^2$ [field energy per unit voulme] (12)

Note: 1. To build up another loop, induced $\mathbf{E}$ will affect the first loop.

2. $w = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2} (\sum_j \mathbf{H}_j) \cdot (\sum_j \mathbf{B}_j) \neq \frac{1}{2} \sum_j (\mathbf{H}_j \cdot \mathbf{B}_j)$

Discussion:

$$1. \begin{cases} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_{free}(\mathbf{x}') + \nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \text{ [(5.78)]} \\ \mathbf{J}_M = \nabla \times \mathbf{M} \text{ [(5.79)]} \\ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_{free} + \nabla \times \mathbf{M}) \text{ [(5.80)]} \\ \nabla \times \mathbf{H} = \mathbf{J}_{free} \text{ [(5.82)], where } \mathbf{H} \equiv \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \text{ [(5.81)]} \end{cases}$$

are applicable to all magnetic materials, but we also need to the relation etween B & H (from exp. or theory of magnetism) in order to have a complete formalism. A simple case is linear materials with $\mathbf{B} = \mu \mathbf{H}$ [(5.84)]. Another simple case is a permanent magnet with a given $\mathbf{M}$ in the absence of an external B-field (e.g. Sec. 5.10).

2. $W = \frac{1}{2} \int (\mathbf{H} \cdot \mathbf{B}) d^3x$ [(5.148)]

is derived for linear magnetic materials obeying $\mathbf{B} = \mu \mathbf{H}$ (p.213). Hence, (5.148) is not applicable to ferromagnetic materials (e.g. a permanent magnet), which are nonlinear.

5.17 Energy and Self- and Mutual Inductances

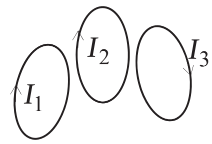
Assume linear relation between **J** and **A**

This & next page will not be covered in class.

$$\delta W = \int \delta \mathbf{A} \cdot \mathbf{J} d^3x = \frac{1}{2}\int \delta(\mathbf{A} \cdot \mathbf{J}) d^3x \quad (5.144)$$

$$\Rightarrow W = \frac{1}{2}\int \mathbf{A} \cdot \mathbf{J} d^3x \quad (5.149)$$

$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ [for $\mu=\mu_0$ medium]



$$= \frac{\mu_0}{8\pi}\int d^3x \int d^3x' \frac{\mathbf{J}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \quad (5.153)$$

for N current-carrying circuits

$$= \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\int d^3x_i \sum_{j=1}^{N}\int d^3x'_j \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|} = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij} I_i I_j, \quad (5.152)$$

where

self-inductance

for a thin wire

$$L_i = \frac{\mu_0}{4\pi I_i^2}\int_{C_i} d^3x_i \int_{C_i} d^3x_i \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_i)}{|\mathbf{x}_i-\mathbf{x}'_i|}\left[= \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_i}\frac{d\ell_i \cdot d\ell'_i}{|\mathbf{x}_i-\mathbf{x}'_i|}\right] \quad (5.154)$$

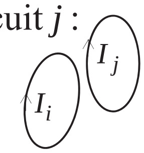
$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{C_i} d^3x_i \int_{C_j} d^3x'_j \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|}\left[= \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\frac{d\ell_i \cdot d\ell'_j}{|\mathbf{x}_i-\mathbf{x}'_j|}\right] \quad (5.155)$$

mutual inductance ($M_{ij} = M_{ji}$)

for thin wires

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad (5.32)$$

$\Rightarrow$ Vector potential at circuit i due to current in circuit j :



$$\mathbf{A}_{ij}(\mathbf{x}_i) = \frac{\mu_0}{4\pi}\oint_{C_j}\frac{\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|} d^3x'_j \quad (13)$$

From (13) and (5.155), we obtain $M_{ij} = \frac{1}{I_i I_j}\int_{C_i}\mathbf{A}_{ij}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}_i) d^3x_i$

Assume **J** flows along wire $d\ell$ of infinistesimal cross section da

$$\Rightarrow \mathbf{J}(\mathbf{x}_i) d^3x_i = J_{\|} da\, d\ell = I_i d\ell$$

magnetic flux from circuit j passing through circuit i

$$\Rightarrow M_{ij} = \frac{1}{I_j}\oint_{C_i}\mathbf{A}_{ij}\cdot d\ell = \frac{1}{I_j}\oint_{S_i}\overbrace{(\nabla\times\mathbf{A}_{ij})}^{\mathbf{B}_{ij}}\cdot \boldsymbol{n} da = \frac{1}{I_j}F_{ij} \quad (5.156)$$

$$\Rightarrow \varepsilon_{ij} \equiv -\frac{d}{dt}F_{ij} = -M_{ij}\frac{d}{dt}I_j$$

ε_{ij}: induced voltage in circuit i due to current variation in circuit j.

The "–" sign implies that the induced ε_{ij} tends to drive a current in circuit i to *inhibit* the flux change caused by circuit j (Lenz's law).