Chapter 11: Special Theory of Relativity

(Ref.: Heald & Marion, “Classical Electromagnetic Radiation,” 3rd ed., Ch. 14)

Einstein’s special theory of relativity is based on two postulates:

1. Laws of physics are invariant in form in all Lorentz frames (In relativity, we often call the inertial frame a Lorentz frame.)

2. The speed of light in vacuum has the same value c in all Lorentz frames, independent of the motion of the source.

The basics of the special theory of relativity are covered in Appendix A with an emphasis on the Lorentz transformation and relativistic mechanics. Here, we examine the theory in the 4-dimensional space of $\mathbf{x}$ and t, which provides a convenient framework to examine postulate 1 for mechanical and EM laws. The lecture notes depart considerably from Ch. 11 of Jackson. Instead, we follow Ch. 14 of Heald & Marion (H & M).

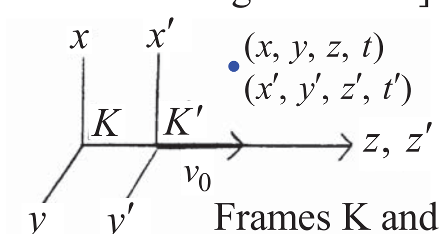
In the lecture notes, section numbers do not follow Jackson.

Section 1: Definitions and Operation Rules of Tensors of Different Ranks in the 4-Dimensional Space

The Lorentz Transformation : Consider two Lorentz frames K and K' (henceforth referred to as K, K'). K' moves along the common z-axis with constant speed v_0 relative to K.

Assume that, at $t = t' = 0$, coordinate axes of K and K' overlap. Postulate 2 leads to the following Lorentz transformation for space and time coordinates [derived in Appendix A, Eq. (A.15), where the relative motion is assumed to be along the x-axis]:

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma_0 (z - v_0 t) \\ t' = \gamma_0 (t - \frac{v_0}{c^2} z) \end{cases} \quad (1)$$



Frames K and K′ coincide at $t = t' = 0$.

where $\gamma_0 \equiv (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}}$ is the Lorentz factor for the transformation.

A note about notation: There are 2 Lorentz factors: one denoting the relative speed between two frames, the other denoting the particle speed in a given frame. In Jackson, the relative speed between two frames is denoted by v and the particle velocity in a given frame is denoted by **u**. This leads to two definitions for the same notation γ:

$$\gamma \equiv (1-\frac{v^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor for the transformation, (11.17)]}$$

$$\gamma \equiv (1-\frac{u^2}{c^2})^{-\frac{1}{2}} \quad \left[\begin{array}{l}\text{Lorentz factor of a particle in a given frame,}\\ \text{(11.46) and (11.51)}\end{array}\right]$$

To avoid confusion about the notation γ (e.g. when we perform a Lorentz transformation of the Lorentz factor of a particle), we will denote the relative speed between two frames by v_0 and the particle velocity by **v** throughout this chapter, and thus define

$$\gamma_0 \equiv (1-\frac{v_0^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor for the transformation]}$$

$$\gamma \equiv (1-\frac{v^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor of a particle in a given frame].}$$

Quantities in 4-Dimensional Space and Operation Rules :

Define a 4-D vector x and a 4-D matrix $a_{\mu\nu}$ in the **x**-t space by

$$\mathsf{x} \equiv (x,\ y,\ z,\ ict) = (\mathbf{x},\ ict)$$

$\mathbf{x}$: 3-D position vector
x : 4-D position vector

and

$$a_{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix},$$

μ : row number
ν : column number

where $\beta_0 = \frac{v_0}{c}$. Then, the Lorentz transformation [(1)] can be written

$$\begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} \quad \text{or} \quad x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \qquad (2)$$

and the inverse Lorentz transformation is: $x_\nu = \sum_{\mu=1}^{4} a_{\mu\nu} x'_\mu$. (3)

An orthogonal transformation is defined by [H & M, Eq. (14.11)]

$$\sum_\mu a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda} \ [\text{definition of orthogonal transformation}] \quad (4a)$$

The Lorentz transformation matrix in (2) satisfies (4a). Thus,

Use $x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu$ [(2)]

for all orthogonal transformations.

$$\sum_\mu x'^2_\mu = \sum_\mu \big(\sum_\nu a_{\mu\nu} x_\nu\big)\big(\sum_\lambda a_{\mu\lambda} x_\lambda\big) = \sum_{\nu,\lambda} \underbrace{\sum_\mu a_{\mu\nu} a_{\mu\lambda}}_{\delta_{\nu\lambda}\ [\text{by } 4(\text{a})]} x_\nu x_\lambda = \sum_\lambda x^2_\lambda \quad (4b)$$

$$\textit{Terminology}: \begin{cases} \text{tensor of the 0-th rank (also called a scalar)} \\ \text{tensor of the 1st rank (also called a vector)} \\ \text{tensor of the 2nd, 3rd rank, etc.} \end{cases}$$

Definition: A tensor of any rank is defined by how its components transform between reference systems. For example, (1 m, 2 m, 3 m) is not necessarily a vector. But if it is the spatial coordinates (x, y, z) of **x**, it is a 3-D vector (3-D tensor of the 1st rank) because its components transform as a vector should under a coordinate rotation (origin fixed).

1. If a quantity Φ is unchanged under the Lorentz transformation, it is a 4-D tensor of the 0-th rank, or a 4-tensor of the 0-th rank, or a 4-scalar, or a Lorentz invariant, or a Lorentz scalar.

   The rest mass m and charge e are both Lorentz invariants [see (24) & (47)]. Some Lorentz invariants are expressed in terms of variables. For example, $\sum_\mu x'^2_\mu = \sum_\lambda x^2_\lambda$ [4(b)] gives the Lorentz invariant: $\quad \Phi = x^2 + y^2 + z^2 - c^2 t^2$

   As a check, sub. the inverse transformation: $x = x'$, $y = y'$, $z = \gamma_0 (z' + v_0 t')$, $t = \gamma_0 (t' + \frac{v_0}{c^2} z')$ into Φ, we obtain

$$\begin{aligned} \Phi &= x'^2 + y'^2 + \gamma_0^2 (z' + v_0 t')^2 - c^2 \gamma_0^2 (t' + \tfrac{v_0}{c^2} z')^2 \\ &= x'^2 + y'^2 + z'^2 (\gamma_0^2 - c^2 \gamma_0^2 \tfrac{v_0^2}{c^4}) - t'^2 (c^2 \gamma_0^2 - \gamma_0^2 v_0^2) \\ &\quad + 2\gamma_0^2 z' v_0 t' - 2c^2 \gamma_0^2 t' \tfrac{v_0}{c^2} z' \\ &= x'^2 + y'^2 + z'^2 - c^2 t'^2 \end{aligned}$$

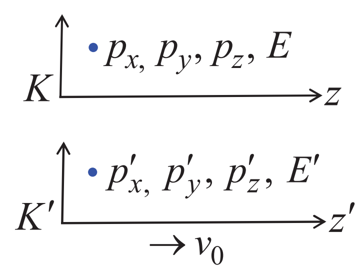
2. If a set of 4 quantities, A_μ $(\mu = 1,2,3.4)$ or $\mathbf{A} = (A_1, A_2, A_3, A_4)$, transforms in the same manner as the Lorentz transformation for $\mathbf{x}$ [$= (\mathbf{x}, ict)$], i.e. $$A'_\mu = \sum_\nu a_{\mu\nu} A_\nu, \qquad (5)$$ then, $\mathbf{A}$ is a 4-vector (4-tensor of the 1st rank).

$\mathbf{x}$ is just one of numerous 4-vectors to be discussed. For example, define the 4-D momentum of a particle as

$$\mathbf{p} \equiv (p_x, p_y, p_z, \frac{iE}{c}) = (\mathbf{p}, \frac{iE}{c}) \quad \text{[4-momentum]}$$

Then, $\mathbf{p}$ transforms as $\mathbf{x}$, i.e. [see Eq. (A.28), Appendix A]:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ \frac{iE'}{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ \frac{iE}{c} \end{bmatrix}$$



Hence, $\mathbf{p}$ is a 4-vector.

3. Define the 4-D operator: $\Box \equiv [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial(ict)}]$ as the counterpart of the 3-D operator ∇ [$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$]. Then, the <u>4-gradient</u> of a Lorentz scalar: $\Box\Phi \equiv [\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}, \frac{\partial\Phi}{\partial(ict)}]$ is a 4-vector.

Proof : $$(\Box'\Phi)_\mu = \frac{\partial\Phi}{\partial x'_\mu} = \sum_\nu \frac{\partial\Phi}{\partial x_\nu} \underbrace{\frac{\partial x_\nu}{\partial x'_\mu}}_{= a_{\mu\nu} \text{ by (3)}} = \sum_\nu a_{\mu\nu} \frac{\partial\Phi}{\partial x_\nu} = \underbrace{\sum_\nu a_{\mu\nu} (\Box\Phi)_\nu}_{\text{Transforms as a 4-vector.}} \qquad (6)$$

4. The <u>4-divergence</u> of a 4-vector: $\Box\cdot\mathbf{A} \equiv \sum_\mu \frac{\partial A_\mu}{\partial x_\mu}$ is a Lorentz scalar.

Proof :

$$\Box'\cdot\mathbf{A}' = \sum_\nu \frac{\partial A'_\nu}{\partial x'_\nu} = \sum_\nu \sum_\mu \underbrace{\frac{\partial x_\mu}{\partial x'_\nu}}_{= a_{\nu\mu} \text{ by (3)}} \frac{\partial \overbrace{A'_\nu}^{= \sum_\lambda a_{\nu\lambda} A_\lambda \text{ by (5)}}}{\partial x_\mu} = \sum_{\mu\lambda} \underbrace{\sum_\nu a_{\nu\mu} a_{\nu\lambda}}_{= \delta_{\mu\lambda} \text{ by (4a)}} \frac{\partial A_\lambda}{\partial x_\mu} = \sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = \Box\cdot\mathbf{A} \qquad (7)$$

$\mathbf{A}$: 4-vector A_μ: component of $\mathbf{A}$

$\Rightarrow$ $\Box\cdot\mathbf{A}$ is unchanged under the Lorentz transformation

5. The <u>4-Laplacian</u> operator, $\Box^2 \equiv \Box \cdot \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, (8)

is a Lorentz scalar operator, i.e. $\Box'^2 \Phi = \Box^2 \Phi$ [Φ : a Lorentz scalar].

Proof : We have shown in item 4 that the divergence of a 4-vector is a Lorentz scalar, i.e. $\Box' \cdot \mathbf{A}' = \Box \cdot \mathbf{A}$. Let Φ be a Lorentz scalar, then $\mathbf{A}' = \Box' \Phi$ and $\mathbf{A} = \Box \Phi$ are both 4-vectors (see item 3). Hence,

$$\Box' \cdot \mathbf{A}' = \Box \cdot \mathbf{A} \Rightarrow \Box' \cdot \Box' \Phi = \Box \cdot \Box \Phi \;\Rightarrow\; \Box'^2 \Phi = \Box^2 \Phi.$$

6. The <u>dot product</u> of two 4-vectors: $\mathbf{A} \cdot \mathbf{B} \equiv \sum_\mu A_\mu B_\mu$ is a Lorentz scalar.

$$Proof: \mathbf{A}' \cdot \mathbf{B}' = \sum_\sigma A'_\sigma B'_\sigma = \sum_\sigma \overbrace{\sum_\nu a_{\sigma\nu} A_\nu}^{A'_\sigma} \overbrace{\sum_\lambda a_{\sigma\lambda} B_\lambda}^{B'_\sigma} = \sum_{\nu\lambda} \overbrace{\sum_\sigma a_{\sigma\nu} a_{\sigma\lambda}}^{\delta_{\nu\lambda} \text{ by (4a)}} A_\nu B_\lambda$$

$$= \sum_\lambda A_\lambda B_\lambda = \mathbf{A} \cdot \mathbf{B} \qquad (9)$$

Example 1:

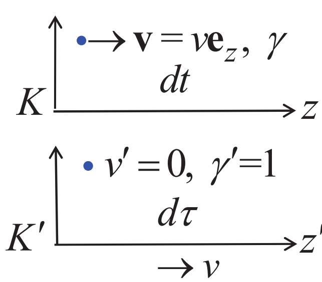
$$\mathbf{A}' \cdot \mathbf{A}' = \mathbf{A} \cdot \mathbf{A} \Rightarrow \sum_\mu A'^2_\mu = \sum_\mu A^2_\mu \quad \left[\begin{array}{l}\text{a consequence of orthogonal} \\ \text{transformation [see (4b)]}\end{array}\right]$$

Example 2: In K, a particle's position changes by $d\mathbf{x}$ in a time interval dt ($dt \to 0$). Then, $d\mathbf{x} = (dx,\ dy,\ dz,\ icdt)$ is a 4-vector. Hence, $d\mathbf{x} \cdot d\mathbf{x}$ is a Lorentz invariant, i.e. $d\mathbf{x}' \cdot d\mathbf{x}' = d\mathbf{x} \cdot d\mathbf{x}$ or

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

$$\Rightarrow (v'^2 / c^2 - 1) dt'^2 = (v^2 / c^2 - 1) dt^2$$

$$\Rightarrow \qquad \frac{dt'}{\gamma'} = \frac{dt}{\gamma} \quad \text{[Lorentz invariant]}$$



In K, let the particle's $\mathbf{v} = v\mathbf{e}_z$ (no loss of generality since z-axis is chosen to be along $\mathbf{v}$). Let K' be the particle's instantaneous rest frame ($v' = 0$). At dt', $\gamma' = [1 - (a'dt'/c)^2]^{-1/2} \approx 1 - (a'dt'/c)^2/2 \Rightarrow$ To 1st order in dt', $\gamma' = 1$. Thus, we let $\gamma' = 1$ on the LHS and denote dt' by $d\tau$ [proper time]. $\Rightarrow$ $\quad d\tau = \frac{dt}{\gamma}$ [Lorentz invariant] (10)

where $d\tau$ ($\to 0$) is the particle's time in its rest frame (K'), and dt is the value of $d\tau$ in K, in which the particle has an instantaneous γ. In (10), we see both $d\tau$ & $\frac{dt}{\gamma}$ are Lorentz invariants [to be used in (47)].

Example 3: Since $\mathbf{p} = (\mathbf{p}, \frac{iE}{c})$ is a 4-vector, we have

$$\mathbf{p}\cdot\mathbf{p} = \mathbf{p}'\cdot\mathbf{p}' \;\Rightarrow\; p^2 - \frac{E^2}{c^2} = p'^2 - \frac{E'^2}{c^2} \tag{11}$$

$\Rightarrow E^2 - p^2c^2$ is a Lorentz scalar (Lorentz invariant). (12)

If K' is the rest frame of the particle, then $p' = 0$ and $E' = mc^2$.

(11) then gives $$E^2 - p^2c^2 = m^2c^4, \tag{13}$$

which is a useful formula relating the particle's E to its p.

Question: Is the rest mass m a Lorentz invariant?

Since $E^2 - p^2c^2$ is a Lorentz invariant, (13) shows that the rest mass m is a Lorentz invariant. This makes the theory self-consistent in (1) the rest mass (m) is a uniquely-defined quantity, and (2) the rest mass energy (mc^2) is the same in all frames.

Note: $\begin{cases} \text{In classical mechanics, } m \text{ and } L \text{ of an object are invariants.} \\ \text{In special relativity, } m \text{ (but not } L\text{) is a Lorentz invariant.} \end{cases}$

7. A <u>4-tensor of the 2nd rank</u> ($\overleftrightarrow{\mathbf{T}}$) is a set of 16 quantities, $T_{\mu\nu}$ ($\mu,\nu = 1-4$), which transform according to

$$T'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma} \tag{14}$$

Note: $a_{\mu\lambda}$ $(\mu,\nu = 1-4)$ is the Lorentz tranformation matrix, not a 4 tensor.

8. The dot product of a 4-tensor of the 2nd rank ($\overleftrightarrow{\mathbf{T}}'$) and a 4-vector ($\mathbf{A}$) is a 4-vector.

$$\textit{Proof}: \; (\overleftrightarrow{\mathbf{T}}'\cdot\mathbf{A}')_\mu = \sum_\nu \overset{\boxed{\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma}}}{T'_{\mu\nu}} \overset{\boxed{\sum_\alpha a_{\nu\alpha}A_\alpha}}{A'_\nu} = \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda} \overbrace{\sum_\nu a_{\nu\sigma}a_{\nu\alpha}}^{\delta_{\sigma\alpha}\ \text{[by (4a)]}} T_{\lambda\sigma}A_\alpha$$

$$= \sum_\lambda a_{\mu\lambda}\sum_\sigma T_{\lambda\sigma}A_\sigma = \underbrace{\sum_\lambda a_{\mu\lambda}(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})_\lambda}_{\text{Transform as a 4-vector.}} \tag{15}$$

9. The 4-divergence of a 4-tensor of the 2nd rank, $(\Box \cdot \overleftrightarrow{\mathbf{T}})_\mu \equiv \sum_\nu \frac{\partial T_{\mu\nu}}{\partial x_\nu}$, is a 4-vector.

Proof :

$$(\Box' \cdot \overleftrightarrow{\mathbf{T}}')_\mu = \sum_\nu \frac{\partial T'_{\mu\nu}}{\partial x'_\nu} = \sum_\nu \frac{\partial}{\partial x'_\nu} \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} T_{\lambda\sigma} = \sum_\nu \sum_\alpha \underbrace{\frac{\partial x_\alpha}{\partial x'_\nu}}_{a_{\nu\alpha}} \frac{\partial}{\partial x_\alpha} \sum_{\lambda\sigma} a_{\mu\lambda} a_{\nu\sigma} T_{\lambda\sigma}$$

$$= \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda} \overbrace{\sum_\nu a_{\nu\alpha} a_{\nu\sigma}}^{\delta_{\alpha\sigma}\ [\text{by (4a)}]} \frac{\partial T_{\lambda\sigma}}{\partial x_\alpha} = \sum_\lambda a_{\mu\lambda} \sum_\sigma \frac{\partial T_{\lambda\sigma}}{\partial x_\sigma} = \underbrace{\sum_\lambda a_{\mu\lambda} (\Box \cdot \overleftrightarrow{\mathbf{T}})_\lambda}_{\text{Transform as a 4-vector.}} \qquad (16)$$

10. A 4-tensor of the 3rd rank is a set of 64 quantities, $G_{\lambda\mu\nu}$ $(\lambda, \mu, \nu = 1\text{-}4)$, which transform according to

$$G'_{\lambda\mu\nu} = \sum_{ijk} a_{\lambda i} a_{\mu j} a_{\nu k} G_{ijk} \qquad (17)$$

Problem 1: If $F_{\mu\nu}$ is a 4-tensor of the 2nd rank, show that $\frac{\partial F_{\mu\nu}}{\partial x_\lambda}$ $(\lambda, \mu, \nu = 1\text{-}4)$ is a 4-tensor of the 3rd rank.

Solution : $F'_{\mu\nu} = \sum_{jk} a_{\mu j} a_{\nu k} F_{jk}$

$$\Rightarrow \frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} = \sum_{jk} a_{\mu j} a_{\nu k} \sum_i \frac{\partial F_{jk}}{\partial x_i} \overbrace{\frac{\partial x_i}{\partial x'_\lambda}}^{a_{\lambda i}} = \underbrace{\sum_{ijk} a_{\lambda i} a_{\mu j} a_{\nu k} \frac{\partial F_{jk}}{\partial x_i}}_{\text{Transform as a 4-tensor of the 3rd rank.}} \qquad (18)$$

Problem 2: Show that the set of 64 equations,

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}=0\ \ [\lambda,\mu,\nu=1\text{-}4] \tag{19}$$

are each invariant in form under the Lorentz transformation, where $\frac{\partial F'_{\mu\nu}}{\partial x'_\lambda}$ is a 4 tensor of the 3rd rank, i.e. $\frac{\partial F'_{\mu\nu}}{\partial x'_\lambda}=\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{jk}}{\partial x_i}$ [(18)]

Solution: Change indices in (15) as follows: $\begin{cases}\lambda\to\nu,\ \mu\to\lambda,\ \nu\to\mu\\ i\to k,\ \ k\to j,\ j\to i\end{cases}$

$$\Rightarrow\ \frac{\partial F'_{\lambda\mu}}{\partial x'_\nu}=\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ij}}{\partial x_k} \tag{20}$$

Change indices in (17) as follows: $\begin{cases}\nu\to\mu,\ \lambda\to\nu,\ \mu\to\lambda\\ k\to j,\ i\to k,\ \ j\to i\end{cases}$

$$\Rightarrow\ \frac{\partial F'_{\nu\lambda}}{\partial x'_\mu}=\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ki}}{\partial x_j} \tag{21}$$

Add (18), (20), & (21)

$$\Rightarrow\frac{\partial F'_{\mu\nu}}{\partial x'_\lambda}+\frac{\partial F'_{\lambda\mu}}{\partial x'_\nu}+\frac{\partial F'_{\nu\lambda}}{\partial x'_\mu}=\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\,\overbrace{\left(\frac{\partial F_{jk}}{\partial x_i}+\frac{\partial F_{ij}}{\partial x_k}+\frac{\partial F_{ki}}{\partial x_j}\right)}^{=0\text{ by }(19)}=0\ \ \begin{bmatrix}\lambda,\mu,\nu\\ =1\text{-}4\end{bmatrix}$$

11. If a physical law can be expressed as a relation between 4-tensors of the same rank, then its form is invariant in all Lorentz frames.

Example 1: If the physical law in K is of the form $\mathbf{A}=\mathbf{B}$, then

$$A'_\nu=\sum_\mu a_{\mu\nu}\underbrace{A_\mu}_{B_\mu}=\sum_\mu a_{\mu\nu}B_\mu=B'_\nu,\ \text{i.e. } \mathbf{A}=\mathbf{B}\Rightarrow\underbrace{\mathbf{A}'=\mathbf{B}'}_{\text{invariant in form}} \tag{22}$$

Example 2: If the physical law in K is of the form $\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}$, then

$$T'_{\mu\nu}=\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}\underbrace{T_{\lambda\sigma}}_{F_{\lambda\sigma}}=\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}F_{\lambda\sigma}=F'_{\mu\nu},\ \text{i.e.}$$

$$\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}\Rightarrow\overleftrightarrow{\mathbf{T}}'=\overleftrightarrow{\mathbf{F}}'\ \ [\text{invariant in form}] \tag{23}$$

Example: We have shown that, for a system of isolated particles,

$$\text{In K,}\ \begin{cases}\sum\mathbf{p}_j(\text{before collision})=\sum\mathbf{p}_j(\text{after collision})\ \textit{and}\\ \sum E_j(\text{before collision})=\sum E_j(\text{after collision})\end{cases}\ \ [(\text{A30a})]$$

$$\text{In K}',\ \begin{cases}\sum\mathbf{p}'_j(\text{before collision})=\sum\mathbf{p}'_j(\text{after collision})\ \textit{and}\\ \sum E'_j(\text{before collision})=\sum E'_j(\text{after collision})\end{cases}\ \ [(\text{A30b})]$$

Since $\mathbf{p} = (\mathbf{p},\ \frac{iE}{c})$ is a 4-vector, the momentum/energy conservation law in (A30a) can be written:

For an *isolated* system of particles,

$$\mathbf{P}\ \text{(before collision)} = \mathbf{P}\ \text{(after collision)} \qquad (24)$$

Since (24a) is in the form of the equality of two 4-vectors, it automatically implies, in any inertial frame, it has the same form, i.e.

$$\mathbf{P}'\ \text{(before collision)} = \mathbf{P}'\ \text{(after collision)} \qquad (25)$$

"Conservation", "Invariance", and "Covariance"

The <u>conservation</u> of a quantity means that it is *unchanged in time,* e.g. $\mathbf{P}$ (before collision) $=$ $\mathbf{P}$ (after collision) [(24)]. A conservation law must have the same form in any inertial frame, e.g. $\mathbf{P}'$ (before collision) $=$ $\mathbf{P}'$ (after collision) [(25)].

The <u>invariance</u> of a quantity means that it is *invariant in value* under a Lorentz transformation. For example, the dot product of two 4-vectors is a Lorentz invariant:

$$\mathbf{p}\cdot\mathbf{p} = \mathbf{p}'\cdot\mathbf{p}' \ \Rightarrow\ p^2 - \frac{E^2}{c^2} = p'^2 - \frac{E'^2}{c^2} \quad [(11)]$$

Note : A Lorentz invariantit is not necessarily a conserved quantity, e.g. if the particle with $\mathbf{p}$ in (11) is under an external force, then (11) is still true at any time, but its value varies in time (not conserved).

The term <u>covariance</u> refers to physical laws. By Einstein's first postulate, laws of physics are <u>covariant</u>, i.e. *invariant in form under the Lorentz transformation*, e.g. (24), (25).

Section 2: Covariance of Electrodynamics

In the special theory of relativity, Newton's law has been radically modified. The electromagnetic laws do not need any modification because they are already covariant. However, the *covariance* of these laws (such as Maxwell equations) is not immediately clear from the equations by which they are usually represented.

Our purpose in this section is to prove that the EM laws are indeed covariant by casting them into relations between 4-tensors of the same rank, e.g. $\mathbf{A} = \mathbf{B}$ [(22)] or $\overleftrightarrow{\mathbf{T}} = \overleftrightarrow{\mathbf{F}}$ [(23)]. We will do this by defining 4-tensors in terms of known EM quantities and forming equations with 4-tensors of the same rank, then show that one or more existing EM laws are implicit in each equation.

Furthermore, Lorentz transformations of these tensors will yield the tranformation equations for various EM quantities.

Note: Jackson switches to the Gausian unit system starting from Ch. 11. From here on, we also adopt the Gaussian unit system.

1. Define a 4-current as

$$\mathbf{J} \equiv (J_x,\ J_y,\ J_z,\ ic\rho) = (\mathbf{J},\ ic\rho) \tag{35}$$

and use it to form a (covariant) 4-scalar relation

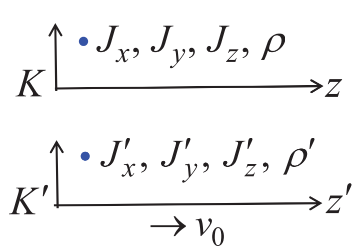
$$\Box \cdot \mathbf{J} = 0 \tag{36}$$

Then, (36) gives the law of conservation of charge

$$\tfrac{\partial}{\partial x} J_x + \tfrac{\partial}{\partial y} J_y + \tfrac{\partial}{\partial z} J_z + \frac{\partial(ic\rho)}{\partial(ict)} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \ [(5.2)]$$

Thus, the definition of $\mathbf{J}$ in (35) as a 4-vector leads to the covariant representation [(36)] of the EM law in (5.2). This in turn justifies the definition of $\mathbf{J}$ as a 4-vector. The Lorentz transformation of $\mathbf{J}$ then gives

$$\begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0 (J_z - v_0 \rho) \\ \rho' = \gamma_0 \left(\rho - \frac{v_0}{c^2} J_z\right) \end{cases} \tag{37}$$



2. Define a <u>4-potential</u> as $\mathbf{A} \equiv (A_x,\ A_y,\ A_z,\ i\Phi)$ (38)

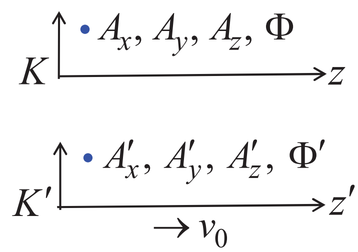
and write the covariant relations: $\begin{cases} \Box^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J} & (39) \\ \Box \cdot \mathbf{A} = 0 & (40) \end{cases}$

$$(39) \Rightarrow \begin{cases} \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} & [(6.15)] \\ \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -4\pi\rho & [(6.16)] \end{cases}$$

$(40) \Rightarrow\ \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\Phi = 0$ [Lorenz condition] [(6.14)]

This again shows the consistency of $\mathbf{A}$ being a 4-vector and (6.14)-(6.16) being covariant laws. The Lorentz transformation

of $\mathbf{A}$ then gives $\begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0\left(A_z - \frac{v_0}{c}\Phi\right) \\ \Phi' = \gamma_0\left(\Phi - \frac{v_0}{c}A_z\right) \end{cases}$ (41)



3. The source-free wave equation can be directly put into the covariant form: $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0 \ \Rightarrow\ \Box^2\psi = 0$ (42)

4. Define a <u>4-wave number</u> as

$$\mathbf{k} \equiv \left(k_x,\ k_y,\ k_z,\ \tfrac{i\omega}{c}\right) = \left(\mathbf{k},\ \tfrac{i\omega}{c}\right) \qquad (43)$$

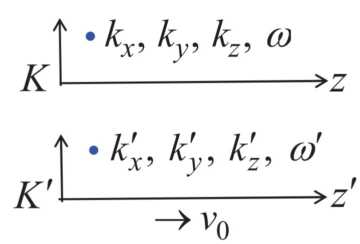
Then, $\mathbf{k}' \cdot \mathbf{x}' = \mathbf{k} \cdot \mathbf{x} \Rightarrow\ \mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t$

$\Rightarrow$ Invariance of the phase

$\Rightarrow$ $\mathbf{k}$ defined in (43) is a legitimate 4-vector.

$\Rightarrow$ Lorentz transformation of $\mathbf{k}$ gives

$$\underbrace{\begin{cases} k'_x = k_x \\ k'_y = k_y \\ k'_z = \gamma_0\left(k_z - \frac{v_0}{c^2}\omega\right) \\ \omega' = \gamma_0(\omega - v_0 k_z) \end{cases}}_{\text{relativistic Doppler shift}} \qquad (44)$$



relativistic Doppler shift $\left[\begin{array}{l} k_z \text{ is the component of } \mathbf{k} \text{ along the} \\ K' \text{ frame velocity } v_0\mathbf{e}_z \text{ relative to } K. \end{array}\right]$

5. Define a <u>field strength tensor of the 2nd rank</u> $\overleftrightarrow{\mathbf{F}}$ [H & M, (14.62)]:

$$\overleftrightarrow{\mathbf{F}} \equiv \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \qquad (45)$$

Then, $\Box\cdot \overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J} \Rightarrow \begin{cases} \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \end{cases}$

In the set of covariant equations [see (19)]

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \quad (\lambda,\mu,\nu = 1\text{-}4) \quad \left[\begin{array}{l} F_{\mu\nu}\text{'s are elements} \\ \text{of } \overleftrightarrow{\mathbf{F}} \text{ in (45).} \end{array}\right],$$

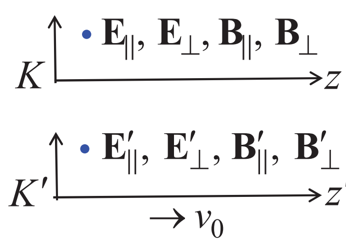
set $(\lambda,\mu,\nu) = (1,2,3) \Rightarrow \nabla\cdot\mathbf{B} = 0$

set $(\lambda,\mu,\nu) = (1,2,4),\ (1,3,4),$ and $(2,3,4) \Rightarrow \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0.$

11.3 Covariance of Electrodynamics (*continued*)

The covariant equations, $\Box\cdot \overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J}$ and $\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0,$ give the set of Maxwell equations in free space. This shows that Maxwell equatins are covariant and also justifies the definition of $\overleftrightarrow{\mathbf{F}}$ as a tensor of the second rank. Thus, $F'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma}$ gives the transformation equations for $\mathbf{E}$ and $\mathbf{B}$ (H and M, Sec. 14.6.):

$$\begin{cases} \mathbf{E}'_{\|} = \mathbf{E}_{\|} \\ \mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c}\times\mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\|} = \mathbf{B}_{\|} \\ \mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c}\times\mathbf{E}_{\perp}\right) \end{cases} \qquad (46)$$



In (46), $\mathbf{v}_0$ is the velocity of frame K' relative to frame K, and "$\|$" and "$\perp$" are with respect to the direction of $\mathbf{v}_0$.

See Appendix B for a summary of transformation equations.

6. Since $d\tau = \frac{dt}{\gamma}$ is a Lorentz scalar [see (10)], $\frac{d}{d\tau}\mathsf{p} = \frac{e}{mc}\vec{\mathsf{F}}\cdot\mathsf{p}$ (47)

is a covariant eq.*. With $\mathsf{p} = (\mathbf{p},\ i\frac{E}{c})$, it gives (H & M, p. 519)

$$\begin{cases} \dfrac{d}{dt}\mathbf{p} = e\left(\mathbf{E} + \dfrac{1}{c}\mathbf{v}\times\mathbf{B}\right) & \begin{bmatrix}\text{relativistic eq. of motion} \\ \mathbf{p} = \gamma m\mathbf{v}, \text{ cf. (A.37)}\end{bmatrix} \quad (48a) \\ mc^2\dfrac{d}{dt}\gamma = e\mathbf{v}\cdot\mathbf{E} & \begin{bmatrix}\text{work done} = \text{rate of change} \\ \text{of relativistic energy}\end{bmatrix} \quad (48b) \end{cases}$$

*In (13), we have shown the rest mass m is a Lorentz invariant. Here we postulate that charge e is also a Lorentz invariant. This has been experimentally verified to high accuracy (p. 554).

7. In a similar manner (see H & M, Sec. 14.12), we can demonstrate the covariance of the 2 conservation laws in Ch. 6 for an isolated system of EM fields and mechanical momentum/energy:

$$\begin{cases} \dfrac{d}{dt}\left(E_{\text{mech}} + E_{\text{field}}\right) = -\oint_s \mathbf{n}\cdot\mathbf{S}da & [(6.111)] \\ \dfrac{d}{dt}\left(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{field}}\right) = \oint_s \sum\limits_{\beta} T_{\alpha\beta}n_\beta da & [(6.122)] \end{cases}$$

Problem 1: A police radar of freq. ω is at rest. A car is moving at speed v_0 away from the radar in a straight road along $\mathbf{e}_z$ [Fig. (1)]. What is the frequenvt of the reflected wave detected by the police?

By (44), the freq. ω' in the car frame [Fig. (2)] is

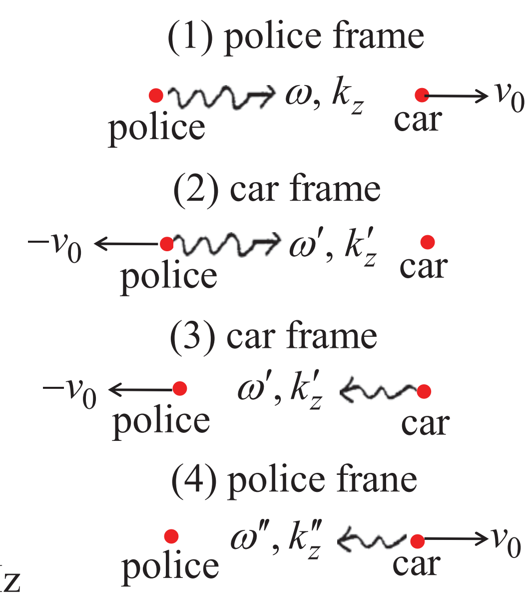
$\omega' = \gamma_0(\omega - v_0 k_z)$, (49)

which is also the reflected wave freq. in the car frame [Fig. (3)].

Use (44) again to tranform ω' to the police frame [Fig. (4)],

$\Rightarrow \omega'' = \gamma_0(\omega' - k_z' v_0) = \gamma_0\omega'(1 - \frac{v_0}{c})$

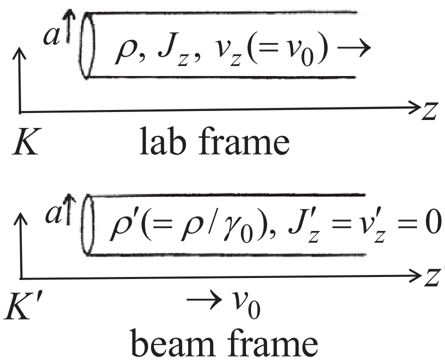
$= \gamma_0^2\omega(1 - \frac{v_0}{c})^2 \approx \omega(1 - 2\frac{v_0}{c})$ if $v_0 \ll c$



If the radar freq. is $f\,(= \frac{\omega}{2\pi}) = 10^9$ Hz and $v_0 = 150$ km/hr, the police would detect $f'' = (10^9 - 278)$ Hz

Problem 2: An observer in the laboratory sees an infinite electron beam of radius a and uniform charge density ρ, moving axially at velocity v_0. What force does he see on an electron at a distance r ($\le a$) from the axis? Assume the electron moves axially at the velocity v_0.

Solution: The problem can be readily solved in the lab frame. Here, we will take a long route as an exercise on some of the transformation equations just derived.



J_z (current density) in the lab frame is

$J_z = \rho v_0$ [ρ has a negative value.]

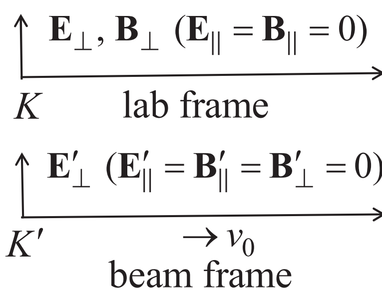
By (37), we have (in the beam frame)

$$\begin{cases} J_z' = \gamma_0\left(J_z - v_0\rho\right) = 0, \\ \rho' = \gamma_0\left(\rho - \frac{v_0}{c^2}J_z\right) = \gamma_0\rho\left(1-\frac{v_0^2}{c^2}\right) = \frac{\rho}{\gamma_0} \end{cases}$$

We see that the lab-frame ρ is greater than the beam-frame ρ' by the factor γ_0. This is because every unit length of the beam in the beam frame (where it has zero velocity) is contracted by the factor γ_0 when viewed in the lab frame.

In the beam frame, $J_z' = 0$, $\rho' = \rho/\gamma_0$.

$\Rightarrow$ There is only a radial E-field (no B-field).



Gauss law: $\oint_{S'} \mathbf{E}'\cdot d\mathbf{a}' = 4\pi\int_{V'}\rho' d^3x'$

then gives $2\pi r' E_r' = 4\pi(\pi\rho' r'^2)$, for $r' \le a$

$\Rightarrow E_r' = 2\pi\rho' r' = \frac{2\pi\rho r}{\gamma_0}$ $\quad [r' = r,\ \rho' = \rho/\gamma_0]$

We now transform $\mathbf{E}_\perp'$ ($= E_r'\mathbf{e}_r$) into lab-frame $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ by using the reverse transformation equations in (46), in which we set $\mathbf{v}_0 = v_0\mathbf{e}_z$.

$$\begin{cases} \mathbf{E}_\perp = \gamma_0\left(\mathbf{E}_\perp' - \frac{\mathbf{v}_0}{c}\times\cancel{\mathbf{B}_\perp'}\right) = \gamma_0\mathbf{E}_\perp' = \gamma_0\frac{2\pi\rho r}{\gamma_0}\mathbf{e}_r = 2\pi\rho r\mathbf{e}_r \\ \mathbf{B}_\perp = \gamma_0\left(\cancel{\mathbf{B}_\perp'} + \frac{\mathbf{v}_0}{c}\times\mathbf{E}_\perp'\right) = \gamma_0\left(\frac{v_0\mathbf{e}_z}{c}\right)\times\frac{2\pi\rho r}{\gamma_0}\mathbf{e}_r = \frac{v_0}{c}2\pi\rho r\mathbf{e}_\theta \end{cases}$$

Thus, the force $\mathbf{f}$ on an electron (in the lab frame) is

$$\mathbf{f} = -e\left(\mathbf{E}+\tfrac{1}{c}\mathbf{v}\times\mathbf{B}\right) = -e\left[2\pi\rho r\mathbf{e}_r + \tfrac{1}{c}(v_0\mathbf{e}_z)\times\left(\tfrac{v_0}{c}2\pi\rho r\mathbf{e}_\theta\right)\right]$$

$$= -2\pi e\rho r\left(1-\frac{v_0^2}{c^2}\right)\mathbf{e}_r = -\frac{2\pi e\rho r}{\gamma_0^2}\mathbf{e}_r$$

[$e \equiv |e|$ is positive. For an electron beam, ρ is negative.]

Appendix A: Relativity in College Physics

(Ref. Halliday, Resnick, and Walker, "Fundamentals of Physics")

Section 1: The Lorentz Transformation

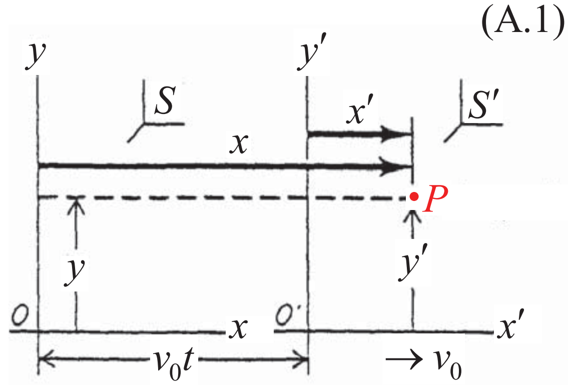
The Galilean Transformation: Consider 2 *inertial frames* S and S' (henceforth called S and S'), whose coordinates coincide at $t = 0$. S is stationary and S' moves at a constant velocity $v_0\mathbf{e}_x$ relative to S.

At time t, the position of an arbitrary point P is (x, y, z) in S and (x', y', z') in S'. Then the Galilean transformation gives

$$x' = x - v_0 t,\ y' = y,\ z' = z,\ t' = t \tag{A.1}$$

Note: The time coordinate (t) is unchanged in the Galilean transformation.

Question: How do you determine a reference frame is inertial?



A Problem and Two Possible Remedies: The laws of classical mehanics do not vary in form under the Galilean transformation, e.g. by (A.1), $\mathbf{F} = m\mathbf{a}$ in S transforms to $\mathbf{F} = m\mathbf{a}'$ in S'. However, when the same transformation is applied to the free-space wave equation: $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0,$ its form changes completely (p. 516.)

So, in order to make *all* the laws of physics invariant in form in all inertial frames, one might

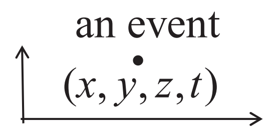
(1) Modify the EM laws so that they are invariant in form under the Galilean transformation; or

(2) Modify the Galilean transformation and the laws of mechanics to make all laws invariant in form under the new transformation.

Einstein's Postulates: Einstein took the second approach. His special theory of relativity is based on 2 postulates:

1. Laws of physics are invariant in form in all inertial frames.
2. The speed of light in vacuum has the same value c in all inertial frames, independent of the motion of the source.

Event: An <u>event</u> is something (e.g. the emission of a flashlight) which happens at a single position (x, y, z) and at a single time instant t. . In special relativity, It is described collectively by four coordinates (x, y, z, t) in the (inertial) frame it is measured



In a given frame, clocks everywhere are synchronized (same as we do today). Consider an event at (x, y, z, t). A stationary observer at (x, y, z) will find t of this event on the clock at (x, y, z) as it occurs. If he is at a distance d away from (x, y, z), he will detect the event at time $t + d/c$ on his local clock (d/c is the travel time of the event's signal). In both cases, he concludes the event's occurs at time t.

Note that in the above example, the (x, y, z, t) of an event refer to the frame in which the observer and all clocks are at rest. However, the source (e.g. a flashlight) which generates the event (x, y, z, t) is not necessarily at rest (the signal travels at speed c even if the source is in motion). It only has to be at (x, y, z) at the time instant t.

Simultaneity: Two events are <u>simultaneous</u> in a frame if they have the same time coordinate in that frame, whether or not they have the same spatial coordinates. Simultaneity can be measured as follows. If two events are detected at the same time by a stationary observer located midway, they are simultaneous in the observer's frame.

Within a given frame, the concept of space and time in the special theory of relativity is not different from our usual concept of space and time. However, we will soon find that space and time transform in a way contrary to our common intuition. For example, two simultaneous events occurring at different positions in frame S will no longer be simultaneous in frame S'.

The algebra of the theory is so simple (high school level) that you can fully understand this fascinating subject by carefully going through all the derivations, examples, problems, and exercises in subsequent pages.

Time Dilation: Consider a pulse of light emitted by a source on a train (event 1). It travels vertically upward for a distance D, then is reflected back by a mirror, and later detected at the source (event 2).

In the train frame (Fig. 1), the time interval between the 2 events is

$$\Delta t_0 = \frac{2D}{c} \text{ [in train frame]}. \quad (A.2)$$

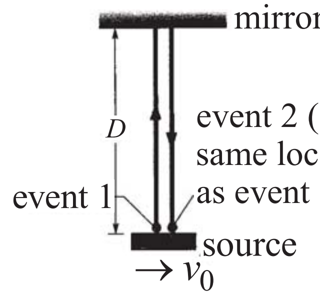


Fig. 1

In the lab frame (Fig. 2), the train, mirror, and source all move at v_0, but the light speed is still c (postulate 2). So

$$\Delta t = \frac{2L}{c} \text{ [in lab frame]}, \quad (A.3)$$

where $L = \sqrt{(\frac{1}{2}v_0\Delta t)^2 + D^2} \quad (A.4)$

Eliminating D, L in (A.2-4) gives

$$\Delta t = \gamma_0 \Delta t_0 \text{ [time dilation]}, \quad (A.5)$$

where $\gamma_0 \equiv [1 - v_0^2 / c^2]^{-\frac{1}{2}} \quad (A.6)$

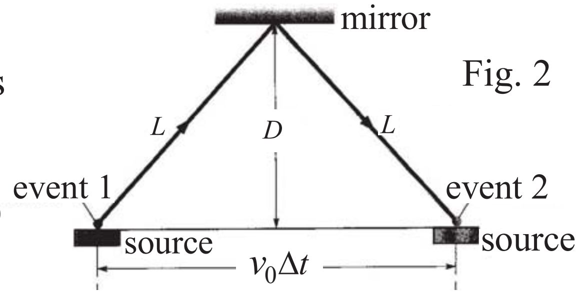


Fig. 2

Questions: 1. What if $v_0 \geq c$?

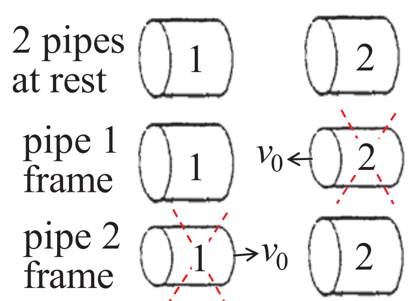
2. Why is D the same in both frames? (see next page)

Lengths perpendicular to the direction of the relative motion of 2 inertial frames (the y, z coordinates) transform as:

$$y' = y, \quad z' = z \quad (A.7)$$

We justify it by the method of contradiction (H. C. Ohanian, "Physics").

Let pipes 1 and 2 be identical in size when both are at rest (upper figure). They cannot fit inside each other because they have the same radius.



Now suppose the 2 pipes are approaching each other along their common axis. If viewed in pipe 1's rest frame (where pipe 1 is at rest and pipe 2 is approaching, middle figure), there were a transverse contraction of pipe 2, then pipe 2 would pass through the inside of pipe 1. However, by symmetry, when viewed the pipe 2's rest frame (lower figure), pipe 1 would pass through the inside of pipe 2. These two observations are contradictory; hence, (A.7) is justified.

Rewtite $\Delta t = \gamma_0 \Delta t_0$ [time dilation] [(A.5)]

Δt_0 is called the <u>proper time</u>. It is the time interval of 2 events measured in a "unique" frame, where the 2 events occur at the same position. Viewed in any other frame, these 2 events will occur at different positions and, by (A.5), their time interval Δt $(= \gamma_0 \Delta t_0)$ will be greater than the proper time by a factor of γ_0.

This is known as the effect of <u>time dilation</u>.

Example 1: The muon's average lifetime is 2.2 μsec (between birth and decay) in its rest frame. In 1977 at CERN, muons were accelerated to γ_0 = 28.87 (v_0 = 0.9994c). Within experimental error, their lab-frame average lifetime was indeed 28.87×2.2 = 63.5 μsec.

Example 2: Two synchronized clocks with near perfect precision showed slightly different readings after one had been flown around the world. The difference was again in agreement with (A.5).

Time dilation runs counter to our intuition, because it is rooted in a postulate which also runs counter to our intuition.

The twin paradox:

(See https://www.youtube.com/watch?v=n2s1-RHuljo):

Suppose someone travels on a spaceship with γ_0 = 20 and his twin brother stays on earth. By "time dilation", every day spent by the traveling twin in the spaceship ("proper time" in his frame) will be 20 days spent by the earth twin as measured in the earth frame. So the earth twin finds himself aging faster and his traveling brother will be 19 years younger when he returns to earth after an one-year journey. The paradox is:

If the traveling twin makes the same measurements, will he find himself aging faster by the same argument?

There is no paradox at all. Only the earth twin's measurement is correct, because he is always in an inertial frame. The traveling twin will have to be accelerated and decelerated in the spaceship. During these periods, he cannot use the special theory of relativity because he is not in an inertial frame. See Problem 4 below.

Length Contraction: Assume that planet neptune is stationary in the earth frame and at a distance L_0 from earth (Fig. 1). A spaceship is traveling at speed v_0 to neptune. The duration of the trip, measured on earth, is

$$\Delta t = L_0 / v_0. \tag{A.8}$$

In the spaceship's rest frame (Fig. 2), both earth and neptune move at speed v_0. The interval between earth's departure and neptune's arrival is Δt_0, which is the spaceship's "proper time" because both events occur at the same position. Thus, by (A.5)

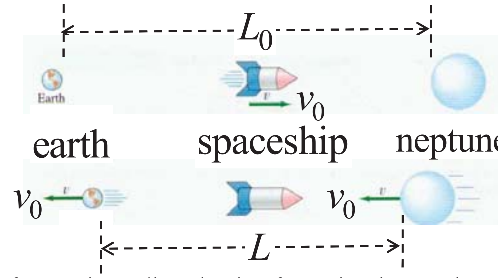


Fig. 1

Fig. 2

from Giancoli, "Physics for Scientists and Engineers"

$$\Delta t_0 = \Delta t / \gamma_0, \tag{A.9}$$

Δt_0 can be used to calculate the earth-naptune distance as viewed on the spaceship

$$L = v_0 \Delta t_0. \tag{A.10}$$

Eliminating Δt and Δt_0 from (A.8)-(A.10), we obtain

$$L = \frac{L_0}{\gamma_0} \quad \text{[length contraction]} \tag{A.11}$$

Rewrite $\quad L = \dfrac{L_0}{\gamma_0}$ [length contraction] [(A.11)]

L_0 is the length of an object (or a distance betwwen 2 objects) as measured in the rest frame of the object(s). There is only one frame where the object is at rest. Length measured in this "unique" frame is called the proper length. Viewed in any other frame, the object will be moving and, by (A.11), its length L $(= L_0/\gamma_0)$ will be less than the proper length by a factor of γ_0.

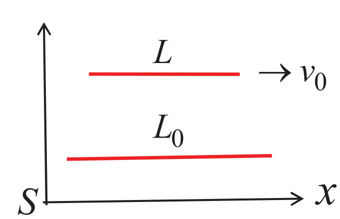
This is known as the effect of length contraction.

Note: "Length contraction" applies only to lengths along the direction of motion and it is a direct consequence of time dilation.

An illustration of length contraction:

The figure shows lengths of the same object, measured in the same frame.

If the object is moving, it is shorter than when it is at rest.



The Lorentz Transformation: Assume S and S' coincide at $t=0$ and S' moves along the common x-axis with speed v_0 relative to S. A point P has coordinates (x, y, z, t) in S and (x', y', z', t') in S'. The length x', when measured in S, is x'/γ_0 (length contraction). So,

$$x = v_0 t + \frac{x'}{\gamma_0} \quad \text{or} \quad x' = \gamma_0(x - v_0 t). \tag{A.12}$$

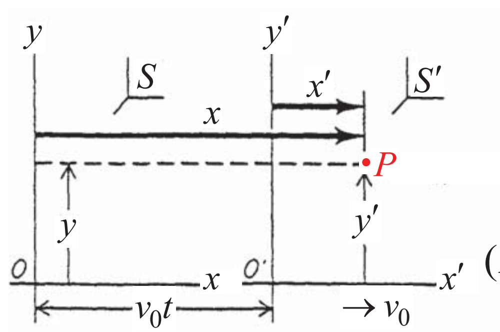
By symmetry or by similar argument, $\quad x = \gamma_0(x' + v_0 t') \qquad$ (A.13)

Eliminating x from (A.12), (A.13) [using $\gamma_0^2 - 1 = \gamma_0^2 v_0^2 / c^2$]

$$\Rightarrow \qquad t' = \gamma_0\left(t - \frac{v_0}{c^2} x\right) \tag{A.14}$$

(A.12), (A.14) together with $y' = y$, $z' = z$ [(A.7)] form the Lorentz transformation:

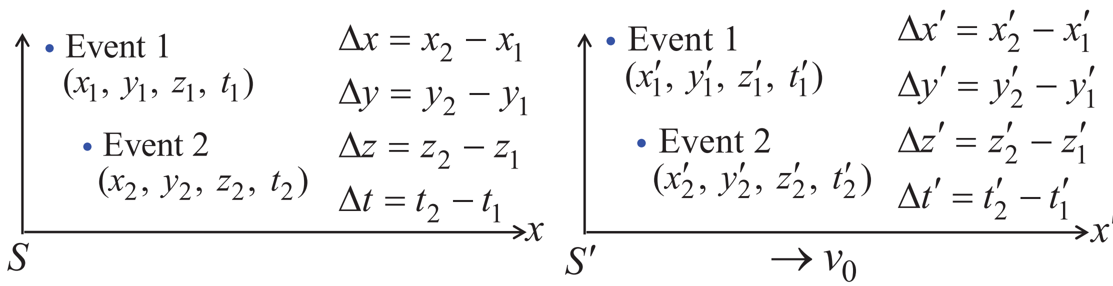
$$\begin{cases} x' = \gamma_0(x - v_0 t) \\ y' = y \\ z' = z \\ t' = \gamma_0\left(t - \dfrac{v_0}{c^2} x\right) \end{cases} \tag{A.15}$$



Transformation of Coordinate Difference between 2 Events :

The coordinate differences between 2 events are

$$\begin{cases} \text{in } S\text{:} \;\; \Delta x = x_2 - x_1, \;\; \Delta y = y_2 - y_1, \;\; \Delta z = z_2 - z_1, \;\; \Delta t = t_2 - t_1 & \text{(A.16)} \\ \text{in } S'\text{:} \;\; \Delta x' = x_2' - x_1', \;\; \Delta y' = y_2' - y_1', \;\; \Delta z' = z_2' - z_1', \;\; \Delta t' = t_2' - t_1' & \text{(A.17)} \end{cases}$$



The Lorentz transformation is linear. $\Rightarrow$ It also applies to coordinate difference transformation. Thus,

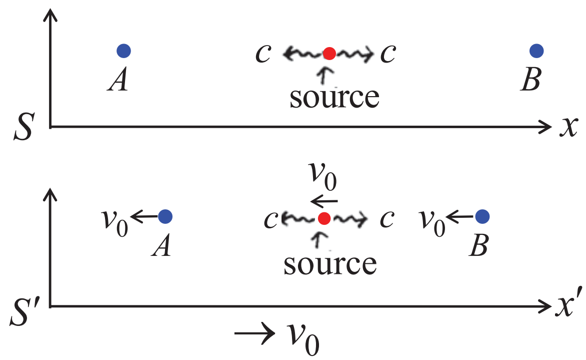
$$\begin{cases} \Delta x' = \gamma_0(\Delta x - v_0 \Delta t) \\ \Delta y' = \Delta y \\ \Delta z' = \Delta z \\ \Delta t' = \gamma_0\left(\Delta t - \dfrac{v_0}{c^2} \Delta x\right) \end{cases} \tag{A.18}$$

Discussion on simultaneity:

Rewrite $\Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)$ [(A.18)]

(A.18) indicates that, if 2 simultaneous events (i.e. $\Delta t = 0$) occur at different positions (i.e. $\Delta x \neq 0$) in S, they will not be simultaneous in S' (i.e. $\Delta t' \neq 0$).

Example: In S, a pulse of light emitted midway between points A and B (both at rest) will reach A and B at the same time, i.e. the 2 events (arrivals of signals at A and B) are simultaneous.

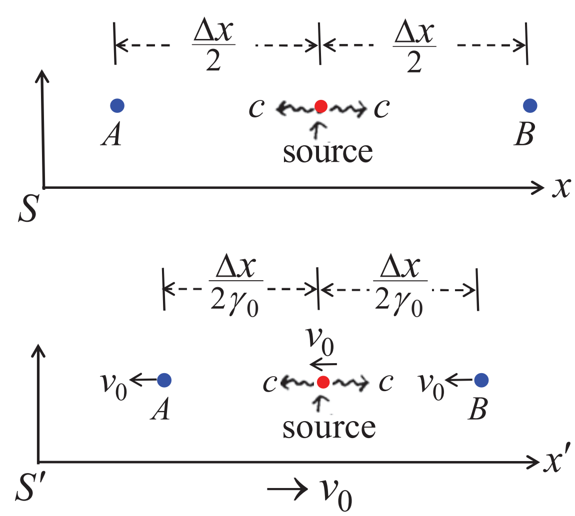


In S', the signal still travels at speed c in both directions, but B is moving toward the source and A away from it. So, the signal will reach B first. Thus, the 2 events are no longer simultaneous in S'.

Note: Relative velocity can be $> c$. e.g. between B & signal in S'.

The example discussed above can be examined quantitatively as follows.

Assume that, in S, the two events are spatially separated by a distance Δx. Observed in S', the distance is shorter by a factor of γ_0 due to length contraction, i.e. $\Delta x' = \frac{\Delta x}{\gamma_0}$



Thus, in S', the signals reach A and B by a time difference of

$$\Delta t' = t'_B - t'_A = \frac{\frac{\Delta x}{2\gamma_0}}{c+v_0} - \frac{\frac{\Delta x}{2\gamma_0}}{c-v_0} = -\frac{\Delta x}{\gamma_0}\frac{v_0}{c^2-v_0^2} = -\frac{\gamma_0 v_0}{c^2}\Delta x.$$

This is precisely the prediction of (A.18):

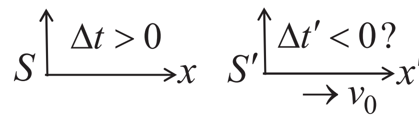
$$\Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x) = -\frac{\gamma_0 v_0}{c^2}\Delta x. \quad [\Delta t = 0 \text{ in frame } S]$$

11.A.1 The Lorentz Transformation (*continued*)

Problem 1: In S, events A and B occur at different positions, and event B occurs after event A. Is it possible for event B to precede event A in another frame S' moving at speed v_0 relative to frame S? If so, does this mean that an effect can precede its cause?

Solution: In S, let the spatial and time intervals of the 2 events be $\Delta x = x_B - x_A$ and $\Delta t = t_B - t_A$. Then, in S',

$$\Delta t' = t'_B - t'_A = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x) \text{ [(A.18)].}$$



We see that if $\Delta t < v_0 \Delta x / c^2$, then $\Delta t' < 0$, which means that the order of *independent* events in S may be reversed in S'.

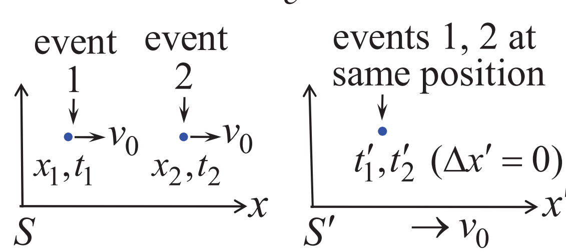
Suppose, however, that the events are connected, i.e. event B is caused by event A. This would require a message to travel from A to B. Rewrite (A.18) as $\Delta t' = \gamma_0 \Delta t (1 - \frac{v_0}{c^2}\frac{\Delta x}{\Delta t})$. Since the fastest speed for a message to travel from A to B is $\Delta x / \Delta t = c$, we must have $v_0 > c$ in order for $\Delta t' < 0$. This is not possible and thus the order of connected events (cause and effect) cannot be reversed.

11.A.1 The Lorentz Transformation (*continued*)

Problem 2: Show that "time dilation" is implicit in the Lorentz transformation: $\Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)$ or $\Delta t = \gamma_0(\Delta t' + \frac{v_0}{c^2}\Delta x')$

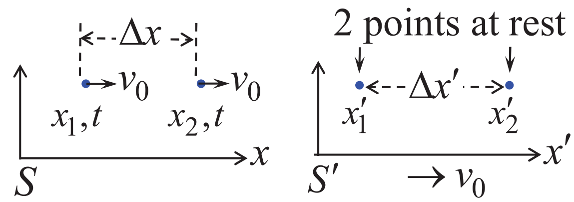
Let $\Delta t'$ be the "proper time" in S', i.e. 2 events occur at the same x' (or $\Delta x' = 0$). So, we use the 2nd eq. & set $\Delta x' = 0$ to obtain

$\Delta t = \gamma_0 \Delta t'$ [time dilation].

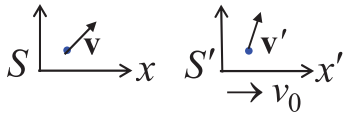


Problem 3: Show that "length contraction" is implicit in the Lorentz transformation: $\Delta x' = \gamma_0(\Delta x - v_0 \Delta t)$ or $\Delta x = \gamma_0(\Delta x' + v_0 \Delta t')$

In S', let points x'_1, x'_2 be at rest. To find the "proper length" $\Delta x'$, we can measure x'_1, x'_2 at any time (i.e. $\Delta t'$ is unknown). But x_1, x_2 are moving in S. To find Δx, x_1, x_2 must be measured simultaneously. So we use the 1st eq. & set $\Delta t = 0$ to obtain $\Delta x = \frac{\Delta x'}{\gamma_0}$ (length contraction).



Transformation of Velocity : $\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{x}}{\Delta t}$ (in S); $\mathbf{v}' = \lim_{\Delta t' \to 0} \frac{\Delta \mathbf{x}'}{\Delta t'}$ (in S')

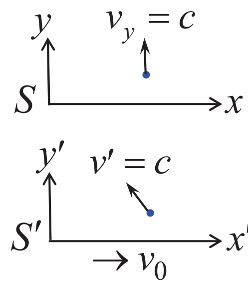


$$\begin{cases} \Delta x' = \gamma_0(\Delta x - v_0 \Delta t) \\ \Delta y' = \Delta y \\ \Delta z' = \Delta z \\ \Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x) \end{cases} \Rightarrow \begin{cases} v_x' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma_0(\Delta x - v_0 \Delta t)}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} \overset{\boxed{\Delta t,\ \Delta t' \to 0}}{=} \frac{v_x - v_0}{1 - \frac{v_0}{c^2} v_x} \\ v_y' = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} = \frac{v_y}{\gamma_0(1 - \frac{v_0}{c^2} v_x)} \\ v_z' = \frac{\Delta z'}{\Delta t'} = \frac{\Delta z}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} = \frac{v_z}{\gamma_0(1 - \frac{v_0}{c^2} v_x)} \end{cases} \quad \text{(A.20)}$$

Example 1 : In S, a flash light is emitted vertically. In S', its velocity is

$$v_x' = \frac{\cancel{v_x} - v_0}{1 - \frac{v_0}{c^2}\cancel{v_x}} = -v_0;\ v_y' = \frac{v_y}{\gamma_0(1 - \frac{v_0}{c^2}\cancel{v_x})} = \frac{c}{\gamma_0} \quad \begin{bmatrix} \textit{Note}: \\ v_y' < c \end{bmatrix}$$

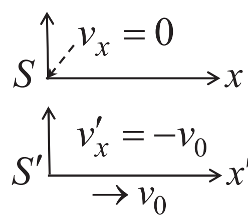
$$\Rightarrow v' = [v_0^2 + c^2(1 - v_0^2/c^2)]^{1/2} = c \text{ [as expected]}$$



Exercise : Relate this example to the derivation of (A.5).

Example 2 : The origin of S is at rest ($v_x = 0$) in S.

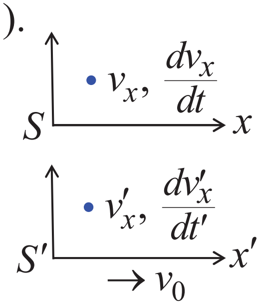
$$\Rightarrow \text{In } S', \text{ the origin of } S \text{ moves at } v_x' = \frac{\cancel{v_x} - v_0}{1 - \frac{v_0}{c^2}\cancel{v_x}} = -v_0$$



Transformation of Acceleration :

We first consider the transformation of $\frac{dv_x}{dt}$ ($\mathbf{a} \parallel \mathbf{e}_x$).

$$v_x' = \frac{v_x - v_0}{1 - \frac{v_0}{c^2} v_x} \quad \text{[(A.20)]}$$



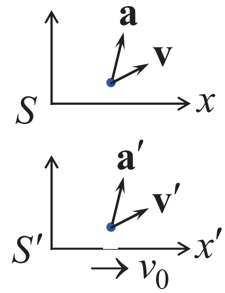
$$\Rightarrow dv_x' = \frac{dv_x}{1 - \frac{v_0}{c^2} v_x} - \frac{(v_x - v_0)(-\frac{v_0}{c^2})dv_x}{(1 - \frac{v_0}{c^2} v_x)^2} = \frac{dv_x}{\gamma_0^2 (1 - \frac{v_0}{c^2} v_x)^2}$$

$$t' = \gamma_0(t - \frac{v_0}{c^2} x) \text{ [(A.15)]} \Rightarrow dt' = \gamma_0(dt - \frac{v_0}{c^2} dx) = \gamma_0(1 - \frac{v_0}{c^2} v_x) dt$$

$$\text{Hence, } \frac{dv_x'}{dt'} = \frac{1}{\gamma_0(1 - \frac{v_0}{c^2} v_x) dt} \frac{dv_x}{\gamma_0^2 (1 - \frac{v_0}{c^2} v_x)^2} = \frac{1}{\gamma_0^3 (1 - \frac{v_0}{c^2} v_x)^3} \frac{dv_x}{dt} \quad \text{(A.21)}$$

By the same method, we may obtain the transformation equations for acceleration in arbitrary directions (see Jackson Problem 11.5).

$$\begin{cases} \mathbf{a}'_{\parallel} = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\mathbf{a}_{\parallel} \\ \mathbf{a}'_{\perp} = \dfrac{1}{\gamma_0^2\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2}\times(\mathbf{a}\times\mathbf{v})\right] \end{cases} \tag{A.22}$$

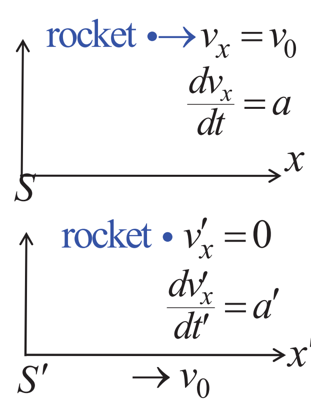


where "$\parallel$" and "$\perp$" is with respect to the direction of $\mathbf{v}_0$.

Discussion on the relative speed v_0 in $\gamma_0 = \dfrac{1}{\sqrt{1-v_0^2/c^2}}$:

From the definition of γ_0, we find that, for the transformed quantities to be physical (i.e. not an infinite or complex value), the theory of special relativity requires $v_0 < c$. As shown in (A.33) below and in all other cases later, the theory is self-consistent in that it forbids any object to be accelerated to speed c.

Problem 4: A rocket is launched from the earth into outer space. It moves on a straight line with a *constant* acceleration (a') in the *rocket's rest frame*. Calculate the time required for it to accelerate from 0 speed to the final speed v_f, according to earth and rocket clocks.



Solution: We first relate the rocket's earth-frame acceleration (a) to a' (a *const*).

Let S be the earth frame in which the rocket moves at the speed v_x. Let S' be an inertial frame moving at speed $v_0 (= v_x)$, so that the rocket is instantaneously at rest in S'.

Note: (1) The rocket is powered by the same engine. So an observer in any S' will see a constant rocket accelartaion a'.

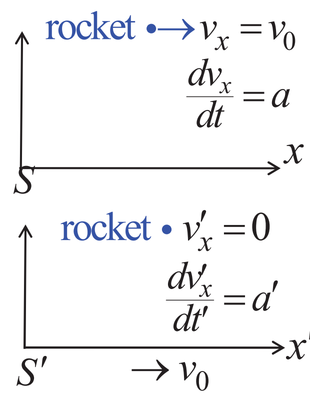
(2) S' is an *instantaneous* frame. Its speed v_0 always equals v_x (rocket's speed in S) so the rocket is always instantaneously at rest in S' with a constant acceleration a'. At every instant, we use a new instantaneous frame S' ($v_0 = v_x$) for Lorentz transformation.

Rewrite $\dfrac{dv_x'}{dt'} = \dfrac{1}{\gamma_0^3(1-\frac{v_0}{c^2}v_x)^3}\dfrac{dv_x}{dt}$ [(A.21)]

$$\Rightarrow \frac{dv_x}{dt} = \frac{1}{\gamma_0^3(1+\frac{v_0}{c^2}v_x')^3}\frac{dv_x'}{dt'} \quad \begin{bmatrix}\text{inverse Lorentz}\\ \text{transformation}\end{bmatrix}$$

In S', $v_x' = 0$ and $\dfrac{dv_x'}{dt'} = a'$

$$\Rightarrow \frac{dv_x}{dt} = \frac{1}{\gamma_0^3}a' \Rightarrow \ a = \frac{a'}{\gamma_0^3} \ \text{ with } \gamma_0 = (1-\frac{v_0^2}{c^2})^{-\frac{1}{2}}$$



v_0 (relative speed between S and S') $= v_x$ (rocket speed in S)

$$\Rightarrow \gamma_0 = (1-\frac{v_0^2}{c^2})^{-\frac{1}{2}} \begin{bmatrix}\text{Lorentz factor}\\ \text{btn. } S \text{ and } S'\end{bmatrix} = \gamma = (1-\frac{v_x^2}{c^2})^{-\frac{1}{2}} \begin{bmatrix}\text{Lorentz factor}\\ \text{of rocket in } S\end{bmatrix}$$

$$\Rightarrow a = \frac{a'}{\gamma^3} = a'(1-\frac{v_x^2}{c^2})^{\frac{3}{2}} \begin{bmatrix}\text{accelaration } a \text{ (in } S\text{) in terms of speed}\\ v_x \text{ in } S \text{ and accelaration } a' \text{ in } S'\end{bmatrix} \quad \text{(A.23)}$$

Rewrite $\qquad a = a'/\gamma^3 = a'(1-v_x^2/c^2)^{3/2}$ [(A.23)]

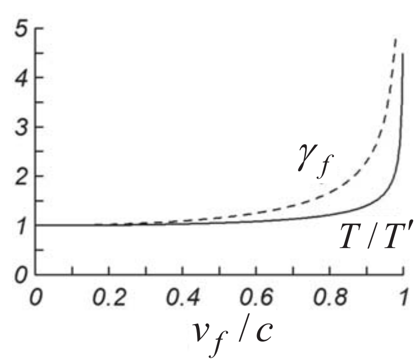
The total acceleration time as measured on earth is (use $dv_x = adt$)

$$T = \int_0^T dt = \int_0^{v_f}\frac{1}{a}dv_x = \int_0^{v_f}\frac{\gamma^3}{a'}dv_x = \frac{1}{a'}\int_0^{v_f}\frac{dv_x}{(1-v_x^2/c^2)^{3/2}} = \frac{v_f}{a'(1-v_f^2/c^2)^{1/2}}$$

The final speed of the rocket is specified in the earth frame. Thus, to calculate the acceleration time measured in the rocket, we convert dt' to dt through $dt' = dt/\gamma$ [$\gamma(v_x)$: instantaneous time dilation factor]

$$T' = \int_0^{T'} dt' = \int_0^T \frac{1}{\gamma}dt = \int_0^{v_f}\frac{1}{\gamma a}dv_x = \frac{1}{a'}\int_0^{v_f}\frac{dv_x}{1-v_x^2/c^2} = \frac{c}{2a'}In(\frac{1+v_f/c}{1-v_f/c}).$$

1. $\frac{v_f}{c} \to 0 \Rightarrow T \approx T'$ (non-relativistic case)
2. $\frac{v_f}{c} \to 1 \Rightarrow \frac{T}{T'}$ increases rapidly with $\frac{v_f}{c}$.
3. T is always $> T' \Rightarrow$ Acceleration time is longer viewed on earth than on rocket. This is why the earth twin ages faster during the traveling twin's journey.



Section 2: Relativistic Mechanics

(Ref.: H. C. Ohanian, “Physics,” 2nd ed., pp.1013-1014.)

Conservation of momentum in classical mechanics: For an isolated system of particles, $\sum m_j \mathbf{v}_j$(before collision) $= \sum m_j \mathbf{u}_j$(after collision).

Under the Galilean transformation, the statement is true in all (inertial) frames. However, under the Lorentz transformation, even if $\sum m_j \mathbf{v}_j$ is conserved in one frame, it will in general not be conserved in another frame. Thus, postulate 1 is violated if we continue to define the momentum as $m\mathbf{v}$. The theory of special relativity takes a major step by redefining (or postulating) the momentum and energy as

$$\begin{cases} \mathbf{p} = \gamma m \mathbf{v} & \text{(A.24)} \\ E = \gamma m c^2 & \text{(A.25)} \end{cases}$$

m : rest mass

Note: $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor of a particle. It is to be distinguished from the Lorentz factor γ_0 for the transformation.

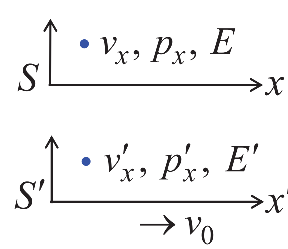
For simplicity, we will consider only one-dimentional motion along the x-axis. The momentum and energy of a particle are then

$$p_x = \frac{m v_x}{\sqrt{1 - v_x^2/c^2}} \quad \text{and} \quad E = \frac{mc^2}{\sqrt{1 - v_x^2/c^2}}$$

We can find out how $p_x = \frac{m v_x}{\sqrt{1 - v_x^2/c^2}}$ and $E = \frac{mc^2}{\sqrt{1 - v_x^2/c^2}}$ transform because we know v_x transforms as $v_x' = \frac{v_x - v_0}{1 - v_x v_0/c^2}$ [(A.20)]. Write p_x of the particle in S' as (rest mass m is the same in all frames)

$$p_x' = \frac{m v_x'}{\sqrt{1 - v_x'^2/c^2}} = \frac{m(v_x - v_0)}{1 - v_x v_0/c^2} \frac{1}{\sqrt{1 - (1/c^2)[(v_x - v_0)/(1 - v_x v_0/c^2)]^2}}$$

$$= \frac{m(v_x - v_0)}{\sqrt{(1 - v_x v_0/c^2)^2 - (v_x - v_0)^2/c^2}}$$



$$\boxed{\begin{aligned} &(1 - v_x v_0/c^2)^2 - (v_x - v_0)^2/c^2 \\ &= (1 - v_0^2/c^2)(1 - v_x^2/c^2) \end{aligned}}$$

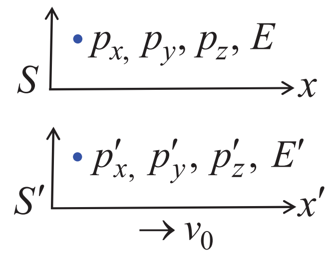
$$\Rightarrow p_x' = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{m v_x}{\sqrt{1 - v_x^2/c^2}} - \frac{v_0}{\sqrt{1 - v_0^2/c^2}} \frac{m}{\sqrt{1 - v_x^2/c^2}} = \gamma_0 (p_x - \frac{v_0}{c^2} E) \quad \text{(A.26)}$$

Similarly, we derive the Lorentz transformation equation for E:

$$E' = \gamma_0 (E - v_0 p_x) \quad \text{(A.27)}$$

By the same method, we can extend the motion to 3 dimensions and derive the Lorentz transformation equations for **p** and E. The result is

$$\begin{cases} p'_x = \gamma_0 (p_x - \frac{v_0}{c^2} E) \\ p'_y = p_y \\ p'_z = p_z \\ E' = \gamma_0 (E - v_0 p_x) \end{cases} \qquad \text{(A.28)}$$



(A.28) shows that $\mathbf{p}'$ and E' in S' is a *linear* combination of **p** and E in S, with constant coefficients (i.e. the coefficients are independent of **p** and E of the particle). The same equations therefore hold true for the *total* momentum and energy ($\Sigma \mathbf{p}_j$, ΣE_j) of a system of particles,

$$\begin{cases} \Sigma p'_{jx} = \gamma_0 (\Sigma p_{jx} - \frac{v_0}{c^2} \Sigma E_j) \\ \Sigma p'_{jy} = \Sigma p_{jy} \\ \Sigma p'_{jz} = \Sigma p_{jz} \\ \Sigma E'_j = \gamma_0 (\Sigma E_j - v_0 \Sigma p_{jx}) \end{cases} \qquad \begin{bmatrix} \mathbf{p} = \gamma m \mathbf{v} \\ E = \gamma m c^2 \end{bmatrix} \qquad \text{(A.29)}$$

$$\text{Rewrite} \begin{cases} \Sigma p'_{jx} = \gamma_0 (\Sigma p_{jx} - \frac{v_0}{c^2} \Sigma E_j) \\ \Sigma p'_{jy} = \Sigma p_{jy} \\ \Sigma p'_{jz} = \Sigma p_{jz} \\ \Sigma E'_j = \gamma_0 (\Sigma E_j - v_0 \Sigma p_{jx}) \end{cases} \Rightarrow \begin{bmatrix} \text{In } S\text{, if } \Sigma E_j \text{ is conserved,} \\ \text{but } \Sigma p_{jx} \text{ is unconserved,} \\ \text{then in } S'\text{, both } \Sigma p'_{jx} \text{ and} \\ \Sigma E'_j \text{ are unconserved.} \end{bmatrix}$$

$\Rightarrow$ Momentum (Σp_{jx}, Σp_{jy}, Σp_{jz},) and energy (ΣE_j) combine into a single conservation law:

For a system of isolated particles, we have in S

$$\begin{cases} \Sigma \mathbf{p}_j (\text{before collision}) = \Sigma \mathbf{p}_j (\text{after collision}) \textit{ and} \\ \Sigma E_j (\text{before collision}) = \Sigma E_j (\text{after collision}) \end{cases} \qquad \text{(A30a)}$$

Then, in S', the law has the same form:

$$\begin{cases} \Sigma \mathbf{p}'_j (\text{before collision}) = \Sigma \mathbf{p}'_j (\text{after collision}) \textit{ and} \\ \Sigma E'_j (\text{before collision}) = \Sigma E'_j (\text{after collision}) \end{cases} \qquad \text{(A30b)}$$

Discussion:

1. Here, by postulating $\begin{cases} \mathbf{p} = \gamma m \mathbf{v} & [(A.24)] \\ E = \gamma mc^2 & [(A.25)] \end{cases}$, we show that the law of conservation of momentum and energy is independent in form in all inertial frames. Conversely, from the requirement of frame -independence of this conservation law, we can deduce that $\mathbf{p} = \gamma m \mathbf{v}$ & $E = \gamma mc^2$, but in a much more complicated way (Jackson Sec. 11.5).

2. Writing $E = \gamma mc^2 = (\gamma - 1)mc^2 + mc^2$, we may divide the total energy into the kinetic enery $(\gamma - 1)mc^2$ (due to motion) and a new form of energy mc^2 called the rest-mass energy (an intrinsic energy).

$$\text{kinetic energy} = (\gamma - 1)mc^2 \approx \tfrac{1}{2}mv^2(1 + \tfrac{3}{4}\tfrac{v^2}{c^2} + \cdots) \text{ if } v \ll c \qquad (A.31)$$

$\Rightarrow$ Relativistically, a particle with the same kinetic energy has a smaller v than given by classical mechanics: kinetic energy $= \frac{1}{2}mv^2$.

Problem 5: Two identical particles of rest mass m and equal and opposite velocities $\pm \mathbf{v}$ collide head-on inelastically to form a single particle. Find the mass and velocity of the new particle.

Solution: $\quad m,\ \gamma \bullet \rightarrow \mathbf{v} \qquad \mathbf{v} \leftarrow \bullet\ m,\ \gamma$ (before)

$M_{cm} \bullet$ (after)

The total momentum before the collision is $\gamma m \mathbf{v} - \gamma m \mathbf{v} = 0$. So the collision occurs in the center-of-momentum (CM) frame, i.e. the frame in which the total momentum of all particles vanishes.

For later comparison with the result in problem 6, we denote the mass of the new particle by M_{cm} to indicate that it is formed in the CM frame.

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \text{The new particle is stationary.} \\ \text{Conservation of energy} \Rightarrow \gamma m + \gamma m = M_{cm} \Rightarrow M_{cm} = 2\gamma m \end{cases}$$

In classical mechanics, $M_{cm} = 2m$. In relativity, $M_{cm} = 2\gamma m$. Which is correct? A precise $F = M_{cm}a$ experiment can prove $M_{cm} = 2\gamma m$.

Discussion: In this problem, we find $M_{cm} = 2\gamma m > 2m$, i.e. rest mass has been created from the kinetic energies $[2(\gamma-1)mc^2]$ of the 2 colliding particles. There is no need to know what's inside the new particle (e.g. its chemical or thermal energy). We only need to know its rest mass and hence the energy associated with it. For the present problem, we may deduce that the lost kinetic energy has become thermal energy inside M_{cm} (hence the interesting fact that a hotter object has a greater rest mass). In many other cases, it's not possible to know what's inside (See Feynman Lectures, Vol. 1, Sec. 16-5).

$$\boxed{\begin{array}{lr} m,\ \gamma \bullet\!\rightarrow \mathbf{v} \quad \mathbf{v} \leftarrow\! \bullet\ m,\ \gamma & \text{(before)} \\ M_{cm} \bullet & \text{(after)} \end{array}}$$

Nuclear fusion and fission are examples of non-conservation of rest mass, in which the total rest mass is reduced after the reaction and the mass deficit appears as kinetic energies and radiation.

In fact, all reactions (chemical or nuclear) in which energy is absorbed (e.g. photosynthesis) or released (e.g. digestion of food, battery discharge) involve a change of the reactants' total rest mass.

Problem 6: A particle of rest mass m and velocity $\mathbf{v}$ collides with a stationary particle of the same rest mass and is absorbed by it. Find the rest mass and velocity of the new particle.

The collision occurs in the stationary-target (ST) frame. So, we denote the new particle mass by M_{st}, velocity by $\mathbf{V}_{st}$, and Lorentz factor by γ_{st} $[=(1-V_{st}^2/c^2)^{-1/2}]$ (γ & $\mathbf{v}$ are also ST frame quantities).

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \ \gamma m\mathbf{v} = \gamma_{st} M_{st} \mathbf{V}_{st} & \text{(A.32)} \\ \text{Conservation of energy} \quad\ \Rightarrow \ (\gamma+1)m = \gamma_{st} M_{st} & \text{(A.33)} \end{cases}$$

$$\frac{\text{(A.32)}}{\text{(A.33)}} \Rightarrow \mathbf{V}_{st} = \frac{\gamma}{\gamma+1}\mathbf{v}$$

$$\text{(A.33)} \Rightarrow M_{st} = \frac{\gamma+1}{\gamma_{st}} m$$

$$\boxed{\begin{array}{lr} m,\ \gamma \bullet\!\longrightarrow \mathbf{v} \qquad \bullet\ m & \text{(before)} \\ M_{st},\ \gamma_{st} \ \bullet\!\longrightarrow \mathbf{V}_{st} & \text{(after)} \end{array}}$$

$$\Rightarrow M_{st}^2 = m^2 \frac{(\gamma+1)^2}{\gamma_{st}^2} = m^2(\gamma+1)^2(1-\frac{V_{st}^2}{c^2})^2 = m^2(\gamma+1)^2[1-\frac{\gamma^2 v^2}{c^2(\gamma+1)^2}]$$

$$= m^2(\gamma^2+2\gamma+1-\gamma^2\frac{v^2}{c^2}) = m^2[\gamma^2(1-\frac{v^2}{c^2})+2\gamma+1] = 2m^2(\gamma+1)$$

$$\Rightarrow M_{st} = \sqrt{2(\gamma+1)}m$$

Discussion:

In problem 5 (CM frame), the new particle mass is $M_{cm} = 2\gamma m$ (A.34)

$m,\ \gamma \bullet\!\rightarrow \mathbf{v} \quad \mathbf{v} \leftarrow\!\bullet\ m,\ \gamma$ (before)
$M_{cm}\ \bullet$ (after)

In problem 6 (ST frame), the new particle mass is $M_{st} = \sqrt{2(1+\gamma)}m$ (A.35)

$m,\ \gamma \bullet\!\longrightarrow \mathbf{v} \quad \bullet\ m$ (before)
$M_{st},\ \gamma_{st}\ \bullet\!\longrightarrow \mathbf{V}_{st}$ (after)

Note: γ is the Lorentz factor of each particle before the collision.

In particle physics exp., $M_{cm}c^2$ or $M_{st}c^2$ is the rest mass energy available for the creation of new particles. In $\gamma_{st}M_{st}c^2$, the motion energy $[(\gamma_{st}-1)M_{st}c^2]$ is unavailable. It vanishes in the rest frame.

The rest mass energy of either e^- or e^+ is $mc^2 = 0.511$ MeV. If 2 TeV of energy is needed for particle creation (i.e. $M_{cm}c^2 = 2$ TeV or $M_{st}c^2 = 2$ TeV), then the required γ of the colliding e^- and e^+ is

$$\begin{cases} \text{by (A.34), } M_{cm}c^2 = 2\gamma mc^2 = 2 \text{ TeV} \Rightarrow \gamma \approx 1.96\times10^6 & \text{[CM frame]} \\ \text{by (A.35), } M_{st}c^2 = \sqrt{2(1+\gamma)}mc^2 = 2 \text{ TeV} \Rightarrow \gamma \approx 7.7\times10^{12} & \text{[ST frame]} \end{cases}$$

Thus,

$$\frac{\text{kinetic energy needed in CM frame}}{\text{kinetic energy needed in ST frame}} = \frac{2\times(1.957\times10^6-1)}{7.66\times10^{12}-1} \approx 5\times10^{-7}$$

This shows that far less kinetic energy is needed in the CM frame than in the ST frame. In fact, all the kinetic energy of the 2 colliding particles $[2\times(1.96\times10^6-1)\times0.511 \text{ MeV} = 2 \text{ TeV}]$ is put in use (i.e. generate new particles) in the CM frame, while in the ST frame, 99.99995% of the kinetic energy of the incident particle is wasted! This is why the International Linear Collider (ILC) project plans to accelerate both the e^- and e^+ to energies up to 1 TeV so that the collision occurs in the CM frame. The ILC design is ~30 km long and costs ~US$7 billion (expected to be completed in the 2040s).

Question: Why use a long linear accelerator instead of a more compact circular accelerator? (to avoid radiation loss, see Ch. 14)

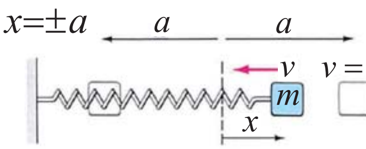
Problem 7: A rest mass m oscillates on the x-axis with amplitude a under the force of a spring : $m\omega^2 x$ ($\omega = const$), i.e. exchange between kinetic and poetntial energies. Express the *relativistic* oscillation period τ as an integral, and obtain the 2 leading terms of this integral for $v \ll c$.

Solution: The period τ is given by $\tau = 4\int_0^a \frac{dx}{v}$ (A.36a)

The velocity v is given by the conservation of relativistic energy :

$$mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} + \tfrac{1}{2}m\omega^2 x^2 = \underbrace{mc^2 + \tfrac{1}{2}m\omega^2 a^2}_{\gamma=1 \text{ at } x=\pm a} \qquad \text{(A.36b)}$$

Sub. v from (A.36b) into (A.36a)



$$\Rightarrow \tau = \frac{4}{\omega}\int_0^a dx \frac{1+\omega^2(a^2-x^2)/2c^2}{(a^2-x^2)^{1/2}[1+\omega^2(a^2-x^2)/(4c^2)]^{1/2}}$$

Expand the integrand in powers of $\omega^2(a^2-x^2)/c^2$ ($\ll 1$) and use

$$\int_0^b dy \frac{1}{(b^2-y^2)^{1/2}} = \frac{\pi}{2} \text{ and } \int_0^b dy(b^2-y^2)^{1/2} = \frac{b^2\pi}{4} \text{ (for } b^2 > 0)$$

$$\Rightarrow \tau = \frac{2\pi}{\omega}\left(1+\frac{3\omega^2 a^2}{16c^2}+\cdots\right)$$

[*Question*: Why is τ larger relativistically? See (A.31).]

$\mathbf{p} = \gamma m\mathbf{v}$ suggests that we postulate the relativistic eq. of motion:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F} \quad [\mathbf{p} = \gamma m\mathbf{v};\ \mathbf{F}\text{: force}] \qquad \text{(A.37)}$$

Example 1: $\mathbf{F} \parallel \mathbf{v}$ (1-D motion) $\bullet \to v,\ \to F$

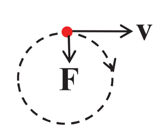
$$F = \frac{d}{dt}(\gamma m v) = mv\underbrace{\frac{d\gamma}{dt}} + \gamma m\frac{dv}{dt} = \gamma m\frac{dv}{dt}\Big(\underbrace{\gamma^2\frac{v^2}{c^2}+1}\Big) = \overbrace{\gamma^3 m\frac{dv}{dt}}^{\text{cf. (A.23)}} \qquad \text{(A.38)}$$

$$\frac{d}{dt}\gamma = \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt} = \gamma^3\frac{v}{c^2}\frac{dv}{dt}$$

$$\gamma^2\frac{v^2}{c^2}+1 = \frac{v^2/c^2}{1-v^2/c^2}+1 = \frac{v^2/c^2+1-v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2$$

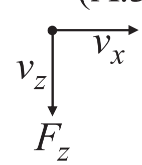
$$F = \gamma^3 m\frac{dv}{dt} \Rightarrow \begin{cases}\text{Const. } F \text{ does not cause const. acceleration } (dv/dt).\\ \gamma \to \infty \text{ as } v \to c \Rightarrow v \text{ can never be accelerated to } c.\end{cases}$$

Example 2: $\mathbf{F} \perp \mathbf{v}$ (centripetal force $\Rightarrow \gamma = const.$)



$$\mathbf{F} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(\gamma m\mathbf{v}) = \gamma m\frac{d}{dt}\mathbf{v} \qquad \text{(A.39)}$$

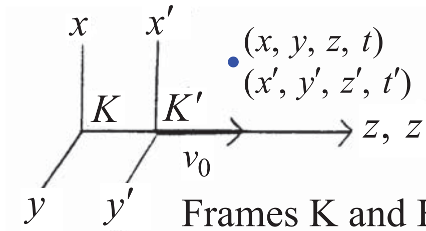
Question : A particle with $\mathbf{v} = v_x\mathbf{e}_x + v_z\mathbf{e}_z$ is under a force $F_z\mathbf{e}_z$. As v_z is accelerated by F_z, v_x will decelerate. Why?



Appendix B: Summary of Lorentz Transformation Equations

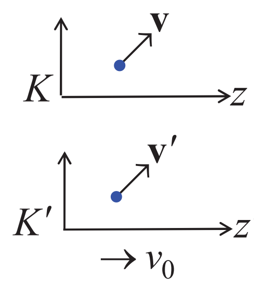
For all equations, $\gamma_0 \equiv (1-\frac{v_0^2}{c^2})^{-\frac{1}{2}}$. By symmetry, equations for the inverse transformation differ only by the sign of v_0 (or $\mathbf{v}_0$).

1. $$\begin{cases} x' = x \\ y' = y \\ z' = \gamma_0\left(z - v_0 t\right) \\ t' = \gamma_0\left(t - \frac{v_0}{c^2} z\right) \end{cases}$$

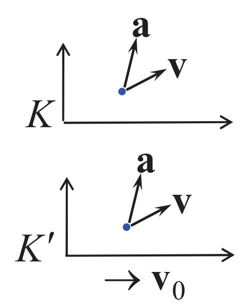


Frames K and K′ coincide at $t = t' = 0$.

2. $$\begin{cases} v_x' = \dfrac{v_x}{\gamma_0(1-\frac{v_0}{c^2}v_z)} \\ v_y' = \dfrac{v_y}{\gamma_0(1-\frac{v_0}{c^2}v_z)} \\ v_z' = \dfrac{v_z - v_0}{1-\frac{v_0}{c^2}v_z} \end{cases}$$



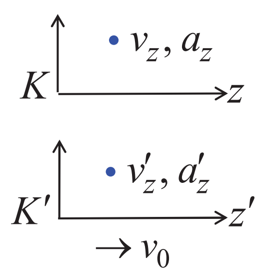
3. $$\begin{cases} \mathbf{a}_{\|}' = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\mathbf{a}_{\|} \\ \mathbf{a}_{\perp}' = \dfrac{1}{\gamma_0^2\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2}\times\left(\mathbf{a}\times\mathbf{v}\right)\right] \end{cases}$$



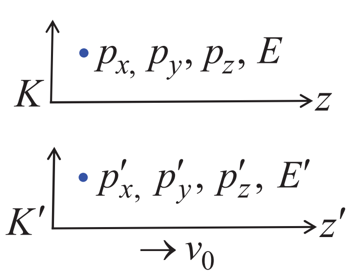
where "$\|$" and "$\perp$" are with respect to the direction of $\mathbf{v}_0$.

Special case: one dimensional motion

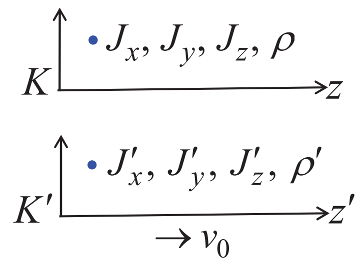
$$a_z' = \frac{1}{\gamma_0^3(1-\frac{v_0}{c^2}v_z)^3} a_z$$



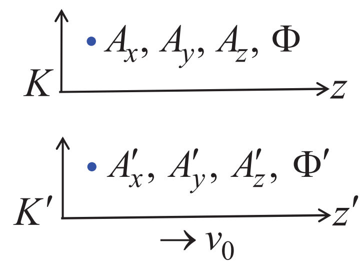
4. $$\begin{cases} p_x' = p_x \\ p_y' = p_y \\ p_z' = \gamma_0(p_z - \frac{v_0}{c^2}E) \\ E' = \gamma_0(E - v_0 p_z) \end{cases}$$



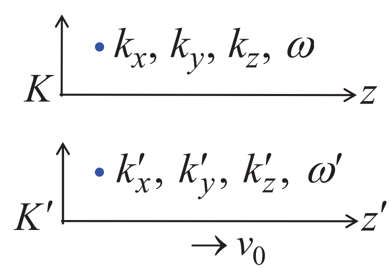
$$
5.\quad \begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0 (J_z - v_0 \rho) \\ \rho' = \gamma_0 \left(\rho - \frac{v_0}{c^2} J_z\right) \end{cases}
$$



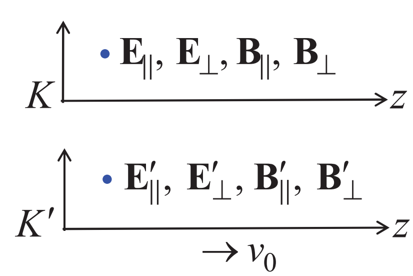
$$
6.\quad \begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0 \left(A_z - \frac{v_0}{c} \Phi\right) \\ \Phi' = \gamma_0 \left(\Phi - \frac{v_0}{c} A_z\right) \end{cases}
$$



$$
7.\quad \begin{cases} k'_x = k_x \\ k'_y = k_y \\ k'_z = \gamma_0 \left(k_z - \frac{v_0}{c^2} \omega\right) \\ \omega' = \gamma_0 (\omega - v_0 k_z) \end{cases}
$$



$$
8.\quad \begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0 \left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0 \left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c} \times \mathbf{E}_{\perp}\right) \end{cases}
$$



where "$\parallel$" and "$\perp$" are with respect to the direction of $\mathbf{v}_0$.