Chapter 6: Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

6.1 Mawell's Displacement Current; Maxwell Equations

The Displacement Current :

So far, we have the following set of laws :

$$\nabla \cdot \mathbf{D} = \rho,\ \nabla \times \mathbf{H} = \mathbf{J},\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \text{and}\ \nabla \cdot \mathbf{B} = 0 \qquad (6.1)$$

Note: From here on, we return to Jackson's notations, i.e. ρ_{free} & $\mathbf{J}_{free}$ (due to free charges) in lecture notes (Chs. 4 & 5) are now ρ & $\mathbf{J}$.

Taking the divergence of $\nabla \times \mathbf{H} = \mathbf{J}$, we obtain

$$\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{0} = \nabla \cdot \mathbf{J} = 0 \qquad (6.2)$$

$$\Rightarrow\ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \neq 0 \text{ if } \frac{\partial \rho}{\partial t} \neq 0$$

in violation of conservation of charge: $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$ [(5.2)].

Maxwell in 1865 (34 years after Faraday's law and 80 years after Coulomb's law) modified Ampere's law by postulating

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad (6.5)$$

Then, $\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{0} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\underbrace{\nabla \cdot \mathbf{D}}_{\rho} \Rightarrow \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

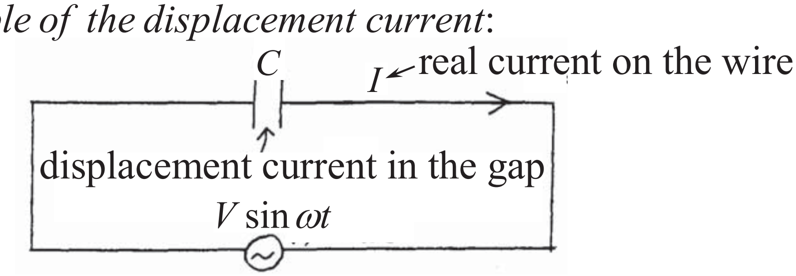
$\Rightarrow$ (6.5) is consistent with conservation of charge.

The immediate significance of (6.5) is that it predicts a new mechanism to generate **B**, i.e. by a time-varying **E**.

(6.5) can be written: $\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_D$,

where $\mathbf{J}_D \equiv \frac{\partial \mathbf{D}}{\partial t}$ is called the <u>displacement current</u> by Maxwell.

Example of the displacement current:



The Maxwell Equations :

Replacing the static equation $\nabla\times\mathbf{H}=\mathbf{J}$ [(6.1)] with the dynamic equation $\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$ [(6.5)], we obtain a set of equations called

the Maxwell equations:
$$\begin{cases}\nabla\cdot\mathbf{B}=0 \\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{D}=\rho \\ \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\end{cases} \tag{6.6}$$

homogeneous eqs.

inhomogeneous eqs.

where ρ and $\mathbf{J}$ are due to free electrons.

These 4 equations form the basis of all classical electromagnetic phenomena. As discussed in Ch. 5, Faraday's law connects $\mathbf{E}$ and $\mathbf{B}$. As will be shown in Ch. 7, (6.6) lead to EM waves. Thus, Maxwell's theory connects "optics" and "electromagnetism". On the other hand, the Lorentz force equation, $\mathbf{f}=\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$, connects "mechanics" and "electromagnetism".

Review of Laws & Equations Obtained under Static Conditions :

Physical laws:

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}' \quad \text{(a)} \quad \underset{\text{(pp. 27-30)}}{\Leftrightarrow} \quad \begin{cases}\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0} & \text{(b)} \\ \nabla\times\mathbf{E}=0 & \text{(c)}\end{cases}$$

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}' \quad \text{(d)} \quad \underset{\text{(pp. 178-9)}}{\Leftrightarrow} \quad \begin{cases}\nabla\cdot\mathbf{B}=0 & \text{(e)} \\ \nabla\times\mathbf{B}=\mu_0\mathbf{J} & \text{(f)}\end{cases}$$

Scalar and vector potentials:

$$\mathbf{E}=-\nabla\phi \text{ with } \phi=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \text{ and } \nabla^2\phi=-\frac{\rho}{\varepsilon_0}$$

$$\mathbf{B}=\nabla\times\mathbf{A} \text{ with } \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \text{ and } \nabla^2\mathbf{A}=-\mu_0\mathbf{J}$$

Question: Which of the above laws/equations still hold if $\frac{\partial}{\partial t}\neq 0$? Why?

Helmholtz's Theorem: "If a vector field goes to 0 at infinity, it is uniquely determined by its divergence and curl." (Griffiths, Sec. 1.6.1)

Field energy:

$$W_E = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}d^3x \qquad \left[\begin{array}{l}\text{for linear medium}\\ \mathbf{D}=\varepsilon\mathbf{E},\ \mathbf{B}=\mu\mathbf{H}\end{array}\right] \qquad (4.89)$$

$$W_B = \tfrac{1}{2}\int \mathbf{B}\cdot\mathbf{H}d^3x \qquad (5.148)$$

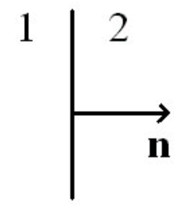
Forces:

$$\mathbf{f}_E = \int \rho\mathbf{E}d^3x$$

$$\mathbf{f}_B = \int \mathbf{J}\times\mathbf{B}d^3x$$

Boundary conditions:

$$\begin{cases}(\mathbf{D}_2-\mathbf{D}_1)\cdot\mathbf{n}=\sigma\\ (\mathbf{E}_2-\mathbf{E}_1)\times\mathbf{n}=0\end{cases} \qquad (4.40)$$



$$\begin{cases}(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0 & (5.86)\\ \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K} & (5.87)\end{cases}$$

Question: All of the above equations still hold if $\frac{\partial}{\partial t}\neq 0$.
Why?

6.2 Vector and Scalar Potentials

From the 2 homogeneous Maxwell equations, we may define a vector potential $\mathbf{A}$ and a scalar potential ϕ to represent $\mathbf{E}$ and $\mathbf{B}$.

$$\nabla\cdot\mathbf{B}=0 \qquad \Rightarrow \mathbf{B}=\nabla\times\mathbf{A} \qquad (6.7)$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \quad \Rightarrow \nabla\times\left(\mathbf{E}+\tfrac{\partial}{\partial t}\mathbf{A}\right)=0 \;\Rightarrow\; \mathbf{E}+\tfrac{\partial}{\partial t}\mathbf{A}=-\nabla\phi$$

$$\Rightarrow \mathbf{E}=-\nabla\phi-\tfrac{\partial}{\partial t}\mathbf{A} \qquad (6.9)$$

With (6.7) and (6.9), we may write the 2 inhomogeneous Maxwell equations (for the *free space* medium) in terms of $\mathbf{A}$ and ϕ as follows

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0} \qquad \Rightarrow \nabla^2\phi+\tfrac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-\frac{\rho}{\varepsilon_0} \qquad (6.10)$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \quad \Rightarrow \nabla^2\mathbf{A}-\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\mathbf{A}-\nabla\left(\nabla\cdot\mathbf{A}+\tfrac{1}{c^2}\tfrac{\partial}{\partial t}\phi\right)$$

$$=-\mu_0\mathbf{J} \qquad (6.11)$$

Thus, the set of 4 Maxwell equations for $\mathbf{E}$ and $\mathbf{B}$ have been reduced to 2 coupled equations for $\mathbf{A}$ and ϕ.

Rewrite
$$\begin{cases} \nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} & [(6.10)] \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial t^2}\mathbf{A} - \nabla(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi) = -\mu_0\mathbf{J} & [(6.11)] \end{cases}$$

If the potentials $\mathbf{A}$ and ϕ satisfy the Lorenz condition:

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0, \tag{6.14}$$

then, (6.10) and (6.11) are uncoupled to give the equations:

$$\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho}{\varepsilon_0} & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

Equations (6.15) and (6.16), under the Lorenz condition, are equivalent in all respects to the Maxwell equations in free space.

If $\mathbf{A}$ and ϕ do not satisfy the Lorenz condition, then through the gauge transformation discussed below, we may obtain a new set of potentials $\mathbf{A}'$ and ϕ', which satisfy the Lorenz condition.

6.3 Gauge Transformations, Lorenz Gauge, Coulomb Gauge

Gauge Transformations :

Rewrite
$$\begin{cases} \mathbf{B} = \nabla\times\mathbf{A} & [(6.7)] \\ \mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} & [(6.9)] \end{cases}$$

If ($\mathbf{A}$, ϕ) are transformed to ($\mathbf{A}'$, ϕ') according to

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \nabla\Lambda & (6.12) \\ \phi' = \phi - \frac{\partial}{\partial t}\Lambda & (6.13) \end{cases}$$

Λ : an arbitrary scalar function of $\mathbf{x}$ and t

then $\mathbf{A}'$ and ϕ' will give the same $\mathbf{E}$ and $\mathbf{B}$, i.e.

$$\begin{cases} \mathbf{B} = \nabla\times\mathbf{A}' \\ \mathbf{E} = -\nabla\phi' - \frac{\partial}{\partial t}\mathbf{A}' \end{cases}$$

The transformation defined by (6.12) and (6.13) is called the gauge transformation. The invariance of $\mathbf{E}$ and $\mathbf{B}$ under the gauge transformation is called gauge invariance.

Lorenz Gauge :

$(\mathbf{A}', \phi')$ under the gauge tranformation give the same $\mathbf{E}$ and $\mathbf{B}$ as $(\mathbf{A}, \phi)$. Hence,

$$\begin{cases} \nabla^2\phi' + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}') = -\rho/\varepsilon_0 & (6.10) \\ \nabla^2\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}' - \nabla(\nabla\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\phi') = -\mu_0\mathbf{J} & (6.11) \end{cases}$$

If the original $(\mathbf{A}, \phi)$ do not satisfy the Lorenz condition, we may choose a gauge function Λ and demand that the new $(\mathbf{A}', \phi')$ satisfy:

$$\nabla\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\phi' = 0 \qquad (1)$$

This then uncouples $\mathbf{A}'$ and ϕ' to give the same equations as in (6.15) and (6.16):

$$\begin{cases} \nabla^2\phi' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi' = -\frac{\rho}{\varepsilon_0} & (6.15) \\ \nabla^2\mathbf{A}' - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}' = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

Using $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$ and $\phi' = \phi - \frac{\partial}{\partial t}\Lambda$, we obatin from (1) the equation for Λ: $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = -\nabla\cdot\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\phi$ (6.18)

Rewrite $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = -\nabla\cdot\mathbf{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\phi$ [(6.18)]

If $(\mathbf{A}, \phi)$ already satisfy the Lorenz condition [i.e. the RHS of $(6.18) = 0$], a restricted gauge transformation with Λ given by

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda = 0 \qquad (6.20)$$

can preserve the Lorenz condition.

All $(\mathbf{A}, \phi)$ in this restricted class are said to belong to the Lorenz gauge. The Lorenz gauge is commonly used because it gives the set of equations:

$$\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho}{\varepsilon_0} & [(6.15)] \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & [(6.16)] \end{cases},$$

which treats $\mathbf{A}$ and ϕ on an equal footing. Furthermore, as will be shown in the next section, these 2 equations lead to a general expression for signal generation and transmission in free space.

Coulomb Gauge: (also called radiation gauge, transverse gauge, or solenoid gauge)

In the Coulomb gauge, we have $\nabla \cdot \mathbf{A} = 0$ (6.21)

then, $\begin{cases} (6.10) \Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} & (6.22) \\ (6.11) \Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla \frac{\partial \phi}{\partial t} & (6.24) \end{cases}$

To uncouple $\mathbf{A}$ and ϕ, we write $\mathbf{J} = \mathbf{J}_l + \mathbf{J}_t$ and demand

$\begin{cases} \nabla \times \mathbf{J}_l = 0 & [\mathbf{J}_l \text{ is called longitudinal or irrotational current}] \\ \nabla \cdot \mathbf{J}_t = 0 & [\mathbf{J}_t \text{ is called transverse or solenoidal current}] \end{cases}$

This may be achieved by constructing $\mathbf{J}_l$ and $\mathbf{J}_t$ from $\mathbf{J}$ as follows:

$\begin{cases} \mathbf{J}_l = -\frac{1}{4\pi}\nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' & (6.27) \\ \mathbf{J}_t = \frac{1}{4\pi}\nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' & (6.28) \end{cases}$

See proof at the end of this section.

Rewrite $\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$ [(6.22)]

The solution is (same math as in electrostatics)

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \quad \begin{bmatrix}\text{called the instantaneous} \\ \text{Coulomb potential}\end{bmatrix} \quad (6.23)$$

Rewrite $\mathbf{J}_l = -\frac{1}{4\pi}\nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ [(6.27)].

Let $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

$$\Rightarrow \mathbf{J}_l = \frac{1}{4\pi}\frac{\partial}{\partial t}\nabla \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \overset{\boxed{(6.23)}}{=} \varepsilon_0 \nabla \frac{\partial \phi}{\partial t}.$$

$$\Rightarrow \frac{1}{c^2}\nabla \frac{\partial \phi}{\partial t} = \mu_0 \mathbf{J}_l \quad [\tfrac{1}{c^2} = \varepsilon_0 \mu_0] \quad \boxed{= \frac{1}{\mu_0 c^2}\nabla \frac{\partial \phi}{\partial t} \text{ [by (6.29)]}} \quad (6.29)$$

Thus, $\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0(\mathbf{J}_l + \mathbf{J}_t) + \frac{1}{c^2}\nabla \frac{\partial \phi}{\partial t}$ [(6.24)]

$$\Rightarrow \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0 \mathbf{J}_t \quad [\mathbf{A} \text{ is now uncoupled from } \phi] \quad (6.30)$$

Discussion:

(i) As in Lorenz gauge, $\mathbf{B}=\nabla\times\mathbf{A}$ [(6.7)] & $\mathbf{E}=-\nabla\phi-\frac{\partial}{\partial t}\mathbf{A}$ [(6.9)]

(ii) $\nabla\cdot\mathbf{J}_t=0 \Rightarrow \mathbf{J}_t$ does not lead to time variation of ρ.

$\because \nabla\cdot\mathbf{J}_t$ doesn't contribute to $\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0$

(iii) $\phi\propto\frac{1}{r} \Rightarrow \underbrace{\nabla\phi}_{\text{part of }\mathbf{E}}\propto\frac{1}{r^2} \Rightarrow$
1. ϕ contributes only to the near fields.
2. Radiation fields are given only by $\mathbf{A}$ ($\because$ Radiation $\mathbf{E}$ should be $\propto\frac{1}{r}$).
3. Coulomb gauge allows separation of "near" and "radiation" fields.

(iv) The Coulomb gauge is often used when there is no source.

Then, $\phi=0$ and $\mathbf{A}$ satisfies the homogeneous equation:

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=0.$$

with the fields given by $\quad \mathbf{E}=-\frac{\partial}{\partial t}\mathbf{A},\quad \mathbf{B}=\nabla\times\mathbf{A} \qquad (6.31)$

subject to $\nabla\cdot\mathbf{A}=0$ (Coulomb gauge).

Exercise: Prove $\mathbf{J}_l$ [in (6.27)] $+\,\mathbf{J}_t$ [in (6.28)] $=\mathbf{J}$

Proof: $\mathbf{J}_t=\frac{1}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ [(6.28)]

$$=\frac{1}{4\pi}[\nabla\underbrace{\left(\nabla\cdot\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right)}_{\text{(A)}}-\underbrace{\nabla^2\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{\text{(B)}}]$$

$$\text{(A)}=\int\nabla\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'=\int\mathbf{J}(\mathbf{x}')\cdot\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-\int\mathbf{J}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$=\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'-\underbrace{\int\nabla'\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{0\text{ (by the divergence thm.)}}$$

$$\text{(B)}=\int\mathbf{J}(\mathbf{x}')\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'=-4\pi\int\mathbf{J}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d^3x'=-4\pi\mathbf{J}(\mathbf{x})$$

$$\Rightarrow\mathbf{J}_t=\frac{1}{4\pi}[\underbrace{\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}_{-4\pi\mathbf{J}_l}+4\pi\mathbf{J}(\mathbf{x})]=-\mathbf{J}_l+\mathbf{J} \qquad \text{QED}$$

$\because \mathbf{J}_l=-\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ [(6.27)]

6.4 Green's Function for the Wave Equation

Rewrite the wave equations: $\begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho}{\varepsilon_0} & [(6.15)] \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & [(6.16)] \end{cases}$

These two equations are applicable to free space and have the basic form:

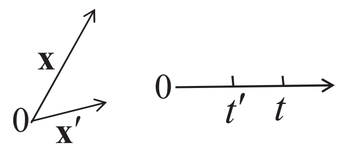
$$\nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = -4\pi f(\mathbf{x},t) \qquad (6.32)$$

distributed source of interest

with

$$f(\mathbf{x},t) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\rho(\mathbf{x},t) \text{ for } \psi = \phi \\ \frac{\mu_0}{4\pi}J_{x,y,z}(\mathbf{x},t) \text{ for } \psi = A_{x,y,z} \end{cases} \qquad (2)$$

To solve (6.32), we first obtain the Green function $G(\mathbf{x},t,\mathbf{x}',t')$ for a point source in *both* space and time, i.e. $G(\mathbf{x},t,\mathbf{x}',t')$ satisfies



$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\mathbf{x},t,\mathbf{x}',t') = -4\pi \underbrace{\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')} \qquad (6.41)$$

point source in space and time

Then, by the principle of linear superposition, we have $\psi(\mathbf{x},t) = \int d^3x' \int dt' G(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t')$

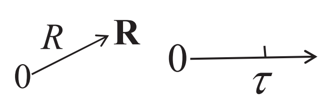
Rewrite $\quad (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad (6.41)$

For an *unbounded* free-space medium, G will be spherically symmetric if the point source is at the origin of spatial coordinates. Thus, define $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. Also, let $\tau \equiv t - t'$. So G depends only on R $(= |\mathbf{R}| = |\mathbf{x}-\mathbf{x}'|)$ and τ.

$$\Rightarrow \begin{cases} \nabla^2 G = \frac{1}{R}\frac{\partial^2}{\partial R^2}(RG) \quad (3) \\ \frac{\partial^2}{\partial t^2}G = \frac{\partial^2}{\partial \tau^2}G \\ G(\mathbf{x},t,\mathbf{x}',t') = G(R,\tau) \end{cases}$$

$\Rightarrow$ (6.41) gives $\frac{1}{R}\frac{\partial^2}{\partial R^2}[RG(R,\tau)] - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}G(R,\tau) = -4\pi\delta(\mathbf{R})\delta(\tau) \qquad (4)$

Question: "G depends only on R" is under the condition of an unbounded medium. Why?



Fourier transform (4) into ω-space $[G(R,\omega) = \int_{-\infty}^{\infty} G(R,\tau)e^{i\omega\tau}d\tau]$

$\Rightarrow \frac{1}{R}\frac{d^2}{dR^2}[RG(R,\omega)] + \frac{\omega^2}{c^2}G(R,\omega) = -4\pi\delta(\mathbf{R})$ [See (14b), Ch. 3], (6.37)

where index ω shows that $G(R,\omega)$ is an ω-space quantity. Jackson defines $k = \frac{\omega}{c}$ (p. 243) and denotes $G(R,\omega)$ here by $G_k(R)$ (p. 244).

Rewrite $\underbrace{\frac{1}{R}\frac{d^2}{dR^2}\left[RG(R,\omega)\right]}_{\text{1st term } (\underbrace{=}_{(3)} \nabla^2 G)} + \underbrace{\frac{\omega^2}{c^2}G(R,\omega)}_{\text{2nd term}} = -4\pi\delta(\mathbf{R})$ [(6.37)]

$0 \xrightarrow{R} \mathbf{R}$

Let $\omega R/c \to 0$ and neglect the 2nd term on LHS (justified below).

$$\Rightarrow \nabla^2 G(R,\omega) = -4\pi\delta(\mathbf{R}) \quad \Rightarrow \quad \underbrace{\lim}_{\omega R/c \to 0} G(R,\omega) = \frac{1}{R} \qquad (6.38)$$

Questions: $\begin{cases} \text{1. Which of } \delta(\mathbf{R}) \text{ \& } \frac{1}{R} \text{ is more divergent as } R \to 0? \\ \text{2. Why take the limit } \frac{\omega R}{c} \to 0 \text{ (instead of } R \to 0)? \end{cases}$

To answer these 2 questions, sub. $G(R,\omega) = 1/R$ [(6.38)] into (6.37) and integrate the LHS over a sphere of infinistesimal radius a.

$$\Rightarrow \begin{cases} \text{1st-term integral: } \int_v \nabla^2 G(R,\omega) d^3x = -4\pi \int_v \delta(\mathbf{R}) d^3x = -4\pi \\ \text{2nd-term integral: } \frac{\omega^2}{c^2}\int_v G(R,\omega) d^3x = \frac{\omega^2}{c^2}\int_0^a \frac{1}{R} 4\pi R^2 dR = 2\pi \frac{\omega^2 a^2}{c^2} \end{cases}$$

$\Rightarrow$ The 2nd-term integral $\to 0$ if $\frac{\omega a}{c} \to 0$. This justifies the neglect of the 2nd term in the limit $\frac{\omega R}{c} \to 0$ and hence the validity of (6.38).

In the region $R > 0$, (6.37) $\Rightarrow \frac{1}{R}\frac{d^2}{dR^2}\left[RG(R,\omega)\right] + \frac{\omega^2}{c^2}G(R,\omega) = 0$

$$\Rightarrow \; G(R,\omega) = A\frac{e^{i\frac{\omega}{c}R}}{R} + B\frac{e^{-i\frac{\omega}{c}R}}{R} \quad \text{[exact solution for } R > 0] \qquad (5)$$

For (5) to match the solution $\underbrace{\lim}_{\omega R/c \to 0} G(R,\omega) = \frac{1}{R}$ [(6.38)], we must have $A + B = 1$. Thus,

$$G(R,\omega) = AG^+(R,\omega) + BG^-(R,\omega) \text{ [exact solu. in all space]} \quad (6.39)$$

subject to the condition $A + B = 1$, where $G^{\pm}(R,\omega) \equiv \frac{e^{\pm i\frac{\omega}{c}R}}{R}$ (6.40)

Question: What's the difference between $\frac{\omega R}{c} \to 0$ and $R \to 0$?

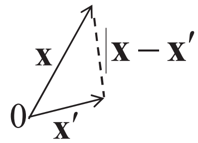
In τ-space, $G^{\pm}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^{\pm}(R,\omega) e^{-i\omega\tau} d\omega$ [inverse transform]

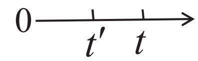
$$\boxed{\begin{aligned} R &= |\mathbf{x}-\mathbf{x}'| \\ \tau &= t - t' \end{aligned}} \longrightarrow \; = \frac{1}{2\pi R}\int_{-\infty}^{\infty} e^{-i\omega(\tau \mp \frac{R}{c})} d\omega = \frac{\delta(\tau \mp \frac{R}{c})}{R} \qquad (6.43)$$

$$\Rightarrow G^{\pm}(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\left[t' - (t \mp \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} \begin{bmatrix} G^+\text{: retarded Green function} \\ G^-\text{: advanced Green function} \end{bmatrix} \quad (6.44)$$

We have obtained 2 solutions:

$$G^{\pm}(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\left[t'-(t\mp\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} \qquad (6.44)$$





for the equation:

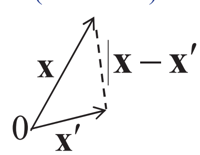
$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad [(6.41)]$$

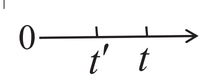
The solution G^+ indicates that an effect at ($\mathbf{x}$, t) is caused by the action of a point source a distance $|\mathbf{x}-\mathbf{x}'|$ away at an *earlier* time $t' = t - |\mathbf{x}-\mathbf{x}'|/c$. This is a physical solution because the time (t') of the cause *precedes* the time (t) of the effect. However, for the G^- solution, the time of the cause ($t' = t + |\mathbf{x}-\mathbf{x}'|/c$) would be *after* the time (t) of the effect. This is not physically possible. Thus, "causality" requires that we reject the G^- solution and set $A = 1,\ B = 0$ in (5) or (6.39). Then, the physical solution of (6.41) is

$$G(\mathbf{x},t,\mathbf{x}',t') = G^+(\mathbf{x},t,\mathbf{x}',t') = \frac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} \qquad (6)$$

Discussion : Rewrite

$$G^+(\mathbf{x},t,\mathbf{x}',t') = \frac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} \quad [(6)]$$





(i) The relation $t' = t - |\mathbf{x}-\mathbf{x}'|/c$ shows it takes $\Delta t = |\mathbf{x}-\mathbf{x}'|/c$ for a signal to travel from $\mathbf{x}'$ to $\mathbf{x}$, i.e. the signal travels at speed c.

(ii) $G^+(\mathbf{x},t,\mathbf{x}',t')$ is the signal at ($\mathbf{x}$, t) due to the point source $\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. Since $\delta(t-t')$ contains components of all ω, if the medium is dispersive (i.e. signal speed varys with ω), different components will reach $\mathbf{x}$ at different times. Thus, the signal at $\mathbf{x}$ will be a pulse of finite duration, rather than a delta function in time as in G^+. This explains why the solution for G^+ is valid only for the free space or a non-dispersive medium [see p. 243 (top) and p. 245], in which all the components propagate toward $\mathbf{x}$ at the same speed and hence reach $\mathbf{x}$ at the same time.

(iii) All quantities refer to the same reference frame (lab frame). The obsevation point $\mathbf{x}$ is fixed. The source point $\mathbf{x}'$ can be moving.

Solution for a Distributed Source

Rewrite $\begin{cases} \nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = -4\pi \underbrace{f(\mathbf{x},t)}_{\text{distributed source}} & [(6.32)] \\ (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\mathbf{x},t,\mathbf{x}',t') = -4\pi \underbrace{\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')}_{\text{point source}} & [(6.41)] \end{cases}$

$\Rightarrow \psi$ is a superpositin of $G^+(\mathbf{x},t,\mathbf{x}',t')$ over $f(\mathbf{x}',t')$:

$$\psi(\mathbf{x},t) = \int d^3x' \int dt' G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t')$$

$$= \int d^3x' \int dt' \frac{\delta\left[t' - (t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}',t')$$

[See Mathews and Walker (pp. 278-280) for another method to derive this eq.]

Sub. $\psi(\mathbf{x},t)$ into (6.32) and use (6.41), we can verify $\psi(\mathbf{x},t)$ is a solution of (6.32).

For completeness, we may add a complementary solution $\psi_{in}(\mathbf{x},t)$, which satisfies the homog. eq.: $\nabla^2\psi_{in} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi_{in} = 0$, e.g. $\psi_{in}(\mathbf{x},t)$ can be fields generated by other sources.

Thus, $$\psi(\mathbf{x},t) = \psi_{in}(\mathbf{x},t) + \int d^3x' \int dt' G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t') \qquad (6.45)$$

Rewrite $\begin{cases} \nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = -4\pi f(\mathbf{x},t) & [(6.32)] \\ \psi(\mathbf{x},t) = \int d^3x' \int dt' \frac{\delta\left[t' - (t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}',t') & [(6.45)] \end{cases}$

where $\psi_{in}(\mathbf{x},t)$ [not due to $f(\mathbf{x},t)$] has been neglected.

In (6.45), the order of dt'- and d^3x'-integrations depends on the form of $f(\mathbf{x}',t')$. For sources stationary in space but varying in time (e.g. an antenna), $\mathbf{x}'$ in (6.45) is a fixed point in space (not a function of t'). Thus, the t'-integration may be immediately carried out to give

$$\psi(\mathbf{x},t) = \int \frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \quad \begin{bmatrix}\text{stationary,} \\ \text{time-varying} \\ \text{sources}\end{bmatrix}, \qquad (6.47)$$

where the symbol $[\cdots]_{ret}$ implies that t' in the bracket is to be evaluated at the retarded time given by $t' = t - |\mathbf{x}-\mathbf{x}'|/c$.

The case of a moving source (where $\mathbf{x}'$ is a function of t') will be considered at the end of the following section.

Discussion: Rewrite $\psi(\mathbf{x},t)=\int\frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ [(6.47)]

(6.47) is valid for unbounded space (see p. 244, bottom). If there are boundary surfaces, boundary conditions must be considered in order to account for sources on the boundary (e.g. due to reflections).

A similar integral solution exists in electrostatics, where

$$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad [(1.23)]$$

is valid for unbounded space, while in the "formal solution" for a finite volume:

$$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V\rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x'-\frac{1}{4\pi}\oint_S\phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da' \quad [(1.44)]$$

the non-zero boundary values are accounted for by the surface integral.

We have no counterpart of (1.44) for (6.47). However, Ch. 8 solves the source-free Maxwell eqs. for **E** and **B** in waveguides and cavities as a boundary value problem.

6.5 Retarded Solution for the Fields…

Rewrite
$$\begin{cases}\nabla^2\phi(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\mathbf{x},t)=-\frac{\rho(\mathbf{x},t)}{\varepsilon_0} & [(6.15)]\\ \nabla^2\mathbf{A}(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x},t)=-\mu_0\mathbf{J}(\mathbf{x},t) & [(6.16)]\end{cases}$$

Each Cartesian component of (6.15) and (6.16) is in the form:

$$\nabla^2\psi(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=-4\pi f(\mathbf{x},t) \quad [(6.32)]$$

with the solution: $\psi(\mathbf{x},t)=\int d^3x'\int dt'\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}f(\mathbf{x}',t')$ [(6.45)]

In term of ρ and **J**: $f(\mathbf{x},t)=\begin{cases}\frac{1}{4\pi\varepsilon_0}\rho(\mathbf{x},t) \text{ for } \psi=\phi\\ \frac{\mu_0}{4\pi}J_{x,y,z}(\mathbf{x},t) \text{ for } \psi=A_{x,y,z}\end{cases}$

$$\Rightarrow\begin{cases}\phi(\mathbf{x},t)=\frac{1}{4\pi\varepsilon_0}\int d^3x'\int dt'\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}',t')\\ \mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int d^3x'\int dt'\frac{\delta\left[t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}',t')\end{cases}\left[\begin{array}{l}\text{stationary}\\ \text{\& moving}\\ \text{sources}\end{array}\right] \quad (7)$$

Rewrite
$$\begin{cases} \phi(\mathbf{x},t) = \dfrac{1}{4\pi\varepsilon_0}\int d^3x' \int dt' \dfrac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}',t') \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0}{4\pi}\int d^3x' \int dt' \dfrac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}',t') \end{cases} \quad [(7)]$$

Stationary, time-varying sources (e.g. an antenna): $\mathbf{x}'$ in (7) is not a function of t'. Thus, the t'-integration immediately gives

$$\begin{cases} \phi(\mathbf{x},t) = \dfrac{1}{4\pi\varepsilon_0}\int \dfrac{1}{|\mathbf{x}-\mathbf{x}'|}[\rho(\mathbf{x}',t')]_{ret}\, d^3x' \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0}{4\pi}\int \dfrac{1}{|\mathbf{x}-\mathbf{x}'|}[\mathbf{J}(\mathbf{x}',t')]_{ret}\, d^3x' \end{cases} \quad \begin{bmatrix}\text{stationary,} \\ \text{time-varying} \\ \text{sources}\end{bmatrix} \quad (6.48)$$

Note: (i) Griffiths (pp.422-424) obtains (6.48) by a heuristic argument.

(ii) ϕ & $\mathbf{A}$ in (6.48) satisfy the Lorenz gauge (Griffiths, Prob. 10.8).

(iii) In the static case, $[\rho(\mathbf{x}',t')]_{ret} = \rho(\mathbf{x}')$ and $[\mathbf{J}(\mathbf{x}',t')]_{ret} = \mathbf{J}(\mathbf{x}')$.

$$\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \ [(1.17)]; \ \ \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \ [(5.32)]$$

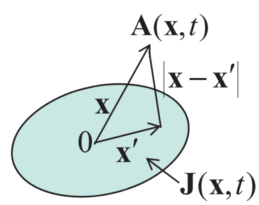
Problem: Find $\mathbf{A}(\mathbf{x},t)$ for $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})\cos\omega t$. ← a stationary, time-varying source

Solution: $\mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{1}{|\mathbf{x}-\mathbf{x}'|}[\mathbf{J}(\mathbf{x}',t')]_{ret}\, d^3x'$ [(6.48)]

$$\Rightarrow \quad \mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{[\mathbf{J}(\mathbf{x}')\cos\omega t']_{ret}}{|\mathbf{x}-\mathbf{x}'|}$$

$[\cdots]_{ret}$ means $t' = t - \dfrac{|\mathbf{x}-\mathbf{x}'|}{c}$. →

$$= \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')\cos\omega(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})}{|\mathbf{x}-\mathbf{x}'|} \quad (8)$$



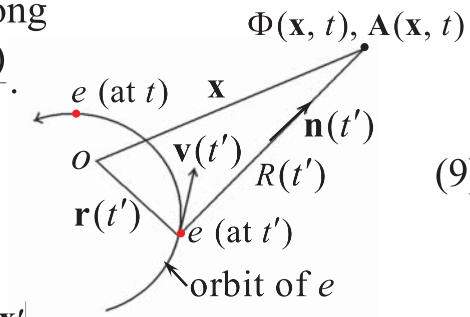
$\Rightarrow$ $\mathbf{A}(\mathbf{x},t)$ is the superposition of an infinite number of components originating from all points [denoted by $\mathbf{x}'$ in (8)] of the source, all having the frequency ω. The component reaching point $\mathbf{x}$ at time t was emitted from point $\mathbf{x}'$ at the retarded time t' $[= t - |\mathbf{x}-\mathbf{x}'|/c]$.

The effect of "time retardation" is reflected in relative phases. In the travel time ($|\mathbf{x}-\mathbf{x}'|/c$) from $\mathbf{x}'$ to $\mathbf{x}$, the source has advanced in phase by $\omega|\mathbf{x}-\mathbf{x}'|/c$. Thus, at any t, the phase of the component at point $\mathbf{x}$ $[\omega(t-|\mathbf{x}-\mathbf{x}'|/c)]$ lags that of the source (ωt) by $\omega|\mathbf{x}-\mathbf{x}'|/c$.

Note: ϕ is given by ρ, which can be obtained from $\nabla\cdot\mathbf{J} + \dfrac{\partial\rho}{\partial t} = 0$

Fields of a moving point charge (e.g. synchrotron radiation):

Consider a point charge e moving along any orbit $\mathbf{r}(t')$ with velocity $\mathbf{v}(t') = \dfrac{d\mathbf{r}(t')}{dt'}$.



$$\Rightarrow \begin{cases} \rho(\mathbf{x}',t') = e\delta[\mathbf{x}' - \mathbf{r}(t')] \\ \mathbf{J}(\mathbf{x}',t') = e\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')] \end{cases} \quad (9)$$

Sub. (9) into (7), we obtain

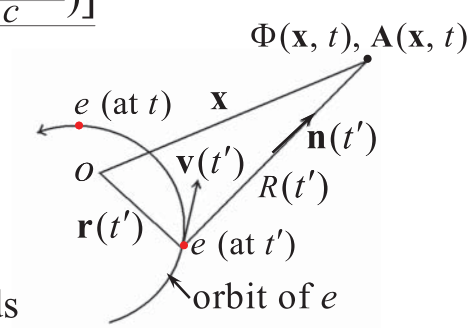
$$\begin{cases} \phi(\mathbf{x},t) = \dfrac{e}{4\pi\varepsilon_0}\displaystyle\int d^3x' \int dt' \dfrac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\delta[\mathbf{x}' - \mathbf{r}(t')] \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0 e}{4\pi}\displaystyle\int d^3x' \int dt' \dfrac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')] \end{cases} \quad (10)$$

In (10), $\delta[t' - (t - \frac{|\mathbf{x}-\mathbf{x}'|}{c})] = \delta[t' - (t - \frac{|\mathbf{x}-\mathbf{r}(t')|}{c})]$ because $\mathbf{x}' = \mathbf{r}(t')$.

$\Rightarrow$ The t'-integration cannot be easily carried out as in obtaining (6.48).

However, $\delta[\mathbf{x}' - \mathbf{r}(t')]$ allows an immediate d^3x'-integration. Thus,

$$(10) \Rightarrow \begin{cases} \phi(\mathbf{x},t) = \dfrac{e}{4\pi\varepsilon_0}\displaystyle\int dt' \dfrac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{r}(t')|}{c}\right)\right]}{|\mathbf{x}-\mathbf{r}(t')|} \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0 e}{4\pi}\displaystyle\int dt' \dfrac{\mathbf{v}(t')\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{r}(t')|}{c}\right)\right]}{|\mathbf{x}-\mathbf{r}(t')|} \end{cases} \quad \begin{bmatrix}\text{for a moving} \\ \text{point charge}\end{bmatrix} \quad (11)$$



Discussion:

1. $\mathbf{x}$ (observation pt.) is indep. of time. $\mathbf{r}$ (position of charge e) varies with time. $\mathbf{v}$ $(= d\mathbf{r}/dt')$ can be any value less than c.

2. Regardless of the velocity of e, fields emitted by e always travel toward $\mathbf{x}$ at the speed c.

Note: This is a postulate of the theory of special relativity.

3. As an example, (11) is used in Appendix A to derive the Lienard-Wiechert potentials. More examples are given in Ch. 14.

4. (6.48) can also be applied to a moving point charge, but not conveniently (see Griffiths, pp. 430-433).

Equations for E and B

E and **B** can be expressed in terms of ϕ and **A** just obtained. We may also express **E** and **B** directly in the form of (6.48) as follows.

$$\begin{cases} \nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \\ \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\mathbf{E} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} \end{cases} \quad \begin{bmatrix}\text{Maxwell equations} \\ \text{in free space}\end{bmatrix}$$

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} \;\Rightarrow\; \nabla\underbrace{(\nabla\cdot\mathbf{E})}_{\rho/\varepsilon_0} - \nabla^2\mathbf{E} = -\mu_0\frac{\partial}{\partial t}\mathbf{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\Rightarrow \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{1}{\varepsilon_0}\nabla\rho + \mu_0\frac{\partial}{\partial t}\mathbf{J} = -\frac{1}{\varepsilon_0}\left(-\nabla\rho - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}\right) \tag{6.49}$$

$$\nabla\times\nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\mathbf{E}$$

$$\Rightarrow \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = \mu_0\nabla\times\mathbf{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}$$

$$\Rightarrow \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B} = -\mu_0\nabla\times\mathbf{J} \tag{6.50}$$

Rewrite

$$\begin{cases} \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{1}{\varepsilon_0}(-\nabla\rho - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}) & (6.49) \\ \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B} = -\mu_0\nabla\times\mathbf{J} & (6.50) \end{cases}$$

(6.49) & (6.50) are in form: $\nabla^2\psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = -4\pi f(\mathbf{x},t)$

Following same steps leading to (6.48), we obtain

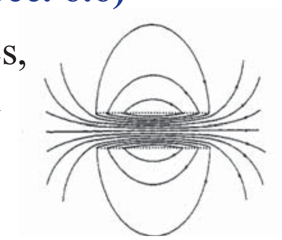
$$\psi(\mathbf{x},t) = \int\frac{[f(\mathbf{x}',t')]_{ret}}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{6.47}$$

$$\Rightarrow \begin{cases} \mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\frac{1}{|\mathbf{x}-\mathbf{x}'|}[-\nabla'\rho - \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}]_{ret} & (6.51) \\ \mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\frac{1}{|\mathbf{x}-\mathbf{x}'|}[\nabla'\times\mathbf{J}]_{ret} & (6.52) \end{cases}$$

(6.51) and (6.52) can be converted into the Jefimenko formulae [see (6.55) and (6.56)], which explicitly show the reduction to the static equations (1.5) and (5.14).

6.7 Poynting's Theorem and Conservation of Energy and Momentum for a System of Particles and Electromagnetic Fields

(See Appendix B for a brief discussion of Sec. 6.6)



We have shown in Sec. 5.7 that, in magnetostatics, manipulation of two familiar relations, $\mathbf{f} = \mathbf{J} \times \mathbf{B}$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, leads to the new concepts of "magnetic pressure" and "magnetic tenson":

$$\mathbf{f}\ \left(\frac{\text{magnetic force}}{\text{unit volume}}\right) = \underset{\uparrow}{\mathbf{J}} \times \mathbf{B} = -\frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) = \underbrace{-\nabla \frac{B^2}{2\mu_0}}_{\substack{\text{magnetic}\\ \text{pressure}}} + \underbrace{\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\substack{\text{magnetic}\\ \text{tension}}}$$

$$\boxed{= \frac{1}{\mu_0}\nabla \times \mathbf{B}}$$

In this section, we will show that, for time-dependent fields, manipulations of other familar relations can lead to other useful new concepts, such as the Poynting vector, the electromagnetic field momemtum, and the radiation pressure.

Poynting's Theorem; Conservation of Energy of Combined System of Particles and Fields:

Let $\mathbf{f} = \dfrac{\text{electric and magnetic forces}}{\text{unit volume}} = \rho\mathbf{E} + \overbrace{\mathbf{J}}^{\rho\mathbf{v}} \times \mathbf{B}\ \left[\begin{matrix}\rho \text{ is moving}\\ \text{at velocity } \mathbf{v}\end{matrix}\right]$

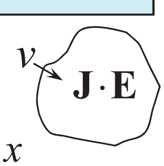
$\Rightarrow \mathbf{f}\cdot\mathbf{v}\ \left(\dfrac{\text{power}}{\text{unit volume}}\text{delivered by the fields to a medium}\right)$

$= \rho\mathbf{E}\cdot\mathbf{v} + \rho\,\cancel{(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}} = \mathbf{J}\cdot\mathbf{E}$

In a volume v, the total power delivered by $\mathbf{J}\cdot\mathbf{E}$ is

$$\boxed{\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D}} \qquad \boxed{= \mathbf{H}\cdot\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial}{\partial t}\mathbf{B}} - \nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} - \nabla\cdot(\mathbf{E}\times\mathbf{H})}$$

$$\underbrace{\int_v \mathbf{J}\cdot\mathbf{E}\,d^3x}_{\uparrow} = \int_v (\overbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}} - \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D})\,d^3x$$



$$= -\int_v \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x \qquad (6.105)$$

Rate of conversion of EM field energy into mechanical energies. A negative value means reverse conversion.

Rewrite (6.105): $\int_v \mathbf{J}\cdot\mathbf{E}d^3x = -\int_v\left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

The terms $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}$ and $\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$ in the integrand can be interpreted physically if we make the following assumptions (p. 259):

Assumption 1: The medium is *linear* with *negligible dispersion* and *negligible losses*.

We can then write (reasons given in Ch. 7 of lecture notes, also see Sec. 6.8 for the case of a dispersive and lossy medium)

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t)$$

$$\Rightarrow\ \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}) \text{ and } \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\cdot\mathbf{B}) \qquad (12)$$

Assumption 2: The *static* field energy density

$$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) = \frac{1}{2}(\varepsilon E^2 + \frac{B^2}{\mu}) \qquad (6.106)$$

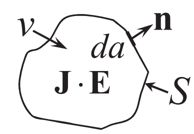
represents the field energy density even for *time-dependent* fields.

(12) and (6.106) give

$$\frac{\partial u}{\partial t} = \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B} = \begin{bmatrix}\text{rate of change of}\\ \text{field energy density}\end{bmatrix} \qquad (13)$$

Rewrite (6.105): $\int_v \mathbf{J}\cdot\mathbf{E}d^3x = -\int_v\left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

Sub. $\frac{\partial u}{\partial t}$ for $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D} + \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$, we obtain



$$\int_v \mathbf{J}\cdot\mathbf{E}d^3x + \int_v \frac{\partial u}{\partial t}d^3x + \int_v \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x = 0 \qquad (6.107)$$

and, by divergence thm., $\underbrace{\int_v \mathbf{J}\cdot\mathbf{E}d^3x}_{\frac{d}{dt}E_{mech}} + \underbrace{\int_v \frac{\partial u}{\partial t}d^3x}_{\frac{d}{dt}E_{field}} + \underbrace{\int_v \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3x}_{\oint_S \mathbf{S}\cdot\mathbf{n}da} = 0$

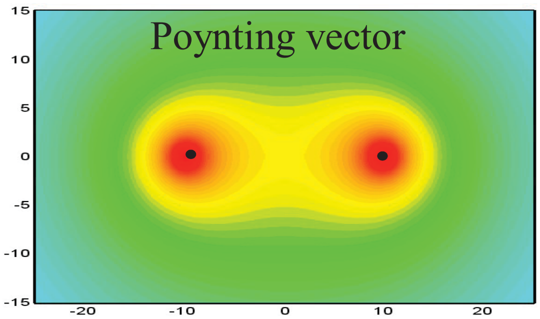
where $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ [Poynting vector] $\qquad \boxed{E_{field} \equiv \int_v u d^3x}$ (6.109)

$$\Rightarrow \frac{d}{dt}(E_{mech} + E_{field}) = -\oint_S \mathbf{S}\cdot\mathbf{n}da \text{ [Poynting's theorem]} \qquad (6.111)$$

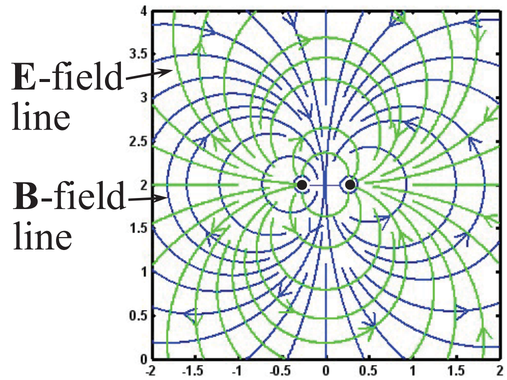
where E_{mech} is the total mechanical (thermal and motion) energy in v, E_{field} is the total field energy in v, and no particles move into or out of v. By conservation of energy, $\oint_S \mathbf{S}\cdot\mathbf{n}da$ must be the total power flow into or out of v. Further, $\mathbf{S}$ gives the $\frac{\text{power}}{\text{unit area}}$ at any point (see p. 259).

By (6.107), we have $\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$ $\begin{bmatrix}\text{Poynting's thm. in}\\ \text{differential form}\end{bmatrix}$ (6.108)

Example 1: power lines



The voltage difference between the two power lines produces the **E**-field. The opposite currents on the two power lines produce the **B**-field.



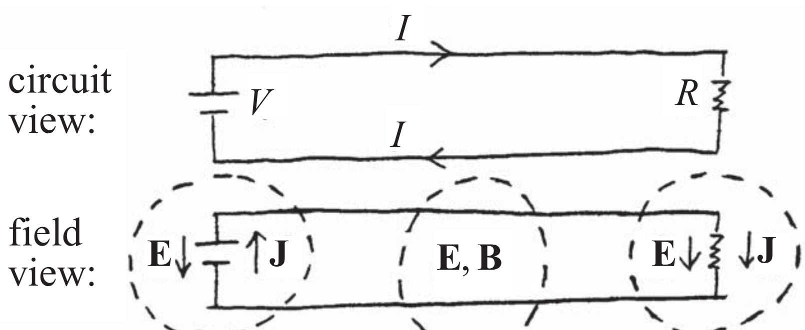
(calculated by C. Y. Kao)

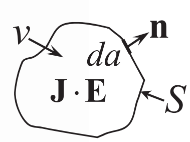
Note: (1) **E** (along a power line) $\ll$ **E** (between two power lines)

(2) $\mathbf{S} = \mathbf{E} \times \mathbf{H} = (\sum_j \mathbf{E}_j) \times (\sum_j \mathbf{H}_j) [\neq \sum_j (\mathbf{E}_j \times \mathbf{H}_j)]$

Example 2: a DC circuit

$$\int_V \mathbf{J}\cdot\mathbf{E}d^3x + \int_V \frac{\partial u}{\partial t} d^3x + \oint_S \mathbf{S}\cdot\mathbf{n}da = 0 \xRightarrow{\text{steady state}} \int_V \mathbf{J}\cdot\mathbf{E}d^3x + \oint_S \mathbf{S}\cdot\mathbf{n}da = 0$$





$\int_V \mathbf{J}\cdot\mathbf{E}d^3x < 0$

$\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}da > 0$

$\Rightarrow$ Power flows out of battery.

Power transmission by Poynting vector as on power lines

$\int_V \mathbf{J}\cdot\mathbf{E}d^3x > 0$

$\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}da < 0$

$\Rightarrow$ Power flows into resistor.

Conservation of Linear Momentum of Combined System of Particles and Fields :

In free space (where $c^2 = \frac{1}{\mu_0\varepsilon_0}$), Maxwell eqs. give

$$\begin{cases} \rho = \varepsilon_0 \nabla\cdot\mathbf{E} \\ \mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B} - \varepsilon_0\frac{\partial}{\partial t}\mathbf{E} \end{cases}$$

$$\boxed{\begin{aligned} \mathbf{B}\times\tfrac{\partial}{\partial t}\mathbf{E} &= -\tfrac{\partial}{\partial t}\mathbf{E}\times\mathbf{B} + \mathbf{E}\times\tfrac{\partial}{\partial t}\mathbf{B} \\ &= -\tfrac{\partial}{\partial t}\mathbf{E}\times\mathbf{B} - \mathbf{E}\times(\nabla\times\mathbf{E}) \end{aligned}}$$

force per unit volume

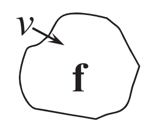
$$\Rightarrow \mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = \varepsilon_0\mathbf{E}(\nabla\cdot\mathbf{E}) - \frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B}) + \varepsilon_0\overbrace{\mathbf{B}\times\frac{\partial}{\partial t}\mathbf{E}}$$

$$= \varepsilon_0[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E}) + \underbrace{c^2\mathbf{B}(\nabla\cdot\mathbf{B})} - c^2\mathbf{B}\times(\nabla\times\mathbf{B})] - \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}\times\mathbf{B}$$

This term, which equals 0, is added for later manipulation.

Applying Newton's 2nd law to a volume v: $\frac{d}{dt}\mathbf{P}_{mech} = \int_v \mathbf{f}d^3x$,

where $\mathbf{P}_{mech}$ is the total mechanical momentum in v.



$$\Rightarrow \frac{d}{dt}\mathbf{P}_{mech} + \frac{d}{dt}\int_v \varepsilon_0(\mathbf{E}\times\mathbf{B})d^3x$$

$$= \varepsilon_0\int_v d^3x\ [\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E}) + c^2\mathbf{B}(\nabla\cdot\mathbf{B}) - c^2\mathbf{B}\times(\nabla\times\mathbf{B})] \quad (6.116)$$

Define $\mathbf{g} \equiv \frac{1}{c^2}\mathbf{E}\times\mathbf{H}$ [electromagnetic momentum density] (6.118)

Then, in (6.116), $\frac{d}{dt}\int_v \varepsilon_0(\mathbf{E}\times\mathbf{B})d^3x = \frac{d}{dt}\int_v \frac{1}{c^2}(\mathbf{E}\times\mathbf{H})d^3x = \frac{d}{dt}\mathbf{P}_{field}$,

where $\mathbf{P}_{field} = \int_v \mathbf{g}d^3x$ [total electromagnetic momentum in v] (14)

(6.116) can thus be written

$$\frac{d}{dt}\mathbf{P}_{mech} + \frac{d}{dt}\int_v \varepsilon_0(\mathbf{E}\times\mathbf{B})d^3x = \frac{d}{dt}\mathbf{P}_{mech} + \frac{d}{dt}\mathbf{P}_{field}$$

$$= \varepsilon_0\int_v d^3x[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E}) + c^2\mathbf{B}(\nabla\cdot\mathbf{B}) - c^2\mathbf{B}\times(\nabla\times\mathbf{B})] \quad (15)$$

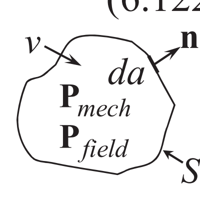
Define a tensor $T_{\alpha\beta}$ (called the Maxwell stress tensor) :

$$T_{\alpha\beta} \equiv \varepsilon_0[E_\alpha E_\beta + c^2B_\alpha B_\beta - \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\delta_{\alpha\beta}] \quad (6.120)$$

The α-component of (15) can be put in the form (see p.261) :

$$\frac{d}{dt}(\mathbf{P}_{mech} + \mathbf{P}_{field})_\alpha = \int_v \sum_\beta \frac{\partial}{\partial x_\beta}T_{\alpha\beta}d^3x = \oint_S \sum_\beta T_{\alpha\beta}n_\beta da, \quad (6.122)$$

Treating the three $\alpha = const$ components of tensor $T_{\alpha\beta}$ (β=1,2,3) as a vector and apply the divergence theorem.

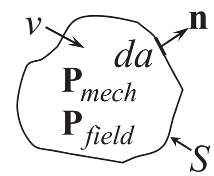


where $n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 = \mathbf{n}$ (outward unit normal).

Rewrite $\frac{d}{dt}(\mathbf{P}_{mech}+\mathbf{P}_{field})_\alpha = \oint_s \sum_\beta T_{\alpha\beta} n_\beta da \quad [(6.122)]$

We may write the 3 scalar eqs. in (6.122) ($\alpha = 1,2,3$) as a vector equation: $\frac{d}{dt}(\mathbf{P}_{mech}+\mathbf{P}_{field}) = \oint_s \overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da,$ (16)

where $\underbrace{\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}}_{\text{a vector}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ [See Griffiths Eq. (1.31) for meaning of RHS.]



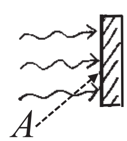
and $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ (outward unit normal) with $n_1^2+n_2^2+n_3^2=1$.

Discussion: 1. $\mathbf{P}_{field}$ (EM momentum) is a new concept developed from familiar laws.

2. (16) is a new form of the old law: $\frac{d}{dt}\mathbf{P}_{mech} = \overbrace{\int_v(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})d^3x}^{\mathbf{F}\text{ on particles}}$.
Now, the vector $\overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}$ is the "force" per unit area transmitted across S to act on both *particles* ($\mathbf{P}_{mech}$) *and fields* ($\mathbf{P}_{field}$) inside volume v.

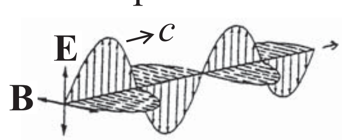
3. The force $\mathbf{F}$ in the old law $\frac{d}{dt}\mathbf{P}_{mech} = \mathbf{F}$ (included in $\oint_s \overleftrightarrow{\mathbf{T}}\cdot\mathbf{n}da$, see problem below) is still a valid and more intuitive concept.

Problem 1: A plane wave is incident normally from free space on a flat object of surface area A and is totally absorbed by its surface. Find the force (radiation pressure) on the object.



Note: For simplicity, here we assume total wave absoption within a negligible surface depth. In reality, the wave will be partially reflected and the absorption is spreaded over a finite depth.

Consider a plane EM wave propagating along $\mathbf{e}_z$ in free space (see college physics):



$$\begin{cases} \mathbf{E}(z,t) = E_0\cos(kz-\omega t)\mathbf{e}_x \\ \mathbf{B}(z,t) = B_0\cos(kz-\omega t)\mathbf{e}_y \end{cases} \quad [\text{instantaneous fields}] \qquad (17)$$

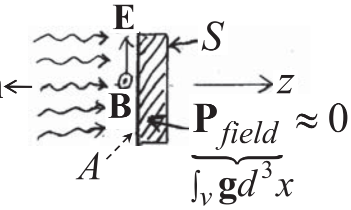
The *instantaneous* and *time-averaged* field energy densities are

$$\begin{cases} u(z,t) = \frac{1}{2}[\varepsilon_0 E^2(z,t) + B^2(z,t)/\mu_0] & (18a) \\ \langle u\rangle_t = \frac{1}{4}(\varepsilon_0 E_0^2 + B_0^2/\mu_0) & (18b) \end{cases} \quad \begin{bmatrix} \langle\cdots\rangle_t:\ \text{averaging} \\ \text{over a wave period.} \end{bmatrix}$$

$$\Rightarrow \begin{cases} P(z,t) = u(z,t)c & (19a) \\ \langle P\rangle_t = \langle u\rangle_t c & (19b) \end{cases} \quad \left[\text{instantaneous \& average } \frac{\text{power}}{\text{unit area}}\right]$$

Let S be the boundary of the object. On the left side of S, we have

$\mathbf{n} = -\mathbf{e}_z = (0, 0, -1)$ [outward unit normal]



$$\begin{cases} \mathbf{E}(z,t) = (E_x, 0, 0) \\ \mathbf{B}(z,t) = (0, B_y, 0) \end{cases} \left[\begin{array}{l}\text{instantaneous fields} \\ \text{on the left side (no} \\ \text{field on other sides)}\end{array}\right]$$

$$\Rightarrow \underbrace{\overleftrightarrow{\mathbf{T}} \cdot \mathbf{n}}_{\substack{\text{on} \\ \text{left side}}} = \varepsilon_0 \begin{bmatrix} E_x^2 - \frac{1}{2}(E^2 + c^2B^2) & 0 & 0 \\ 0 & c^2B_y^2 - \frac{1}{2}(E^2 + c^2B^2) & 0 \\ 0 & 0 & -\frac{1}{2}(E^2 + c^2B^2) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \tfrac{1}{2}\varepsilon_0(E^2 + c^2B^2)\mathbf{e}_z = \tfrac{1}{2}(\varepsilon_0 E^2 + \tfrac{B^2}{\mu_0})\mathbf{e}_z = u\mathbf{e}_z \quad [u \text{ given by (18a)}]$$

$$\tfrac{d}{dt}\mathbf{P}_{mech} + \tfrac{d}{dt}\cancel{\mathbf{P}_{field}} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot \mathbf{n} da \Rightarrow \mathbf{F} = \tfrac{d}{dt}\mathbf{P}_{mech} = uA\mathbf{e}_z \quad [A: \text{area}] \quad (20)$$

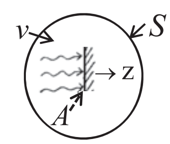
$$\Rightarrow \frac{\mathbf{F}}{A}\left[\begin{array}{c}\text{instantaneous} \\ \text{radiation pressure}\end{array}\right] = u\mathbf{e}_z = \frac{P}{c}\mathbf{e}_z \quad [P \text{ given by (19a)}] \quad (21)$$

(21) $\Rightarrow$ average radiation pressure, if u, P are replaced by $<u>_t$, $<P>_t$.

Question: The radiation pressure is due to the $\mathbf{J} \times \mathbf{B}$ force. How?

Alternative solution: Consider a finite length of wave with cross section A. Enclose the whole wave and the object into S to form an *isolated* "electromagnetic + mechanical" system (see figure). *Note*: This is not an isolated mechanical system ($\because$ EM force on the object).

Hence, by (16), $\frac{d}{dt}(\mathbf{P}_{mech} + \mathbf{P}_{field}) = \oint_S \cancel{\overleftrightarrow{\mathbf{T}}} \cdot \mathbf{n} da = 0$ (22)



The object can have any shape.

(22) $\Rightarrow \mathbf{P}_{mech} + \mathbf{P}_{field}$ (not $\mathbf{P}_{mech}$ or $\mathbf{P}_{field}$ individually) in an isolated system is conserved (i.e. const. in time).

Average $\mathbf{F}$, $\mathbf{P}_{mech}$, $\mathbf{P}_{field}$ and $\mathbf{g}$ over a wave period.

$$\Rightarrow <\mathbf{F}>_t = \frac{d<\mathbf{P}_{mech}>_t}{dt} = -\frac{d<\mathbf{P}_{field}>_t}{dt} \underset{(14)}{=} -\frac{d}{dt}\int_v <\mathbf{g}>_t d^3x, \text{ where}$$

$$<\mathbf{g}>_t \overset{(6.118)}{=} \frac{\overbrace{<\mathbf{E}\times\mathbf{H}>_t}^{P\mathbf{e}_z \text{(Poynting vector)}}}{c^2} = \frac{<P>_t}{c^2}\mathbf{e}_z \left[\begin{array}{l} <\cdots>_t : \text{average over a wave period} \\ \frac{d}{dt}: \text{time variation of total } <\cdots>_t \text{ in } v \end{array}\right]$$

$<\mathbf{g}>_t$ moves at speed c and is totally absorbed on hitting area A.

$\Rightarrow$ Total $<\mathbf{g}>_t$ in volume v varies as $\frac{d}{dt}\int_v <\mathbf{g}>_t d^3x = -<\mathbf{g}>_t cA$

$$\Rightarrow \frac{<\mathbf{F}>_t}{A} = <\mathbf{g}>_t c = \frac{<P>_t}{c}\mathbf{e}_z \overset{(19b)}{=} \langle u \rangle_t \mathbf{e}_z \quad [\text{average radiation pressure}]$$

Problem 2: A spherical particle in the outer space with radius a and mass density $\rho_m = 3.5 \times 10^3$ kg/m^3 absorbs all the sunlight it intercepts. For what value of a is the sun's radiation force on the particle equal to the sun's gravitational force on the particle?

$$\begin{cases} I = \text{sunlight intensity (average power/unit area) at the particle} \\ P_s = 3.9 \times 10^{26} \text{ W (total power radiated by sun)} \\ R = \text{sun-to-particle distance} \end{cases}$$

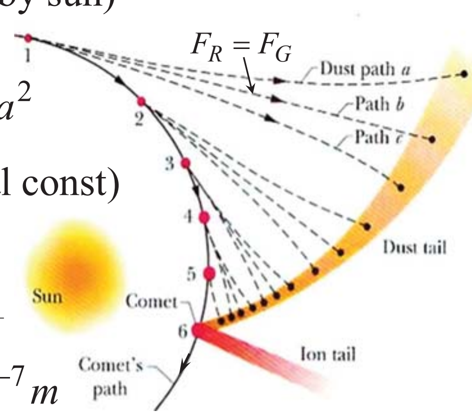
$$F_R \text{ (radiation force)} = \frac{I}{c}\pi a^2 = \frac{P_s}{4\pi R^2 c}\pi a^2$$

$$\begin{cases} G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \text{ (gravitational const)} \\ M_s = 1.99 \times 10^{30} \text{ kg (sun's mass)} \end{cases}$$

$$F_G \text{ (gravitational force)} = \frac{GM_s}{R^2}\frac{4\pi a^3 \rho_m}{3}$$

$$F_R = F_G \Rightarrow a = \frac{3P_s}{16\pi c \rho_m G M_s} = 1.7 \times 10^{-7} m$$

$$\Rightarrow F_G \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} F_R \quad \text{if } a \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1.7 \times 10^{-7} \text{ m}$$



from Haliday, Resnick, and Walker

6.9 Poynting's Theorem for Harmonic Fields; Field Definitions of Impedance and Admittance

Phasors :

In linear equations, harmonic quantities can be represented by complex variables as follows:

$$\underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x},t) \\ \mathbf{D}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) \\ \mathbf{H}(\mathbf{x},t) \\ \mathbf{J}(\mathbf{x},t) \\ \rho(\mathbf{x},t) \end{Bmatrix}}_{\text{real}} = \underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{D}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \\ \mathbf{J}(\mathbf{x}) \\ \rho(\mathbf{x}) \end{Bmatrix}}_{\text{complex (called the \underline{phasor})}} e^{-i\omega t}$$

It is assumed that the LHS is given by the real part of the RHS.

Representation of Time-Averaged Quantities by Phasors :

To express nonlinear quantities by phasors, such as the product of 2 harmonic quantities, we write the quantities as

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}[\mathbf{E}(\mathbf{x})e^{-i\omega t}] = \tfrac{1}{2}[\mathbf{E}(\mathbf{x})e^{-i\omega t} + \mathbf{E}^*(\mathbf{x})e^{i\omega t}]$$

$$\mathbf{J}(\mathbf{x},t) = \mathrm{Re}[\mathbf{J}(\mathbf{x})e^{-i\omega t}] = \tfrac{1}{2}[\mathbf{J}(\mathbf{x})e^{-i\omega t} + \mathbf{J}^*(\mathbf{x})e^{i\omega t}]$$

Below we derive an expression (frequently used in later chapters) for the time average of $\mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)$ over one wave period.

$$\mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)$$

$$= \tfrac{1}{4}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})e^{-2i\omega t} + \mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x})e^{2i\omega t}]$$

$$= \tfrac{1}{2}\mathrm{Re}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})e^{-2i\omega t}]$$

and the time average can be written in terms of phasors as

$$\langle \mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\rangle_t = \tfrac{1}{2}\mathrm{Re}[\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})] \quad \text{[valid for real } \omega] \qquad (23)$$

Similary, $\langle \mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\rangle_t = \tfrac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{x})^*\times\mathbf{H}(\mathbf{x})]$ [for real ω] (24)

Maxwell Equations in Terms of Phasors :

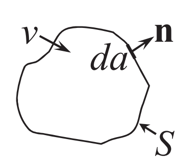
$$\begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x},t) = 0 \\ \nabla\times\mathbf{E}(\mathbf{x},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\cdot\mathbf{D}(\mathbf{x},t) = \rho(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t) = \mathbf{J}(\mathbf{x},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \quad \begin{matrix}\text{variables} \\ \sim e^{-i\omega t} \\ \Rightarrow\end{matrix} \quad \begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x}) = 0 \\ \nabla\times\mathbf{E}(\mathbf{x}) = i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\cdot\mathbf{D}(\mathbf{x}) = \rho(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) - i\omega\mathbf{D}(\mathbf{x}) \end{cases}$$

(Applicable to any time dependence) (Applicale to $e^{-i\omega t}$ dependence)

Note: 1. $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are due to free charges. 2. $\mathbf{D}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x}) = \mu\mathbf{H}(\mathbf{x})$. 3. effects of bound charges are implicit in ε and μ.

Complex Poynting's Theorem :

Apply the phasor representation of Maxwell equations to volume v



$$\Rightarrow \tfrac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}\,d^3x = \tfrac{1}{2}\int_v[\overbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}^*}^{-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\mathbf{H}^*\cdot\overbrace{\nabla\times\mathbf{E}}^{i\omega\mathbf{B}}} - i\omega\mathbf{E}\cdot\mathbf{D}^*]d^3x$$

$$= \tfrac{1}{2}\int_v[-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) - i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)]d^3x \quad (6.131)$$

Rewrite

$$\frac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x = \frac{1}{2}\int_v [-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) - i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)] d^3x \quad [(6.131)]$$

This equation gives the complex Poynting theorem:

$$\frac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x + 2i\omega \int_v (w_e - w_m) d^3x + \oint_s \mathbf{S}\cdot\mathbf{n} da = 0 \qquad (6.134)$$

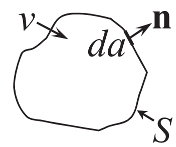
where $\mathbf{S} \equiv \frac{1}{2}\mathbf{E}\times\mathbf{H}^*$ [called the complex Poynting vector] (6.132)

and the real part of $\mathbf{S}$ is the time-averaged power [see (23)].

In (6.134), w_e and w_m are defined as

$$\begin{cases} w_e \equiv \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* = \frac{\varepsilon}{4}|E|^2 \\ w_m \equiv \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* = \frac{\mu}{4}|H|^2 \end{cases} \left[\begin{array}{l}\text{The real part of } w_e \ (w_m) \text{ is the time} \\ \text{averaged E (B) field energy density.}\end{array}\right] \qquad (6.133)$$

For real ε & μ, the real part of (6.134) gives



$$\frac{1}{2}\int_v \mathrm{Re}[\mathbf{J}^*\cdot\mathbf{E}] d^3x + \oint_s \mathrm{Re}[\mathbf{S}\cdot\mathbf{n}] da = 0,$$

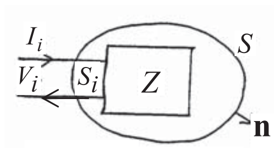
which is the time average of (6.107) applicable to constant-amplitude harmonic fields (for which the field energy remains constant).

Field Definition of Impedance :

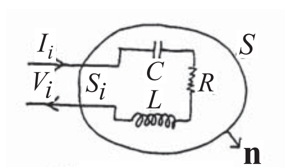
We now apply (6.134) (complex Poynting's thm):

$$\frac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x + 2i\omega \int_v (w_e - w_m) d^3x + \oint_s \mathbf{S}\cdot\mathbf{n} da = 0$$

to a 2-terminal circuit (see figures). Draw a closed surface S surrounding the circuit. Let the complex I_i, V_i be the input current & voltage, respectively.



general circuit



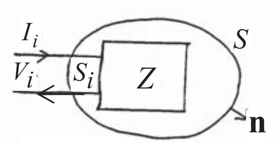
a specific example

$\mathbf{S}$ contains 2 mixed (but clearly identifiable) parts: $\mathbf{S}$ due to input I_i, V_i and $\mathbf{S}$ due to outward circuit radiation. Let the area symbols "S_i" & "$S - S_i$" denote these 2 parts as if fields due to I_i, V_i were confined to a small area S_i and the circuit radiation confined to the area $S - S_i$ (they are actually *mixed*). Then,

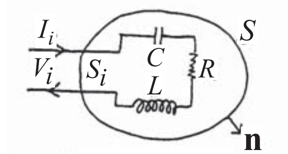
$$\begin{cases} \int_{S_i} \mathbf{S}\cdot\mathbf{n} da = -\frac{1}{2} I_i^* V_i \quad [\frac{1}{2} I_i^* V : \text{complex power input}] \\ \int_{S-S_i} \mathbf{S}\cdot\mathbf{n} da = \text{complex Poynting flux due to circuit radiation} \end{cases} \qquad (6.135)$$

$$\Rightarrow \int_S \mathbf{S}\cdot\mathbf{n} da = \int_{S_i} \mathbf{S}\cdot\mathbf{n} da + \int_{S-S_i} \mathbf{S}\cdot\mathbf{n} da = -\frac{1}{2} I_i^* V_i + \int_{S-S_i} \mathbf{S}\cdot\mathbf{n} da$$

Note : $\frac{1}{2}\mathrm{Re}(I_i^* V_i)$ is the average power sent by I_i, V_i *into* the circuit.



general circuit



a specific example

Sub. $\int_S \mathbf{S}\cdot\mathbf{n}da = -\frac{1}{2}I_i^* V + \int_{S-S_i}\mathbf{S}\cdot\mathbf{n}da$ into the complex Poynting's theorem:

$$\frac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 2i\omega\int_v (w_e - w_m)d^3x + \oint_S \mathbf{S}\cdot\mathbf{n}da = 0 \qquad (6.134)$$

$$\Rightarrow \frac{1}{2}I_i^* V_i = \frac{1}{2}\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 2i\omega\int_v (w_e - w_m)d^3x + \int_{S-S_i}\mathbf{S}\cdot\mathbf{n}da = \frac{1}{2}|I_i|^2 Z,$$

where Z is the field definition of <u>impedance</u>:

$$Z \equiv \frac{V_i}{I_i} = \frac{1}{|I_i|^2}\Big[\int_v \mathbf{J}^*\cdot\mathbf{E}d^3x + 4i\omega\int_v (w_e - w_m)d^3x + 2\overbrace{\int_{S-S_i}\mathbf{S}\cdot\mathbf{n}da}^{\text{Real part is radiation loss}}\Big]$$

$$= R - iX \ \ [R = \mathrm{Re}\,Z \text{ (resistance)}, X = \mathrm{Im}\,Z \text{ (reactance)}] \quad (6.137\text{-}138)$$

The radiation loss term is not included in commonly-used circuit equations, but can be modeled by an effective resistance (pp. 412-3).

Appendix A: Liénard-Wiechert Potentials for a Point Charge

Rewrite (11):
$$\begin{cases} \phi(\mathbf{x},t) = \dfrac{e}{4\pi\varepsilon_0}\displaystyle\int dt' \frac{\delta[t' - (t - \frac{R(t')}{c})]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0 e}{4\pi}\displaystyle\int dt' \frac{\mathbf{v}(t')\delta[t' - (t - \frac{R(t')}{c})]}{R(t')} \end{cases} \qquad \begin{bmatrix} R(t') \equiv \\ |\mathbf{x} - \mathbf{r}(t')| \end{bmatrix}$$

$$\delta[f(x)] = \frac{\delta(x - x_i)}{|f'(x)|} \ [f(x_i) = 0, \text{ p.26}] \Rightarrow \delta[t' + \frac{R(t')}{c} - t] = \frac{\delta(t' - t_{ret})}{\left|\frac{d}{dt'}[t' + \frac{R(t')}{c} - t]\right|},$$

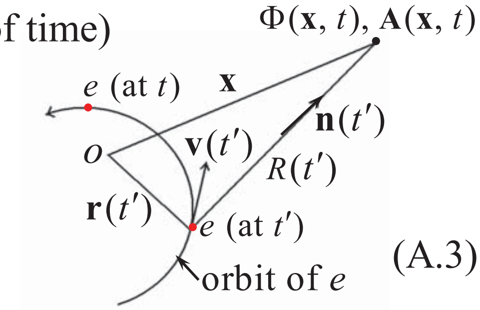
where t_{ret} (retarded time) is the solution of $t' + \frac{R(t')}{c} - t = 0$, we obtain

$$\begin{cases} \Phi(\mathbf{x},t) = \dfrac{e}{4\pi\varepsilon_0}\displaystyle\int dt' \frac{\delta(t' - t_{ret})}{R(t')\left|\frac{d}{dt'}[t' + \frac{R(t')}{c} - t]\right|} = \frac{e}{4\pi\varepsilon_0}\Big[\frac{1}{R(t')\left|\frac{d}{dt'}f(t')\right|}\Big]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0 e}{4\pi}\displaystyle\int dt' \frac{\mathbf{v}(t')\delta(t' - t_{ret})}{R(t')\left|\frac{d}{dt'}[t' + \frac{R(t')}{c} - t]\right|} = \frac{\mu_0 e}{4\pi}\Big[\frac{\mathbf{v}(t')}{R(t')\left|\frac{d}{dt'}f(t')\right|}\Big]_{ret} \end{cases} \quad (A.1)$$

where $f(t') \equiv t' + \dfrac{R(t')}{c}$ (A.2)

$$\frac{dR(t')}{dt'} = \frac{d\left|\mathbf{x}-\mathbf{r}(t')\right|}{dt'} = \frac{d}{dt'}[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}^2(t')]^{\frac{1}{2}}$$

($\mathbf{x}$: observation point, indep. of time)



$$= \frac{-2\mathbf{x}\cdot\frac{d}{dt'}\mathbf{r}(t') + 2\mathbf{r}(t')\cdot\frac{d}{dt'}\mathbf{r}(t')}{2[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}^2(t')]^{\frac{1}{2}}}$$

$$= -\frac{\mathbf{v}(t')\cdot[\mathbf{x}-\mathbf{r}(t')]}{R(t')} = -\mathbf{v}(t')\cdot\mathbf{n}(t') \qquad \text{(A.3)}$$

$$\Rightarrow \frac{d}{dt'} f(t') \overset{\boxed{\text{(A.2)}}}{=} \frac{d}{dt'}[t' + \frac{R(t')}{c}] \overset{\boxed{\text{(A.3)}}}{=} 1 - \frac{\mathbf{v}(t')\cdot\mathbf{n}(t')}{c} \quad \left[\mathbf{n}(t') = \frac{\mathbf{x}-\mathbf{r}(t')}{R(t')}\right] \qquad \text{(A.4)}$$

Sub. (A.4) into (A.1) gives the <u>Lienard-Wiechert potentials</u>

$$\begin{cases} \Phi(\mathbf{x},t) = \dfrac{e}{4\pi\varepsilon_0}\left[\dfrac{1}{(1-\frac{\mathbf{v}\cdot\mathbf{n}}{c})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0 e}{4\pi}\left[\dfrac{\mathbf{v}}{(1-\frac{\mathbf{v}\cdot\mathbf{n}}{c})R}\right]_{ret} \end{cases}$$

[This is identical to Griffiths, Eqs. (10.39) & (10.40); also identical to Jackson (14.8) if converted into Gaussian units.]

Appendix B. A Brief Discussion of Sec. 6.6

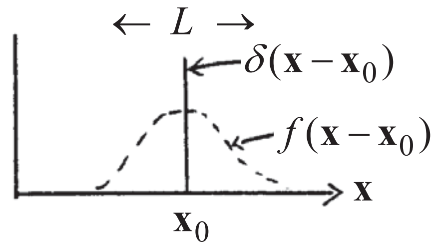
We limit the scope of our consideration of Sec. 6.6 to a general discussion of the averaging method and the derivation of (6.65).

Microscopically, the matter is composed of electrons and nuclei, in which the spatial variations of charge/current distribution functions and electromagnetic field functions occur over the atomic distances (of the order of 10^{-10} m). These functions can be regarded as sums of delta functions. However, macroscopic instruments only measure the averaged quantity. Hence there is a need to develop an averaging method to reduce microscopically fluctuating functions to macroscopically smooth functions, and thereby obtain a set of macroscopic Maxwell equations.

As early as Ch. 1, we have defined a macroscopic charge density $\rho = Nq$ [N: no of point charges q per unit volume]. Sec. 6.6 gives a more formal derivation of macroscopic quantities and eqs.

If we replace each delta function, e.g. $\delta(\mathbf{x}-\mathbf{x}_0)$, in the microscopic distribution function (of charges, etc.) with a smooth function $f(\mathbf{x}-\mathbf{x}_0)$ (see figure) subject to the condition

$$\int f(\mathbf{x}-\mathbf{x}_0)d^3x = 1$$



and if the width L of $f(\mathbf{x}-\mathbf{x}_0)$ is much greater than the atomic distances (e.g. $L \approx 10^{-8}$ m), then the sum of many such functions (each representing a delta function in the microscopic distribution function) will become a smooth function representing the spatially averaged microscopic distribution function. This is the method used in Sec. 6.6 for the derivation of macroscopic equations.

6.6 Derivation of the Equations of Macroscopic Electromagnetism (*continued*)

We may look at the above averaging procedure as follows. A delta function $\delta(\mathbf{x}-\mathbf{x}_0)$ generates a smooth function $f(\mathbf{x}-\mathbf{x}_0)$. Thus, for a distribution function $F(\mathbf{x})$ composed of a large number of point sources (delta functions), the response [denoted by $\langle F(\mathbf{x})\rangle$] will be the superposition of the responses from all points:

$$\langle F(\mathbf{x})\rangle = \int f(\mathbf{x}-\mathbf{x}_0)F(\mathbf{x}_0)d^3x_0 \text{..... spatial average of } F(\mathbf{x})$$

In the integrand, replacing $\mathbf{x}_0$ with $\mathbf{x}-\mathbf{x}'$, we obtain (6.65):

$$\langle F(\mathbf{x})\rangle = \int f(\mathbf{x}')F(\mathbf{x}-\mathbf{x}')d^3x', \tag{6.65}$$

where $f(\mathbf{x})$ is now a smooth function centered at $\mathbf{x}=0$.

As an example, we let $F(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_0)$ and sub. it into (6.65)

$$\langle \delta(\mathbf{x}-\mathbf{x}_0)\rangle = \int f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{x}')d^3x' = f(\mathbf{x}-\mathbf{x}_0)$$

Thus, we have recovered our assumption that the delta function $\delta(\mathbf{x}-\mathbf{x}_0)$ generates a smooth function $f(\mathbf{x}-\mathbf{x}_0)$ centered at $\mathbf{x}_0$.