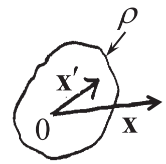
CHAPTER 4: Multipoles, Electrostatics of Macroscopic Media, Dielectrics

4.1 Multipole Expansion

In Ch. 3, we developed various methods of expansion to solve the Poisson equation for $\phi(\mathbf{x})$, with an emphasis on mathematics.

Here, we continue the subject of electrostatics by taking a closer look at $\rho(\mathbf{x})$, with an emphasis on applications. By the method of expansion, we first decompose the RHS of



$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad \begin{bmatrix}\text{Applicable to}\\ \text{infinite space}\end{bmatrix} \qquad (1.17)$$

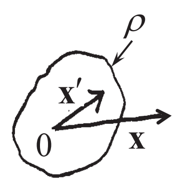
into multipole fields and thereby express $\rho(\mathbf{x})$ in multipole moments.

Applying the results to materials, we find that the atomic/molecular dipole moments account for the electrical properties of a dielectric medium. We then show that these properties can be characterized by a single parameter: the dielectric constant.

Multipole Expansion in Spherical Coordinates :

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.70)$$

For $\mathbf{x}$ outside the sphere enclosing ρ, $r_< = r'$, $r_> = r$.



$$\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$= \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}[\underbrace{\int_V Y_{lm}^*(\theta',\varphi')r'^l\rho(\mathbf{x}')d^3x'}_{\equiv q_{lm}\text{ (multipole moments)}}]\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \qquad (4.2)$$

A conceptual picture of multipole moments:

monopole $(l=0)$ $\quad [+] \quad \Rightarrow \phi \propto \frac{1}{r}$

dipole $(l=1)$ $\quad [+ \quad -] \quad \Rightarrow \phi \propto \frac{1}{r^2}$ ← $\boxed{\text{partial cancellation of monopoles}}$

quadrupole $(l=2)$ $\quad \begin{bmatrix} + & - \\ - & + \end{bmatrix} \quad \Rightarrow \phi \propto \frac{1}{r^3}$ ← $\boxed{\text{partial cancellation of dipoles}}$

Multipole Expansion in Cartesian (Rectangular) Coordinates :

Expansion in Cartesian coordinates is more useful for our purposes. We first summarize the formulae needed for the expansion.

Taylor expansion about point $\mathbf{x}$: [see Appendix A]

$$f(\mathbf{x}+\mathbf{a}) = f(\mathbf{x}) + (\mathbf{a}\cdot\nabla)f(\mathbf{x}) + \tfrac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla)f(\mathbf{x}) + \cdots, \tag{1}$$

where

$$\mathbf{a}\cdot\nabla = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i\frac{\partial}{\partial x_i} \tag{2a}$$

$$(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) = \sum_i a_i\frac{\partial}{\partial x_i}\sum_j a_j\frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j\frac{\partial^2}{\partial x_i\partial x_j} \tag{2b}$$

and all derivatives are evaluated at $\mathbf{x}$.

Other useful relations:

$$\nabla|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \qquad \begin{bmatrix}\text{Eq. (1), lecture}\\ \text{notes, Ch. 1}\end{bmatrix} \tag{3a}$$

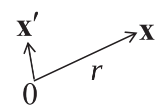
$$\frac{\partial}{\partial x_i}|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(x_i - x_i') \tag{3b}$$

Rewrite

$$f(\mathbf{x}+\mathbf{a}) = f(\mathbf{x}) + (\mathbf{a}\cdot\nabla)f(\mathbf{x}) + \tfrac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla)f(\mathbf{x}) + \cdots \tag{1}$$

$$(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) = \sum_i a_i\frac{\partial}{\partial x_i}\sum_j a_j\frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j\frac{\partial^2}{\partial x_i\partial x_j} \tag{2b}$$

$$\nabla|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \tag{3a}$$



Use (1). Let $f(\mathbf{x}) = 1/|\mathbf{x}| = 1/r$ and $\mathbf{a} = -\mathbf{x}'$.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} - \mathbf{x}'\cdot\nabla\frac{1}{r} + \frac{1}{2}(\mathbf{x}'\cdot\nabla)(\mathbf{x}'\cdot\nabla)\frac{1}{r} + \cdots$$

Use (2b), (3a)

$$= \frac{1}{r} + \frac{\mathbf{x}'\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{ij} x_i' x_j' \underbrace{\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}} + \cdots$$

$$= -\frac{\partial}{\partial x_i}\left(\frac{x_j}{r^3}\right) = -x_j\frac{\partial}{\partial x_i}\frac{1}{r^3} - \frac{1}{r^3}\frac{\partial x_j}{\partial x_i} = x_j\frac{3x_i}{r^5} - \frac{\delta_{ij}}{r^3}$$

$$r^2 = \sum_{ij} x_i x_j \delta_{ij}$$

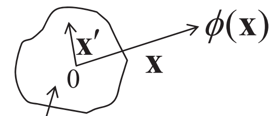
$$= \frac{1}{r} + \frac{\mathbf{x}'\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\sum_{ij} 3x_i' x_j' x_i x_j - \frac{1}{2r^5} r^2 \overbrace{\sum_{ij} x_i' x_j' \delta_{ij}}^{r'^2} + \cdots$$

$$= \frac{1}{r} + \frac{\mathbf{x}'\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\sum_{ij} x_i x_j \left(3x_i' x_j' - r'^2\delta_{ij}\right) + \cdots \tag{4}$$

See (3.70), (3.148), & (3.164) for other expressions of $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$ in ∞ space.

Multipole moments with respect to (w.r.t.) $\mathbf{x}=0$:

Rewrite $$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{r}+\frac{\mathbf{x}'\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}x_ix_j\left(3x_i'x_j'-r'^2\delta_{ij}\right)+\cdots \quad (4)$$



$\rho(\mathbf{x}')$ (characterized by q, $\mathbf{p}$, Q_{ij}, $\cdots$)

$$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

Sub. (4) for $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$

$$=\frac{1}{4\pi\varepsilon_0}\Big[\frac{\overbrace{\int\rho(\mathbf{x}')d^3x'}^{\equiv q \text{ (monopole moment)}}}{r}+\frac{\mathbf{x}\cdot\overbrace{\int\mathbf{x}'\rho(\mathbf{x}')d^3x'}^{\equiv\mathbf{p} \text{ (dipole moment)}}}{r^3}$$
$$+\frac{1}{2r^5}\sum_{ij}x_ix_j\overbrace{\int(3x_i'x_j'-r'^2\delta_{ij})\rho(\mathbf{x}')d^3x'}^{\equiv Q_{ij} \text{ (quadrupole moment)}}+\cdots\Big]$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}Q_{ij}x_ix_j+\cdots\right) \quad (4.10)$$

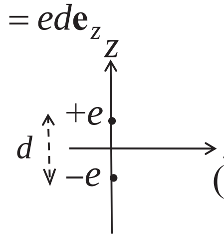
Note:

1. q, $\mathbf{p}$, Q_{ij}, $\cdots$ are *constants* characterizing the charge distribution $\rho(\mathbf{x})$.
2. $\mathbf{p}$, Q_{ij}, $\cdots$ are defined w.r.t. a reference point [it is $\mathbf{x}=0$ in (4.10)].

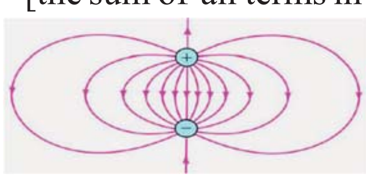
Question: What are the merit and limitation of (4.10)? See next page.

Example: For point charges $\pm e$ separated by distance d as in the fig., we have $q=\int\rho(\mathbf{x}')d^3x'=0$, $\mathbf{p}=\int\mathbf{x}'\rho(\mathbf{x}')d^3x'=ed\mathbf{e}_z$

and $$\phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$
$$=\frac{1}{4\pi\varepsilon_0}\left(\frac{\cancel{q}}{r}+\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}Q_{ij}x_ix_j+\cdots\right) \quad (4.10)$$

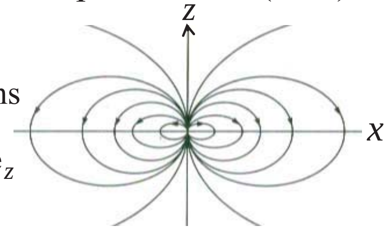


Exact **E** of point charges $\pm e$ [the sum of all terms in (4.10)]



Benson Fig. 30.9

E of the dipole term in (4.10)



Griffiths p. 154 $\mathbf{p}=p\mathbf{e}_z$

Note: A pair of $\pm e$ consists of all multipole moments (not just $\mathbf{p}$).

Merit of (4.10): In the far zone, **E**(dipole) accurately gives **E**(exact).

Limitation: In the near zone, all terms are needed to give **E**(exact). Amazingly, each term diverges as $r\to 0$, but their sum diverges at $\pm e$.

Multipole moments with respect to $\mathbf{x}=\mathbf{x}_0$:

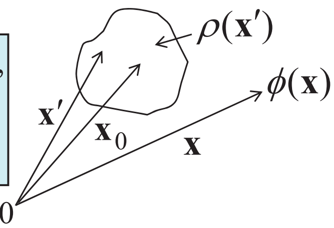
Values of $\mathbf{p}$, $Q_{ij},\cdots$ depend on the choice of the reference point (exception noted in the exercise below), but the sum of all multipole fileds gives the same $\phi(\mathbf{x})$. It is thus desirable to choose a reference point to minimize higher moments so that the lowest moment can give a good approximation of the exact $\phi(\mathbf{x})$.

We may convert the reference point from $\mathbf{x}=0$ in (4.10) to $\mathbf{x}=\mathbf{x}_0$

by using $$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{r}+\frac{\mathbf{x}'\cdot\mathbf{x}}{r^3}+\frac{1}{2r^5}\sum_{ij}x_i x_j(3x_i'x_j'-r'^2\delta_{ij})+\cdots \tag{4}$$

$$\Rightarrow \frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|}$$

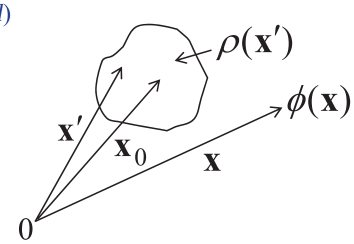
On the RHS of (4), let $\begin{cases}\mathbf{x}\rightarrow\mathbf{x}-\mathbf{x}_0\\ \mathbf{x}'\rightarrow\mathbf{x}'-\mathbf{x}_0\end{cases}$



$$=\frac{1}{|\mathbf{x}-\mathbf{x}_0|}+\frac{(\mathbf{x}'-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}$$

$$+\frac{1}{2}\sum_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}\left[3(x_i'-x_{0i})(x_j'-x_{0j})-|\mathbf{x}'-\mathbf{x}_0|^2\delta_{ij}\right]+\cdots \tag{4a}$$

$$\Rightarrow \phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$



$$\overset{(4a)}{=}\frac{1}{4\pi\varepsilon_0}\Bigg[\frac{\overbrace{\int\rho(\mathbf{x}')d^3x'}^{q}}{|\mathbf{x}-\mathbf{x}_0|}+\frac{(\mathbf{x}-\mathbf{x}_0)\cdot\overbrace{\int(\mathbf{x}'-\mathbf{x}_0)\rho(\mathbf{x}')d^3x'}^{\mathbf{p}\text{ w.r.t. }\mathbf{x}_0}}{|\mathbf{x}-\mathbf{x}_0|^3}$$

$$+\frac{1}{2}\sum_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}\overbrace{\int\{3(x_i'-x_{0i})(x_j'-x_{0j})-|\mathbf{x}'-\mathbf{x}_0|^2\delta_{ij}\}\rho(\mathbf{x}')d^3x'}^{Q_{ij}\text{ w.r.t. }\mathbf{x}_0}+\cdots\Bigg]$$

$$=\frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\frac{q}{|\mathbf{x}-\mathbf{x}_0|}}_{\text{due to net charge }q}+\underbrace{\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}}_{\text{due to dipole }\mathbf{p}\text{ at }\mathbf{x}=\mathbf{x}_0}+\underbrace{\frac{1}{2}\sum_{ij}Q_{ij}\frac{(x_i-x_{0i})(x_j-x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5}}_{\text{due to quadrupole }Q_{ij}\text{ at }\mathbf{x}=\mathbf{x}_0}+\cdots\Bigg] \tag{5}$$

In (5), $\mathbf{p}$, $Q_{ij},\cdots$ are defined w.r.t. $\mathbf{x}_0$ and we may regard q, $\mathbf{p}$, Q_{ij}, etc. to be located at $\mathbf{x}_0$ [see p. 147, statement above (4.13)].

$\mathbf{x}_0$ at or near the center of ρ can minimize higher moments. Why?

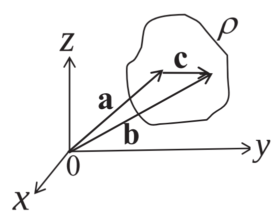
At a large enough $\mathbf{x}$, $\phi(\mathbf{x})$ of the lowest moment dominates. Why?

Exercise: Prove that the lowest non-vanishing multipole moment is independent of the reference point (see pp. 147-8).

Solution: Each component of the ℓ-th multipole moment w.r.t. reference points **a** & **b** consists, respectively, of integrals of the form

$$I_{ijk}^{(\mathbf{a})} = \int \rho(\mathbf{x})(x-a_x)^i (y-a_y)^j (z-a_z)^k d^3x \quad \text{and}$$

$$\begin{aligned} I_{ijk}^{(\mathbf{b})} &= \int \rho(\mathbf{x})(x-b_x)^i (y-b_y)^j (z-b_z)^k d^3x \\ &= \int \rho(\mathbf{x})(x-a_x-c_x)^i (y-a_y-c_y)^j (z-a_z-c_z)^k d^3x, \end{aligned}$$

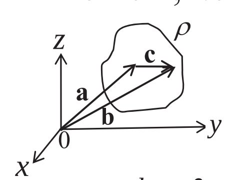


where $i, j,$ and k are zero or positive integers $(i+j+k=\ell)$, $\mathbf{a}=(a_x,a_y,a_z)$, $\mathbf{b}=(b_x,b_y,b_z)$, and $\mathbf{b}=\mathbf{a}+\mathbf{c}$ with $\mathbf{c}$ given by $\mathbf{c}=(c_x,c_y,c_z)$.

For example, the monopole moment has only one term $(i=j=k=0)$, each component of the dipole moment consists of one term (i or j or $k=1$), and each component of the quadrupole moment consists of multiple terms (all having $i+j+k=2$).

The monopole moment q $[=\int \rho(\mathbf{x})d^3x]$ is clearly indep. of the reference point. If $q=0$ and if the lowest nonvanishing multipole moment w.r.t. reference point **a** is the ℓ-th moment, i.e.

$$I_{ijk}^{(\mathbf{a})} = \begin{cases} =0\ , & i+j+k<l \\ \neq 0\ , & i+j+k=l \end{cases},$$



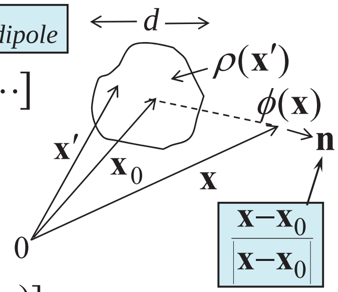
then, w.r.t. reference point **b**, we have

$$\begin{aligned} \underset{i+j+k=l}{I_{ijk}^{(\mathbf{b})}} &= \int \rho(\mathbf{x}) \underbrace{(x-a_x-c_x)^i}_{\substack{=(x-a_x)^i - \\ ic_x(x-a_x)^{i-1}+..}} \underbrace{(y-a_y-c_y)^j}_{\substack{=(y-a_y)^j - \\ jc_y(y-a_y)^{j-1}+..}} \underbrace{(z-a_z-c_z)^k}_{\substack{=(z-a_z)^k - \\ kc_z(z-a_z)^{k-1}+..}} d^3x \\ &= \int_{i+j+k=l} \rho(\mathbf{x})(x-a_x)^i (y-a_y)^j (z-a_z)^k d^3x \\ &\quad + \sum_{\alpha\beta\gamma} \underbrace{C_{\alpha\beta\gamma}}_{\boxed{\substack{\text{multiplications} \\ \text{of } c_x,\ c_y,\ \&\ c_z}}} \underbrace{\int \rho(\mathbf{x})(x-a_x)^\alpha (y-a_y)^\beta (z-a_z)^\gamma d^3x}_{\substack{=I_{\alpha\beta\gamma}^{(\mathbf{a})}=0 \\ \alpha+\beta+\gamma<l}} \\ &= \underset{i+j+k=l}{I_{ijk}^{(\mathbf{a})}} \end{aligned}$$

Q.E.D.

Dipole electric field ($\mathbf{E}_{dipole}$):

Rewrite (5): $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{x}_0|} + \overbrace{\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}}^{\phi_{dipole}} + \cdots\right]$



$$\mathbf{E}_{dipole}(\mathbf{x}) = -\nabla\phi_{dipole}(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\nabla\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}$$

$$= -\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{4\pi\varepsilon_0}\nabla\underbrace{\frac{1}{|\mathbf{x}-\mathbf{x}_0|^3}}_{\frac{-3(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^5}} - \frac{1}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|^3}\underbrace{\nabla[\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)]}$$

Use (3a)

$= (\mathbf{p}\cdot\nabla)(\mathbf{x}-\mathbf{x}_0) + [\cancel{(\mathbf{x}-\mathbf{x}_0)\cdot\nabla}]\mathbf{p}$ ←-- a const.
$+\,\mathbf{p}\times[\nabla\times\cancel{(\mathbf{x}-\mathbf{x}_0)}] + (\mathbf{x}-\mathbf{x}_0)\times(\cancel{\nabla\times\mathbf{p}})$
$= (\mathbf{p}\cdot\nabla)(\mathbf{x}-\mathbf{x}_0) = \mathbf{p}$ [use (2a)]

$$\Rightarrow \underbrace{\mathbf{E}_{dipole}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|^3}}, \quad \mathbf{n} = \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \quad \left[\begin{array}{l}\mathbf{p}\text{ defined w.r.t. }\mathbf{x}_0;\\ \mathbf{E}_{dipole}\text{ due to }\mathbf{p}\text{ at }\mathbf{x}_0\end{array}\right] \quad (4.13)$$

Good approx. of the exact $\mathbf{E}$ if $q = 0$ and $|\mathbf{x}-\mathbf{x}_0| \gg d$ (size of ρ)
Divergent (invalid) as $|\mathbf{x}-\mathbf{x}_0| \to 0$

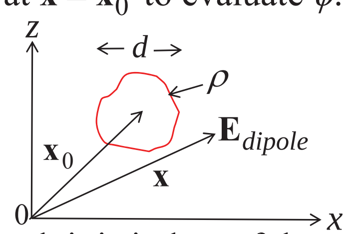
Discussion on reference point choice and field evaluation:

1. In $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\sum_{ij} Q_{ij}x_i x_j + \cdots\right)$ [(4.10)], $\mathbf{p}$, Q_{ij}, etc. are defined w.r.t. $\mathbf{x} = 0$ and are put at $\mathbf{x} = 0$ to evaluate ϕ. In

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{x}_0|} + \frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{(x_i - x_{0i})(x_j - x_{0j})}{|\mathbf{x}-\mathbf{x}_0|^5} + \cdots\right] \quad [(5)],$$

$\mathbf{p}$, Q_{ij}, etc. are defined w.r.t. $\mathbf{x} = \mathbf{x}_0$ and are put at $\mathbf{x} = \mathbf{x}_0$ to evaluate ϕ.

2. Define $\mathbf{p}$, Q_{ij}, etc. w.r.t. a reference pt. ($\mathbf{x}_0$, right figure) located at or near the center of ρ so as to minimize higher moments.

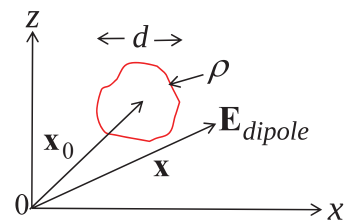


3. The lowest non-vanishing moment gives the dominant contribution in the far zone. Although it is indep. of the reference pt., it is still desirable to put it at or near the center of ρ to evaluate ϕ. For example, if $q = 0$, (4.10) and (5) give the same $\mathbf{p}$, but

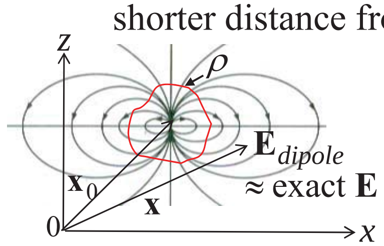
$$\mathbf{E}_{dipole}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}|^3}\ \left[\begin{array}{l}\text{from}\\(4.10)\end{array}\right]; \quad \mathbf{E}_{dipole}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_0|^3}\ \left[\begin{array}{l}\text{from}\\(5)\end{array}\right]$$

(*continued*) Consider the same ρ as in last page. Since $q=0$, $\mathbf{p}$ is indep. of the reference pt. (let it be $\mathbf{p}=p\mathbf{e}_z$).

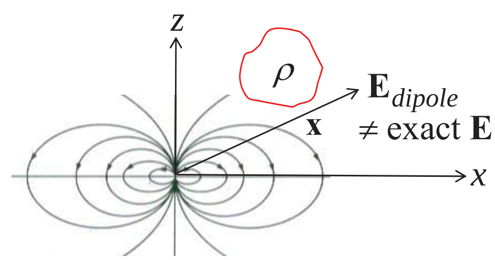
$$\begin{cases} \text{Case A: Put } \mathbf{p}\ (=p\mathbf{e}_z) \text{ at } \mathbf{x}=\mathbf{x}_0 \\ \Rightarrow \mathbf{E}_{dipole}(\mathbf{x}) = \dfrac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}_0|^3} \\ \text{Case B: Put } \mathbf{p}\ (=p\mathbf{e}_z) \text{ at } \mathbf{x}=0 \\ \Rightarrow \mathbf{E}_{dipole}(\mathbf{x}) = \dfrac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n})-\mathbf{p}}{4\pi\varepsilon_0 |\mathbf{x}|^3} \end{cases}$$



Case A: $\mathbf{E}_{dipole} \to$ exact $\mathbf{E}$ at a shorter distance from ρ



Case B: $\mathbf{E}_{dipole} \to$ exact $\mathbf{E}$ at a much larger distance from ρ



4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field

In (1.53), we have $W=\frac{1}{2}\int \rho(\mathbf{x})\phi(\mathbf{x})d^3x$ [$\phi(\mathbf{x})$ due to $\rho(\mathbf{x})$]

$\phi(\mathbf{x})$ due to $\rho(\mathbf{x})$ in the integrand; $\nabla^2\phi(\mathbf{x}) = -\rho(\mathbf{x})/\varepsilon_0$

Here, we consider the energy of $\rho(\mathbf{x})$ in $\phi(\mathbf{x})$, where $\phi(\mathbf{x})$ is due to external charges:

$$W=\int \rho(\mathbf{x})\phi(\mathbf{x})d^3x \tag{4.21}$$

$\phi(\mathbf{x})$ due to external charges; $\nabla^2\phi(\mathbf{x}) = 0$ in region of $\rho(\mathbf{x})$

Expand the external field $\phi(\mathbf{x})$ [Use (A.3) in appendix A]:

$$\begin{aligned}\phi(\mathbf{x}) &= \phi(0) + \mathbf{x}\cdot\nabla\phi\Big|_{\mathbf{x}=0} + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2\phi}{\partial x_i \partial x_j}\Big|_{\mathbf{x}=0} + \cdots \\ &= \phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j}{\partial x_i}\Big|_{\mathbf{x}=0} + \cdots \\ &= \phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j}{\partial x_i}\Big|_{\mathbf{x}=0} + \cdots \end{aligned} \tag{4.23}$$

add $\underbrace{\frac{1}{6}\sum_{ij} r^2 \frac{\partial E_j}{\partial x_i}\Big|_{\mathbf{x}=0}\delta_{ij}}_{=\frac{1}{6}r^2\nabla\cdot\mathbf{E}(0)=0}$

Rewrite $\phi(\mathbf{x}) = \phi(0) - \mathbf{x}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\underbrace{\frac{\partial E_j(0)}{\partial x_i}}_{\text{i.e. } \left.\frac{\partial E_j(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=0}} + \cdots$ (4.23)

$$\Rightarrow W = \int \rho(\mathbf{x})\phi(\mathbf{x})d^3x$$

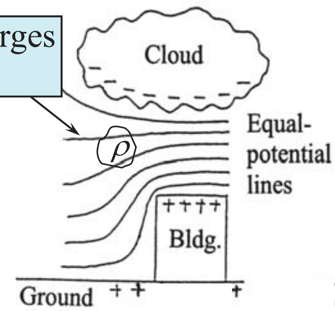
$$= q\phi(0) - \underbrace{\mathbf{p}}_{} \cdot \mathbf{E}(0) - \frac{1}{6}\sum_{ij}\underbrace{Q_{ij}}_{}\frac{\partial E_j(0)}{\partial x_i} + \cdots \quad (4.24)$$

$\boxed{\mathbf{p} = \int \mathbf{x}\rho(\mathbf{x})d^3x \text{ (w.r.t. } \mathbf{x}=0)}$

$\boxed{Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{x})d^3x \text{ (w.r.t. } \mathbf{x}=0)}$

$$\Rightarrow \begin{cases} q \quad [+] & \text{interacts with } \phi \\ \mathbf{p} \quad [+ \ \ -] & \text{interacts with } \mathbf{E} \text{ (non-uniform } \phi) \\ Q_{ij}\begin{bmatrix} + & - \\ - & + \end{bmatrix} & \text{interacts with non-uniform } \mathbf{E} \end{cases}$$

$\phi(\mathbf{x})$ due to charges external to ρ.



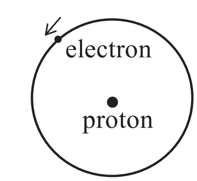
Questions:

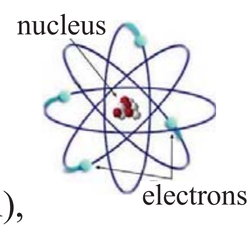
1. Higher order moments can "see" finer structure of $\phi(\mathbf{x})$. Why?
2. How does a charged rod attract a piece of paper?
3. How does a microwave oven heat food?

4.6 Models for the Molecular Polarizability

Induced Dipole Moment:

Molecular (atomic) electrons obey the laws of quantum mechanics. In the simplest case of a hydrogen atom, an electron (described by a wavefunction) rapidly moves around a proton. We ignore its orbital motion and consider only the motion of the orbital center, which coincides with the proton when there is no external **E**.

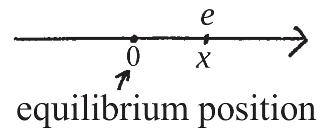




For most molecules/atoms (with few exceptions, e.g. water), there is 0 dipole moment in the absence of **E**(external). For simplicity, we may regard charges inside as e^-/proton pairs. In the absence of **E**(external), each pair are at the same equilibrium position, like a hydrogen atom.

Electrons and ions in a molecule (atom) are bound charges. In the presence of an external **E**, they (predominantly electrons) tend to be very slightly displaced from their equilibrium positions. The molecule (atom) is thus *polarized* (i.e. +−) and dipole moments are *induced*.

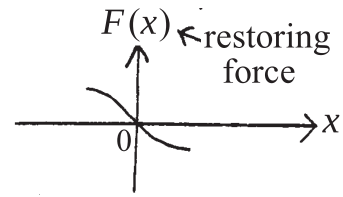
When a charge e is displaced from its equilibrium position ($x = 0$), it will be subject to a restoring force $F(x)$, much like an object tied to a spring. We expand $F(x)$ as



$$F(x) = \underbrace{F(0)}_{=0} + \underbrace{F'(0)}_{=-m\omega_0^2} x + \underbrace{\tfrac{1}{2}F''(0)x^2 + \cdots}_{\text{nonlinear effects, negligible if } x \to 0}$$

because $x = 0$ is the equilibrium position.

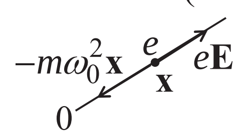
ω_0 is the natural frequency of e if it oscillates as a simple harmonic oscillator.



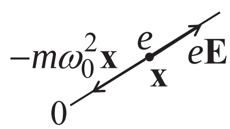
Consider a charge e displaced from position 0 to $\mathbf{x}$ by force $e\mathbf{E}$ (e carries a sign). Assume $\mathbf{F}$ (restoring force) and $\mathbf{x}$ are along the same line (true for *isotropic* media), but in opposite directions.

For small $\mathbf{x}$, we have $\quad \mathbf{F}(\mathbf{x}) \approx -m\omega_0^2\mathbf{x}$ (4.71)

Static balance $\Rightarrow \underbrace{e\mathbf{E}}_{\text{electric force}} + \underbrace{(-m\omega_0^2\mathbf{x})}_{\text{restoring force}} = 0 \Rightarrow \mathbf{x} = \dfrac{e}{m\omega_0^2}\mathbf{E}$



Rewrite $\mathbf{x} = \dfrac{e}{m\omega_0^2}\mathbf{E}$



This results in a dipole moment $\mathbf{p}$:

$$\mathbf{p} \equiv \int \mathbf{x}'\rho(\mathbf{x}')d^3x' = \int \mathbf{x}'e\delta(\mathbf{x}'-\mathbf{x})d^3x = e\mathbf{x} = \frac{e^2}{m\omega_0^2}\mathbf{E} = \varepsilon_0\gamma\mathbf{E}, \quad (4.72)$$

where $\gamma \equiv e^2/(\varepsilon_0 m\omega_0^2)$ is the polarizability for a *single* charge.

Note: $\mathbf{p}$ of a single charge (let it be an electron) depends on the reference point (it is $\mathbf{x} = 0$ here). However, as discussed before, we match each electron with a proton (initially also at $\mathbf{x} = 0$). Since the monopole is 0, the dipole moment is indep. of the reference pt.

$\mathbf{p}_{mol}$ (molecular dipole moment) is the sum of all $\mathbf{p}$ in a molecule:

$$\mathbf{p}_{mol} = \sum_j e_j\mathbf{x}_j = \sum_j \frac{e_j^2}{m_j\omega_j^2}\mathbf{E} = \varepsilon_0\gamma_{mol}\mathbf{E} \quad \left[\begin{array}{l}\text{mostly due}\\ \text{to electrons}\end{array}\right] \quad (4.72)+(4.73)$$

where $\gamma_{mol} \equiv \dfrac{1}{\varepsilon_0}\sum_j \dfrac{e_j^2}{m_j\omega_j^2}$ [molecular polarizability]. (4.73)

Note: An *induced* $\mathbf{p}_{mol}$ is $\parallel$ to $\mathbf{E}$. $\Rightarrow$ no touque on $\mathbf{p}_{mol}$ (by $\mathbf{E}$)

Discussion:

(1) $\mathbf{p} = e\mathbf{x}$ as calculated above for a single charge is with respect to the equilibrium position at $\mathbf{x} = 0$. Different charges have different equilibrium positions. Thus, in $\mathbf{p}_{mol} = \sum_j e_j \mathbf{x}_j$, each $e_j \mathbf{x}_j$ is with respect to a different reference point.

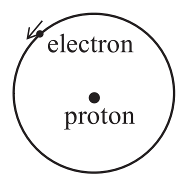
This will not cause any difficulty if the total $+$ & $-$ charges in the molecule are equal. Since the net charge is 0, $\mathbf{p}_{mol}$ is the lowest non-vanishing moment, hence indep. of the reference point.

"Independent of the reference point" is a key to subsequent calculations. Thus, we assume that $\mathbf{p}_{mol}$ is contributed by an equal amount of $+/-$ charges in the molecule. If there is a net molecular charge, it will be treated separately [see (4.29) below].

(ii) The approximation made in (4.71), $\mathbf{F}(\mathbf{x}) \approx -m\omega_0^2 \mathbf{x}$, has led to a linear relation between $\mathbf{p}_{mol}$ and $\mathbf{E}$, i.e. $\mathbf{p}_{mol} = \varepsilon_0 \gamma_{mol} \mathbf{E}$.

4.6 Models for the Molecular Polarizability (*continued*)

Problem: A hydrogen atom is in an external $E = 10^6$ V/m. Use $p = \varepsilon_0 \gamma E$ [(4.72)] to get a crude estimate of the dipole moment p of this atom and the the electron displacement from its equilibrium position.



$$\begin{cases} \varepsilon_0 \approx 8.854 \times 10^{-12} \text{ F/m (p. 782)} \\ \gamma \approx 10^{-29} \text{ m}^3 \text{ (p. 163)} \end{cases} \Rightarrow \begin{cases} p \approx 8.854 \times 10^{-12} \times 10^{-29} \times 10^6 \\ \quad \approx 8.9 \times 10^{-35} \text{ C-m} \end{cases}$$

Let d = the electron displacement from its equilibrium position.

$p = ed = 1.6 \times 10^{-19} d$ C-m (for $E = 10^6$ V/m)

$$\Rightarrow d = p/(1.6 \times 10^{-19}) \approx 8.9 \times 10^{-35}/(1.6 \times 10^{-19}) \approx \overbrace{5.6 \times 10^{-16} \text{ m}}^{\text{extremely small!}}$$

The water molecule has a permanent dipole moment of

$p_{water} \approx 6.2 \times 10^{-30}$ C-m [7×10^4 times the p in $E = 10^6$ V/m]

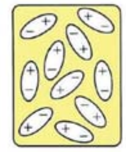
$\Rightarrow$ To *induce* a p equal to p_{water} on the hydrogen atom, it requires an external $E > 10^{10}$ V/m, which is $\gg$ typical interatomic fields (10^5-10^8 V/m). So, well before E reaches this value, the hydrogen atom would be ionized (p would become a nonlinear function of E even sooner).

Permanent Dipole Moment under an E-Field:

Some molecules (like water) have a permanent dipole moment ($\mathbf{p}$). These molecules are randomly oriented in the absence of an applied $\mathbf{E}$ so the *vector* sum of all permanent $\mathbf{p}$ over a small volume is 0, i.e.

$\sum_j \mathbf{p}_j = 0$ [$\mathbf{p}_j$: same magnitude; random directions]

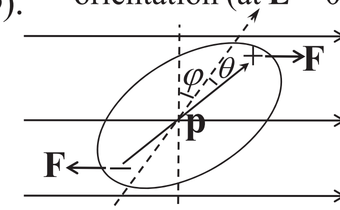
$$\Rightarrow \langle \mathbf{p} \rangle = \frac{\sum_j \mathbf{p}_j}{\text{total no. of molecules}} = 0 \quad [\text{applied } \mathbf{E} = 0]$$



from Benson

If the applied $\mathbf{E} \neq 0$, there will be a torque ($\mathbf{\Gamma}_E$) on $\mathbf{p}$, which tends to align $\mathbf{p}$ along $\mathbf{E}$ (lower fig.). But a restoring torque (Γ_r) will restrict the rotation angle θ to $\theta \ll 1$ and the new equilibrium angle is $\varphi + \theta$, where φ is the original equilibrium angle. With $\theta \ll 1$, we have $\Gamma_r \propto \theta$ (Taylor expansion) and $\Gamma_E \propto E$. Thus, $\Gamma_r = \Gamma_E$ (new equilibrium) $\Rightarrow \theta = \alpha E$ (α depends on φ). Unless $\varphi = 90^{\circ}$, a rotation tilts each $\mathbf{p}$ toward $\mathbf{E}$. With $\theta \ll 1$, it's easy to show $\mathbf{p}$ thus gains a $\|$ component which is $\propto \theta = \alpha E$ (How?). So $\langle \mathbf{p} \rangle = (+const) \cdot \mathbf{E}$ [a linear relation as (4.72)]

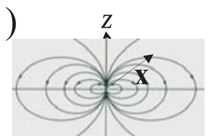
original equilibrium orientation (at $\mathbf{E} = 0$)



Electric Polarization , Polarization Charge, and Free Charge :

By (4.13), $\mathbf{E}$ of a *microscopic* molecular dipole ($\mathbf{p}_{mol}$) located at $\mathbf{x} = 0$ is

$$\underbrace{\mathbf{E}_{dipole} = \frac{3\mathbf{n}(\mathbf{p}_{mol} \cdot \mathbf{n}) - \mathbf{p}_{mol}}{4\pi\varepsilon_0 |\mathbf{x}|^3}}_{\boxed{\text{Good approx. of the exact } \mathbf{E} \text{ if } |\mathbf{x}| \gg \text{dipole size; divergent as } |\mathbf{x}| \to 0}} \quad \left[\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}\right]$$



$\mathbf{p}_{dipole} = p_{dipole}\mathbf{e}_z$

We now define $\mathbf{P}$ (electric polarization, a *mascroscopic* quantity) as the total dipole mement per unit volume (due to bound charges):

$$\mathbf{P}(\mathbf{x}) = \underbrace{\sum_i}_{\text{sum over all types of molecules}} \underbrace{N_i(\mathbf{x})}_{\text{number density of type } i \text{ molecules}} \langle \mathbf{p}_i \rangle \qquad (4.28)$$

$\boxed{\langle \mathbf{p}_i \rangle\text{: type-}i\ \mathbf{p}_i \text{ averaged over a small volume at } \mathbf{x}.\ \langle \mathbf{p}_i \rangle \text{ can be} \ll \mathbf{p}_i \text{ due to vector cancellation of, e.g. permanent } \mathbf{p}_i.}$

$\mathbf{P}(\mathbf{x})$ gives no divergent $\mathbf{E}$ at any $\mathbf{x}$, just like a macroscopic $\rho(\mathbf{x})$ defined by $\rho(\mathbf{x}) = N(\mathbf{x})q$ [$N(\mathbf{x})$: number density of point charges q] gives no divergent $\mathbf{E}$ at any $\mathbf{x}$ [see Exercise 2, Sec. 1.3].

Rewrite $\quad \mathbf{P}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) \langle \mathbf{p}_i \rangle \qquad (4.28)$

We now divide the charge density in a dielectric medium into two categories: $\begin{cases} \rho_{pol} & \text{(polarization charge density)} \\ \rho_{free} & \text{(free charge density)} \end{cases}$

where ρ_{pol} results from the separation of (equal) $+/-$ bound charges in molecules, while ρ_{free} (if any) consists of (1) net molecular charges and (2) excess charges (such as free electrons) in the medium, i.e.

$$\rho_{free}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) \langle e_i \rangle + \rho_{excess}(\mathbf{x}) \quad \left[\begin{array}{r} \langle e_i \rangle\text{: average net charge} \\ \text{per type-}i\text{ molecule} \end{array} \right] \qquad (4.29)$$

Example: Some air molecules may contain a net charge e_i [$\Rightarrow \langle e_i \rangle$]. There are also a small number of free electrons in air [$\Rightarrow \rho_{excess}(\mathbf{x})$].

Note: 1. $\langle e_i \rangle$ can be $\ll e_i$ (e.g. 1 in 10^6 molecules contains a net e_i.)

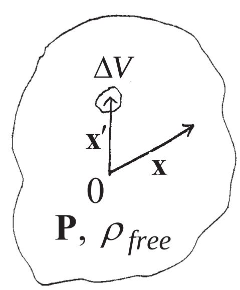
2. We have used the notation ρ_{free} to distinguish it from ρ_{pol}. ρ_{free} here is denoted by ρ in Jackson [in (4.29), (4.35), etc.]

4.3 Elementary Treatment of Electrostatics with Ponderable Media

Macroscopic Version of $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ (for Dielectric Media):

Consider a medium with $\mathbf{P}$ (due to ρ_{pol}) and ρ_{free}. We treat $\mathbf{P}$ (hence ρ_{pol}) and ρ_{free} separately by expressing the potential [$\Delta\phi(\mathbf{x})$] due to charges in a small volume ΔV at $\mathbf{x}'$ as

$$\Delta\phi(\mathbf{x}) = \Delta\phi_{free}(\mathbf{x}) + \Delta\phi_{pol}(\mathbf{x}),$$



where $\Delta\phi_{free}$ is due to $\rho_{free}(\mathbf{x}')\Delta V$ and $\Delta\phi_{pol}$ is due to $\rho_{pol}(\mathbf{x}')\Delta V$. Clearly, $\Delta\phi_{free}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \frac{\rho_{free}(\mathbf{x}')\Delta V}{|\mathbf{x}-\mathbf{x}'|}$.

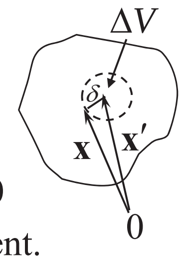
For $\Delta\phi_{pol}$, we don't yet have an expression for ρ_{pol}. So we use (5):

$$\Delta\phi_{pol}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\Big[\frac{\overset{\boxed{q_{pol} = \int \rho_{pol}(\mathbf{x}) d^3x = 0}}{\cancel{q_{pol}}}}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{P}(\mathbf{x}')\Delta V \cdot (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} + \cdots\Big] \approx \frac{\overbrace{\mathbf{P}(\mathbf{x}')\Delta V}^{\boxed{\text{total } \mathbf{p} \text{ in } \Delta V, \text{ indep. of ref. point}}} \cdot (\mathbf{x}-\mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}'|^3} \qquad (6)$$

Rewrite $\Delta\phi_{pol}(\mathbf{x}) \approx \dfrac{\cancel{q_{pol}}}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|} + \dfrac{\mathbf{P}(\mathbf{x}')\Delta V\cdot(\mathbf{x}-\mathbf{x}')}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|^3} + \left(\cancel{\text{higher order terms}}\right)$ [(6)]

Question 1: Will $\Delta\phi_{pol}(\mathbf{x})$ be divergent as $\mathbf{x}\to\mathbf{x}'$?

Ans.: Let $|\mathbf{x}-\mathbf{x}'| = \delta$ with $\delta\to 0$. Let ΔV be a sphere of radius δ centered at $\mathbf{x}'$. So $\mathbf{x}$ is a point on the spherical surface. Then, in (6), $\Delta V \propto \delta^3$ and $\dfrac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \propto \dfrac{1}{\delta^2}$



$\Rightarrow \Delta\phi_{pol}(\mathbf{x})$ (due to nearby dipoles in the sphere) $\propto \delta \to 0$

$\Rightarrow$ The macroscopic $\mathbf{P}$ makes the dipole term non-divergent.

$\Rightarrow \phi_{pol}(\mathbf{x})$ are contributed mostly by dipoles farther away from $\mathbf{x}$.

Question 2: What's the meaning of neglecting higher-order terms?

Ans.: Higher-order terms contribute mostly to the extremely large *microscopic* fields *inside* molecules, which are of no interest to us. Fields outside the charge-neutral molecules are much smaller. Outside the molecules, the dipole term is dominant. Higher-order terms fall off much faster with distance than the dipole term, so we just neglect them.

$$\Delta\phi_{free}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\rho_{free}(\mathbf{x}')\Delta V}{|\mathbf{x}-\mathbf{x}'|} \text{ and } \Delta\phi_{pol}(\mathbf{x}) = \frac{\mathbf{P}(\mathbf{x}')\Delta V\cdot(\mathbf{x}-\mathbf{x}')}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|^3}$$

$$\Rightarrow \Delta\phi(\mathbf{x}) = \Delta\phi_{free} + \Delta\phi_{pol} = \frac{1}{4\pi\varepsilon_0}\left[\frac{\rho_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathbf{P}(\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]\Delta V \quad (4.30)$$

Let $\Delta V \to d^3x'$, use $\dfrac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$, and integrate over volume

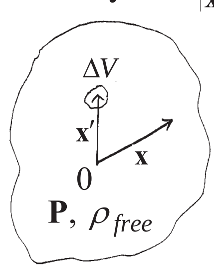
$$\Rightarrow \phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{\rho_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \mathbf{P}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x',$$

$\boxed{\mathbf{a}\cdot\nabla f = -f\nabla\cdot\mathbf{a} + \nabla\cdot(f\mathbf{a})}$

$\boxed{= \oint_S \frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\cdot d\mathbf{a}' = 0 \quad (\mathbf{P} = 0 \text{ on } S \text{ as } S\to\infty)}$

$$\text{where } \int \mathbf{P}(\mathbf{x}')\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' \;\overset{\downarrow}{=}\; -\int\frac{\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \underbrace{\int\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'}_{\text{integrated over all space}}$$

$$= -\frac{1}{4\pi\varepsilon_0}\int\frac{\nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$



$$\Rightarrow \phi(\mathbf{x}) = \phi_{free}(\mathbf{x}) + \phi_{pol}(\mathbf{x})$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_{free}(\mathbf{x}') - \nabla'\cdot\mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad (4.32)$$

Rewrite: $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_{free}(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ (4.32)

$$\Rightarrow \underbrace{\nabla^2 \phi(\mathbf{x})}_{-\nabla\cdot\mathbf{E}(\mathbf{x})} = \frac{1}{4\pi\varepsilon_0}\int [\rho_{free}(\mathbf{x}') - \nabla' \cdot \mathbf{P}(\mathbf{x}')]\, \underbrace{\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|}}_{-4\pi\delta(\mathbf{x}-\mathbf{x}')} d^3x'$$

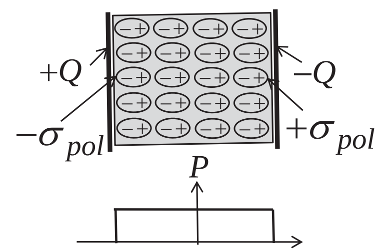
$$\Rightarrow \nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon_0}[\rho_{free}(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})] \quad (4.33)$$

In electrostatics, only electric charges can produce $\mathbf{E}$. The equal footing of ρ_{free} and $-\nabla \cdot \mathbf{P}$ in (4.32) and (4.33) suggests that $-\nabla \cdot \mathbf{P}$ must represent a charge density (see p. 153 and p. 156). Thus, (4.33) can be written

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon_0}[\rho_{free}(\mathbf{x}) + \rho_{pol}(\mathbf{x})]$$

with $\rho_{pol}(\mathbf{x}) = -\nabla \cdot \mathbf{P}(\mathbf{x})$ $\begin{bmatrix}\text{polarization} \\ \text{charge density}\end{bmatrix}$ (7)

example:



A direct derivation of (7) can be found in Appendix B [see Eq. (B.2)].

Define an electric displacement ($\mathbf{D}$) as $\quad \mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$ (4.34)

We may put $\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon_0}[\rho_{free}(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x})]$ [(4.33)] in the form:

$$\nabla \cdot \mathbf{D}(\mathbf{x}) = \rho_{free}(\mathbf{x}) \quad \text{[applicable to dielectric media]} \quad (4.35)$$

Dielectric Constant: For an isotropic medium, the (approximate) linear relation $\mathbf{F}(x) \approx -m\omega_0^2 \mathbf{x}$ [(4.71)] has led to the linear relation of $\mathbf{p}_{mol} = \varepsilon_0 \gamma_{mol} \mathbf{E}$ [(4.72)+(4.73)] for a *single* molecule. So, $\mathbf{P}$ (the sum of all $\mathbf{p}_{mol}$ per unit volume) and $\mathbf{E}$ must also have a linear relation:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad (4.36)$$

where χ_e in the proportionality constant is the electric susceptibility.

Sub. $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ [(4.36)] into $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ [(4.34)], we obtain

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad (4.37)$$

where $\varepsilon = \varepsilon_0(1+\chi_e)$ (4.38)

$\left\{\begin{matrix}(4.34) \\ (4.37)\end{matrix}\right. \Rightarrow \mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}$

ε: electric permittivity $\varepsilon/\varepsilon_0$: dielectric constant or relative electric permittivity

(8)

Question: Does $\mathbf{D}$ have a physical meaning?

Discussion: ε is a *macroscopic* property of a medium derived from the *microscopic* behavior of molecular/atomic electrons. It is usually obtained by measurement and tabulated in handbooks.

Examples: $\varepsilon(\text{air}) \approx 1.0006\varepsilon_0$, $\varepsilon(\text{teflon}) \approx 2\varepsilon_0$, $\varepsilon(\text{water}) \approx 80\varepsilon_0$

Special case: For a uniform medium, ε is independent of $\mathbf{x}$.

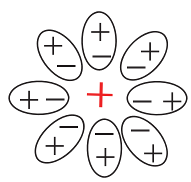
$$\Rightarrow \qquad \nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \varepsilon \nabla \cdot \mathbf{E} = \rho_{free}$$

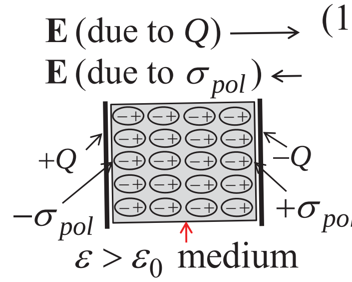
$$\Rightarrow \qquad \nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\varepsilon} \begin{bmatrix} \text{for } \rho_{free} \text{ in a uniform} \\ \text{dielectric medium} \end{bmatrix} \tag{4.39}$$

Compare (4.39) with (1.13):

$$\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\varepsilon_0} \begin{bmatrix} \text{for } \rho_{free} \text{ in} \\ \text{free space} \end{bmatrix} \tag{1.13}$$

$\Rightarrow$ In a uniform medium with $\varepsilon > \varepsilon_0$ (usual case), ρ_{free} produces $\frac{\varepsilon_0}{\varepsilon}$ of the $\mathbf{E}$ it produces in free space.





Conversion of ε to the Gaussian System:

ε in the SI system is called the electric permittivity ($\varepsilon = \varepsilon_0$ for the free-space). It has no counterpart in the Gaussian system. However, $\varepsilon/\varepsilon_0$ in the SI system (called dielectric constant or relative permittivity, see p.154) has a counterpart denoted by ε in the Gaussian system (a dimensionless quantity, $\varepsilon = 1$ for the free-space).

P. 782 gives the conversion formula:

$$\begin{bmatrix} \text{Gaussian} \\ \varepsilon \end{bmatrix} \Leftrightarrow \begin{bmatrix} \text{SI} \\ \varepsilon/\varepsilon_0 \end{bmatrix}$$

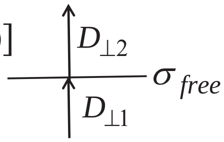
Although ε in the Gaussian system has the same notation as the electric permittivity of the SI system, it is really the dielectric constant $\varepsilon/\varepsilon_0$ of the SI system. Thus, ε in these two systems are related, but not the same physical quantity.

4.4 Boundary-Value Problems with Dielectrics

b.c.'s: [Derived as (1.22) in lecture notes. σ_{free} here is σ in (4.40)]

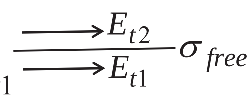
(i) $\nabla \cdot \mathbf{D} = \rho_{free}$ [(4.35)]

$\Rightarrow D_{\perp 2} - D_{\perp 1} = \sigma_{free}$



(ii) $\nabla \times \mathbf{E} = 0$

$\Rightarrow E_{t2} = E_{t1}$



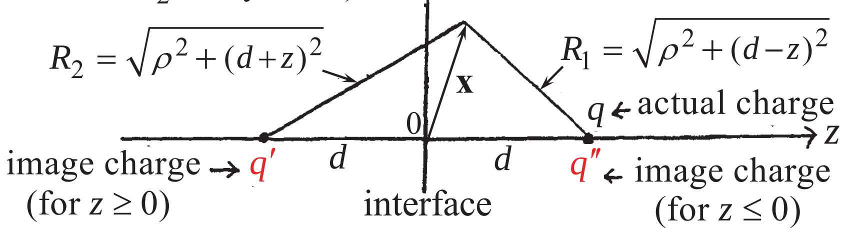
Problem 1: Two semi-infinite dielectrics have an interface plane at $z = 0$. A point charge q is at $z = d$. Find ϕ, σ_{pol}, and ρ_{pol} everywhere.

Calculating ϕ: For $\phi(z \geq 0)$, put an image charge q' at $z = -d$ on the z-axis. For $\phi(z \leq 0)$, put an image charge q'' at $z = d$ on the z-axis.

$\boxed{\rho: \text{distance from } z\text{-axis}}$

Let $\phi_2(z \leq 0) = \frac{1}{4\pi\varepsilon_2}\frac{q''}{R_1}$ (assume $\varepsilon = \varepsilon_2$ everywhere)

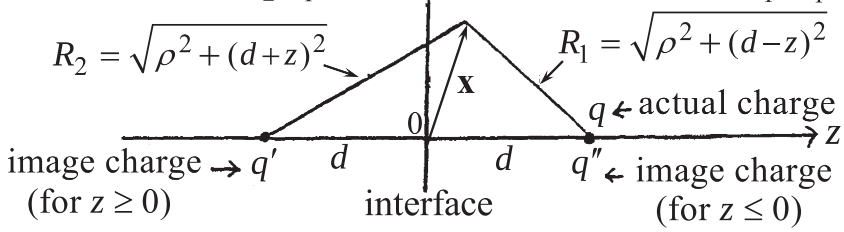
Let $\phi_1(z \geq 0) = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$ (assume $\varepsilon = \varepsilon_1$ everywhere)



ρ ← $\boxed{\text{distance from } z\text{-axis}}$

$\phi_2(z \leq 0) = \frac{1}{4\pi\varepsilon_2}\frac{q''}{R_1}$

$\phi_1(z \geq 0) = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$



Apply b.c.'s at $z = 0$.

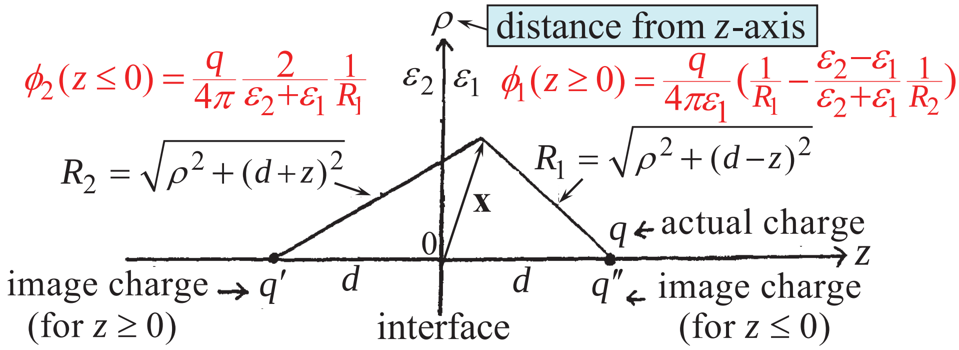
b.c. 1: $\varepsilon_1 E_{\perp 1} - \varepsilon_2 E_{\perp 2} = \sigma_{free} = 0 \Rightarrow \varepsilon_1 \left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} = \varepsilon_2 \left.\frac{\partial \phi_2}{\partial z}\right|_{z=0}$

$$\Rightarrow \left[q \frac{\partial}{\partial z}\frac{1}{R_1} + q' \frac{\partial}{\partial z}\frac{1}{R_2} \right]_{z=0} = q'' \left.\frac{\partial}{\partial z}\frac{1}{R_1}\right|_{z=0} \Rightarrow q - q' = q''$$

b.c. 2: $E_{t1} = E_{t2} \Rightarrow \left.\frac{\partial \phi_1}{\partial \rho}\right|_{z=0} = \left.\frac{\partial \phi_2}{\partial \rho}\right|_{z=0}$

$\boxed{\begin{array}{l} E_\perp: \text{normal to interface} \\ E_t: \text{tangential to interface} \end{array}}$

$$\Rightarrow \frac{1}{\varepsilon_1}\left[q \frac{\partial}{\partial \rho}\frac{1}{R_1} + q' \frac{\partial}{\partial \rho}\frac{1}{R_2} \right]_{z=0} = \frac{1}{\varepsilon_2} q'' \left.\frac{\partial}{\partial \rho}\frac{1}{R_1}\right|_{z=0} \Rightarrow \frac{1}{\varepsilon_1}(q + q') = \frac{1}{\varepsilon_2} q''$$



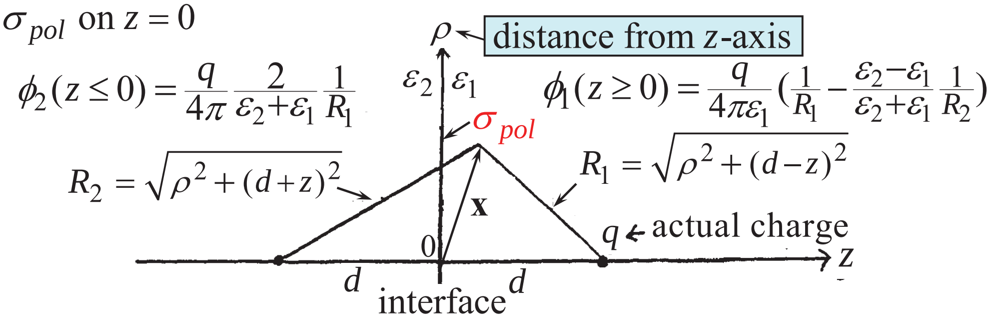
$$\left.\begin{cases} q-q'=q'' \\ \frac{1}{\varepsilon_1}(q+q')=\frac{1}{\varepsilon_2}q'' \end{cases}\right\}\Rightarrow q'=-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}q \ \ \& \ \ q''=\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}q \qquad (4.45)$$

$$\Rightarrow\begin{cases} \phi_1(z\ge 0)=\frac{q}{4\pi\varepsilon_1}\left(\frac{1}{R_1}-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\frac{1}{R_2}\right) \\ \phi_2(z\le 0)=\frac{q}{4\pi}\frac{2}{\varepsilon_2+\varepsilon_1}\frac{1}{R_1}\end{cases}\quad \left[\begin{array}{l}\text{At } z=0,\ R_1=R_2. \Rightarrow \phi_1=\phi_2 \\ \because E \text{ is finite at the interface.}\end{array}\right]$$

ϕ_1, ϕ_2 obviously satisfy the Poisson Eq., hence are unique solutions.

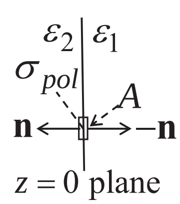
Calculating surface and volume polarization charges:

1. σ_{pol} on $z=0$



$$\text{From (8):}\begin{cases} \mathbf{P}_1=(\varepsilon_1-\varepsilon_0)\mathbf{E}_1=-(\varepsilon_1-\varepsilon_0)\nabla\phi_1 \\ \mathbf{P}_2=(\varepsilon_2-\varepsilon_0)\mathbf{E}_2=-(\varepsilon_2-\varepsilon_0)\nabla\phi_2\end{cases}\left[\begin{array}{l}\mathbf{P}_1,\mathbf{P}_2 \text{ are given} \\ \text{by } \phi_1,\phi_2 \text{ above.}\end{array}\right]$$

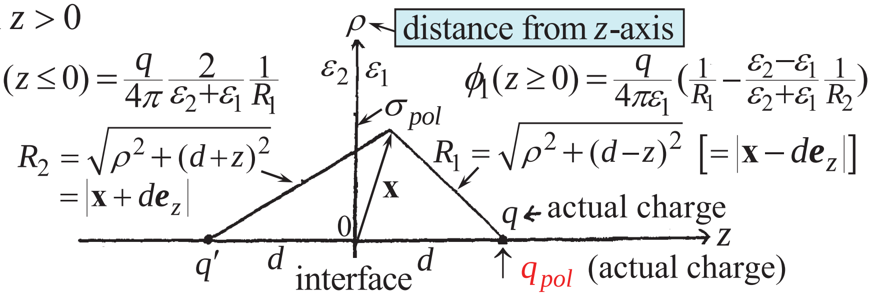
To find σ_{pol} on the interface, draw a thin pillbox of small area A on the interface and apply the divergence thm.:



$$\int_V \underbrace{\nabla\cdot\mathbf{P}}_{-\rho_{pol}}\, d^3x=\oint_S \mathbf{P}\cdot\mathbf{n}da \ \Rightarrow\ -Q_{pol}=[\mathbf{P}_2-\mathbf{P}_1]_{z=0}\cdot\mathbf{n}A$$

$$\Rightarrow \sigma_{pol}=\frac{Q_{pol}}{A}=-[\mathbf{P}_2-\mathbf{P}_1]_{z=0}\cdot\mathbf{n}=-\frac{q}{2\pi}\frac{\varepsilon_0(\varepsilon_2-\varepsilon_1)}{\varepsilon_1(\varepsilon_2+\varepsilon_1)}\frac{d}{(\rho^2+d^2)^{3/2}} \qquad (4.47)$$

2. ρ_{pol} in $z>0$

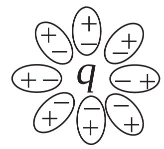


$$\phi_1=\frac{q}{4\pi\varepsilon_1}\left(\frac{1}{R_1}-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\frac{1}{R_2}\right)=\frac{q}{4\pi\varepsilon_1}\left(\frac{1}{|\mathbf{x}-d\mathbf{e}_z|}-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\frac{1}{|\mathbf{x}+d\mathbf{e}_z|}\right)$$

$$\Rightarrow \rho_{pol}(z>0)=-\nabla\cdot\mathbf{P}_1\overset{(8)}{=}-(\varepsilon_1-\varepsilon_0)\nabla\cdot\mathbf{E}_1=(\varepsilon_1-\varepsilon_0)\nabla^2\phi_1$$

$$\boxed{\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-4\pi\delta(\mathbf{x}-\mathbf{x}')} \longrightarrow =-q\frac{\varepsilon_1-\varepsilon_0}{\varepsilon_1}\Big[\delta(\mathbf{x}-d\mathbf{e}_z)-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\underbrace{\delta(\mathbf{x}+d\mathbf{e}_z)}_{=0\text{ for } z>0}\Big]$$

$$=-q\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right)\delta(\mathbf{x}-d\mathbf{e}_z)$$



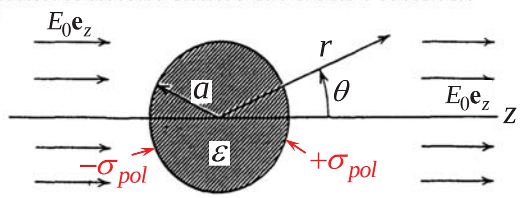
$\Rightarrow$ There is a $q_{pol}=-q(1-\frac{\varepsilon_0}{\varepsilon_1})$ at the same position ($\mathbf{x}=d\mathbf{e}_z$) as q.

$\Rightarrow q_{pol}+q=q\frac{\varepsilon_0}{\varepsilon_1}\Rightarrow$ If $\varepsilon_1>\varepsilon_0$ (usual case), q_{pol} partially offsets q.

Similarly, we can show $\rho_{pol}=0$ everywhere in the region $z<0$.

Problem 2: A uniform dielectric sphere is placed in a uniform electric field $E_0\mathbf{e}_z$. Find ϕ and polarization charges everywhere.

$\boxed{\text{Expect } E_0\mathbf{e}_z \text{ to induce } \sigma_{pol} \text{ on the sphere, which produces } \mathbf{E}_{pol}.}$



Divide the space into 2 regions: $r<a$ and $r>a$. In both regions, we have $\nabla\cdot\mathbf{E}=\begin{cases}\rho_{free}/\varepsilon\ (r<a)\ [(4.39)]\\ \rho_{free}/\varepsilon_0(r>a)\ [(1.13)]\end{cases}$. $\rho_{free}=0\Rightarrow\nabla^2\phi=0$ $\begin{bmatrix}\text{for } r<a\\ \&\ r>a\end{bmatrix}$

$$\Rightarrow\quad \phi=\begin{Bmatrix}r^l\\ r^{-l-1}\end{Bmatrix}\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}e^{im\varphi}\\ e^{-im\varphi}\end{Bmatrix}\quad\begin{bmatrix}\text{in spherical coordinates;}\\ \text{see Sec. 3.1 of lecture notes}\end{bmatrix}$$

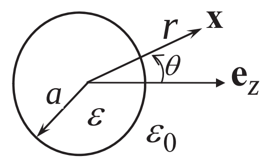
$$\text{b.c.}\begin{Bmatrix}\phi \text{ is independent of } \varphi.\\ \phi \text{ is finite at } \cos\theta=\pm1.\\ \phi_{in} \text{ is finite at } r=0.\end{Bmatrix}\Rightarrow\begin{cases}\phi_{in}=\sum\limits_{l=0}^{\infty}A_l r^l P_l(\cos\theta)\\ \phi_{out}=\sum\limits_{l=0}^{\infty}\left[B_l r^l+C_l r^{-l-1}\right]P_l(\cos\theta)\end{cases}$$

Rewrite: $\phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$, $\phi_{out} = \sum_{l=0}^{\infty}\left[B_l r^l + C_l r^{-l-1}\right] P_l(\cos\theta)$

b.c. (i): $\phi_{out}(\infty) = -E_0 r\cos\theta \Rightarrow B_1 = -E_0;\ B_l(l \neq 1) = 0$ $\boxed{\begin{aligned}&P_1(\cos\theta)\\&=\cos\theta\end{aligned}}$

b.c. (ii): $\phi_{in}(a) = \phi_{out}(a)$ $[\Rightarrow E_t^{in}(a) = E_t^{out}(a)]$

$$\Rightarrow A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \Rightarrow \begin{cases} A_1 = -E_0 + \dfrac{C_1}{a^3} & (9) \\ A_l = \dfrac{C_l}{a^{2l+1}},\ l \neq 1 & (10)\end{cases}$$



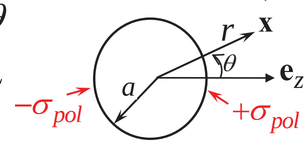
b.c. (iii): $\varepsilon E_r^{in}(a) = \varepsilon_0 E_r^{out}(a) \Rightarrow -\varepsilon \frac{\partial}{\partial r}\phi_{in}\Big|_{r=a} = -\varepsilon_0 \frac{\partial}{\partial r}\phi_{out}\Big|_{r=a}$

$$\Rightarrow \varepsilon l A_l a^{l-1} = \varepsilon_0\left[l B_l a^{l-1} - \frac{(l+1)C_l}{a^{l+2}}\right] \Rightarrow \begin{cases} \varepsilon A_1 = -\varepsilon_0[E_0 + \dfrac{2C_1}{a^3}] & (11) \\ \varepsilon l A_l = -\varepsilon_0 \dfrac{(l+1)C_l}{a^{2l+1}},\ l \neq 1 & (12)\end{cases}$$

$(9), (11) \Rightarrow A_1 = -\frac{3}{\varepsilon/\varepsilon_0 + 2}E_0;\ C_1 = (\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2})a^3 E_0$

$(10), (12) \Rightarrow A_l = C_l = 0$ for $l \neq 1$

$$\Rightarrow \begin{cases} \phi_{in} = \phi(r \le a) = -\dfrac{3}{\frac{\varepsilon}{\varepsilon_0}+2}E_0 r\cos\theta\ [= \phi \text{ of } E_0\mathbf{e}_z + \phi \text{ due to } \sigma_{pol}] \\ \phi_{out} = \phi(r \ge a) = \underbrace{-E_0 r\cos\theta}_{\phi \text{ of } E_0\mathbf{e}_z} + \underbrace{\dfrac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}E_0\dfrac{a^3}{r^2}\cos\theta}_{\phi \text{ due to } \sigma_{pol}} \end{cases} \quad (4.54)$$



Comparing (4.54) with (4.10): $\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2r^5}\overleftrightarrow{\mathbf{Q}}\cdot\mathbf{xx} + \cdots)$,

we find $\phi(r \ge a)$ due to σ_{pol} is *exactly* the ϕ of the dipole term with

$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}E_0\mathbf{e}_z \quad [\text{with } q, \overleftrightarrow{\mathbf{Q}}\cdots, etc. = 0] \quad (4.56)$$

$\Rightarrow$ The dipole field (for $r \ge a$) is an exact expression even as $r \to a$.

For $r \le a$, $\phi_{in} \Rightarrow \mathbf{E}_{in} = \frac{3E_0}{\frac{\varepsilon}{\varepsilon_0}+2}\mathbf{e}_z \Rightarrow \sigma_{pol}$ on the sphere produces a *uniform* E_z inside the sphere. If $\varepsilon > \varepsilon_0$ (usual case), we have $\mathbf{E}_{in} < E_0\mathbf{e}_z$.

$\Rightarrow$ $\mathbf{E}$ due to σ_{pol} cancels part of the external $E_0\mathbf{e}_z$.

$$\phi \text{ in (4.54)} \Rightarrow \begin{cases} \mathbf{E}_{in} = \dfrac{3E_0}{\varepsilon/\varepsilon_0+2}\mathbf{e}_z \text{ (uniform)} & (4.55) \\ \mathbf{E}_{out} = E_0\mathbf{e}_z + \dfrac{3\mathbf{e}_r(\mathbf{p}\cdot\mathbf{e}_r)-\mathbf{p}}{4\pi\varepsilon_0 r^3} \begin{bmatrix}\text{same form} \\ \text{as (4.13)}\end{bmatrix} \text{ with } \mathbf{e}_r = \dfrac{\mathbf{x}}{|\mathbf{x}|} \end{cases}$$

$$\Rightarrow \mathbf{P} \overset{\boxed{(8)}}{=} (\varepsilon-\varepsilon_0)\mathbf{E}_{in} = 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\mathbf{e}_z \quad \begin{bmatrix}\mathbf{P} \text{ (total } \mathbf{p} \text{ per unit} \\ \text{volume) is uniform.}\end{bmatrix} \quad (4.57)$$

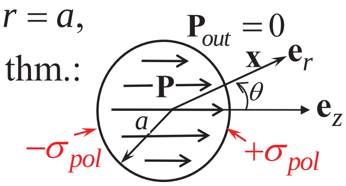
Exercise: Show that $\mathbf{p}$ in (4.56) is indeed the total dipole moment.

$$\mathbf{p} = \frac{4\pi a^3}{3}\mathbf{P} \Rightarrow \mathbf{p} = 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\mathbf{e}_z \text{ [same as (4.56)]}$$

Rewrite $\quad \nabla\cdot\mathbf{P} = -\rho_{pol}$ [(7)]

$\mathbf{P}$ is uniform inside the sphere. $\Rightarrow \rho_{pol}(r<a) = -\nabla\cdot\mathbf{P} = 0$.

To find σ_{pol} due to the discontinuity of $\mathbf{P}$ at $r=a$, draw a pillbox at $r=a$ and apply the divergence thm.:



$$\int_V \underbrace{\nabla\cdot\mathbf{P}}_{-\rho_{pol}}\, d^3x = \oint_S \mathbf{P}\cdot\mathbf{e}_r da.$$

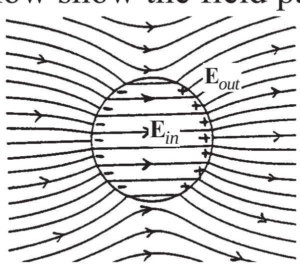
$$\Rightarrow \sigma_{pol} = -(\cancel{\mathbf{P}_{out}} - \mathbf{P})\cdot\mathbf{e}_r = P\cos\theta = 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\cos\theta \quad (4.58)$$

Figures below show the field patterns just calculated (for $\varepsilon > \varepsilon_0$).

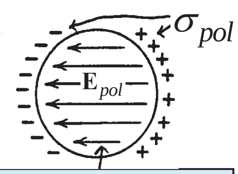
$\mathbf{E}_{total}$ $\longrightarrow$

$(E_0\mathbf{e}_z + \mathbf{E}_{pol})$

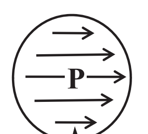
a pure dipole field outside



$\rightarrow E_0\mathbf{e}_z$

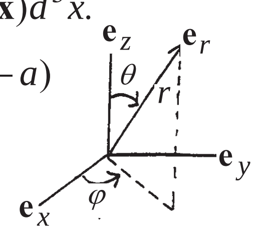


$\mathbf{E}_{pol}$ (due to σ_{pol}) is uniform inside



uniform $\mathbf{P}$ inside

Exercise: Show that $\mathbf{p}$ in (4.56) agrees with $\mathbf{p} = \int \mathbf{x}\rho(\mathbf{x})d^3x$.



$$\text{Let } \begin{cases} \rho_{pol}(\mathbf{x}) = \sigma_{pol}\delta(r-a) = 3\varepsilon_0\left(\dfrac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\cos\theta\delta(r-a) \\ \mathbf{x} = r\mathbf{e}_r = r[\cos\theta\mathbf{e}_z + \sin\theta(\cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y)] \end{cases}$$

$$\Rightarrow \mathbf{p} = \int \mathbf{x}\rho_{pol}(\mathbf{x})d^3x$$

$$= \int_0^\infty r^2dr\int_{-1}^1 d(\cos\theta)\int_0^{2\pi} d\varphi\, r[\cos\theta\mathbf{e}_z + \sin\theta\underbrace{(\cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y)}_{\text{vanishes upon } \varphi\text{-integration}}]$$

$$\cdot 3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\cos\theta\delta(r-a)$$

$$= 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right)E_0\mathbf{e}_z \quad \text{[same as (4.56)]}$$

4.7 Electrostatic Energy in Dielectric Media

Let $\phi(\mathbf{x})$ be the field due to ρ_{free} and ρ_{pol} in a dielectric medium. The work done to add $\delta\rho_{free}$ is

$$\delta W = \int \delta\rho_{free}(\mathbf{x})\phi(\mathbf{x})d^3x$$

$\boxed{\delta\rho_{free} = \nabla\cdot\delta\mathbf{D}}$

$$= \int \phi\nabla\cdot\delta\mathbf{D}(\mathbf{x})d^3x$$

Using $\nabla\cdot\psi\mathbf{a} = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$ we obtain
$\phi\nabla\cdot\delta\mathbf{D} = \nabla\cdot(\phi\delta\mathbf{D}) - \delta\mathbf{D}\cdot\nabla\phi$
$= \nabla\cdot(\phi\delta\mathbf{D}) + \mathbf{E}\cdot\delta\mathbf{D}$

$$= \int \nabla\cdot(\phi\delta\mathbf{D})d^3x + \int \mathbf{E}\cdot\delta\mathbf{D}d^3x$$

$= \oint_S \underbrace{\phi}_{\frac{1}{r}} \underbrace{\delta\mathbf{D}}_{\frac{1}{r^2}} \cdot \underbrace{d\mathbf{a}}_{r^2} = 0$, as $r \to \infty$

Note: For this integral to vanish, the integration must be over all space.

$$= \int \mathbf{E}\cdot\delta\mathbf{D}d^3x \;\; \text{[integrated over all space]} \qquad (4.86)$$

Note: (1) $\rho_{free}(\mathbf{x})$ here is denoted by $\rho(\mathbf{x})$ in Jackson (4.84).

(2) In a dielectric medium, the addition of $\delta\rho_{free}(\mathbf{x})$ will induce $\delta\rho_{pol}(\mathbf{x})$. Hence, $\phi(\mathbf{x})$ in the above equation is due to both ρ_{free} and ρ_{pol}. The effect of ρ_{pol} is implicit in $\mathbf{D}$ $(=\varepsilon\mathbf{E})$.

$$\delta W = \int \mathbf{E}\cdot\delta\mathbf{D}d^3x \; [(4.86)] \;\Rightarrow\; W = \int d^3x \int_0^D \mathbf{E}\cdot\delta\mathbf{D} \qquad (4.87)$$

Linear and isotropic media ($\mathbf{D} = \varepsilon\mathbf{E}$; ε indep. of $\mathbf{E}$):

$$\mathbf{E}\cdot\delta\mathbf{D} = \mathbf{E}\cdot\delta(\varepsilon\mathbf{E}) = \varepsilon\mathbf{E}\cdot\delta\mathbf{E} = \tfrac{1}{2}\varepsilon\delta(\mathbf{E}\cdot\mathbf{E}) = \tfrac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})$$

Linear and anisotropic media ($\mathbf{D} = \overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}$; $\overleftrightarrow{\boldsymbol{\varepsilon}}$ indep. of $\mathbf{E}$):

$$\mathbf{E}\cdot\delta\mathbf{D} = \mathbf{E}\cdot\delta(\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}) = \mathbf{E}\cdot\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\delta\mathbf{E} = \tfrac{1}{2}\delta(\mathbf{E}\cdot\overleftrightarrow{\boldsymbol{\varepsilon}}\cdot\mathbf{E}) = \tfrac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D})$$

[**E** is the variable at a fixed point **x**.]

$$\Rightarrow W = \tfrac{1}{2}\int d^3x \int_0^D \delta(\mathbf{E}\cdot\mathbf{D}) = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}d^3x \quad \text{[for linear media]} \qquad (4.89)$$

$$\Rightarrow w = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} \quad \text{[energy/unit volume; for linear media]} \qquad (13)$$

$$\mathbf{E}\cdot\mathbf{D} = -\mathbf{D}\cdot\nabla\phi = -\nabla\cdot(\phi\mathbf{D}) + \phi\nabla\cdot\mathbf{D} = -\nabla\cdot(\phi\mathbf{D}) + \rho_{free}\phi$$

$$\Rightarrow W = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}d^3x = \tfrac{1}{2}\int \rho_{free}(\mathbf{x})\phi(\mathbf{x})d^3x - \cancel{\tfrac{1}{2}\oint_S \underbrace{\phi}_{\frac{1}{r}} \underbrace{\mathbf{D}}_{\frac{1}{r^2}} \cdot \underbrace{d\mathbf{a}}_{r^2}}$$

$$= \tfrac{1}{2}\int \rho_{free}(\mathbf{x})\phi(\mathbf{x})d^3x \; \text{[for linear media]} \qquad (14)$$

Note:

$W = \frac{1}{2}\int \rho_{free}(\mathbf{x})\phi(\mathbf{x})d^3x$ [(14)]; ϕ is due to ρ_{free} & ρ_{pol}.

$W = \frac{1}{2}\int \rho(\mathbf{x})\phi(\mathbf{x})d^3x$ [(1.53)]; ϕ is due to ρ in intergrand.

$W = \int \rho(\mathbf{x})\phi(\mathbf{x})d^3x$ [(4.24)]; ϕ is due to external charges.

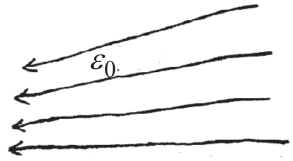
Energy change due to a dielectric object with linear $\varepsilon_1(\mathbf{x})$ in $\mathbf{E}_0$ of a fixed external source. (This and next page will not be covered.)

Without the object:

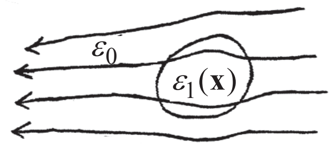
$W_0 = \frac{1}{2}\int \mathbf{E}_0 \cdot \mathbf{D}_0 d^3x$

With the object:

$W_1 = \frac{1}{2}\int \mathbf{E} \cdot \mathbf{D} d^3x$



$\mathbf{E}_0(\mathbf{x}),\ \mathbf{D}_0(\mathbf{x}),\ \phi_0(\mathbf{x})$



$\mathbf{E}(\mathbf{x}),\ \mathbf{D}(\mathbf{x}),\ \phi(\mathbf{x})$

$$\Rightarrow \Delta W = W_1 - W_0 = \tfrac{1}{2}\int (\mathbf{E}\cdot\mathbf{D} - \mathbf{E}_0\cdot\mathbf{D}_0)\, d^3x$$
$$= \tfrac{1}{2}\int (\mathbf{E}\cdot\mathbf{D}_0 - \mathbf{D}\cdot\mathbf{E}_0)\, d^3x + \tfrac{1}{2}\underbrace{\int (\mathbf{E}+\mathbf{E}_0)\cdot(\mathbf{D}-\mathbf{D}_0)\, d^3x}$$
$$= \tfrac{1}{2}\int (\mathbf{E}\cdot\mathbf{D}_0 - \mathbf{D}\cdot\mathbf{E}_0)\, d^3x$$

$$-\int \nabla(\phi+\phi_0)\cdot(\mathbf{D}-\mathbf{D}_0) d^3x = \int (\phi+\phi_0)\, \underbrace{\nabla\cdot(\mathbf{D}-\mathbf{D}_0)}_{=\rho_{free}-\rho_{free}=0}\, d^3x = 0$$

integration by parts

Reason for $\nabla\cdot\mathbf{D}_0 = \nabla\cdot\mathbf{D} = \rho_{free}$: A dielectric object contains no ρ_{free} and the external source is fixed. $\Rightarrow \rho_{free}$ is unchanged before and after the introduction of the object.

$\boxed{\mathbf{D}_0 = \varepsilon_0\mathbf{E}_0}$

$$\Delta W = \tfrac{1}{2}\int (\mathbf{E}\cdot\mathbf{D}_0 - \mathbf{D}\cdot\mathbf{E}_0)\, d^3x \qquad \begin{cases} \text{Outside the object: } \mathbf{D} = \varepsilon_0\mathbf{E} \\ \text{Inside the object: } \mathbf{D} = \varepsilon_1\mathbf{E} \end{cases}$$

$\Rightarrow \Delta W$ (outside the object) $= 0$

$$\Rightarrow \Delta W = -\tfrac{1}{2}\int_{v_1} (\varepsilon_1 - \varepsilon_0)\mathbf{E}\cdot\mathbf{E}_0 d^3x \qquad (4.92)$$

v_1 is the volume of the object.

$\Rightarrow$ The dielectric object tends to move toward (away from) the region of increasing $\mathbf{E}_0$ if $\varepsilon_1 > \varepsilon_0$ ($\varepsilon_1 < \varepsilon_0$).

$\mathbf{D} = \varepsilon_1\mathbf{E}$ & $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\Rightarrow \mathbf{P} = (\varepsilon_1 - \varepsilon_0)\mathbf{E}$

induced polarization of the object

$$\Rightarrow \Delta W = -\tfrac{1}{2}\int_{v_1} \mathbf{P}\cdot\mathbf{E}_0 d^3x \qquad (4.93)$$

$\Rightarrow$ The energy density of a dielectric object placed in the field $\mathbf{E}_0$ of a fixed external source is

$$w = -\tfrac{1}{2}\mathbf{P}\cdot\mathbf{E}_0 \qquad (4.94)$$

Question: Explain the factor $\frac{1}{2}$ which is in (4.94) but not in the 2nd term of $W = q\phi(0) - \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + \cdots$ [(4.24)]

Appendix A. Taylor Expansion

Define $e^{\mathbf{a}\cdot\nabla} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n$ [a translational operator]

Taylor expansion of $f(\mathbf{x}+\mathbf{a})$ and $\mathbf{A}(\mathbf{x}+\mathbf{a})$ about point $\mathbf{x}$:

$$\begin{cases} f(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} f(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n f(\mathbf{x}) \\ \quad = f(\mathbf{x}) + (\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \frac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) f(\mathbf{x}) + \cdots & \text{(A.1)} \\ \mathbf{A}(\mathbf{x}+\mathbf{a}) = e^{\mathbf{a}\cdot\nabla} \mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathbf{a}\cdot\nabla)^n \mathbf{A}(\mathbf{x}) \\ \quad = \mathbf{A}(\mathbf{x}) + (\mathbf{a}\cdot\nabla)\mathbf{A}(\mathbf{x}) + \frac{1}{2}(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla)\mathbf{A}(\mathbf{x}) + \cdots & \text{(A.2)} \end{cases}$$

derivatives evaluated at $\mathbf{x}$

Similarly, operating $f(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ with $e^{(\mathbf{x}-\mathbf{a})\cdot\nabla}$, we obtain the Taylor expansion of $f(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ about point $\mathbf{a}$:

$$\begin{cases} f(\mathbf{x}) = f(\mathbf{a}) + [(\mathbf{x}-\mathbf{a})\cdot\nabla] f(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla] f(\mathbf{a}) + \cdots & \text{(A.3)} \\ \mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{a}) + [(\mathbf{x}-\mathbf{a})\cdot\nabla]\mathbf{A}(\mathbf{a}) + \frac{1}{2}[(\mathbf{x}-\mathbf{a})\cdot\nabla][(\mathbf{x}-\mathbf{a})\cdot\nabla]\mathbf{A}(\mathbf{a}) + \cdots & \text{(A.4)} \end{cases}$$

derivatives evaluated at $\mathbf{a}$

In (A.1) and (A.2), we have [in Cartesian coordinates]

$$\mathbf{a}\cdot\nabla = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} \quad \text{(A.5)}$$

$$(\mathbf{a}\cdot\nabla)(\mathbf{a}\cdot\nabla) = \sum_i a_i \frac{\partial}{\partial x_i} \sum_j a_j \frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{(A.6)}$$

$$(\mathbf{a}\cdot\nabla) f(\mathbf{x}) = \sum_i a_i \frac{\partial}{\partial x_i} f(\mathbf{x}) = \mathbf{a}\cdot\nabla f(\mathbf{x}) \quad \text{(A.7)}$$

$$(\mathbf{a}\cdot\nabla)\mathbf{A}(\mathbf{x}) = \sum_i a_i \frac{\partial}{\partial x_i}(\sum_j A_j \mathbf{e}_j) = \sum_j (\sum_i a_i \frac{\partial}{\partial x_i} A_j)\mathbf{e}_j \quad \text{(A.8)}$$

Example: $(\mathbf{a}\cdot\nabla)(\mathbf{x}-\mathbf{x}') = \sum_j [\sum_i a_i \underbrace{\frac{\partial}{\partial x_i}(x_j - x'_j)}_{\delta_{ij}}]\mathbf{e}_j = \sum_j a_j \mathbf{e}_j = \mathbf{a}$

For scalar functions with a scalar argument, (A.1) & (A.3) reduce to

$$f(x+a) = f(x) + af'(x) + \frac{1}{2}a^2 f''(x) + \cdots \quad \text{(A.9)}$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \cdots \quad \text{(A.10)}$$

Appendix B. Polarization Current Density and Polarization Charge Density in Dielectric Media

We divide the bound charges (electrons and ions) in a dielectric into different groups. The i-th group has N_i identical charged particles per unit volume. Each particle in the group carries a charge e_i and has a dipole moment given by $\mathbf{p}_i = e_i\mathbf{x}_i$, where $\mathbf{x}_i$ is the particle's displacement from its equilibrium position under the influence of a static or time-dependent electric field. We assume that the variation of $\mathbf{x}_i$ of all particles is so small that it will not change N_i. Then, the electric polarization $\mathbf{P}$ as a function of position and time can be written as

$$\mathbf{P}(\mathbf{x},t) = \sum_i N_i(\mathbf{x})\mathbf{p}_i(t) = \sum_i N_i(\mathbf{x})e_i\mathbf{x}_i(t) = \sum_i \overbrace{\rho_i(\mathbf{x})}^{\text{charge density of the } i\text{-th group}}\mathbf{x}_i(t)$$

and the polarization current density is the time derivative of $\mathbf{P}(\mathbf{x},t)$

$$\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) = \sum_i \rho_i(\mathbf{x})\frac{d}{dt}\mathbf{x}_i(t) = \sum_i \rho_i(\mathbf{x})\mathbf{v}_i(t) = \overbrace{\mathbf{J}_{pol}(\mathbf{x},t)}^{\text{polarization current density}} \qquad \text{(B.1)}$$

Let ρ_{pol} be the polarization charge density of the medium, then

$$\frac{\partial}{\partial t}\rho_{pol} + \nabla\cdot\mathbf{J}_{pol} = 0 \qquad \text{(conservation of charge)}$$

$$\Rightarrow \frac{\partial}{\partial t}\rho_{pol} + \nabla\cdot\frac{\partial}{\partial t}\mathbf{P} = 0 \Rightarrow \frac{\partial}{\partial t}\left(\rho_{pol} + \nabla\cdot\mathbf{P}\right) = 0$$

$$\Rightarrow \rho_{pol} + \nabla\cdot\mathbf{P} = K$$

If $\mathbf{P} = 0$, we have $\rho_{pol} = 0$. Hence, $K = 0$.

$$\Rightarrow \rho_{pol} = -\nabla\cdot\mathbf{P} \qquad \text{(B.2)}$$

$\mathbf{J}_{pol}$ is due to the *motion* of bound charges, whereas ρ_{pol} is due to the *displacement* of bound charges. The presence of $\mathbf{J}_{pol}$ does not necessarily imply the presence of ρ_{pol}, and vice versa. For example, in a static electric field $\mathbf{E}$, we have $\mathbf{J}_{pol} = 0$ because bound charges are stationary. But the stationary charges will be displaced by $\mathbf{E}$; hence $\rho_{pol} \neq 0$ if $\nabla\cdot\mathbf{P} \neq 0$. In time-dependent cases, there must be a $\mathbf{J}_{pol}$ if $\mathbf{P} \neq 0$ [hence $\frac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t) \neq 0$] but not necessarily a ρ_{pol} unless $\nabla\cdot\mathbf{P} \neq 0$.