Bertrand's Postulate

For every $n \geq 1$ there is some prime p with $n < p \leq 2n$

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Definition

For every $n \geq 1$ there is some prime number p with $n < p \leq 2n$

History

- **Bertrand (1845)**: Conjectured the statement after verifying it for $n \leq 3,000,000$.
- **Chebyshev (1852)**: Provided the first rigorous proof using factorial and prime properties.
- **Ramanujan (1919)**: Shorter proof using Sterling Formula.
- **Erdős (1932)**: Elegant combinatorial proof using no calculus.

General Approach

For every $n \geq 1$ there is some prime number p with $n < p \leq 2n$

- Choose $f(n)$ that contains these primes p
- Find a lower bound $f_{\min}(n) \leq f(n)$
- If no primes in $(n, 2n]$ then $f_{\min}(n) \leq f(n)$ only holds for $n < N$
- Verify for $n < N$

General Approach

- Choose $f(n)$ that contains these primes p

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

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Example: $n = 4$

$$\binom{8}{4} = \frac{5 \cdot 6 \cdot 7 \cdot 8}{2 \cdot 3 \cdot 4} = \frac{5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3}{2 \cdot 3 \cdot 2^2} = 5 \cdot 7 \cdot 2$$

General Approach

For every $n \geq 1$ there is some prime p with $n < p \leq 2n$

- Find a lower bound $f_{\min}(n) \leq \binom{2n}{n}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
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 & & & & & & \cdots & & & & & &
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$$

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$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\cdots$$

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$$

General Approach

For every $n \geq 1$ there is some prime p with $n < p \leq 2n$

- Find a lower bound $f_{\min}(n) \leq \binom{2n}{n}$

$$\binom{0}{0} = 2^0$$

$$\binom{1}{0} + \binom{1}{1} = 2^1$$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4$$

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5$$

$$\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 2^6$$

$$\cdots$$

General Approach

For every $n \geq 1$ there is some prime p with $n < p \leq 2n$

- Find a lower bound $f_{\min}(n) \leq \binom{2n}{n}$

$$2^{2n} \overset{(1)}{\leq} 2n \cdot \binom{2n}{n} \implies f_{min}(n) = \frac{2^{2n}}{2n}$$

(1): $\binom{2n}{0} + \binom{2n}{2n} \leq \binom{2n}{n}$

General Approach

For every $n \geq 1$ there is some prime p with $n < p \leq 2n$

- $\binom{2n}{n}$ contains all primes in $(n, 2n]$
- $\frac{2^{2n}}{2n} \leq \binom{2n}{n}$
- If no primes in $(n, 2n]$ then $\frac{2^{2n}}{2n} \leq \binom{2n}{n}$ only holds for $n < N$
- Verify for $n < N$

Contradiction Slide

Contradiction:

Assume $\exists n$ such that there is no prime p with $n < p \leq 2n$, then show that $\binom{2n}{n}$ is smaller than some fixed lowerbound.

$$Lowerbound \leq \binom{2n}{n} = \prod_{p_i \leq n} p_i^{r_i} \cdot \prod_{n < p_i \leq 2n} p_i^{r_i}$$

Then our assumption should lead to

$$Lowerbound > \prod_{p_i \leq n} p_i^{r_i} \cdot \underbrace{\prod_{n < p_i \leq 2n} p_i^{r_i}}_{=0}$$

Proof Idea - Summary

$$\frac{4^n}{2n} \leq \binom{2n}{n} = \prod_{p_i \leq n} p_i^{r_i} \cdot \prod_{n < p_i \leq 2n} p_i^{r_i}$$

What can we say about how often the primes appear in $\binom{2n}{n}$ i.e how large are the r_i's ?

Legendre's Theorem

$n!$ contains the prime factor p exactly $\sum_{k\geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$ times

Intuition:

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Intuition:

- Exactly $\left\lfloor \frac{n}{p} \right\rfloor$ of the factors from $n! = 1 \cdot 2 \cdot 3 \cdots n$ are divisible by p since $p, 2p, 3p, \ldots, \left\lfloor \frac{n}{p} \right\rfloor \cdot p \leq n$.

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- iterate over k because higher powers p^2, p^3 etc. contribute additional factors of p that must be counted separately.

Legendre's Theorem - Example

$n!$ contains the prime factor p exactly $\sum_{k\geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$ times

Example: $n = 8$ and $p = 2$

$k = 1$:

$$1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \implies \left\lfloor \frac{n}{p} \right\rfloor = 4$$

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$k = 2$:

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$k = 3$:

$$1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \implies \left\lfloor \frac{n}{p^3} \right\rfloor = 1$$

Legendre's Theorem - Application

How often does prime p appear in $\frac{(2n)!}{n!\,n!}$?

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Accounting for cancellation from the fraction yields:

$$\sum_{k \geq 1} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor \; - \; 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

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Because $\lfloor x \rfloor < x$ and $\lfloor x \rfloor > x - 1$, each summand satisfies:

$$\left\lfloor \frac{2n}{p^k} \right\rfloor \; - \; 2 \left\lfloor \frac{n}{p^k} \right\rfloor \; < \; \frac{2n}{p^k} \; - \; 2\left(\frac{n}{p^k} - 1 \right) \; = \; 2$$

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Each term is 0 whenever $k > r$ and at most 1 otherwise, thus:

$$\sum_{k\geq 1}\left(\left\lfloor \frac{2n}{p^k}\right\rfloor \;-\; 2\left\lfloor \frac{n}{p^k}\right\rfloor\right) \leq \max\{\, r \;\mid\; p^r \leq 2n \,\}.$$

Legendre's Theorem - Observations

prime p appears in $\frac{(2n)!}{n!\,n!}$ at most $\max\{\, r \mid p^r \leq 2n \}$.

- Largest power of p in the factorization cannot exceed $2n$.

$$\frac{(2n)!}{n!\,n!} \leq \prod_{p \leq 2n} 2n$$

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- Primes satisfying $p > \sqrt{2n}$ appear at most once.

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- Largest power of p in the factorization cannot exceed $2n$.
- Primes satisfying $p > \sqrt{2n}$ appear at most once.
- primes p that satisfy $\frac{2}{3}n < p \leq n$ don't appear at all.

$$\frac{(2n)!}{n!n!} \leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{n < p \leq 2n} p$$

Extensiosnsssss

Weil:

- Für $3p > 2n$ (und $n \geq 3$ und damit $p \geq 3$) sind p und $2p$ die einzigen Vielfachen von p, die im Zähler von $\frac{(2n)!}{n!\,n!}$ vorkommen, während wir zwei p-Faktoren im Nenner haben.

Abstract Approach

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- Verify for $n \leq N$

Product of prime numbers

$$\prod_{p \le x} p \le 4^x - 1 \quad \text{for all real } x \ge 2$$

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- We proof now for all primes by induction.

Inductive Base Case

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- We are left with only odd primes $q = 2m + 1$.
- Inductive step for arbitrary odd prime:

Inductive Step

$$\prod_{p \leq 2m+1} p = \prod_{p \leq m+1} p \prod_{m+1 < p \leq 2m+1} p$$

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Using the induction hypothesis: $\prod_{p \le m+1} p \le 4^m$.

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Using the induction hypothesis: $\prod_{p \leq m+1} p \leq 4^m$.

Bounding the Prime Product

Can we bound this $\prod_{m+1<p\leq 2m+1} p$?

- Yes, $\frac{(2m+1)!}{m!(m+1)!}$ is an integer and contains all prime numbers between $m+1$ and $2m+1$.

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- Yes, $\frac{(2m+1)!}{m!(m+1)!}$ is an integer and contains all prime numbers between $m+1$ and $2m+1$.
- Follows since both denominator have factors less than $m+1$.

$$\prod_{p\leq 2m+1} p = \prod_{p\leq m+1} p \prod_{m+1<p\leq 2m+1} p \leq 4^m \binom{2m+1}{m}$$

Binomial Bound

$$\binom{2m+1}{m} \leq 2^{2m} \ ?$$

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$$\binom{2m+1}{m} \cdot 2 \leq \sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1}$$

Binomial Bound

$$\binom{2m+1}{m} \leq 2^{2m} \ ?$$

$$\binom{2m+1}{m} \cdot 2 \leq \sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1}$$

Since

$$\binom{2m+1}{m} = \binom{2m+1}{m+1}$$

Conclusion

Plugging everything together we get:

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$$\prod_{p \le 2m+1} p = \prod_{p \le m+1} p \prod_{m+1 < p \le 2m+1} p \le 4^m \cdot 4^m = 4^{2m}$$

Temporary results

$$4^n \leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3} n} p \cdot \prod_{n < p \leq 2n} p$$

Temporary results

$$4^n \leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \cdot \prod_{n < p \leq 2n} p$$

$$(2n)^{1+\sqrt{2n}} \cdot 4^{\frac{2}{3}n} \cdot \prod_{n < p \leq 2n} p$$

Proof by Contradiction

Assume no prime p between $n < p \leq 2n$

$$4^n \;\leq\; (2n)^{1+\sqrt{2n}} \;\cdot\; 4^{\frac{2}{3}n}$$

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$$4^n \leq (2n)^{1+\sqrt{2n}} \cdot 4^{\frac{2}{3}n}$$

$$4^{\frac{1}{3}n} \leq (2n)^{1+\sqrt{2n}}$$

Intuition

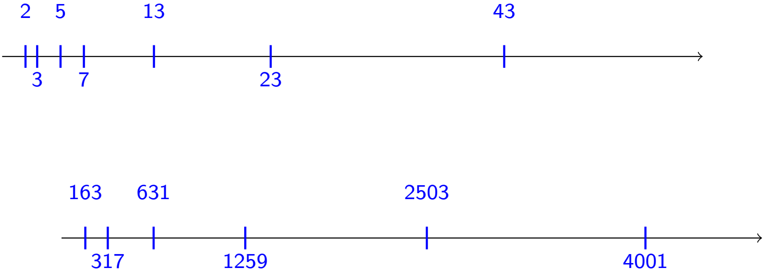
contra.png

Rigorous proof

Using $a + 1 < 2^a$ for $a \geq 2$, we get:

$$2n = \left(\sqrt[6]{2n}\right)^6 < \left(\lfloor\sqrt[6]{2n}\rfloor + 1\right)^6 < 2^{6\lfloor\sqrt[6]{2n}\rfloor} \leq 2^{6\sqrt[6]{2n}}.$$

Verification for Small Values of n



Conclusion: Every interval $\{y : n < y \leq 2n\}$, with $n \leq 4000$, contains one of these 14 primes.

Bounding 2^{2n} and Conclusion

For $n \geq 50$ (hence $18 < 2\sqrt{2n}$), we obtain:

$$2^{2n} \leq (2n)^{3(1+\sqrt{2n})} < 2^{6\sqrt{2n}(18+18\sqrt{2n})} < 2^{20\sqrt{2n}\sqrt{2n}} = 2^{20(2n)^{2/3}}.$$

This implies $(2n)^{1/3} < 20$, and thus $n < 4000$.

Further reads

Is there a prime number between every consecutive perfect squares?