Self Consistent ill-posed Inverse Pronlem With Cross Validation

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1 Introduction and Motivation

The mathematical term well-posed problem stems from a definition given by Jacques Hdamard. He believed that mathematical models of physical phenomena should have the properties that.

1. A solution exists.
2. The solution is unique.
3. The solution's behavior changes continuously with the initial conditions.

Problems that are not well-posed in the sense of Hadamard are termed ill-posed. Inverse problems are often ill-posed.

Continuum models must often be discretized in order to obtain a numerical solution, while solutions may be continuous with respect to the initial conditions, they may suffer from numerical instability when solved with finite precision, or with erros in the data. Even if a problem is well posed, it may still be ill-conditioned, meaning that a small error in the intial data can result in much larger errors in the answers.

In order to solve the ill-posed problem, regularization method is introduced. A simple form of regularization applied to integral equations, generally termed Tikhonov regularization, is essentially a trade-off between fitting the data and reducing a norm of the solution.

Ill-posed inverse problem with the regularization method can give out a very good reconstructed result compared to the exact result, but only if the measured value and the model is accurate enough.

In order to solve the ill-posed inverse problem which the measure value or the model is not accurate enougth to give out a good reconstructed result, we introduce the self consistent regularization method With Cross Validation.

2 Regularization Methods at Work: A Model Problem from Geophysics

We use a very simple model to illustrate how we use regularization method to solve the ill-posed inverse problem. We use a simplified problem from gravity surveying. An unknown mass distribution with density $f(t)$ is located at depth de below the surface. From 0 to 1 on the t axis shown in Figure. 1. We assume there is no mass outside this source, which produce a gravity field everywhere. At the surface, along the s axis in Figure. 1 from 0 to 1, we mreasure the vertical component of the gravity field, which reger to as g(s).

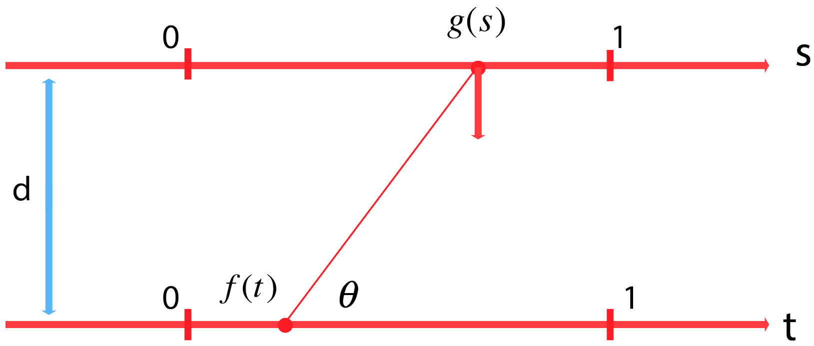


Figure 1: The geometry of the gravity surveying model problem: $f(t)$ is the mass density at t. and g(s) is the vertical component of the gravity field at s.

The two functions f and g are related via a Fredholm integral equation of the first kind. The gravity field from an infinitesmally small part of f(t), of length dt, on the axis is identical to the field from a point mass at t of strength f(t)dt. Hence, the magnitude of the gravity field along s is $f(t)dt/r^{2x}$, where $r = \sqrt{d^2 + (s-t)^2}$ is the distance between the source point at t and the field point at s. The direction of the gravity field is from the field point to he source point, and therefore the measured value of g(s) is

$$dg = \frac{\sin\theta}{r^2} f(t)dt$$

where θ is the angle shwon in Figure. 1. Using that $\sin\theta = d/r$, we obtain

$$\frac{\sin\theta}{r^2} f(t)dt = \frac{d}{\left(d^2 + (s-t)^2\right)^{3/2}} f(t)dt$$

The total value of g(s) for any $0 \le s \le 1$ consists of contributions from all mass along the t axis (from 0 to 1). and it is therefore given by the integral

$$g(s) = \int_0^1 \frac{d}{\left(d^2 + (s-t)^2\right)^{3/2}} f(t)dt$$

This is the forward problem and writing it as

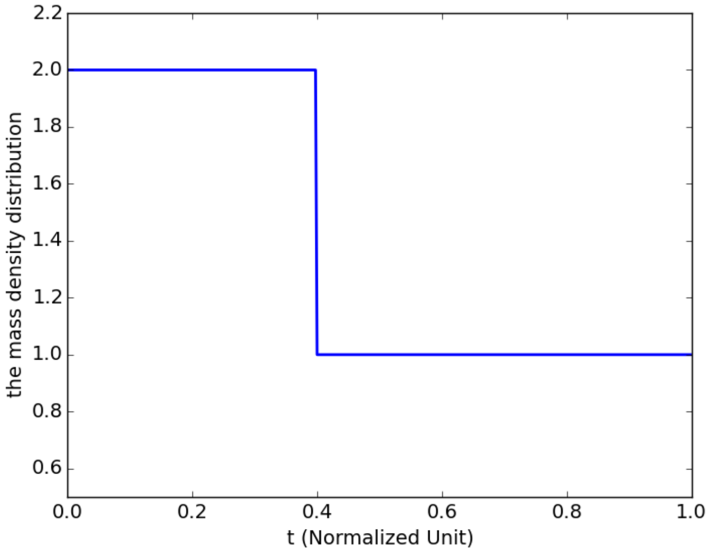
$$\int_0^1 K(s,t)f(t)dt = g(s), \quad 0 \le s \le 1 \tag{1}$$

where the function K, which represents the model, is given by

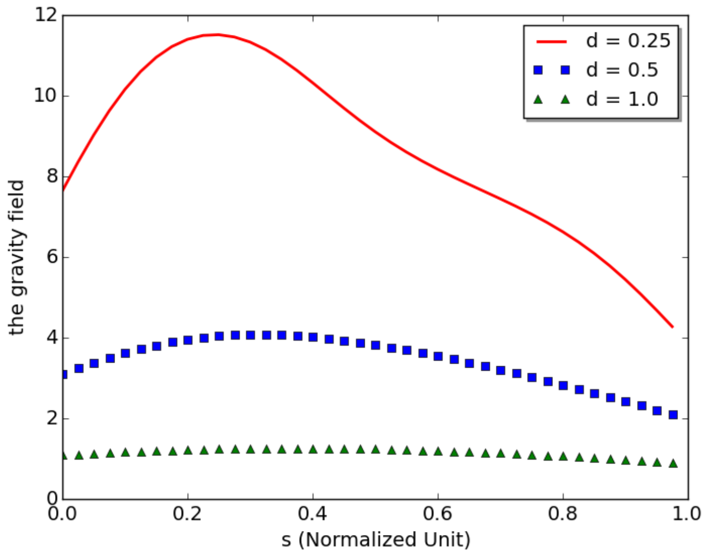
$$k(s,t) = \frac{d}{\left(d^2 + (s-t)^2\right)^{3/2}} \tag{2}$$

and the righ-hand side g is what we are able to measure. The function K is the vertical component of the gravity field, measured at s, from a unit point source locate at t. From K and g we want to compute f, and this is the inverse problem.

Figure.2 shows an example of the computation of the measured signal g(s), given the mass distribution f and three different values of the depth d.



(a) f(t)



(b) g(s)

Figure 2: (a) shows the function f(the mass density distribution), and (b) shows the measured signal g (the gravity field) for three different values of the depth d in Figure. 1

2.1 Discretization of Liner Inverse Problems: Quadrature Methods

We compute approximations $\tilde{f}_j$ to the solution f solely at selected abscissas $t_1, t_2, ...t_n, i.e.,$

$$\tilde{f}_j = \tilde{f}(t_j), \qquad j = 1, 2, ..., n.$$

Quadrature methods–also called Nystrom methods–take their basis in the general quadrature rule of the form

$$\int_0^1 \varphi(t)dt = \sum_{j=1}^{n} \omega_j \varphi(t_j) + E_n$$

where φ is the function whose integral we want to evaluate, E_n is the quadrature error, $t_1, t_2, ..., t_n$ are the abscissas for the quadrature rule, and $\omega_1, \omega_2, \omega_3, ..., \omega_n$ are the corresponding weights. For example, for the midpoint rule in the interval $[0, 1]$ we have

$$t_j = \frac{j - \frac{1}{2}}{n}, \qquad \omega_j = \tfrac{1}{n}, \qquad j = 1, 2, ...n. \tag{3}$$

Apply the quadrature methods on Eq.1, we arrive at the relations

$$\sum_{j=1}^{n} \omega_j K(s_j, t_j) \tilde{f}_j = g(s_i), \qquad i = 1, ...n. \tag{4}$$

The realtion in Eq.7 are just a linear system, which can be also written as

$$\begin{pmatrix} \omega_1 K(s_1, t_1) & \omega_2 K(s_1, t_2) & ... & \omega_n K(s_1, t_n) \\ \omega_1 K(s_2, t_1) & \omega_2 K(s_2, t_2) & ... & \omega_1 K(s_2, t_n) \\ \vdots & \vdots & & \vdots \\ \omega_1 K(s_n, t_1) & \omega_1 K(s_n, t_2) & ... & \omega_1 K(s_n, t_n) \end{pmatrix}$$

or simply $Ax = b$, where A is an $n \times n$ matrix. The elements of the matrix A, the right-hand side b, and the solution vector x are given by

$$\left. \begin{aligned} a_{ij} &= \omega_j K(s_i, t_j) \\ x_j &= \tilde{f}(t_j) \\ b_i &= g(s_i) \end{aligned} \right\} \qquad i, j = 1, ..., n. \tag{5}$$

3 Generalized Cross Validation

The fitness is calculated as:

$$\min_{\lambda} ||Ax_\lambda - b^{exact}||_2^2 \tag{6}$$

However, we can't calculate it since b^{exact} is not available. Generalized Cross Validation is a classical statistical technique that comes into good use here [1].

Using Generalized Cross Validation(GSV), the fitness can be calculated as:

$$\min_{\lambda} \frac{||Ax_\lambda - b||_2^2}{\left(m - \sum_{i=1}^{n} \varphi_i^{[\lambda]}\right)^2} \tag{7}$$

4 Cellular Evolutionary Algorithm

Usually EAs assume that the strucutre of the population is panmictic, which means that any inidividual may interact with any other individual in the population. However, this need not be always the case: we often see population sin the biological and social world in which individuals only interact with a subset of the rest of the population. This situation can usefully be depicated by using the concept of a population graph [2].

5 Result

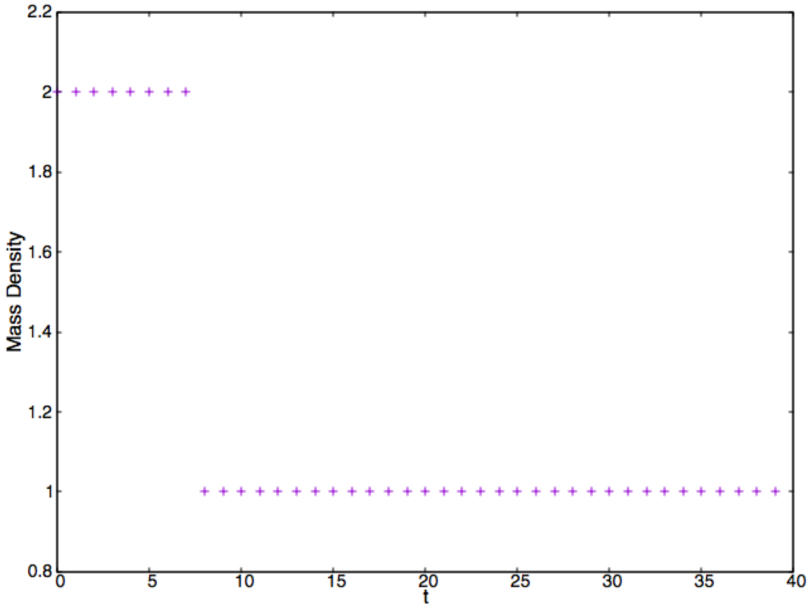


Figure 3: Exact f function (mass density distribution)

References

[1] Per Christian Hansen. *Discrete inverse problems: insight and algorithms*, volume 7. Siam, 2010.

[2] Alfons G Hoekstra, Jiri Kroc, and Peter MA Sloot. *Simulating complex systems by cellular automata*. Springer, 2010.

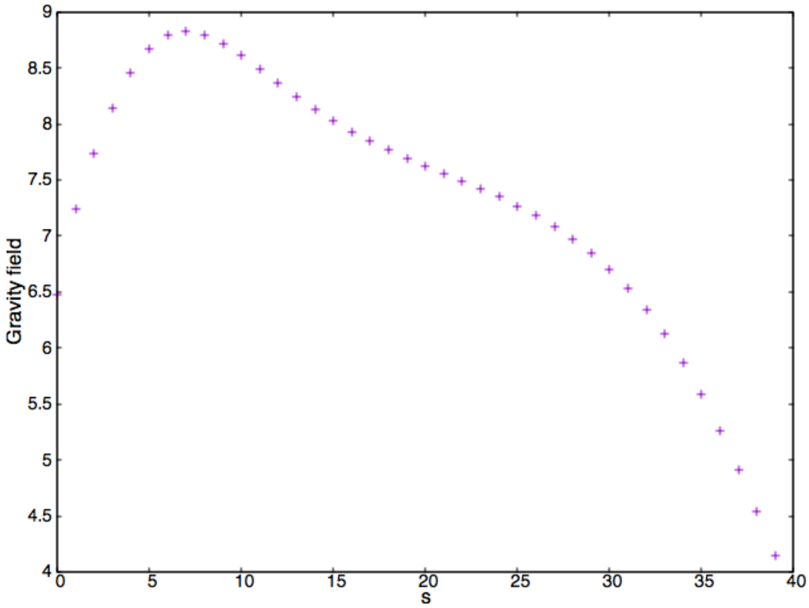


Figure 4: Exact signal g (the gravity field)

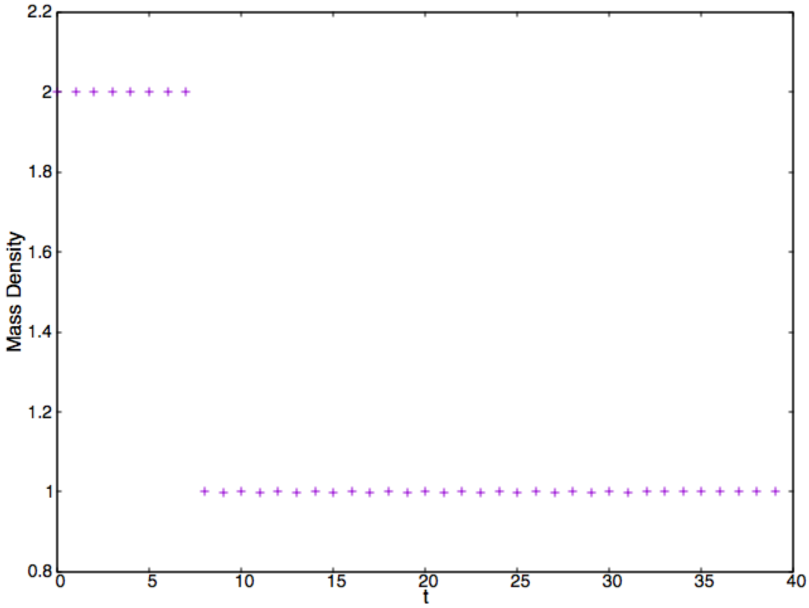


Figure 5: The Reconstructed function f(the mass density distribution) after self-consistent genetic algorithm, at $\lambda = 10^{-12}$, GSV value = 1.65e-25

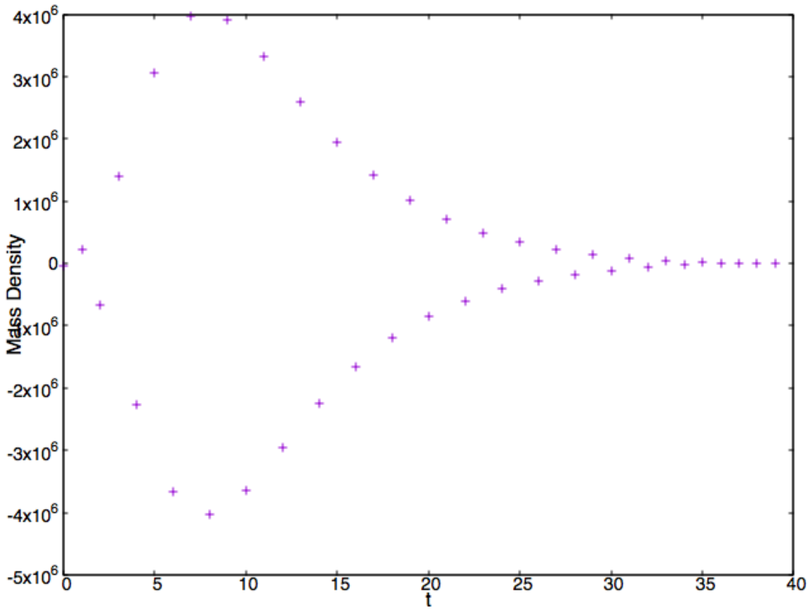


Figure 6: The Reconstructed function f(the mass density distribution) without self-consistent genetic algorithm, at $\lambda = 10^{-12}$, GSV value = 7.22e-10