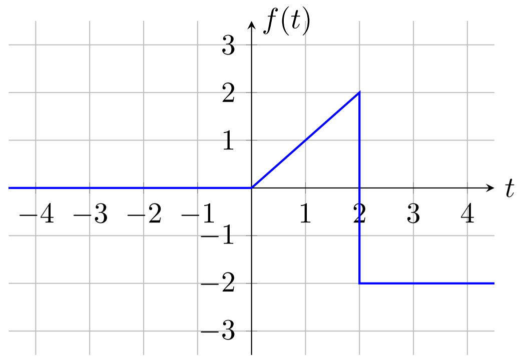
ECE 2260 hw06

1. Laplace Transforms

Find the Laplace transforms for the following functions. You may use direct integration or tables.

(a) Find $F(s)$.



Solution: First, we need to define the function $f(t)$ based on the graph. We can express it using step functions

$$f(t) = t(u(t) - u(t-2)) - 2u(t-2)$$

where $u(t)$ is the unit step function. We can break this apart

$$f(t) = tu(t) - tu(t-2) - 2u(t-2)$$

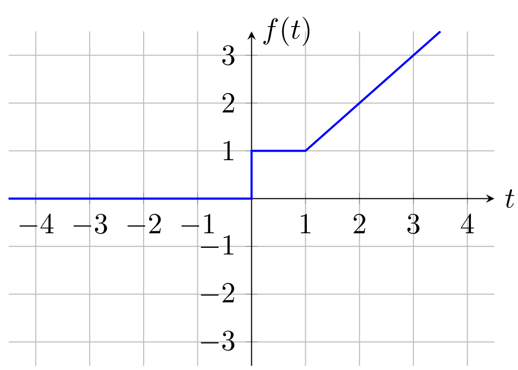
We can't directly apply the shift property quite yet, but we can add and subtract $2u(t-2)$ to get

$$f(t) = tu(t) - (t-2)u(t-2) + 2u(t-2) - 2u(t-2) - 2u(t-2) = tu(t) - (t-2)u(t-2) + -4u(t-2)$$

Now we can apply the shift property to get

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} - 4e^{-2s}\frac{1}{s}$$

(b) Find $F(s)$.



Solution: This is similar. Start by writing an expression for $f(t)$

$$f(t) = u(t) - u(t-1) + tu(t-1).$$

We can't use the shift property yet, so we can rewrite this as

$$f(t) = u(t) + (t-1)u(t-1).$$

Now we can apply the shift property to get

$$F(s) = \frac{1}{s} + e^{-s}\frac{1}{s^2}.$$

(c) Given $F(s) = \frac{5s+2}{s(s+1)}$, find $f(t \to \infty)$.

Solution: We can use the final value theorem to find $f(t \to \infty)$

$$f(t \to \infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s\frac{5s+2}{s(s+1)} = \lim_{s \to 0} \frac{5s+2}{s+1} = 2.$$

2. Laplace transforms

Find the Laplace transform of each of the following functions:

a) $f(t) = te^{-at}$;

Solution: Using the property $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ or knowing $\mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}$

$$F(s) = \frac{1}{(s+a)^2}$$

b) $f(t) = \sin \omega t$;

Solution: From standard tables

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

c) $f(t) = \sin(\omega t + \theta)$;

Solution: Using the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$f(t) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$$

Taking the Laplace transform

$$\begin{aligned} F(s) &= \cos \theta \mathcal{L}\{\sin \omega t\} + \sin \theta \mathcal{L}\{\cos \omega t\} \\ &= \cos \theta \frac{\omega}{s^2 + \omega^2} + \sin \theta \frac{s}{s^2 + \omega^2} \\ &= \frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2} \end{aligned}$$

d) $f(t) = t$;

Solution: From standard tables for t^n with $n = 1$

$$F(s) = \frac{1}{s^2}$$

e) $f(t) = \cosh(t + \theta)$.

Solution: Using the identity $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$

$$f(t) = \cosh t \cosh\theta + \sinh t \sinh\theta$$

Taking the Laplace transform (with $\omega = 1$ for $\cosh t$ and $\sinh t$)

$$\begin{aligned} F(s) &= \cosh\theta \mathcal{L}\{\cosh t\} + \sinh\theta \mathcal{L}\{\sinh t\} \\ &= \cosh\theta \frac{s}{s^2-1} + \sinh\theta \frac{1}{s^2-1} \\ &= \frac{s\cosh\theta + \sinh\theta}{s^2-1} \end{aligned}$$

3. Parallel RLC Circuit with a Switch

There is no energy stored in the circuit shown in Fig. 1 at the time the switch is opened.

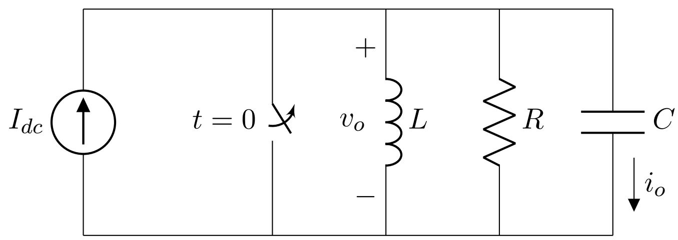


Figure 1: RLC circuit with a switch.

Note: The switch is initially closed and opens at $t = 0$.

a) Derive the integrodifferential equation that governs the behavior of the voltage v_o.

Solution: Apply KCL at the top node for $t > 0$. The switch is open, so the current source I_{dc} supplies the parallel RLC circuit. The currents leaving the top node through the passive elements sum to the entering current I_{dc}.

$$i_C + i_R + i_L = I_{dc}$$

Substituting the V-I relationships for each element in terms of $v_o(t)$:

- Capacitor current: $i_C = C\frac{dv_o}{dt}$
- Resistor current: $i_R = \frac{v_o}{R}$
- Inductor current: $i_L(t) = \frac{1}{L}\int_0^t v_o(\tau)d\tau + i_L(0)$

Since there is no energy stored initially, $i_L(0) = 0$ (and $v_o(0) = 0$). The integrodifferential equation is:

$$C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o(\tau)d\tau = I_{dc}$$

b) Show that

$$V_o(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

Solution: Take the Laplace transform of the integrodifferential equation derived in part (a). Note that $\mathcal{L}\{I_{dc}\} = \frac{I_{dc}}{s}$ (step function starting at $t = 0$). Assuming zero initial conditions ($v_o(0) = 0$):

$$CsV_o(s) + \frac{1}{R}V_o(s) + \frac{1}{Ls}V_o(s) = \frac{I_{dc}}{s}$$

Factor out $V_o(s)$:

$$V_o(s)\left(Cs + \frac{1}{R} + \frac{1}{Ls}\right) = \frac{I_{dc}}{s}$$

Multiply through by s:

$$V_o(s)\left(Cs^2+\frac{s}{R}+\frac{1}{L}\right)=I_{dc}$$

Solve for $V_o(s)$:

$$V_o(s)=\frac{I_{dc}}{Cs^2+\frac{s}{R}+\frac{1}{L}}$$

Divide numerator and denominator by C:

$$V_o(s)=\frac{I_{dc}/C}{s^2+\frac{1}{RC}s+\frac{1}{LC}}$$

c) Show that

$$I_o(s)=\frac{sI_{dc}}{s^2+(1/RC)s+(1/LC)}.$$

Solution: The current $i_o(t)$ is the current through the capacitor, so $i_o(t)=C\frac{dv_o}{dt}$. In the s-domain:

$$I_o(s)=C[sV_o(s)-v_o(0)]$$

Since the initial voltage is zero, $v_o(0)=0$:

$$I_o(s)=sCV_o(s)$$

Substitute the expression for $V_o(s)$ from part (b):

$$I_o(s)=sC\left[\frac{I_{dc}/C}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\right]$$

The C term cancels:

$$I_o(s)=\frac{sI_{dc}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}$$

d) Suppose $R=20\ \Omega$, $L=50$ mH, $C=20$ μF, and $I_{dc}=75$ mA. Find the poles of $V_o(s)$ and $I_o(s)$.

Solution: Plugging in the values

$$V_o(s)=\frac{3750}{s^2+2500s+1000000}.$$

Factoring this

$$V_o(s)=\frac{3750}{(s+2000)(s+500)}.$$

We can do a partial fraction expansion to find $V_o(s)$ in terms of the poles

$$V_o(s)=\frac{2.5}{s+500}-\frac{2.5}{s+2000}.$$

We can do the inverse Laplace transform to find $v_o(t)$

$$v_o(t) = 2.5e^{-500t} - 2.5e^{-2000t} \text{ V}\, t \geq 0.$$

We can also find $I_o(s)$

$$I_o(s) = \frac{0.075s}{(s+500)(s+2000)}.$$

We can do a partial fraction expansion to find $I_o(s)$ in terms of the poles

$$I_o(s) = \frac{-0.025}{s+500} + \frac{0.1}{s+2000}.$$

We can do the inverse Laplace transform to find $i_o(t)$

$$i_o(t) = -25e^{-500t} + 100e^{-2000t} \text{ mA}\, t \geq 0.$$

4. Inverse Laplace Transforms

Find $f(t)$ for each of the following functions:

a)

$$F(s) = \frac{6(s+10)}{(s+5)(s+8)}.$$

Solution: Decompose $F(s)$ into partial fractions:

$$F(s) = \frac{K_1}{s+5} + \frac{K_2}{s+8}$$

Calculate the residues:

$$K_1 = F(s)(s+5)\Big|_{s=-5} = \frac{6(-5+10)}{-5+8} = \frac{6(5)}{3} = 10$$

$$K_2 = F(s)(s+8)\Big|_{s=-8} = \frac{6(-8+10)}{-8+5} = \frac{6(2)}{-3} = -4$$

Thus,

$$F(s) = \frac{10}{s+5} - \frac{4}{s+8}$$

Inverse Laplace transform:

$$f(t) = [10e^{-5t} - 4e^{-8t}]u(t)$$

b)

$$F(s) = \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)}.$$

Solution: The denominator factors as $s(s+3)(s+7)$.

$$F(s) = \frac{20s^2 + 141s + 315}{s(s+3)(s+7)} = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{s+7}$$

Calculate residues:

$$K_1 = \frac{315}{(3)(7)} = \frac{315}{21} = 15$$

$$K_2 = \frac{20(9) + 141(-3) + 315}{(-3)(4)} = \frac{180 - 423 + 315}{-12} = \frac{72}{-12} = -6$$

$$K_3 = \frac{20(49) + 141(-7) + 315}{(-7)(-4)} = \frac{980 - 987 + 315}{28} = \frac{308}{28} = 11$$

$$F(s) = \frac{15}{s} - \frac{6}{s+3} + \frac{11}{s+7}$$

Inverse Laplace transform:

$$f(t) = [15 - 6e^{-3t} + 11e^{-7t}]u(t)$$

c)

$$F(s) = \frac{15s^2 + 112s + 228}{(s+2)(s+4)(s+6)}.$$

Solution:

$$F(s) = \frac{K_1}{s+2} + \frac{K_2}{s+4} + \frac{K_3}{s+6}$$

Calculate residues:

$$K_1 = \frac{15(4) + 112(-2) + 228}{(2)(4)} = \frac{60 - 224 + 228}{8} = \frac{64}{8} = 8$$

$$K_2 = \frac{15(16) + 112(-4) + 228}{(-2)(2)} = \frac{240 - 448 + 228}{-4} = \frac{20}{-4} = -5$$

$$K_3 = \frac{15(36) + 112(-6) + 228}{(-4)(-2)} = \frac{540 - 672 + 228}{8} = \frac{96}{8} = 12$$

$$F(s) = \frac{8}{s+2} - \frac{5}{s+4} + \frac{12}{s+6}$$

Inverse Laplace transform:

$$f(t) = [8e^{-2t} - 5e^{-4t} + 12e^{-6t}]u(t)$$

d)

$$F(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s^2 + 5s + 6)}.$$

Solution: The term $(s^2 + 5s + 6)$ factors as $(s+2)(s+3)$.

$$F(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s+1)(s+2)(s+3)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2} + \frac{K_4}{s+3}$$

Calculate residues:

$$K_1 = \frac{54}{(1)(2)(3)} = \frac{54}{6} = 9$$

$$K_2 = \frac{2(-1) + 33(1) + 93(-1) + 54}{(-1)(1)(2)} = \frac{-2 + 33 - 93 + 54}{-2} = \frac{-8}{-2} = 4$$

$$K_3 = \frac{2(-8) + 33(4) + 93(-2) + 54}{(-2)(-1)(1)} = \frac{-16 + 132 - 186 + 54}{2} = \frac{-16}{2} = -8$$

$$K_4 = \frac{2(-27) + 33(9) + 93(-3) + 54}{(-3)(-2)(-1)} = \frac{-54 + 297 - 279 + 54}{-6} = \frac{18}{-6} = -3$$

$$F(s) = \frac{9}{s} + \frac{4}{s+1} - \frac{8}{s+2} - \frac{3}{s+3}$$

Inverse Laplace transform:

$$f(t) = [9 + 4e^{-t} - 8e^{-2t} - 3e^{-3t}]u(t)$$

5. Inverse Laplace Transforms

Find $f(t)$ for each of the following functions:

a)
$$F(s) = \frac{280}{s^2 + 14s + 245}.$$

Solution: Complete the square in the denominator:

$$s^2 + 14s + 245 = (s^2 + 14s + 49) + 196 = (s + 7)^2 + 14^2$$

Rewrite $F(s)$:
$$F(s) = \frac{280}{(s+7)^2 + 14^2} = \frac{20 \cdot 14}{(s+7)^2 + 14^2}$$

Inverse Laplace transform:
$$f(t) = 20e^{-7t}\sin(14t)u(t)$$

b)
$$F(s) = \frac{-s^2 + 52s + 445}{s(s^2 + 10s + 89)}.$$

Solution: Use partial fraction expansion:

$$F(s) = \frac{5}{s} + \frac{-3 - 2j}{s - (-5 + 8j)} + \frac{-3 + 2j}{s - (-5 - 8j)}.$$

We can take the inverse Laplace transform of each term

$$f(t) = [5 + (-3 - 2j)e^{(-5+8j)t} + (-3 + 2j)e^{(-5-8j)t}]u(t)$$

We can use Euler's formula to rewrite the complex exponentials in terms of sines and cosines

$$f(t) = [5 + e^{-5t}(-6\cos(8t) + 4\sin(8t))]u(t)$$

c)
$$F(s) = \frac{14s^2 + 56s + 152}{(s + 6)(s^2 + 4s + 20)}.$$

Solution: Partial fraction expansion

$$F(s) = \frac{a}{s + 6} + \frac{b}{s - (-2 + 4j)} + \frac{b^*}{s - (-2 - 4j)}.$$

We can find a by calculating the residue at $s = -6$:

$$a = F(s)(s+6)\Big|_{s=-6} = \frac{14(36) + 56(-6) + 152}{36 - 24 + 20} = \frac{320}{32} = 10$$

We can find b by calculating the residue at $s = -2 + 4j$:

$$b = F(s)(s - (-2+4j))\Big|_{s=-2+4j} = 2 + j2$$

Putting it together

$$F(s) = \frac{10}{s+6} + \frac{2+j2}{s-(-2+4j)} + \frac{2-j2}{s-(-2-4j)}.$$

We can take the inverse Laplace transform

$$f(t) = [10e^{-6t} + (2+j2)e^{(-2+4j)t} + (2-j2)e^{(-2-4j)t}]u(t).$$

We can use Euler's formula to rewrite the complex exponentials in terms of sines and cosines

$$f(t) = [10e^{-6t} + 4e^{-2t}\cos(4t) - 4e^{-2t}\sin(4t)]u(t).$$

6. Inverse Laplace Transforms

Find $f(t)$ for each of the following functions:

a)

$$F(s) = \frac{320}{s^2(s+8)}.$$

Solution: Expand into partial fractions

$$F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+8}$$

Calculate residues

$$K_1 = s^2F(s)\Big|_{s=0} = \frac{320}{8} = 40$$

$$K_2 = \frac{d}{ds}\left[\frac{320}{s+8}\right]_{s=0} = \left[\frac{-320}{(s+8)^2}\right]_{s=0} = \frac{-320}{64} = -5$$

$$K_3 = (s+8)F(s)\Big|_{s=-8} = \frac{320}{(-8)^2} = \frac{320}{64} = 5$$

Thus:

$$F(s) = \frac{40}{s^2} - \frac{5}{s} + \frac{5}{s+8}$$

Inverse Laplace transform

$$f(t) = [40t - 5 + 5e^{-8t}]u(t)$$

b)

$$F(s) = \frac{80(s+3)}{s(s+2)^2}.$$

Solution: Partial fraction expansion

$$F(s) = \frac{K_1}{s} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2}$$

Calculate residues

$$K_1 = \frac{80(3)}{(2)^2} = \frac{240}{4} = 60$$

$$K_2 = \frac{80(-2+3)}{-2} = \frac{80}{-2} = -40$$

$$K_3 = \frac{d}{ds}\left[\frac{80(s+3)}{s}\right]_{s=-2} = 80\left[\frac{s(1)-(s+3)(1)}{s^2}\right]_{s=-2} = 80\left[\frac{-2-1}{4}\right] = 80\left[\frac{-3}{4}\right] = -60$$

Thus

$$F(s) = \frac{60}{s} - \frac{40}{(s+2)^2} - \frac{60}{s+2}$$

Inverse Laplace transform

$$f(t) = [60 - 40te^{-2t} - 60e^{-2t}]u(t)$$

c)

$$F(s) = \frac{60(s+5)}{(s+1)^2(s^2+6s+25)}.$$

Solution: Partial fraction expansion form

$$F(s) = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+6s+25}$$

Solving for coefficients yields

$$F(s) = \frac{12}{(s+1)^2} + \frac{0.6}{s+1} - \frac{0.6s+15}{s^2+6s+25}$$

Examine the quadratic term denominator: $s^2+6s+25 = (s+3)^2+16 = (s+3)^2+4^2$. Rewrite the numerator $-(0.6s+15)$ to match the shift $(s+3)$

$$0.6s + 15 = 0.6(s+3) - 1.8 + 15 = 0.6(s+3) + 13.2$$

So the term is

$$-\frac{0.6(s+3)}{(s+3)^2+4^2} - \frac{13.2}{(s+3)^2+4^2}$$

For the sine term, we need 4 in the numerator

$$\frac{13.2}{4} \cdot \frac{4}{(s+3)^2+4^2} = 3.3\frac{4}{(s+3)^2+4^2}$$

Therefore

$$F(s) = \frac{12}{(s+1)^2} + \frac{0.6}{s+1} - 0.6\frac{s+3}{(s+3)^2+4^2} - 3.3\frac{4}{(s+3)^2+4^2}$$

Inverse Laplace transform

$$f(t) = [12te^{-t} + 0.6e^{-t} - 0.6e^{-3t}\cos(4t) - 3.3e^{-3t}\sin(4t)]u(t)$$

d)

$$F(s) = \frac{25(s+4)^2}{s^2(s+5)^2}.$$

Solution: Partial fraction expansion

$$F(s) = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

Calculated coefficients (using partial fraction decomposition)

$$F(s) = \frac{16}{s^2} + \frac{1.6}{s} + \frac{1}{(s+5)^2} - \frac{1.6}{s+5}$$

Inverse Laplace transform

$$f(t) = [16t + 1.6 + te^{-5t} - 1.6e^{-5t}]u(t)$$