ECE 2260 hw04

1. Natural Response of a Parallel RLC Circuit

The circuit elements in the circuit in Fig. 1 are $R = 125\ \Omega$, $L = 200$ mH, and $C = 5$ µF. The initial inductor current is −300 mA and the initial capacitor voltage is 25 V.

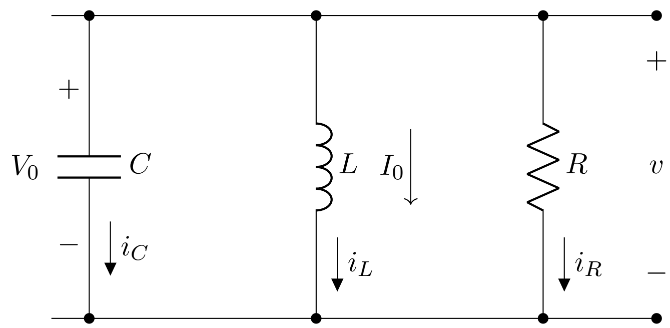


Figure 1: A circuit used to illustrate the natural response of a parallel RLC circuit.

a) Calculate the initial current in each branch of the circuit.

Solution: At $t = 0^+$, the voltage across all parallel elements is $v(0^+) = v(0^-) = 25$ V.

The current through the resistor is:

$$i_R(0^+) = \frac{v(0^+)}{R} = \frac{25}{125} = 0.2\text{ A} = 200\text{ mA}$$

The current through the inductor is continuous:

$$i_L(0^+) = i_L(0^-) = -300\text{ mA}$$

Applying KCL at the top node ($i_C + i_L + i_R = 0$):

$$i_C(0^+) = -i_L(0^+) - i_R(0^+) = -(-300\text{ mA}) - 200\text{ mA} = 300\text{ mA} - 200\text{ mA} = 100\text{ mA}$$

b) Find $v(t)$ for $t \geq 0$.

Solution: First, we find the circuit parameters:

$$\alpha = \frac{1}{2RC} = \frac{1}{2(125)(5\text{ µF})} = \frac{1}{1250 \times 10^{-6}} = 800\text{ Np s}^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.2)(5\text{ µF})}} = \frac{1}{\sqrt{10^{-6}}} = 1000\text{ rad s}^{-1}$$

Since $\alpha < \omega_0$, the response is underdamped. The damped frequency is:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{1000^2 - 800^2} = \sqrt{360000} = 600\text{ rad s}^{-1}$$

The voltage response is of the form:

$$v(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t))$$

Using initial conditions:

$$v(0) = B_1 = 25 \text{ V}$$

We also know that $i_C = C\frac{dv}{dt}$. At $t = 0^+$:

$$\frac{dv}{dt}(0^+) = \frac{i_C(0^+)}{C} = \frac{0.1}{5 \times 10^{-6}} = 20\,000 \text{ V s}^{-1}$$

Differentiating the voltage expression at $t = 0$:

$$\frac{dv}{dt}(0) = -\alpha B_1 + \omega_d B_2$$

$$20000 = -800(25) + 600B_2$$

$$20000 = -20000 + 600B_2 \implies 40000 = 600B_2 \implies B_2 = \frac{400}{6} = \frac{200}{3} \approx 66.67$$

Thus:

$$v(t) = e^{-800t}\left(25\cos(600t) + \frac{200}{3}\sin(600t)\right) \text{ V}$$

c) Find $i_L(t)$ for $t \geq 0$.

Solution: The form of the current response is the same as the voltage response:

$$i_L(t) = e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))$$

Using initial conditions:

$$i_L(0) = A_1 = -300 \text{ mA}$$

Since $v = L\frac{di_L}{dt}$:

$$\frac{di_L}{dt}(0^+) = \frac{v(0^+)}{L} = \frac{25}{0.2} = 125 \text{ A s}^{-1} = 125\,000 \text{ mA s}^{-1}$$

Differentiating the current expression at $t = 0$:

$$\frac{di_L}{dt}(0) = -\alpha A_1 + \omega_d A_2$$

$$125000 = -800(-300) + 600A_2$$

$$125000 = 240000 + 600A_2 \implies 600A_2 = 125000 - 240000 = -115000$$

$$A_2 = \frac{-115000}{600} \approx -191.67$$

Thus:

$$i_L(t) = e^{-800t}\left(-300\cos(600t) - 191.67\sin(600t)\right) \text{mA}$$

2. Parallel RLC Circuit Response

The resistance, inductance, and capacitance in a parallel RLC circuit are 2000 Ω, 250 mH, and 10 nF, respectively.

a) Calculate the roots of the characteristic equation that describe the voltage response of the circuit.

Solution: First, we calculate the neper frequency α and the resonant frequency ω_0 for a parallel RLC circuit.

$$\alpha = \frac{1}{2RC} = \frac{1}{2(2000\ \Omega)(10\ \text{nF})} = 25\,000\ \text{Np}\,\text{s}^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(250\ \text{mH})(10\ \text{nF})}} = 20\,000\ \text{rad}\,\text{s}^{-1}$$

The roots of the characteristic equation are given by $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$.

$$s_{1,2} = -25000 \pm \sqrt{25000^2 - 20000^2} = -25000 \pm 15000$$

$$s_1 = -10\,000\ \text{rad}\,\text{s}^{-1}, \quad s_2 = -40\,000\ \text{rad}\,\text{s}^{-1}$$

b) Will the response be over-, under-, or critically damped?

Solution: Since $\alpha > \omega_0$ ($25\,000 > 20\,000$), the response is **overdamped**.

c) What value of R will yield a damped frequency of 12 $\text{krad}\,\text{s}^{-1}$?

Solution: The damped frequency is given by $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$. Since we have a damped frequency, the response must be underdamped, so $\alpha < \omega_0$.

$$\alpha = \sqrt{\omega_0^2 - \omega_d^2} = \sqrt{20000^2 - 12000^2} = \sqrt{(400 - 144) \times 10^6} = 16\,000\ \text{Np}\,\text{s}^{-1}$$

Now we solve for R using $\alpha = \frac{1}{2RC}$:

$$R = \frac{1}{2C\alpha} = \frac{1}{2(10\ \text{nF})(16\,000)} = 3125\ \Omega$$

d) What are the roots of the characteristic equation for the value of R found in (c)?

Solution: For the underdamped case, the roots are $s_{1,2} = -\alpha \pm j\omega_d$. Using the values from part (c):

$$s_{1,2} = -16000 \pm j12000 \quad \text{rad}\,\text{s}^{-1}$$

e) What value of R will result in a critically damped response?

Solution: For a critically damped response, $\alpha = \omega_0$.

$$\frac{1}{2RC} = \omega_0 \implies R = \frac{1}{2C\omega_0}$$

$$R = \frac{1}{2(10\text{ nF})(20\,000)} = 2.5\text{ k}\Omega$$

3. Damping in Parallel RLC Circuits

A parallel, natural response RLC circuit is in its simplest form (a resistor R, an inductor L, and capacitor C all in parallel). Given the following component values, determine if the response is over-damped, under-damped, or critically damped.

(i) $R = 400\ \Omega$, $L = 25$ mH, and $C = 25$ nF

Solution: For a parallel RLC circuit, we calculate the neper frequency α and the resonant frequency ω_0.

$$\alpha = \frac{1}{2RC} = \frac{1}{2(400)(25 \times 10^{-9})} = \frac{1}{20 \times 10^{-6}} = 50\,000\ \mathrm{Np\,s^{-1}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(25 \times 10^{-3})(25 \times 10^{-9})}} = \frac{1}{\sqrt{625 \times 10^{-12}}} = \frac{1}{25 \times 10^{-6}} = 40\,000\ \mathrm{rad\,s^{-1}}$$

Since $\alpha > \omega_0$ $(50000 > 40000)$, the response is **over-damped**.

(ii) $R = 625\ \Omega$, $L = 25$ mH, and $C = 25$ nF

Solution: We recalculate α with the new resistance value. ω_0 remains effectively unchanged as L and C are the same.

$$\alpha = \frac{1}{2RC} = \frac{1}{2(625)(25 \times 10^{-9})} = \frac{1}{31.25 \times 10^{-6}} = 32\,000\ \mathrm{Np\,s^{-1}}$$

We already found $\omega_0 = 40\,000\ \mathrm{rad\,s^{-1}}$.

Since $\alpha < \omega_0$ $(32000 < 40000)$, the response is **under-damped**.

4. Circuit with Synchronous Switches

The two switches in the circuit seen in Fig. 2 operate synchronously. When switch 1 is in position a, switch 2 is in position d. When switch 1 moves to position b, switch 2 moves to position c. Switch 1 has been in position a for a long time. At $t = 0$, the switches move to their alternate positions. Find $v_o(t)$ for $t \geq 0$.

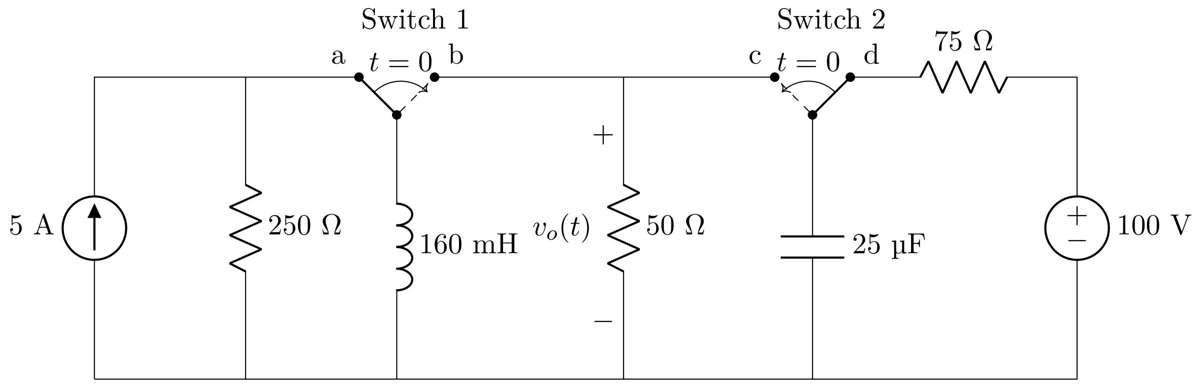


Figure 2: Circuit for problem.

Solution: Step 1: Determine initial conditions at $t = 0^-$.
For $t < 0$, Switch 1 is in position ‘a‘ and Switch 2 is in position ‘d‘. The circuit is in steady state.
Inductor Current $i_L(0^-)$: The inductor is connected to the 5A source and 250Ω resistor. In steady state, the inductor acts as a short circuit, so all of th current from the 5A source flows through the inductor. Thus, $I_0 = i_L(0^-) = 5$ A.

Capacitor Voltage $v_o(0^-)$: Similarly, at steady state, the capacitor acts as an open circuit. The capacitor is connected to the 100V source and 75Ω resistor. No current flows through the resistor, so there is no voltage drop across it. Therefore, the capacitor charges to the full source voltage. Thus, $V_0 = v_o(0^-) = 100$ V.

Step 2: Analyze the circuit at $t = 0^+$ to get ODE initial conditions.
The switches move. Switch 1 to ‘b‘, Switch 2 to ‘c‘. Now we have a parallel RLC circuit consisting of:

- Inductor $L = 160$ mH with initial current $I_0 = 5$ A (down).
- Resistor $R = 50\ \Omega$.
- Capacitor $C = 25\ \mu$F with initial voltage $V_0 = 100$ V (plus on top).

The variable $v_o(t)$ is the voltage across the parallel combination (top to bottom). Since v_C is continuous, $v_o(0^+) = v_C(0^-) = 100$ V.

We will need the derivative of $v_o(t)$ at $t = 0^+$ to solve for the coefficients in the voltage response. We notice that for a parallel RLC circuit:

$$C\frac{dv_o}{dt}(0^+) + \frac{v_o(0^+)}{R} + i_L(0^+) = 0$$

Note directions: v_o is top to bottom. i_R flows down. i_C flows down. i_L flows down. We know all of these pieces except $v_o'(0)$, we can solve for as $v_o'(0) = \frac{-7\text{ A}}{25\times10^{-6}\text{ F}} = -280\,000\text{ V s}^{-1}$.

Step 3: Determine the type of damping.
We can solve for α and ω_0 to determine the damping type. We notice that $\alpha = \frac{1}{2RC} 400\ \mathrm{Np\,s^{-1}}$ and $\omega_0 = \frac{1}{\sqrt{LC}} = 500\ \mathrm{rad\,s^{-1}}$. Since $\alpha < \omega_0$, the response is **underdamped**.
Step 4: Come up with general form.
The general form for the underdamped voltage response is

$$v_o(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t))$$

and with our numbers this is

$$v_o(t) = e^{-400t}(B_1 \cos(300t) + B_2 \sin(300t))$$

Step 5: Find coefficients.
If we let $t = 0$ in the voltage expression, we get

$$v_o(0) = B_1 = 100\ \mathrm{V}.$$

We can use $v_o'(0)$ to find B_2. We take the derivative of the voltage expression

$$\frac{dv_o}{dt} = -\alpha e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)) + e^{-\alpha t}(-B_1 \omega_d \sin(\omega_d t) + B_2 \omega_d \cos(\omega_d t))$$

and let $t = 0$

$$\frac{dv_o}{dt}(0) = -\alpha B_1 + \omega_d B_2.$$

Since we already know $B_1 = 100$ V and $v_o'(0) = -280\,000\ \mathrm{V\,s^{-1}}$, we can solve for B_2

$$B_2 = \frac{v_o'(0) + \alpha B_1}{\omega_d} = \frac{-280000 + 400(100)}{300} = -800.$$

Step 6: Put it together:

$$v_o(t) = e^{-400t}(100 \cos(300t) - 800 \sin(300t))\mathrm{V} \quad \text{for } t \geq 0$$

5. Circuit with Dependent Source

The switch in the circuit of Fig. 3 has been in position a for a long time. At $t = 0$ the switch moves instantaneously to position b. Find $v_o(t)$ for $t \geq 0$.

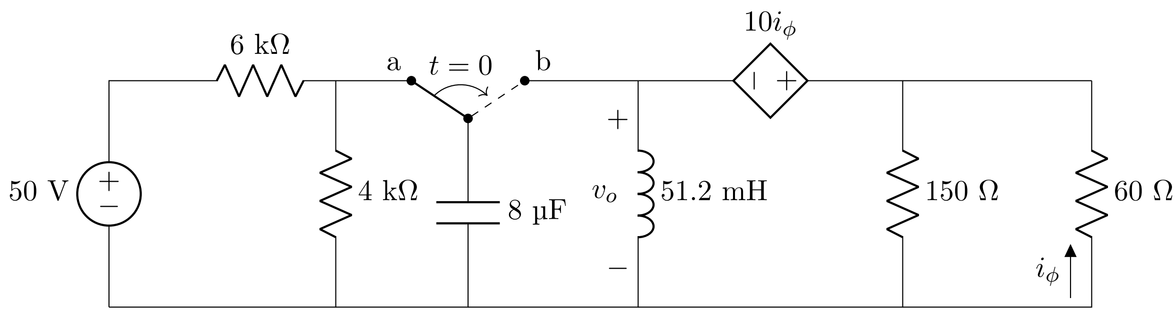


Figure 3: Circuit for problem.

Solution: Step 1: Calculate initial conditions at $t = 0^-$. The switch is in position 'a' for a long time. The capacitor acts as an open circuit to DC. The voltage across the capacitor is the voltage across the 4 kΩ resistor. Using voltage division

$$v_C(0^-) = 50\ \text{V} \times \frac{4\ \text{k}\Omega}{6\ \text{k}\Omega + 4\ \text{k}\Omega} = 20\ \text{V}$$

The inductor is disconnected from any active source (switch is at 'a', inductor connects to 'b'). Without an independent source on the right side, all currents and voltages are zero. So, $i_L(0^-) = 0$. Since voltage and current are continuous for C and L respectively:

$$v_C(0^+) = 20\ \text{V}$$

$$i_L(0^+) = 0$$

Step 2: Analyze the circuit at $t = 0^+$. The switch moves to 'b'. Now the capacitor (with initial voltage) is connected in parallel with the inductor and the rest of the resistive network.

We need to find the equivalent resistance R_{eq} of the network connected to the parallel LC pair to see this as a parallel RLC circuit. We see this would be $R_{Th} = 50\ \Omega$.

We already know that $v_o(0^+) = v_C(0^+) = 20$ V. We need the second condition, $v_o'(0^+)$, and we can get a quick expression for that using KCL at the top node of the parallel RLC circuit

$$Cv_o'(0^+) + \frac{v_o(0^+)}{R_{eq}} + i_L(0^+) = 0$$

Solving for $v_o'(0^+)$

$$v_o'(0^+) = -\frac{1}{C}\left(\frac{v_o(0^+)}{R_{eq}} + i_L(0^+)\right) = -\frac{1}{8 \times 10^{-6}\ \text{F}}\left(\frac{20\ \text{V}}{50\ \Omega} + 0\right) = -50\,000\ \text{V}\,\text{s}^{-1}$$

Step 3: Determine the type of damping. We have a parallel RLC circuit with:

- Inductor $L = 51.2$ mH

- Capacitor $C = 8\ \mu\text{F}$
- Resistor $R_{eq} = 50\ \Omega$

We can calculate α and ω_0 to determine the damping type.

$$\alpha = \frac{1}{2R_{eq}C} = \frac{1}{2(50\ \Omega)(8\times 10^{-6}\ \text{F})} = 1250\ \text{Np}\,\text{s}^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(51.2\times 10^{-3})(8\times 10^{-6})}} = \frac{1}{\sqrt{409.6\times 10^{-9}}} = \frac{1}{2.024\times 10^{-4}} = 4939\ \text{rad}\,\text{s}^{-1}$$

Since $\alpha < \omega_0$, the response is **underdamped**. The damped frequency is

$$\omega_d = 937.5\ \text{rad}\,\text{s}^{-1}$$

Step 4: General form of the solution. The voltage response for an underdamped parallel RLC circuit is

$$v_o(t) = e^{-\alpha t}(B_1\cos(\omega_d t) + B_2\sin(\omega_d t))$$

and using our numbers gives

$$v_o(t) = e^{-1250t}(B_1\cos(937.5t) + B_2\sin(937.5t))$$

Step 5: Solve for coefficients B_1 and B_2. Using the initial condition $v_o(0^+) = 20$ V gives

$$B_1 = 20\ \text{V}$$

We also have $v_o'(0^+) = -50\,000\ \text{V}\,\text{s}^{-1}$. Differentiating the voltage expression:

$$v_o'(t) = -\alpha e^{-\alpha t}(B_1\cos(\omega_d t) + B_2\sin(\omega_d t)) + e^{-\alpha t}(-B_1\omega_d\sin(\omega_d t) + B_2\omega_d\cos(\omega_d t))$$

Evaluating at $t = 0$:

$$v_o'(0) = -\alpha B_1 + \omega_d B_2$$

Plugging in values:

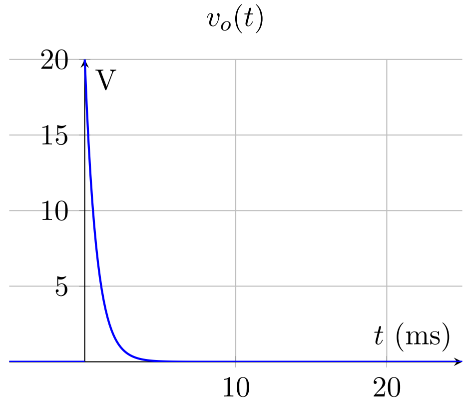
$$-50000 = -1250(20) + 937.5B_2$$

$$-50000 = -25000 + 937.5B_2 \implies -25000 = 937.5B_2 \implies B_2 = \frac{-25000}{937.5} = -26.67$$

Step 6: Put it all together. The final expression for $v_o(t)$ is

$$v_o(t) = \begin{cases} 0\ \text{V} & t < 0 \\ 20e^{-1250t}\cos(937.5t) - 26.67e^{-1250t}\sin(937.5t)\text{V} & \text{for } t \geq 0 \end{cases}$$

It might be helpful to plot this response to see the behavior.


$v_o(t)$
20
15
10
5
V
t (ms)
10
20

6. RLC Step Response

The switch in the circuit in Fig. 4 has been open for a long time before closing at $t = 0$. Find $i_o(t)$.

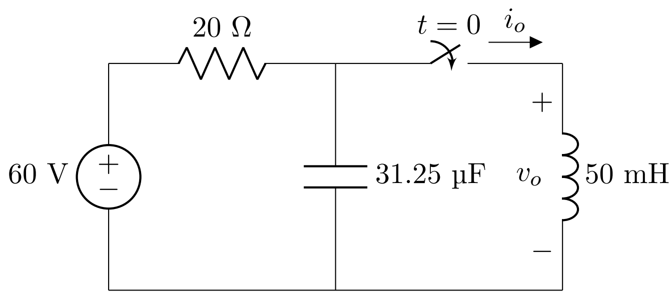


Figure 4: Circuit for problem

Solution: First find the intial conditions when $t < 0$. At this point the switch is open so $i_o(0^-) = 0$ A. The capacitor will act as an open circuit so the voltage across it is equal to the source voltage, $v_C(0^-) = 60$ V. At $t = 0$, the switch closes and we can find the initial conditions for $t \geq 0$. The inductor current cannot change instantaneously, so $i_o(0^+) = i_o(0^-) = 0$ A. The capacitor voltage also cannot change instantaneously, so $v_C(0^+) = v_C(0^-) = 60$ V.

Next, we can find the final conditions as $t \to \infty$. The inductor will act as a short circuit and the capacitor will act as an open circuit. Therefore, $I_f = \frac{60\text{ V}}{20\ \Omega} = 3$ A.

Now we can find the characteristic roots. We see that $\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 20 \cdot 31.25 \times 10^{-6}} = 800\ \text{s}^{-1}$ and $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{50 \times 10^{-3} \cdot 31.25 \times 10^{-6}}} = 800\ \text{s}^{-1}$. Since $\alpha = \omega_0$, we have a critically damped response. The characteristic roots are $s_1 = s_2 = -\alpha = -800\ \text{s}^{-1}$.

The current response is of the form

$$i_o(t) = I_f + A_1 e^{s_1 t} + A_2 t e^{s_2 t} = 3 + A_1 e^{-800t} + A_2 t e^{-800t}.$$

We can find A_1 and A_2 using the initial conditions. We know that $i_o(0) = 0$, so

$$0 = 3 + A_1 \implies A_1 = -3.$$

We will want to find the derivative of $i_o(t)$ to use the other initial condition. Notice that $v_L(t) = v_C(t)$ and that $v_L(t) = L\frac{di_o(t)}{dt}$. Therefore,

$$\frac{di_o(t)}{dt} = \frac{v_C(t)}{L}.$$

At $t = 0$, $v_C(0) = 60$ V, so

$$\frac{di_o(0)}{dt} = \frac{60\text{ V}}{50 \times 10^{-3}\text{ H}} = 1200\ \text{A}\,\text{s}^{-1}.$$

Now we can differentiate $i_o(t)$

$$\frac{di_o(t)}{dt} = -800 A_1 e^{-800t} + A_2 e^{-800t} - 800 A_2 t e^{-800t}.$$

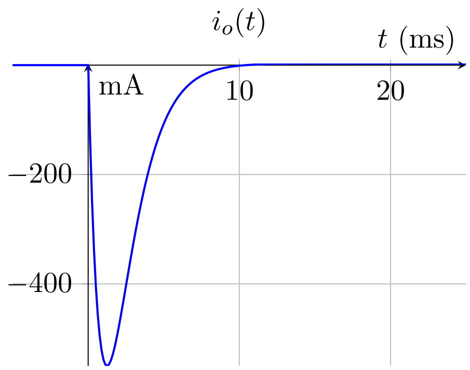
We already know that $A_1 = -3$, so at $t = 0$ we have

$$\frac{di_o(0)}{dt} = 2400 + A_2 = 1200 \implies A_2 = -1200.$$

Therefore, the complete response is

$$i_o(t) = \begin{cases} 0 \text{ A} & t < 0 \\ 3 - 3e^{-800t} - 1200te^{-800t} \text{ A} & t \geq 0. \end{cases}$$

We can do a quick sanity check at $t = 0$ and as $t \to \infty$. At $t = 0$, $i_o(0) = 0$ A, which matches our initial condition. As $t \to \infty$, the exponential terms go to zero and we have $i_o(\infty) = 3$ A, which matches our final condition.
It would be worthwhile to plot this response to visualize it.



7. RLC Step Response

The left switch in the circuit in Fig. 5 has been closed for a long time. The right switch has been open for a long time. At time $t = 0$, the left switch opens, and the right switch closes simultaneously.

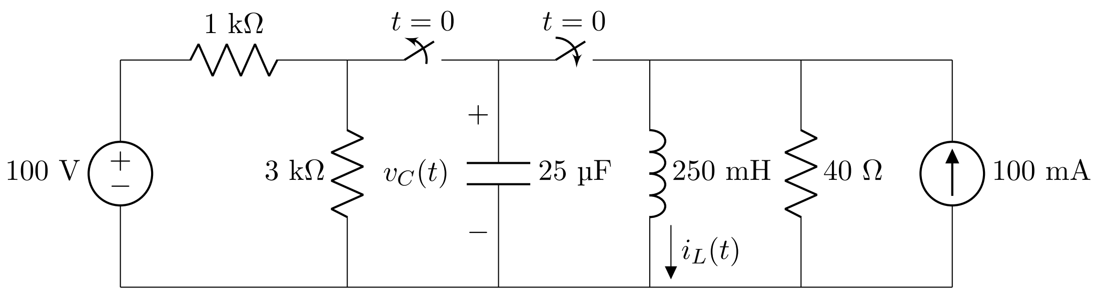


Figure 5: Circuit for problem.

a) $i_L(t)$ for $t \geq 0$,

Solution: First, we need to find the initial voltage over the capacitor. This is a voltage divider

$$v_C(0^-) = v_C(0^+) = \frac{3\ \text{k}\Omega}{1\ \text{k}\Omega + 3\ \text{k}\Omega} \cdot 100\ \text{V} = 75\ \text{V}.$$

We can also find the initial current through the inductor. The current source forces 100 mA through the right side of the circuit, so

$$i_L(0^-) = i_L(0^+) = 100\ \text{mA}.$$

We can also find the characteristic roots of the circuit. We see that $\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 40 \cdot 25 \times 10^{-6}} = 500\ \text{s}^{-1}$, and $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{250 \times 10^{-3} \cdot 25 \times 10^{-6}}} = 400\ \text{s}^{-1}$. Since $\alpha > \omega_0$, we have an over-damped circuit. The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -500 \pm \sqrt{500^2 - 400^2} = -500 \pm 300,$$

so $s_1 = -200$ and $s_2 = -800$. Thus, the response is

$$i_{L,n}(t) = I_f + A_1 e^{-200t} + A_2 e^{-800t}.$$

The final current through the inductor is simply the current source value, so $I_f = 100$ mA. To find A_1 and A_2, we use the initial conditions. At $t = 0$,

$$i_L(0) = I_f + A_1 + A_2 = 100\ \text{mA} + A_1 + A_2 = 100\ \text{mA},$$

so $A_1 + A_2 = 0$. We also know that $v_L(t) = v_C(t)$ and that $v_L(t) = L\frac{di_L(t)}{dt}$. We can say that $\frac{di_L(t)}{dt} = \frac{v_L(t)}{L}$. At $t = 0$ this would be $i_L'(0^+) = \frac{v_C(0^+)}{L} = 300\ \text{A}\,\text{s}^{-1}$. Thus, we see that $i_L'(t) = -200A_1e^{-200t} - 800A_2e^{-800t}$, and at $t = 0$,

$$i_L'(0) = -200A_1 - 800A_2 = 300\ \text{A}\,\text{s}^{-1}.$$

We now have two equations and two unknowns:

$$A_1 + A_2 = 0,$$
$$-200A_1 - 800A_2 = 300\ \text{A}\,\text{s}^{-1}.$$

Solving, we find that $A_1 = 500$ mA and $A_2 = -500$ mA. Thus, the final expression for $i_L(t)$ is

$$i_L(t) = \begin{cases} 100 \text{ mA}, & t < 0 \\ 100 + 500e^{-200t} - 500e^{-800t} \text{ mA}, & t \geq 0 \end{cases}.$$

We know that the current through the inductor cannot change instantaneously, so $i_L(0^-) = i_L(0^+) = 100$ mA, which is confirmed by our expression.

b) Find $v_C(t)$.

Solution: We can find $v_C(t)$ using $v_C(t) = L\frac{di_L(t)}{dt}$. We already have $\frac{di_L(t)}{dt} = -200A_1e^{-200t} - 800A_2e^{-800t}$, so

$$v_C(t) = 250 \times 10^{-3} \left(-200 \cdot 500 \text{ mA}e^{-200t} - 800 \cdot (-500 \text{ mA})e^{-800t}\right),$$

which simplifies to

$$v_C(t) = \begin{cases} 75 \text{ V}, & t < 0 \\ -25e^{-200t} + 100e^{-800t} \text{ V}, & t \geq 0 \end{cases}.$$

We know that the voltage across the capacitor cannot change instantaneously, so $v_C(0^-) = v_C(0^+) = 75$ V, which is confirmed by our expression.

It might be helpful to include some plots of $i_L(t)$ and $v_C(t)$ here.