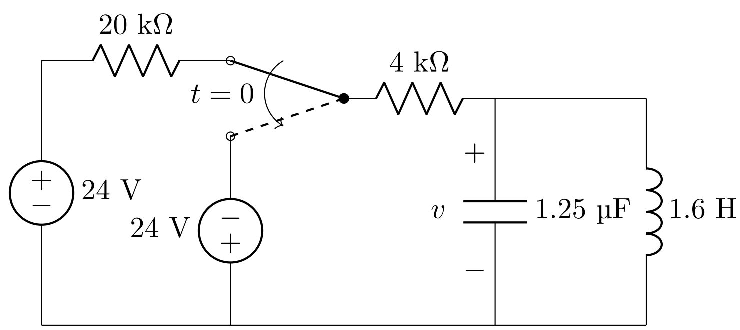
ECE 2260 hw05

1. RLC circuits

For the following circuit, find $v(t)$ for $t \geq 0$.



Solution: First, we should notice that we need to get the circuit in the parallel form with a source transformation after $t \geq 0$. This would give us a current source with a parallel resistor, capacitor, and inductor. The current source value is given by $I_s = V/R = 24V/20k\Omega = 1.2mA$. The parallel resistance is simply $R = 20k\Omega$.
We should figure out some initial conditions. The initial current through the inductor is $i_L(0^-) = \frac{24\text{ V}}{24\text{ k}\Omega} = 1$ mA. The initial voltage across the capacitor is $v_C(0^-) = 0$ V. We can find $i'(0^+) = \frac{v(0^+)}{L} = \frac{0\text{ V}}{1.6\text{ H}} = 0$ A/s. Lastly, we can find $I_f = \frac{-24\text{ V}}{4\text{ k}\Omega} = -6$ mA.
We next need to determine the natural response characteristic roots

$$S_{1,2} = -100 \pm j700\,\text{rad/s}.$$

We can then find the natural response

$$i_L(t) = I_f + e^{-100t}(B_1' \cos(700t) + B_2' \sin(700t)).$$

We can find B_1' using the initial condition

$$i_L(0^+) = I_f + B_1' = 1\text{ mA} \implies B_1' = 7\text{ mA}.$$

We can find B_2' using the derivative initial condition

$$i_L'(0^+) = -100B_1' + 700B_2' = 0\text{ A/s} \implies B_2' = 1\text{ mA}.$$

Putting it together

$$i_L(t) = -6 + e^{-100t}(7\cos(700t) + \sin(700t))\,\text{mA}.$$

However, we are looking for $v(t)$! We can find $v(t)$ using $v(t) = L\frac{di_L(t)}{dt}$.

$$\begin{aligned} v(t) &= 1.6\left(-100e^{-100t}(0.007\cos(700t) + 0.001\sin(700t)) + e^{-100t}(-0.007\cdot 700\sin(700t) + 0.001\cdot 700\cos(700t))\right)\text{ V} \\ &= -8e^{-100t}\sin(700t))\,\text{V} \end{aligned}$$

We can also solve this from the voltage side. The characteristic equation is the same, so we can write the voltage as

$$v(t) = e^{-100t}(B_1\cos(700t) + B_2\sin(700t))$$

We can find B_1 using the initial condition:

$$v(0^+) = B_1 = 0 \text{ V}$$

We can find B_2 using the derivative initial condition

$$v'(0^+) = \frac{i_c(0^+)}{C} = \frac{-7 \text{ mA}}{1.25 \text{ µF}} = -5600 \text{ V/s} \implies 700B_2 = -5600 \text{ V/s} \implies B_2 = -8 \text{ V}$$

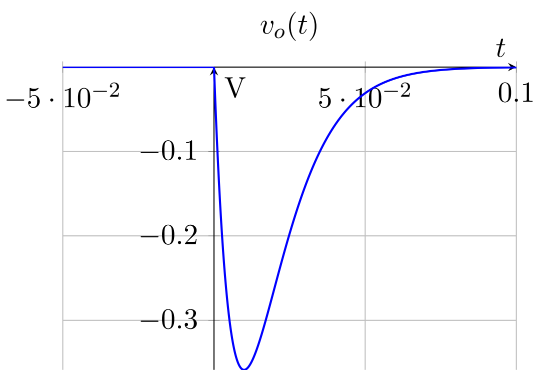
Putting it together

$$v(t) = -8e^{-100t}\sin(700t)\,\text{V}\; t > 0.$$

Let's give a more complete final answer

$$v(t) = \begin{cases} 0, & t < 0 \\ -8e^{-100t}\sin(700t)\,\text{V}, & t \geq 0 \end{cases}$$

It might be helpful to visualize $v(t)$ with a plot.



2. Critical Damping in Series RLC

In the circuit in Fig. 1 the switch closes at $t = 0$. The resistor is adjusted for critical damping. The initial capacitor voltage is 15 V, and the initial inductor current is 6 mA.

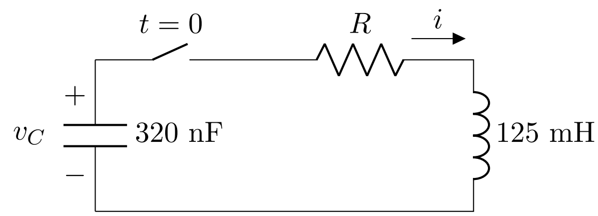


Figure 1: The circuit for problem.

a) Find the numerical value of R.

Solution: For a series RLC circuit, critical damping occurs when $\alpha^2 = \omega_0^2$, or $\alpha = \omega_0$.

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Setting them equal:

$$\frac{R}{2L} = \frac{1}{\sqrt{LC}} \implies R = 2\sqrt{\frac{L}{C}}$$

Substituting the values $L = 125$ mH and $C = 320$ nF:

$$R = 2\sqrt{\frac{125 \times 10^{-3}}{320 \times 10^{-9}}} = 2\sqrt{\frac{125}{320} \times 10^6}$$

$$R = 2 \times 1000 \times \sqrt{\frac{25}{64}} = 2000 \times \frac{5}{8} = 1250\ \Omega$$

b) Find the numerical values of i and $i'(t)$ immediately after the switch is closed.

Solution: The current through the inductor is continuous:

$$i(0^+) = i(0^-) = 6 \text{ mA}$$

To find $di/dt(0^+)$, we use KVL around the loop for $t > 0$. Note that the current i flows clockwise, entering the negative terminal of the capacitor (since v_C is defined positive at the top).

KVL equation (going clockwise, starting from bottom left):

$$-v_C + iR + L\frac{di}{dt} = 0$$
$$L\frac{di}{dt} = v_C - iR$$

At $t = 0^+$:

$$125 \text{ mH} \cdot \frac{di}{dt}(0^+) = 15 \text{ V} - (6 \text{ mA})(1250 \ \Omega)$$

$$0.125 \frac{di}{dt}(0^+) = 15 - 7.5 = 7.5 \text{ V}$$

$$\frac{di}{dt}(0^+) = \frac{7.5}{0.125} = 60 \text{ A s}^{-1}$$

c) Find $v_C(t)$ for $t \geq 0$.

Solution: Since the circuit is critically damped, the response is of the form:

$$v_C(t) = (A_1 + A_2 t)e^{-\alpha t}$$

Calculate α:

$$\alpha = \frac{R}{2L} = \frac{1250}{2(0.125)} = \frac{1250}{0.25} = 5000 \text{ Np s}^{-1}$$

So $v_C(t) = (A_1 + A_2 t)e^{-5000t}$.

Using the initial condition $v_C(0) = 15$:

$$v_C(0) = A_1 = 15$$

To find A_2, we need $dv_C/dt(0^+)$. The current i flows out of the positive terminal of the capacitor (upwards branch current is $-i$). Relationship between capacitor current and voltage: $i_{cap} = C\frac{dv_C}{dt}$. From the diagram, clockwise i enters the negative terminal, so $i = -C\frac{dv_C}{dt}$.

$$i(0^+) = -C\frac{dv_C}{dt}(0^+) \implies \frac{dv_C}{dt}(0^+) = -\frac{i(0^+)}{C}$$

$$\frac{dv_C}{dt}(0^+) = -\frac{6 \times 10^{-3}}{320 \times 10^{-9}} = -18\,750 \text{ V s}^{-1}$$

Differentiating the assumed solution $v_C(t)$:

$$\frac{dv_C}{dt} = A_2 e^{-\alpha t} - \alpha(A_1 + A_2 t)e^{-\alpha t}$$

At $t = 0$:

$$\frac{dv_C}{dt}(0) = A_2 - \alpha A_1$$

$$-18750 = A_2 - 5000(15)$$

$$-18750 = A_2 - 75000$$

$$A_2 = 75000 - 18750 = 56250$$

Thus, the expression for $v_C(t)$ is:

$$v_C(t) = (15 + 56250t)e^{-5000t}\text{V}, \quad t \geq 0$$

3. Step Response of a Series RLC Circuit

The circuit shown in Fig. 2 has been in operation for a long time. At $t = 0$, the source voltage suddenly drops to 150 V. Find $v_o(t)$ for $t \geq 0$.

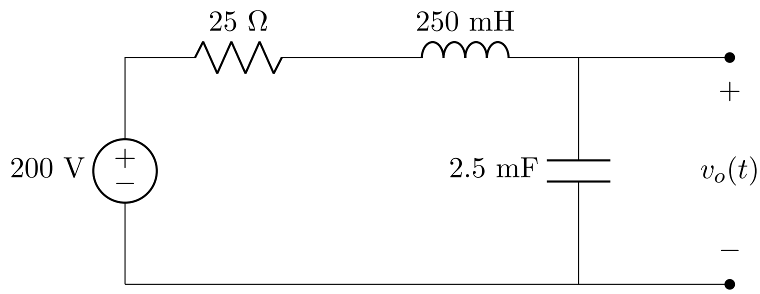


Figure 2: The circuit for problem.

Solution: First, we need to start with the initial conditions. The initial voltage across the capacitor is 200 V since it has been connected to the 200 V source for a long time. The initial current through the inductor is 0 A since the circuit has been in steady state for a long time. Thus, the derivative of the voltage across the capacitor is

$$v_C'(0^+) = \frac{i_L(0^+)}{C} = \frac{0\ \text{A}}{2.5\ \text{mF}} = 0\ \text{V/s}.$$

We can also find the final voltage across the capacitor

$$V_f = 150\ \text{V}.$$

The characteristic roots are

$$S_1 = -20\,\text{rad/s}, \quad S_2 = -80\,\text{rad/s}.$$

Thus, we can write the voltage across the capacitor as

$$v_C(t) = V_f + A_1' e^{-20t} + A_2' e^{-80t}.$$

We can apply initial conditions to get a system of equations

$$\begin{aligned} v_C(0^+) &= 150\ \text{V} + A_1' + A_2' = 200\ \text{V} \\ v_C'(0^+) &= -20A_1' - 80A_2' = 0\ \text{V/s} \end{aligned}$$

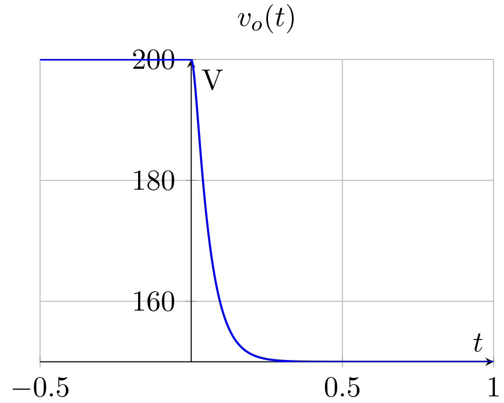
Solving this system of equations gives us

$$A_1' = 66.67\ \text{V}, \quad A_2' = -16.67\ \text{V}.$$

Putting it all together, we have

$$v_o(t) = \begin{cases} v_C(t) = 150\ \text{V} + 66.67\ \text{V}e^{-20t} - 16.67\ \text{V}e^{-80t} & t \geq 0 \\ 200\ \text{V} & t < 0 \end{cases}$$

It might be helpful to visualize $v_o(t)$ with a plot.



$v_o(t)$
200
V
180
160
t
-0.5
0.5
1

4. Multiple Switches in RLC

The two switches in the circuit seen in Fig. 3 operate synchronously. When switch 1 is in position a, switch 2 is closed. When switch 1 is in position b, switch 2 is open. Switch 1 has been in position a for a long time. At $t = 0$, it moves instantaneously to position b. Find $v_c(t)$.

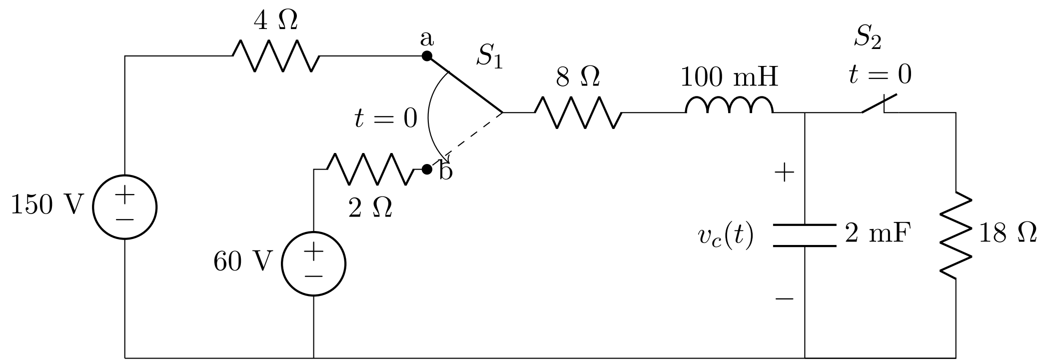


Figure 3: The circuit for problem.

Solution:
Switch 1 is in position a, and Switch 2 is closed. The circuit reaches steady state, so the inductor acts as a short circuit and the capacitor acts as an open circuit.
The active loop consists of the 150 V source (note polarity: negative at top), the 4 Ω resistor, the 8 Ω resistor, and the 18 Ω resistor (since switch 2 is closed).
We can just use Ohm's law to find the current in the loop

$$i_L(0^-) = \frac{V}{R_{total}} = \frac{150\text{ V}}{30\ \Omega} = 5\text{ A}.$$

To find initial voltage, we can just use Ohm's law across the 18 Ω resistor.

$$v_c(0^+) = v_c(0^-) = i_L(0^-) \cdot 18\ \Omega = (5\text{ A})(18\ \Omega) = 90\text{ V}.$$

To find $v'(0^+)$, we can use the fact that the inductor current is continuous and is the same current going through the capacitor

$$v'(0^+) = \frac{i_c(0^+)}{C} = \frac{i_L(0^+)}{C} = \frac{5\text{ A}}{2\text{ mF}} = 2500\text{ V s}^{-1}.$$

After the switches change position at $t = 0$, we see that the final voltage is simply the voltage of the 60 V source, since the capacitor will eventually charge to that voltage.

$$V_f = 60\text{ V}.$$

Next we need the characteristic roots of the natural response.

$$\alpha = \frac{R}{2L} = \frac{10}{2(0.1)} = 50\text{ Np s}^{-1}, \quad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.1)(0.002)}} = \sqrt{5000} \approx 70.7\text{ rad s}^{-1}.$$

Since $\alpha^2 < \omega_0^2$, the response is underdamped.

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{5000 - 2500} = \sqrt{2500} = 50\text{ rad s}^{-1}.$$

The form of the solution is

$$v_c(t) = V_f + e^{-\alpha t}(B_1' \cos(\omega_d t) + B_2' \sin(\omega_d t)).$$

$$v_c(t) = 60 + e^{-50t}(B_1' \cos(50t) + B_2' \sin(50t)).$$

We can find B_1' using the initial condition

$$v_c(0^+) = 60 + B_1' = 90 \text{ V} \implies B_1' = 30 \text{ V}.$$

We can find B_2' using the derivative initial condition

$$v_c'(0^+) = -\alpha B_1' + \omega_d B_2' = 2500 \text{ V s}^{-1} \implies -50(30) + 50B_2' = 2500 \text{ V s}^{-1}$$

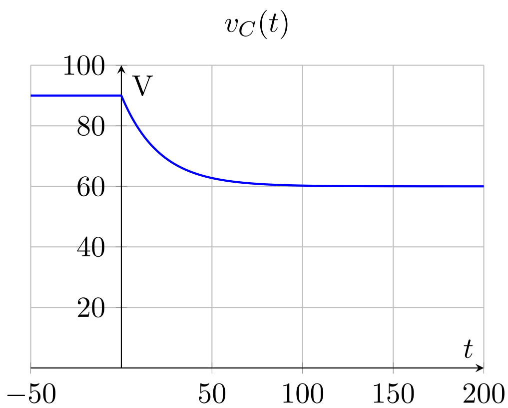
and then

$$B_2' = 80 \text{ V}.$$

Putting it all together, we have

$$v_c(t) = \begin{cases} 90 \text{ V} & t < 0 \\ 60 + e^{-50t}(30\cos(50t) + 80\sin(50t))\text{V} & t \geq 0 \end{cases}.$$

It might be helpful to visualize $v_c(t)$ with a plot.



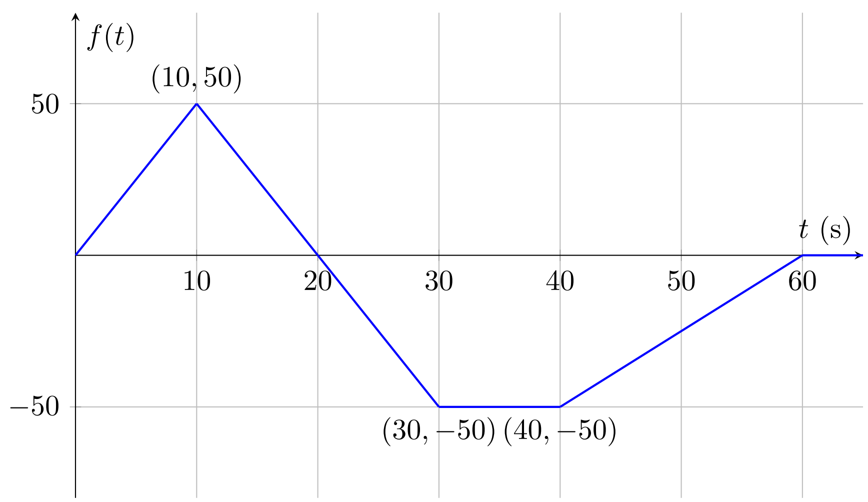
5. Step Functions and Window Functions

Step functions can be used to define a *window function*. Thus $u(t+2) - u(t-3)$ defines a window 1 unit high and 5 units wide located on the time axis between -2 and 3.
A function $f(t)$ is defined as follows:

$$
\begin{aligned}
f(t) &= 0, & t &\le 0 \\
&= 5t, & 0 \le t &\le 10 \text{ s} \\
&= -5t + 100, & 10\text{s} \le t &\le 30 \text{ s} \\
&= -50, & 30\text{s} \le t &\le 40 \text{ s} \\
&= 2.5t - 150, & 40\text{s} \le t &\le 60 \text{ s} \\
&= 0, & 60\text{s} \le t &< \infty
\end{aligned}
$$

a) Sketch $f(t)$ over the interval $0\text{s} \le t \le 60$ s.

Solution:



b) Use the concept of the window function to write an expression for $f(t)$.

Solution: We can construct $f(t)$ by multiplying each segment's function by a window function that is 1 only during that interval.

The intervals are

- $[0, 10]$: $5t[u(t) - u(t-10)]$
- $[10, 30]$: $(-5t + 100)[u(t-10) - u(t-30)]$
- $[30, 40]$: $-50[u(t-30) - u(t-40)]$
- $[40, 60]$: $(2.5t - 150)[u(t-40) - u(t-60)]$

Combining these

$$
\begin{aligned}
f(t) = {} & 5t(u(t) - u(t-10)) \\
& + (-5t + 100)(u(t-10) - u(t-30)) \\
& - 50(u(t-30) - u(t-40)) \\
& + (2.5t - 150)(u(t-40) - u(t-60))
\end{aligned}
$$

6. Step Function Expressions

Use step functions to write the expression for each function shown in Fig. 4.

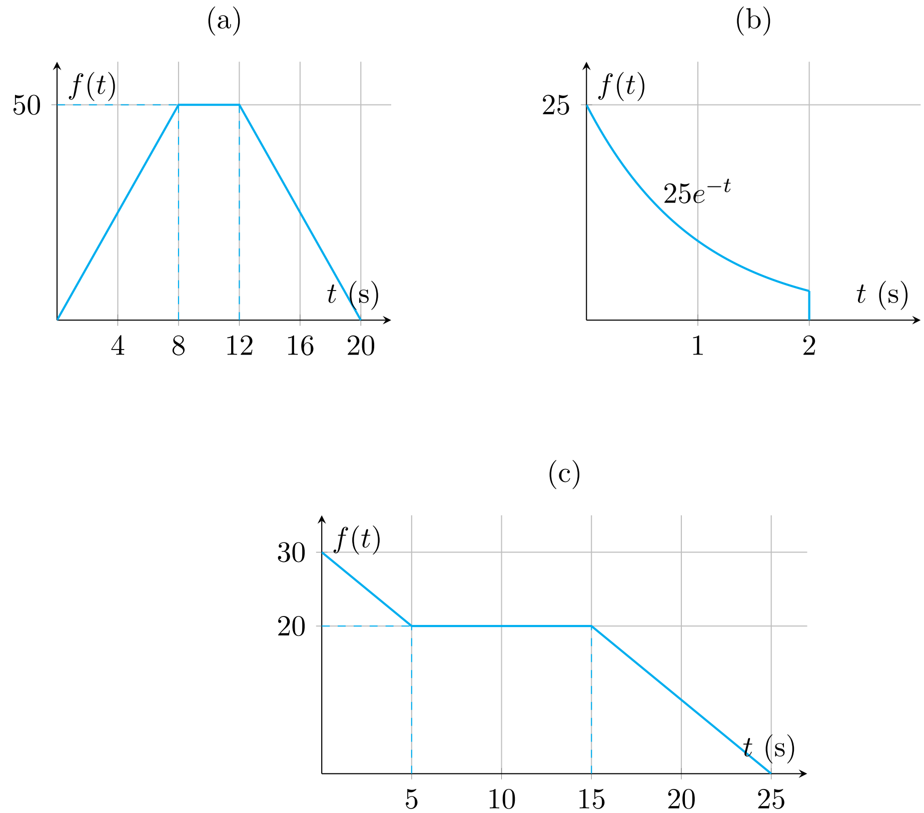


Figure 4: Functions for problem.

a) **Solution:** This function is trapezoidal. We can define it clearly using the singularity function method (ramp functions $r(t) = tu(t)$) by observing changes in slope.

- At $t = 0$, slope changes from 0 to $\frac{50}{8} = 6.25$. Term: $6.25t \cdot u(t)$.
- At $t = 8$, slope changes from 6.25 to 0. Change is -6.25. Term: $-6.25(t-8)u(t-8)$.
- At $t = 12$, slope changes from 0 to $\frac{0-50}{20-12} = -6.25$. Change is -6.25. Term: $-6.25(t-12)u(t-12)$.
- At $t = 20$, slope changes from -6.25 to 0. Change is $+6.25$. Term: $+6.25(t-20)u(t-20)$.

Gated step function form:

$$f(t) = 6.25t[u(t) - u(t-8)] + 50[u(t-8) - u(t-12)] + (-6.25t + 125)[u(t-12) - u(t-20)]$$

b) **Solution:** The function is defined as $25e^{-t}$ for the interval $0 \leq t \leq 2$ and is zero elsewhere. Using the window property of step functions:

$$f(t) = 25e^{-t}[u(t) - u(t-2)]$$

c) **Solution:** This function has an initial step and then linear segments.

- At $t = 0$, the value jumps to 30. Step term: $30u(t)$.
- Initial slope (0 to 5): $\frac{20-30}{5} = -2$. Ramp term: $-2t \cdot u(t)$.
- At $t = 5$, slope changes from -2 to 0. Change is $+2$. Term: $+2(t-5)u(t-5)$.
- At $t = 15$, slope changes from 0 to $\frac{0-20}{25-15} = -2$. Change is -2. Term: $-2(t-15)u(t-15)$.
- At $t = 25$, slope changes from -2 to 0. Change is $+2$. Term: $+2(t-25)u(t-25)$.

Gated step function form:

$$f(t) = (-2t+30)[u(t) - u(t-5)] + 20[u(t-5) - u(t-15)] + (-2t+50)[u(t-15) - u(t-25)]$$

7. Integrals with Impulses

Evaluate the following integrals:

a) $I = \int_{-1}^{3} (t^3 + 2)[\delta(t) + 8\delta(t-1)]dt.$

Solution: Use the sifting property of the impulse function:

$$\int_{a}^{b} f(t)\delta(t - t_0)dt = \begin{cases} f(t_0) & a < t_0 < b \\ 0 & \text{otherwise} \end{cases}$$

Splitting the integral:

$$I = \int_{-1}^{3} (t^3 + 2)\delta(t)dt + \int_{-1}^{3} (t^3 + 2)8\delta(t-1)dt$$

First term ($t_0 = 0$, which is in $[-1, 3]$):

$$f(0) = 0^3 + 2 = 2$$

Second term ($t_0 = 1$, which is in $[-1, 3]$):

$$8f(1) = 8(1^3 + 2) = 8(3) = 24$$

Total:

$$I = 2 + 24 = 26$$

b) $I = \int_{-2}^{2} t^2[\delta(t) + \delta(t + 1.5) + \delta(t-3)]dt.$

Solution: Split the integral into three parts:

$$I = \int_{-2}^{2} t^2\delta(t)dt + \int_{-2}^{2} t^2\delta(t + 1.5)dt + \int_{-2}^{2} t^2\delta(t-3)dt$$

1. For $\delta(t)$, $t_0 = 0$. Since $-2 < 0 < 2$, this term contributes $0^2 = 0$.
2. For $\delta(t + 1.5)$, $t_0 = -1.5$. Since $-2 < -1.5 < 2$, this term contributes $(-1.5)^2 = 2.25$.
3. For $\delta(t-3)$, $t_0 = 3$. Since 3 is outside the interval $[-2, 2]$, this integral is zero.

Total:

$$I = 0 + 2.25 + 0 = 2.25$$