ECE 2260 hw03

1. Natural Response of an RL Circuit with a Switch

The switch in the circuit shown in Fig. 1 has been in position a for a long time before moving to position b at $t = 0$.

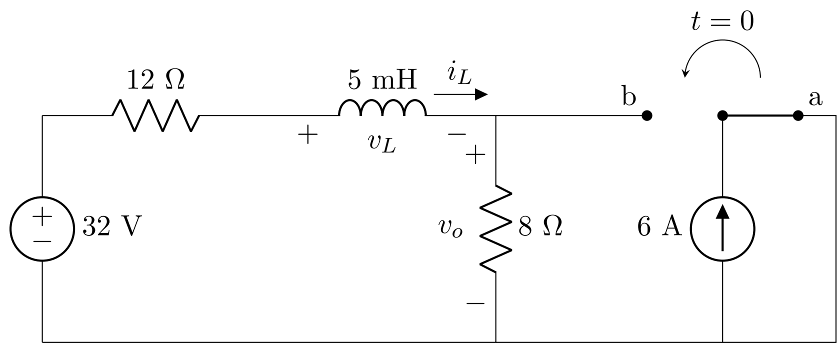

Figure 1: Circuit for problem.

a) Find the numerical expressions for $i_L(t)$ and $v_o(t)$ for $t \geq 0$.

Solution: First, we need to find the initial conditions at $t = 0^-$. Since the switch has been in position a for a long time, the inductor current $i_L(0^-)$ can be found by analyzing the circuit in steady state. This gives us $i_L(0^-) = 1.6$ A.

Next, after the switch moves to position b at $t = 0$, we can analyze the circuit for $t \to \infty$. In steady state, the inductor behaves like a short circuit, so we can find $i_L(\infty) = -0.8$ A.

We need to find the time constant τ of the circuit. The equivalent resistance seen by the inductor when the switch is in position b is $R_{eq} = 20\ \Omega$. Therefore, the time constant is $\tau = \frac{L}{R_{eq}} = \frac{5\text{ mH}}{20\ \Omega} = 250\ \mu\text{s}$.

Putting it all together, we can write the expression for $i_L(t)$

$$i_L(t) = i_L(\infty) + \left(i_L(0^-) - i_L(\infty)\right) e^{-\frac{t}{\tau}} = -0.8 + (1.6 + 0.8)e^{-\frac{t}{250\ \mu\text{s}}} = -0.8 + 2.4e^{-\frac{t}{250\ \mu\text{s}}} \quad \text{A}$$

To find $v_o(t)$, we can use Ohm's law across the resistor $R_o = 8\ \Omega$

$$v_o(t) = R_o \cdot i_L(t) + 48\text{ V} = 8\left(-0.8 + 2.4e^{-\frac{t}{250\ \mu\text{s}}}\right) + 48\text{ V} = 41.6 + 19.2e^{-\frac{t}{250\ \mu\text{s}}} \quad \text{V}$$

b) Find the numerical values of $v_L(0^+)$ and $v_o(0^+)$.

Solution: At $t = 0^+$, the inductor current $i_L(0^+)$ is the same as $i_L(0^-)$ due to the inductor's property of current continuity. Therefore, $i_L(0^+) = 1.6$ A. To find $v_L(0^+)$, we can use the inductor voltage-current relationship:

$$v_L(0^+) = L\frac{di_L}{dt}\bigg|_{t=0^+}$$

We can find $\frac{di_L}{dt}$ at $t = 0^+$ by differentiating $i_L(t)$:

$$\frac{di_L}{dt} = -\frac{2.4}{250\ \mu\text{s}}e^{-\frac{t}{250\ \mu\text{s}}} \quad \Rightarrow \quad \frac{di_L}{dt}\bigg|_{t=0^+} = -\frac{2.4}{250\ \mu\text{s}} = -9600 \quad \text{A}\,\text{s}^{-1}$$

Therefore,

$$v_L(0^+) = 5\ \text{mH} \cdot (-9600) = -48 \quad \text{V}$$

To find $v_o(0^+)$, we can substitute $t = 0$ into the expression for $v_o(t)$:

$$v_o(0^+) = 41.6 + 19.2e^0 = 41.6 + 19.2 = 60.8 \quad \text{V}$$

2. Step Response of an RL Circuit

The switch in the circuit shown in Fig. 2 has been in position a for a long time. At $t = 0$, the switch moves instantaneously to position b.

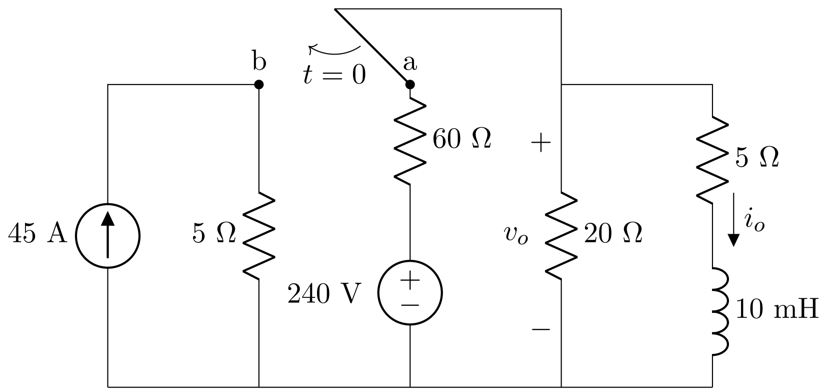

Figure 2: Circuit for step response problem.

a) Find the numerical expression for $i_o(t)$ when $t \geq 0$.

Solution: First, we need to find $i_o(0^-)$, which is the current through the inductor just before the switch moves. This is $i_o(0^-) = 3$ A.

Next, we need to find the Thevenin equivalent circuit seen by the inductor branch after the switch moves to position b at $t = 0$. This will give us $V_{th} = 180$ V and $R_{th} = 9\ \Omega$. This will give us a time constant $\tau = \frac{L}{R_{th}} = \frac{10\text{ mH}}{9\ \Omega} = \frac{1}{900}$ s.

We need the final value of the current $i_o(\infty) = \frac{V_{th}}{R_{th}} = \frac{180}{9} = 20$ A.

Putting this all together, we can write the expression for $i_o(t)$

$$i_o(t) = i_o(\infty) + [i_o(0^-) - i_o(\infty)]e^{-t/\tau} = 20 + (3 - 20)e^{-900t} = 20 - 17e^{-900t} \quad \text{A}, \quad t \geq 0.$$

b) Find the numerical expression for $v_o(t)$ for $t \geq 0^+$.

Solution: The voltage $v_o(t)$ is the voltage across the 20 Ω resistor. However, looking at this, we see this is also the voltage over the 5 Ω resistor, which makes them in parallel with an equivalent resistance of 4 Ω. We know current in this equivalent branch is

$$i_{eq}(t) = 45\text{ A} - i_o(t) = 45 - (20 - 17e^{-900t}) = 25 + 17e^{-900t}\text{ A}.$$

Therefore, the voltage across this equivalent resistance is

$$v_{eq}(t) = 4\ \Omega i_{eq}(t) = 4(25 + 17e^{-900t}) = 100 + 68e^{-900t}\text{ V}, t \geq 0.$$

c) Now assume that the switch has been in position b for a long time before moving to position a at $t = 0$. Repeat parts (a) and (b).

Solution: First determine the initial conditions at $t = 0^-$. Since the switch has been in position b for a long time, the inductor current $i_L(0^-)$ can be found by analyzing the circuit in steady state. This gives us $i_L(0^-) = 20$ A.
Next, after the switch moves to position a at $t = 0$, we can analyze the circuit for $t \to \infty$. In steady state, the inductor behaves like a short circuit, so we can find $i_L(\infty) = 3$ A.
We need to find the new time constant by finding the new Thevenin equivalent circuit which would be seen by the inductor when the switch is in position a. This will give us $R_{eq} = 20\ \Omega$, so the time constant is $\tau = \frac{L}{R_{eq}} = \frac{10\ \text{mH}}{20\ \Omega} = 500\ \mu\text{s}$.
Putting it all together, we can write the expression for $i_L(t)$

$$i_L(t) = i_L(\infty) + \left(i_L(0^-) - i_L(\infty)\right) e^{-\frac{t}{\tau}} = 3 + (20-3)e^{-\frac{t}{500\ \mu\text{s}}} = 3 + 17e^{-\frac{t}{500\ \mu\text{s}}} \quad \text{A}$$

To find $v_o(t)$, we can use the same approach. Let's do a source transformation on the 60 V and 240 V branch such that the current source would be 4 A in parallel with a 60 Ω resistor. We notice the 60 Ω resistor is in parallel with the 20 Ω resistor, giving an equivalent resistance of 15 Ω. The current through this equivalent resistance is

$$i_{eq}(t) = 4\ \text{A} - i_o(t) = 4 - (3 + 17e^{-\frac{t}{500\ \mu\text{s}}}) = 1 - 17e^{-\frac{t}{500\ \mu\text{s}}}\ \text{A}.$$

Therefore, the voltage across this equivalent resistance is

$$v_o(t) = 15\ \Omega \cdot i_{eq}(t) = 15(1 - 17e^{-\frac{t}{500\ \mu\text{s}}}) = 15 - 255e^{-\frac{t}{500\ \mu\text{s}}}\ \text{V}, \quad t \geq 0.$$

3. Step Response with Dependent Source

The switch in the circuit in Fig. 3 has been open a long time before closing at $t = 0$. Find $i_o(t)$ for $t \geq 0$.

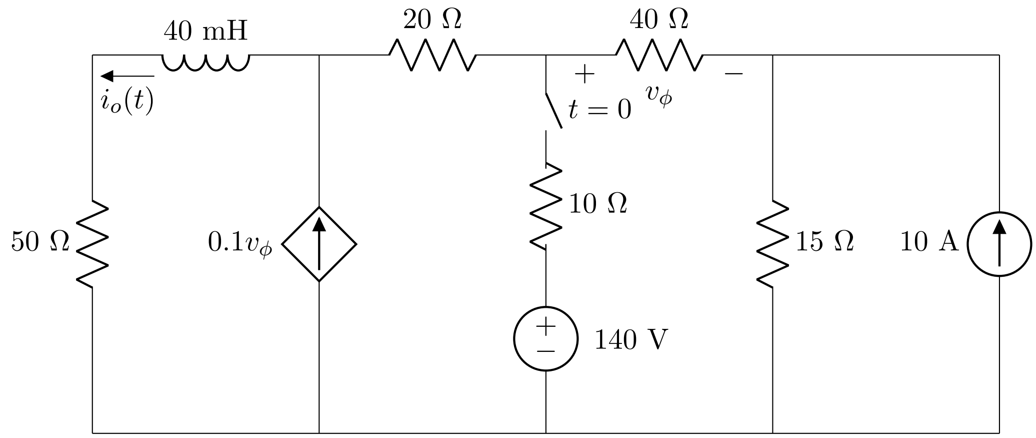

Figure 3: Circuit for problem.

Solution: First, we need to find $i_o(0^-)$, which is the current through the inductor just before the switch closes. Since the switch has been open for a long time, there is no current through the branch with the switch, and the inductor current is determined by the rest of the circuit. Thus we see that $i_o(0^-) = 6$ A.

After the switch closes, we need to find the current as $t \to \infty$. In this case, the inductor acts as a short circuit, so we can analyze the resulting resistive circuit to find $i_o(\infty)$. Using mesh analysis, we find that $i_o(\infty) = 0.774$ A.

We next need to find the Thevenin equivalent circuit seen by the dependent current source and inductor branch. This gives $R_{th} = 74\ \Omega$. However, we cannot forget the additional 50 Ω resistor in series from the leftmost branch. Thus the total resistance seen by the inductor is $R = R_{th} + 50\ \Omega = 124\ \Omega$. This gives us a time constant of $\tau = \frac{L}{R} = \frac{40\text{ mH}}{124\ \Omega} = 0.00032258\text{ s} = 322.6\ \mu\text{s}$.

Finally, we can write the expression for $i_o(t)$

$$i_o(t) = i_o(\infty) + [i_o(0^-) - i_o(\infty)]e^{-t/\tau} = 0.774 + (6 - 0.774)e^{-3100t} = 0.774 + 5.226e^{-3100t} \quad \text{A}, \quad t \geq 0.$$

4. Step Response of an RL Circuit with Multiple Sources

The switch in the circuit in Fig. 4 has been open a long time before closing at $t = 0$. Find $v_o(t)$ for $t \geq 0^+$.

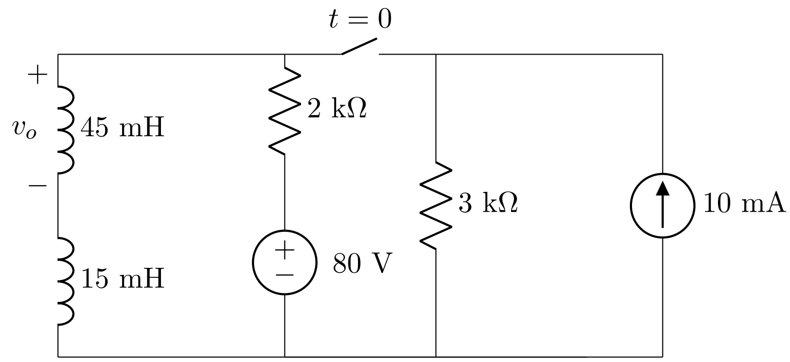

Figure 4: Circuit for problem.

Solution: Because this is an inductor problem, let's find the current through the inductor branch, which we will call $i_L(t)$ and then use it to find $v_o(t)$.

First, find the initial condition at $t = 0^-$. This is $i_L(0^-) = 40$ mA. Next find the current at steay state after the switch closes, $i_L(\infty) = 50$ mA. We will need the Thevenin equivalent circuit seen by the inductor branch to find the time constant. This gives $R_{th} = 1.2$ kΩ. The total inductance is $L_{total} = 60$ mH. Thus the time constant is $\tau = \frac{L_{total}}{R_{th}} = \frac{60 \text{ mH}}{1.2 \text{ k}\Omega} = 50$ µs.

We can put it together to find the expression for $i_L(t)$

$$i_L(t) = i_L(\infty) + [i_L(0^-) - i_L(\infty)]e^{-t/\tau} = 50 + (40 - 50)e^{-t/50 \text{ µs}} = 50 - 10e^{-20000t} \quad \text{mA}, \quad t \geq 0.$$

Now we can find $v_o(t)$, which is the voltage across the 45 mH inductor.

$$v_o(t) = L_1 \frac{di_L}{dt}$$

where $L_1 = 45$ mH $= 0.045$ H. Differentiating $i_L(t)$

$$\frac{di_L}{dt} = \frac{d}{dt}(50 - 10e^{-20000t}) = -10(-20000)e^{-20000t} = 200000e^{-20000t} \text{ mA/s}.$$

Calculating voltage

$$v_o(t) = 0.045 \times (200000e^{-20000t}) = 9000e^{-20000t} \text{ mV} = 9e^{-20000t} \text{ V}.$$

5. RC Circuit with Switch and Multiple Sources

Assume that the switch in the circuit of Fig. 5 has been in position a for a long time and that at $t = 0$ it is moved to position b.

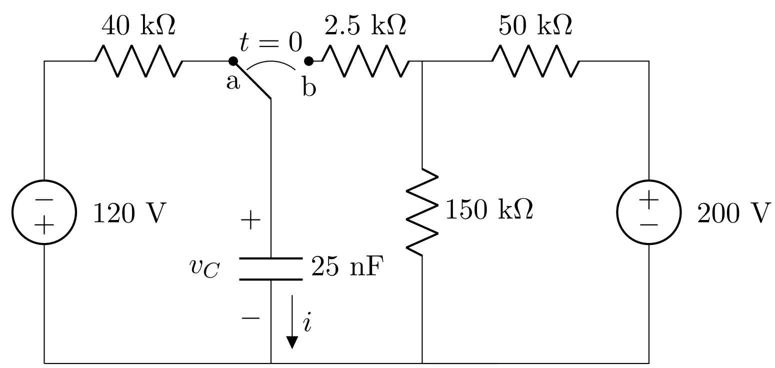

Figure 5: Circuit for problem.

a) Find $v_C(t)$ where $t \geq 0$.

Solution: First, we need to find the initial condition $v_C(0^+)$, which is the voltage across the capacitor just after the switch moves. Since the switch has been in position a for a long time, the capacitor voltage is determined by the left-hand side of the circuit. We see that $v_C(0^+) = -120$ V.
Next, we need to find the final value of the capacitor voltage after a long time in position b, $v_C(\infty)$. In this case, the capacitor acts as an open circuit, so we can analyze the resulting resistive circuit to find $v_C(\infty)$. Using voltage division, we find that $v_C(\infty) = 150$ V.
We next need to find the Thevenin equivalent circuit seen by the capacitor branch. This gives $R_{th} = 40$ kΩ. This gives us a time constant of $\tau = R_{th}C = 40\text{ k}\Omega \times 25\text{ nF} = 1$ ms.
Finally, we can write the expression for $v_C(t)$

$$v_C(t) = v_C(\infty)+[v_C(0^+)-v_C(\infty)]e^{-t/\tau} = 150+(-120-150)e^{-t/1\text{ ms}} = 150-270e^{-1000t}\text{ V},\quad t \geq 0.$$

b) Next find the expression for $i(t)$ where $t \geq 0^+$.

Solution: The current $i(t)$ is the current through the capacitor branch.

$$i(t) = C\frac{dv_C}{dt}$$

Differentiating $v_C(t)$

$$\frac{dv_C}{dt} = \frac{d}{dt}(150 - 270e^{-1000t}) = 0 - 270(-1000)e^{-1000t} = 270000e^{-1000t}\text{ V/s}.$$

Calculating current

$$i(t) = (25 \times 10^{-9}) \times (270000e^{-1000t}) = 6.75 \times 10^{-3}e^{-1000t}\text{ A} = 6.75e^{-1000t}\text{ mA},\quad t \geq 0^+.$$

6. Step Response with Dependent Voltage Source

The switch in the circuit shown in Fig. 6 has been in the OFF position for a long time. At $t = 0$, the switch moves instantaneously to the ON position. Find $v_o(t)$ for $t \geq 0$.

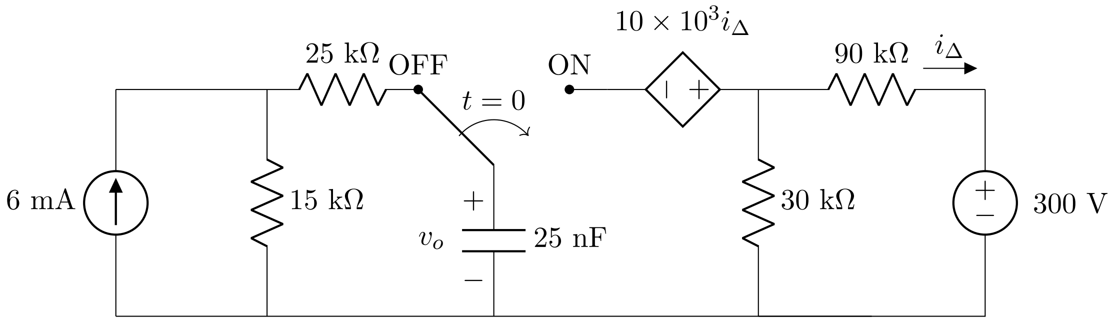

Figure 6: Circuit for problem.

Solution: First, we need to find the voltage v_o when the circuit is in the OFF position. This is The switch is in the OFF position. The capacitor acts as an open circuit in steady state. The left part of the circuit consists of a 6 mA current source in parallel with a 15 kΩ resistor. The 25 kΩ resistor connects this pair to the capacitor. Since the capacitor is open ($i_C = 0$), no current flows through the 25 kΩ resistor. Therefore, the voltage across the capacitor is equal to the voltage across the 15 kΩ resistor.

$$v_o(0^-) = (6 \text{ mA})(15 \text{ k}\Omega) = 90 \text{ V}.$$

Since capacitor voltage is continuous:

$$v_o(0) = 90 \text{ V}.$$

At $t > 0$, we need to find $v_o(\infty)$, the final voltage across the capacitor after a long time in the ON position. When the switch is ON, the capacitor connects to the right-hand network. We see that i_Δ is the current through the 90 kΩ resistor, which we can find using a KVL, on the right-hand loop. Using this, we can do another KVL on the left-hand loop to find $v_o(\infty) = 100$ V.

Next we can find the equivalent resistance seen by the capacitor to find the time constant τ. This gives $R_{Th} = 20$ kΩ. Thus the time constant is

$$\tau = R_{Th}C = 20 \text{ k}\Omega \times 25 \text{ nF} = 0.5 \text{ ms}.$$

Finally, we can write the expression for $v_o(t)$

$$v_o(t) = v_o(\infty) + [v_o(0^+) - v_o(\infty)]e^{-t/\tau} = 100 + (90 - 100)e^{-t/0.5 \text{ ms}} = 100 - 10e^{-2000t} \text{ V}, \quad t \geq 0.$$

7. Sequential Switching in RL Circuit

The action of the two switches in the circuit seen in Fig. 7 is as follows. For $t < 0$, switch 1 is in position a and switch 2 is open. This state has existed for a long time. At $t = 0$, switch 1 moves instantaneously from position a to position b, while switch 2 remains open. Ten milliseconds after switch 1 operates, switch 2 closes, remains closed for 10 ms and then opens. Find $v_o(t)$ 25 ms after switch 1 moves to position b.

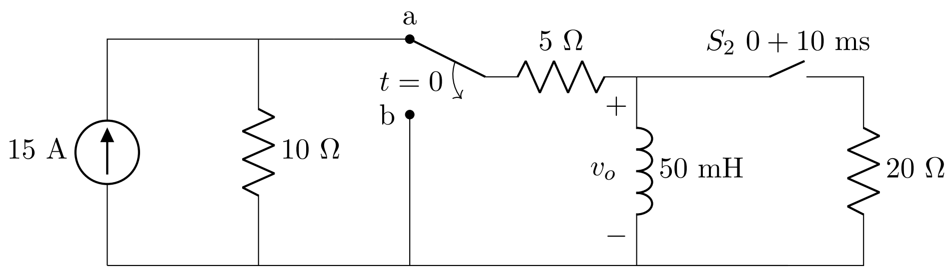

Figure 7: Circuit for problem.

Solution: First, we need to find the current through the inductor $i_L(0^-)$. Since the switch has been in position a for a long time, the circuit is in steady state. The inductor acts as a short circuit. We can use current division to find the current through the inductor branch. This gives $i_L(0^-) = 10$ A.

Next, we need to analyze the circuit when $0 \leq t \leq 10$ ms. In this case, switch 1 is in position b and switch 2 is open. The circuit consists of the inductor and the 5 Ω resistor in series. The time constant for this circuit is $\tau_1 = \frac{L}{R} = \frac{50\text{ mH}}{5\ \Omega} = 10$ ms. The current through the inductor decays exponentially from its initial value:

$$i_L(t) = 10e^{-t/0.01} \quad \text{A} \quad (0 \leq t \leq 0.01).$$

At $t = 10$ ms, the current is:

$$i_L(0.01) = 10e^{-1} \approx 3.679 \text{ A}.$$

Next we need to analyze the circuit when 10 ms $\leq t \leq$ 20 ms. In this case, switch 2 is closed, placing the 20 Ω resistor in parallel with the inductor. The equivalent resistance seen by the inductor is

$$R_{eq2} = 5\ \Omega \parallel 20\ \Omega = 4\ \Omega.$$

The new time constant is $\tau_2 = \frac{L}{R_{eq2}} = \frac{50\text{ mH}}{4\ \Omega} = 12.5$ ms. The current through the inductor decays exponentially from its value at $t = 10$ ms

$$i_L(t) = i_L(10\text{ ms})e^{-100t} = 1.65e^{-100t} \quad 20\text{ ms} \leq t < \infty.$$

To find the voltage $v_o(t)$ at $t = 25$ ms, we first find an expression for the voltage

$$v_o(t) = L\frac{di_L}{dt}.$$

Differentiating $i_L(t)$ for 10 ms $\leq t < \infty$

$$\frac{di_L}{dt} = 1.65 \times (-100)e^{-100t} = -165e^{-100t} \text{ A/s}.$$

Calculating $v_o(t)$

$$v_o(t) = 50\text{ mH} \times (-165e^{-100t}) = -8.25e^{-100t} \text{ V}, \quad t \geq 10\text{ ms}.$$

Finally, evaluating at $t = 25$ ms

$$v_o(25\text{ ms}) = -8.25e^{-100\times 0.025-0.02} = -8.25e^{-0.5} \approx -5.013 \text{ V}.$$

8. Sequential Switching with Capacitor

There is no energy stored in the capacitor (i.e., both switches are open) in the circuit in Fig. 8 when switch 1 closes at $t = 0$. Switch 2 closes 2.5 ms later. Find $v_o(t)$ for $t \geq 0$.

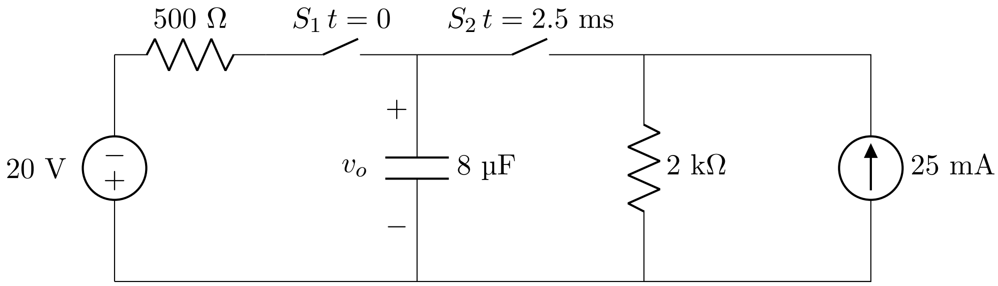

Figure 8: Circuit for problem.

Solution: Initial State ($t < 0$) Both switches are open. The capacitor has no initial energy, so $v_o(0) = 0$ V.

Interval 1: $0 \leq t < 2.5$ ms

We see at $t = 0$, Switch 1 closes. The right-hand network is disconnected because Switch 2 is still open. We see that if Switch 2 remained open, the steady-state voltage would be $v_0(\infty) = -20$ V. **Case:** $0 \leq t < 2.5$ ms. We need to find the Thevenin equivalent of the left-hand side of the circuit connected. This is trivial since there is only a single voltage source and resistor. Thus $R_{th} = 500\ \Omega$. The time constant for this interval is

$$\tau_1 = R_{th}C = 500 \times 8 \times 10^{-6} = 0.004 \text{ s} = 4 \text{ ms}.$$

The inverse time constant is $1/\tau_1 = 1/0.004 = 250\ \text{s}^{-1}$. Putting this together, we have for $0 \leq t < 2.5$ ms

$$v_o(t) = v_o(\infty) + (v_o(0) - v_o(\infty))e^{-t/\tau_1} = -20 + (0 - (-20))e^{-250t} = -20 + 20e^{-250t} \text{ V}.$$

Interval 2: $t \geq 2.5$ ms
At $t = 2.5$ ms, we can find the voltage just before Switch 2 closes:

$$v_o(2.5 \text{ ms}) = -20 + 20e^{-250 \times 0.0025} = -20 + 20e^{-0.625} = -9.295 \text{ V}.$$

Taken to steady-state, the voltage would be $v_o(\infty) = -6$ V if Switch 2 remained open. We also need the time constant τ_2. The Thevenin resistance is $R_{th} = 400\ \Omega$. The time constant is then $\tau_2 = 400\ \Omega \times 8\ \mu\text{F} = 3.2$ ms. The inverse time constant is $1/\tau_2 = 1/0.0032 = 312.5\ \text{s}^{-1}$. Putting this together, we have for 2.5 ms $< t$

$$v_o(t) = -6 + (-9.295 - (-6))e^{-312.5(t-0.0025)} = -6 - 3.295e^{-312.5(t-0.0025)} \text{ V}.$$

Final Answer: We can put each of the cases together to get the final answer:

$$v_o(t) = \begin{cases} -20 + 20e^{-250t} \text{ V}, & 0 \leq t < 2.5 \text{ ms} \\ -6 - 3.295e^{-312.5(t-0.0025)} \text{ V}, & t \geq 2.5 \text{ ms} \end{cases}.$$

9. Unstable RL Circuit with Dependent Source

The inductor current in the circuit in Fig. 9 is 25 mA at the instant the switch is opened. The inductor will malfunction whenever the magnitude of the inductor current equals or exceeds 5 A. How long after the switch is opened does the inductor malfunction?

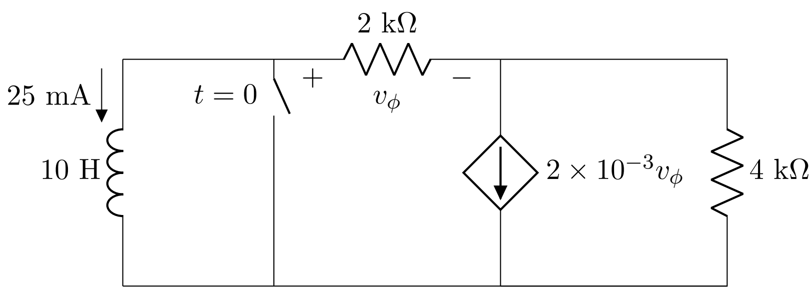

Figure 9: Circuit for problem.

Solution: Step 1: Equivalent Circuit Analysis We determine the Thévenin equivalent resistance seen by the inductor terminals after the switch is opened ($t > 0$). We remove the inductor and apply a test current i_t flowing *out* of the top terminal of the inductor, into the rest of the circuit.
The test current flows through the 2 kΩ resistor.

$$v_\phi = i_t(2000\ \Omega).$$

At the node above the dependent source, apply KCL. The current entering from the left is i_t. The current leaving downwards through the dependent source is $2 \times 10^{-3} v_\phi$. The current leaving to the right through the 4 kΩ resistor is i_R.

$$i_t = 2 \times 10^{-3} v_\phi + i_R$$

Substitute $v_\phi = 2000 i_t$:

$$i_t = (2 \times 10^{-3})(2000 i_t) + i_R$$

$$i_t = 4 i_t + i_R \implies i_R = -3 i_t.$$

The voltage across the parallel branch (dependent source and 4 kΩ resistor) is:

$$v_{right} = i_R(4000\ \Omega) = (-3 i_t)(4000) = -12\,000 i_t.$$

The total voltage v_t across the terminals (looking in) is the sum of the voltage across the 2 kΩ resistor and v_{right}:

$$v_t = v_\phi + v_{right} = 2000 i_t - 12000 i_t = -10000 i_t.$$

The equivalent resistance is:

$$R_{eq} = \frac{v_t}{i_t} = -10\ \text{k}\Omega.$$

Step 2: Differential Equation The circuit equation for the inductor connected to this equivalent resistance is:

$$L\frac{di}{dt} + R_{eq} i = 0$$

Note: The current i here is the inductor current. Based on the passive sign convention relative to the resistance calculation, if i flows out of the inductor into the resistance, the voltage drop is iR_{eq}. $v_L = -v_{terminals}$.

$$10\frac{di}{dt} + (-10000) i = 0$$

$$\frac{di}{dt} = 1000i.$$

The solution to this first-order differential equation is:

$$i(t) = i(0)e^{1000t}.$$

Given $i(0) = 25$ mA $= 0.025$ A:

$$i(t) = 0.025e^{1000t} \text{ A}.$$

The positive exponent confirms that the current will grow (unstable circuit due to negative resistance).
Step 3: Calculate Time to Malfunction We need to find t when $i(t) = 5$ A.

$$5 = 0.025e^{1000t}$$

$$e^{1000t} = \frac{5}{0.025} = \frac{5000}{25} = 200.$$

Taking the natural logarithm:

$$1000t = \ln(200).$$

$$t = \frac{\ln(200)}{1000} \approx \frac{5.2983}{1000} \text{ s}.$$

$$t \approx 5.3 \text{ ms}.$$

The inductor will malfunction after 5.3 ms.